Quantum field theories of relativistic Luttinger fermions

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We propose relativistic Luttinger fermions as a new ingredient for the construction of fundamental quantum field theories. We construct the corresponding Clifford algebra and the spin metric for relativistic invariance of the action using the spin-base invariant formalism. The corresponding minimal spinor has 32 complex components, matching with the degrees of freedom of a standard-model generation including a right-handed neutrino. The resulting fermion fields exhibit a canonical scaling different from Dirac fermions and thus support the construction of novel relativistic and perturbatively renormalizable, interacting quantum field theories. In particular, new asymptotically free self-interacting field theories can be constructed, representing first examples of high-energy complete quantum field theories based on pure matter degrees of freedom. Gauge theories with relativistic Luttinger fermions exhibit a strong paramagnetic dominance, requiring large non-Abelian gauge groups to maintain asymptotic freedom. We comment on the possibility to use Luttinger fermions for particle physics model building and the expected naturalness properties of such models.

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I. INTRODUCTION

One of the remarkable features of quantum field theories is given by the interconnection of fields as representations of the Lorentz group [1,2], their powercounting dimensionality and the renormalizability of interacting field theories in $d = 3 + 1$-dimensional spacetime [3–5]. These interconnections are particularly obvious in the standard model containing spin $0, \frac{1}{2}, 1$ fields and accommodating all possible renormalizable interactions allowed by the symmetries.

In addition to the experimental searches for further degrees of freedom, theoretical studies of consistent quantum field theories have paved the way for new concepts, e.g., with a consistent quantization of interacting spin $\frac{3}{2}$ particles requiring supersymmetry.

In the present paper, we explore the possibility to construct relativistic versions of Luttinger fermions and perform first perturbative studies of corresponding interacting quantum field theories. This type of fermions has been discovered by Luttinger while searching for the most general form of a nonrelativistic Hamiltonian of a semiconductor excitation in a magnetic field [6]. In recent solid-state research, these nonrelativistic degrees of freedom find extensive application in spin-orbit coupled materials with quadratic band touching/crossing points (e.g., inverted band gap semiconductors, pyrochlore iridates) [7–9]; such systems can give rise to interesting quantum critical phenomena [10–19]. Also gauged versions have been studied recently in the context of quantum spin liquids [20].

While the generalization of the underlying algebra to the relativistic case is, in principle, straightforward, we find that the construction of a fully relativistic action requires a reducible representation in terms of the related Dirac algebra with interesting consequences for the construction of interacting quantum field theories. From the viewpoint of the propagator pole structure, the theories exhibit typical features of a higher-derivative theory [21–24], whereas the ultraviolet (UV) properties of loop integrals resembles that of standard scalar field theories with the decisive difference that self-interacting theories can be asymptotically free.

II. RELATIVISTIC LUTTINGER FERMIONS

We propose the kinetic action for a relativistic theory with Luttinger fermions in d dimensional spacetime to be of the form

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$$S = \int d^d x [\bar{\psi} G_{\mu\nu} (i\partial^\mu)(i\partial^\nu)\psi], \tag{1}$$

where ψ denotes the Grassmann-valued spinor, $\bar{\psi}$ its conjugate to be constructed, and $G_{\mu\nu}$ is a $d_\gamma \times d_\gamma$ dimensional matrix for each fixed pair of Lorentz indices $\mu, \nu = 0, \ldots, (d-1)$. In order to remove the Lorentz-reducible part proportional to a trivial Klein-Gordon operator, the Luttinger matrices $G_{\mu\nu}$ are Lorentz traceless, $G^\mu{}_\mu = g^{\mu\nu} G_{\mu\nu} = 0$. Also, they are Lorentz symmetric $G_{\mu\nu} = G_{\nu\mu}$ as is obvious from Eq. (1), satisfying the anticommuting algebra

$$\{G_{\mu\nu}, G_{\kappa\lambda}\} = -\frac{2}{d-1} g_{\mu\nu} g_{\kappa\lambda} + \frac{d}{d-1}(g_{\mu\kappa} g_{\nu\lambda} + g_{\mu\lambda} g_{\nu\kappa}), \tag{2}$$

generalizing the Abrikosov algebra for the spatial Euclidean [11,25] to the Lorentzian case. The tensor structure of the right-hand side is fixed by Lorentz and index symmetries. The prefactors follow from the Lorentz tracelessness of $G_{\mu\nu}$ and from the requirement that the Luttinger operator should square to the square of the Klein-Gordon operator,

$$G_{\mu\nu}(i\partial^\mu)(i\partial^\nu) G_{\kappa\lambda}(i\partial^\kappa)(i\partial^\lambda) = (\partial^2)^2. \tag{3}$$

With $G_{\mu\nu}$ being symmetric in μ, ν and traceless, we need at least $d_\mathrm{e} = \frac{1}{2} d(d+1) - 1$ linearly independent elements to span the algebra (2). Since Eq. (2) defines a Clifford algebra, the dimension d_γ must at least be that of the irreducible representation $d_{\gamma,\mathrm{irr}} = 2^{\lfloor d_\mathrm{e}/2 \rfloor}$. Naively, this suggests that we need a $d_{\gamma,\mathrm{irr}} = 16$ dimensional representation for the required $d_\mathrm{e} = 9$ elements $G_{\mu\nu}$ in $d = 4$ spacetime dimensions. Using a metric $g = \mathrm{diag}(+, -, -, \ldots)$, the G_{0i} can be chosen anti-hermitean with respect to their spin indices while all other G_{ik} and G_{00} are hermitean. In $d = 4$ spacetime dimensions, we can use a corresponding d_γ dimensional Euclidean Dirac algebra $\{\gamma_A, \gamma_B\} = 2\delta_{AB}$ with $A, B \in 1, \ldots, 9$ to construct an explicit representation of the $G_{\mu\nu}$ as linear combinations of the γ_A (see Appendix A for an in-depth discussion of the relativistic Abrikosov algebra and its representation). So far, the construction is analogous to Luttinger fermion applications in condensed matter physics, replacing the spatial Euclidean metric by the Minkowski metric, cf. [11].

For the relativistic action, we also need the definition of the conjugate spinor $\bar{\psi}$. A unitary time evolution requires a real action which suggests to write $\bar{\psi} = \psi^\dagger h$, where h denotes the spin metric. For its construction, we note that the Abrikosov algebra (2) is invariant under Lorentz transformations with respect to the Lorentz indices as well as invariant under $\mathrm{SL}(d_\gamma, \mathbb{C})$ spin-base transformations [26–29]

$$G_{\mu\nu} \to \mathcal{S} G_{\mu\nu} \mathcal{S}^{-1}, \qquad \mathcal{S} \in \mathrm{SL}(d_\gamma, \mathbb{C}). \tag{4}$$

The action is spin-base invariant with $\psi \to \mathcal{S}\psi$, provided that the spin metric transforms as $h \to (\mathcal{S}^\dagger)^{-1} h \mathcal{S}^{-1}$. The condition that $\bar{\psi}\psi$ should form a real scalar (mass term) implies $h^\dagger = h$. The kinetic term (1) is real if

$$\{h, G_{0i}\} = 0, \qquad [h, G_{ij}] = 0, \qquad [h, G_{\underline{\mu}\underline{\mu}}] = 0, \tag{5}$$

where underscored indices are not summed over. Now, the important point is that no solution for h, satisfying Eq. (5) exists in the irreducible representation $d_{\gamma,\mathrm{irr}} = 16$ for $d = 4$. The construction of a relativistic theory of Luttinger fermions thus requires the use of a reducible representation of the corresponding Euclidean Dirac Clifford algebra, e.g., $d_\gamma = 32$ as the minimal possibility in $d = 4$ dimensional spacetime. For instance, for a representation with $G_{0i} = i\sqrt{\frac{2}{3}}\gamma_{A=i}$, a suitable spin metric is given by $h = \gamma_1\gamma_2\gamma_3\gamma_{10}$. (Another linearly independent solution is given by replacing γ_{10} by γ_{11}.)

In summary, the action (1) defines a free theory of massless propagating relativistic Luttinger fermions. These fermions have $d_\gamma = 32$ complex components that obey the classical equation of motion $G_{\mu\nu}\partial^\mu\partial^\nu\psi = 0$. Because of Eq. (3), each component satisfies the Klein-Gordon equation. The theory is $\mathrm{SL}(d_\gamma, \mathbb{C})$ spin-base invariant.

III. SELF-INTERACTING QUANTUM LUTTINGER FIELDS

Because of the kinetic term being quadratic in the derivatives, the canonical mass dimension of the fermions is $[\psi] = 1$ in $d = 4$ analogous to a scalar field. Therefore, quartic self-interactions are perturbatively renormalizable. With 1024 independent bilinears (compared to 16 for Dirac fermions), there are plenty of opportunities for model building. Here, we concentrate on a few selected channels, in the Euclidean domain, starting with the Luttinger variant of the Gross-Neveu model [30] in $d = 4$. In contradistinction to Dirac fermions, the Gross-Neveu channel generates also a tensor channel such that a Fierz-complete basis of local interactions is given by

$$S = \int d^4x \left[-\bar{\psi} G_{\mu\nu} \partial^\mu \partial^\nu \psi + \frac{\lambda_0}{2} (\bar{\psi}\psi)^2 + \frac{\lambda_t}{2} (\bar{\psi} G_{\mu\nu} \psi)^2 \right]. \tag{6}$$

As a first step towards the quantum analysis of this field theory, we compute the one-loop β functions, yielding

$$\partial_t \lambda_0 = -\frac{1}{(12\pi)^2} (540\lambda_0^2 - 528\lambda_0\lambda_t), \tag{7}$$

$$\partial_t \lambda_t = -\frac{1}{(12\pi)^2} (-992\lambda_t^2 + 28\lambda_0\lambda_t - 3\lambda_0^2). \tag{8}$$

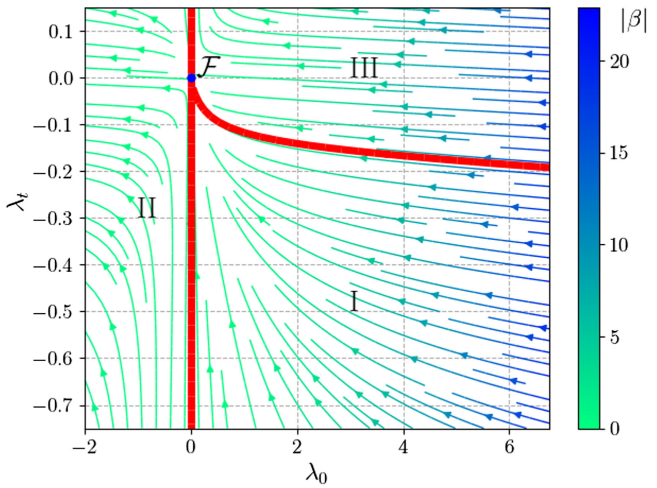

FIG. 1. Phase diagram of the Gross-Neveu-like model with relativistic Luttinger fermions in $d = 4$ for $N_\mathrm{f} = 1$ and $d_\gamma = 32$ with arrows indicating the flow towards the UV. In region I, The model is asymptotically free in both the scalar coupling λ_0 as well as the tensor channel λ_t, approaching the Gaussian fixed point $\mathcal{F}$ logarithmically.

Details of the calculation are provided in Appendix B. For the pure Gross-Neveu case, $\lambda_t = 0$, the scalar coupling λ_0 has a negative β function (7) exhibiting asymptotic freedom. This is in agreement with naive expectations, as the standard Gross-Neveu model is asymptotically free in its critical dimension. For Dirac fermions, the critical dimension is $d_\mathrm{cr} = 2$; but for Luttinger fermions, we have $d_\mathrm{cr} = 4$. In comparison to a scalar ϕ^4 theory which is not asymptotically free, we deal here with diagrams of the same topology, but the fermion loop comes with another minus sign.

However, the tensor channel is generated by the scalar channel, cf. Eq. (8). Therefore, the analysis should be performed in the (λ_0, λ_t) plane. The phase diagram reveals that the Gaussian fixed point is UV attractive in a large region with $\lambda_0 > 0$ and λ_t sufficiently negative, see Fig. 1. Defining the angle $\alpha = \arctan\frac{\lambda_t}{\lambda_0}$, the model is asymptotically free for $-90° \leq \alpha \lesssim 0°$ with the scalar coupling dominating for $\alpha > -45.8°$. To our knowledge, this is the first example of an asymptotically free, UV complete, pure fermionic matter theory.

As a second example, we consider a fermionic model with a continuous chiral/axial symmetry. Analogously to the Dirac case, we need an algebra element that anticommutes with all $G_{\mu\nu}$ as well as with h. For the present case, this element is given by γ_{10}, and it is straightforward to check that $\psi \to e^{i\vartheta\gamma_{10}}\psi$, $\bar\psi \to \bar\psi e^{i\vartheta\gamma_{10}}$ is a U(1) "axial" symmetry of the kinetic action. A self-interacting model with this symmetry is given by the analogue of the Nambu–Jona-Lasinio (NJL) model [31],

$$S = \int d^4x \left[-\bar\psi G_{\mu\nu}\partial^\mu\partial^\nu\psi + \frac{\lambda}{2}\left[(\bar\psi\psi)^2 - (\bar\psi\gamma_{10}\psi)^2\right]\right]. \quad (9)$$

The one-loop β function for the NJL coupling yields for $d_\gamma = 32$, cf. Appendix B,

$$\partial_t\lambda = -\frac{4N_\mathrm{f}}{\pi^2}\lambda^2. \quad (10)$$

Again, we observe that the coupling is asymptotically free. Here, we have neglected further vector/axial-vector channels that are generated by the NJL coupling. This is justified in the limit of large fermion flavor number N_f where the channels decouple.

In summary, self-interacting quantum field theories with Luttinger fermions give rise to asymptotically free RG trajectories emanating from the Gaussian fixed point thus representing UV complete theories. Diagrammatically, this is similar to Dirac fermionic models in their critical dimension. For Luttinger fermions, we have $d_\mathrm{cr} = 4$ guaranteeing perturbative renormalizability. Conversely, the theories become strongly interacting towards the infrared (IR) such that phenomena such as dynamical symmetry breaking and mass generation can be expected. The exploration of the long-range phase structure of the present and similar models will be an interesting field of future research.

IV. CONNECTION TO DIRAC FERMIONS

Counting the degrees of freedom, a 32-component Luttinger spinor contains eight 4-component Dirac spinors. Noteworthily, this covers the number of spinor degrees of freedom in one standard-model family: up and down quarks with 3 colors each, an electron and a neutrino (including a possible right-handed component). This raises the question as to whether a UV-completion of the standard model in terms of a model with Luttinger-fermionic matter can be constructed.

For this, a minimum requirement is that the spin-base symmetry $\mathrm{SL}(32,\mathbb{C})$ needs to be broken down to the Dirac spin-base symmetry $\mathrm{SL}(4,\mathbb{C})$ (possibly times some residual Dirac flavor symmetry). The Gross-Neveu- or NJL-type models presumably preserve spin-base symmetry also across possible strong-coupling transitions. Therefore, an explicit symmetry breaking mechanism appears more attractive—also in order to avoid a potentially large number of Goldstone bosons.

Explicit breaking terms could be formulated on the level of RG marginal four-fermion interactions. However, such marginal terms then induce also an RG relevant term which is given by the Dirac kinetic term $\sim\zeta_\mathrm{D}\int_x\bar\psi P_\mathrm{D}(i\partial\!\!\!/)\psi$, where $P_\mathrm{D}(i\partial\!\!\!/)$ is a suitable Abrikosov algebra element linear in the Dirac operator $i\partial\!\!\!/$ projecting the Luttinger components onto Dirac components. The prefactor ζ_D is a coupling of mass dimension one. Powercounting suggests that the UV remains still dominated by the RG behavior of relativistic Luttinger theory, whereas the fermion propagators become Dirac-like at momenta below ζ_D. Such a transition from

Luttinger to Dirac fermions has been studied in the nonrelativistic case in [16] for Bernal-stacked bilayer honeycomb lattices such as bilayer graphene; for a scalar analogue, see [32].

The explicit Dirac term also comes with another advantage: While we have studied theories of massless Luttinger fermions so far, a massive generalization of the free field equation reads

$$(-G_{\mu\nu}\partial^\mu\partial^\nu - m^2)\psi = 0. \tag{11}$$

Transition to momentum space and using Eq. (3) yields $0 = (p^4 - m^4)\psi(p) = (p^2 - m^2)(p^2 + m^2)\psi(p)$. This illustrates that each component of the Luttinger spinor satisfies the classical field equation of a higher-derivative theory [21–24,33]. This observation also suggests that the massive theory features tachyonic solutions with a negative spectral weight in addition to conventional massive modes. These are consequences of Ostrogradsky's theorem implying that Hamiltonians of higher-derivative theories are unbounded from below [34]. Whereas this appears to point to instabilities (or nonunitarity) at the quantum level, such theories are nevertheless discussed intensely in the literature in a variety of contexts both on the classical and the quantum level. Concrete proposals for a consistent treatment on the quantum level have been suggested [22,24,35–45]. Interestingly, explicit proofs exist for classical example systems that their motion remains stable for all initial conditions, despite interactions with the seemingly unstable modes [46,47].

Even if instabilities from negative-energy modes persist for the present class of models in a detailed quantum analysis, they can still be discussed from the perspective of effective field theory (EFT), as long as the rate of instability is small enough to satisfy all relevant phenomenological bounds. For instance, a fairly universal bound arises from gravity-mediated vacuum decay (into photons and negative-energy modes) which would contribute to the astrophysical diffuse photon background [48,49]. In fact, cosmology and its puzzles involving dark energy and dark matter have been a fruitful field for the application of field theories with negative energy modes [50]. A quantum treatment of such theories in an EFT framework bears the possibility of ameliorating tensions in current cosmological data [51,52], or provides mechanisms to produce dark-matter candidates accessible to direct detection [49,52]. Whether or not theories with Luttinger fermions can be useful in this context is a subject for future research.

For the purpose of this work, we emphasize that we have paid careful attention to possible problems arising from the in-principle unboundedness of the Hamiltonian, see Appendixes B and C, but found no impact on the quantities under study. In this context, we note that an explicit Dirac kinetic term has the potential to decouple the tachyonic poles from the real momentum axis, as the Dirac equation admits only classical solutions with $p^2 = m^2$. We thus consider an explicit Dirac kinetic term as a useful ingredient for future model building as well as for an investigation of the implications of Ostrogradsky's theorem.

V. GAUGE THEORIES WITH LUTTINGER FERMIONS

Luttinger fermions can straightforwardly be coupled to gauge fields by minimal coupling. For instance, the action for quantum electrodynamics (QED) with Luttinger fermions reads

$$S = \int_x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \bar\psi G_{\mu\nu}D^\mu D^\nu\psi - m^2\bar\psi\psi\right], \tag{12}$$

where $D^\mu = \partial^\mu - ieA^\mu$. We have included here also a mass term. As we are ultimately interested in the one-loop β function in a mass-independent scheme, the tachyonic modes mentioned in the previous section are of no relevance for the present study.

A convenient way to compute the one-loop β function of the running gauge coupling proceeds via the one-loop effective action upon integrating out the fermions:

$$\begin{aligned}\Gamma_{1\ell}[A] &= -i\ln\ \det(-G_{\mu\nu}D^\mu D^\nu - m^2)\\ &= -\frac{i}{2}\ln\ \det[-(G_{\mu\nu}D^\mu D^\nu)^2 + m^4)],\end{aligned} \tag{13}$$

where in the last step we have used γ_{10}-hermiticity of the kinetic term, $\gamma_{10}G_{\mu\nu}D^\mu D^\nu\gamma_{10} = -G_{\mu\nu}D^\mu D^\nu$. An interesting and relevant structure arising from the minimally coupled kinetic term is given by the spin-field coupling, reading $\sim\frac{ie}{2}[G_{\mu\nu}, G_{\kappa\lambda}]F^{\nu\lambda}\{D^\mu, D^\kappa\} + \mathcal{O}(\partial F)$, see Appendix C for details. This is the analogue of the Pauli term $\sim -\frac{e}{2}\sigma_{\mu\nu}F^{\mu\nu}$ for Dirac spinors, leading to an enhancement of "paramagnetic" contributions.

Expanding the determinant in powers of the field strength, the leading order term $\sim F_{\mu\nu}F^{\mu\nu}$ contains the information about the renormalization of the photon wave function; this computation is presented in Appendix. C. Within the background field formalism, the wave-function renormalization is connected to the renormalization of the coupling [53,54], yielding the β function

$$\partial_t e^2 = \frac{4\cdot 19}{9\pi^2}e^4 = \frac{4}{9\pi^2}(22|_{\text{para}} - 3|_{\text{dia}})e^4. \tag{14}$$

In the last expression, we have decomposed the result into a "diamagnetic" contribution from the Klein-Gordon operator contained in Eq. (13) and a "paramagnetic" contribution arising from the remaining terms including the spin-field coupling. Obviously, we observe *paramagnetic dominance*, i.e. the paramagnetic contributions dominate the final result and are also responsible for the sign of the β

function, as is known for many theories including non-Abelian gauge theories and even gravity [55].

Due to this strong paramagnetic dominance, the β function is positive. As in ordinary QED, the theory is thus not asymptotically free, and the vicinity of the Gaussian fixed point does not support RG trajectories that are UV complete and yield an interacting theory in the long-range limit. As for the standard model, a different mechanism is needed to establish high-energy completeness.

Let us finally turn to non-Abelian gauge theories, generalizing the action (12) to a non-Abelian $\mathrm{SU}(N_\mathrm{c})$ gauge group with N_c colors. Since the Yang-Mills sector remains unmodified, the resulting one-loop β function for the coupling g of QCD with N_f relativistic Luttinger quarks can immediately be written down,

$$\partial_t g^2 = -\frac{1}{3\pi^2}\left(\frac{11}{8}N_\mathrm{c} - \frac{2\cdot 19}{3}N_\mathrm{f}\right)g^4. \tag{15}$$

As is the standard case, asymptotic freedom of QCD can get lost for a large number of fermionic degrees of freedom relative to the number of colors. Due to the strong paramagnetic dominance of the Luttinger fermions, the critical color number below which asymptotic freedom is lost is comparatively large:

$$N_{\mathrm{c,cr}} = \frac{304}{33}N_\mathrm{f} \simeq 9.21 N_\mathrm{f}. \tag{16}$$

Hence, for $N_\mathrm{f} = 1$ Luttinger flavors, QCD is asymptotically free for $\mathrm{SU}(N_\mathrm{c} \geq 10)$.

An inclusion of all fermionic matter of the standard model (including right-handed neutrinos) would require $N_\mathrm{f} = 3$, since a standard-model generation fits into a single Luttinger flavor, i.e., the number of generations equals the number of flavors. Correspondingly, asymptotically free grand unified models based on a simple Lie group can be constructed for all $\mathrm{SU}(N_\mathrm{c} \geq 28)$, provided all obstructions can be met [56–59]. In principle, it is conceivable that this gauge group can be broken dynamically by suitable fermionic condensates that are seeded by four-fermion interactions instead of explicit independent Higgs fields. While this would be similar in spirit to models of top-quark condensation [60–63], the essential difference is that the four fermion interactions are RG marginal and perturbatively renormalizable—and potentially asymptotically free.

VI. CONCLUSIONS

We have constructed relativistic versions of Luttinger fermions in analogy to effective low-energy degrees of freedom of nonrelativistic solid-state systems. We propose to use these relativistic versions as fundamental degrees of freedom of interacting quantum field theories which can straightforwardly be constructed using perturbative quantization. Owing to their powercounting properties, models with quartic self-interactions of relativistic Luttinger fermions can be asymptotically free in $d = 4$ dimensional spacetime and thus have a chance to exist as fundamental quantum field theories on all scales. We provided perturbative evidence for this for the analogues of Gross-Neveu and NJL models. Since $d = 4$ is the RG critical dimension for our new models, the perturbative RG running of the couplings is logarithmic and a correspondingly large degree of UV insensitivity is obtained. Conversely, these models become strongly interacting in the long-range limit and their phase structure presumably characterized by dimensional transmutation and condensate formation deserves to be explored in detail.

A crucial ingredient for the construction of these relativistic models is given by the spin metric which requires a $d_\gamma = 32$ dimensional representation of the Abrikosov algebra. It is fascinating to see that one flavor of Luttinger fermions can thus host a whole standard-model generation. A corresponding perturbative study of gauge theories with Luttinger fermion matter reveals that comparatively large gauge groups are needed in order to preserve asymptotic freedom.

A suitable combination of gauge and fermionic self-interactions holds the promise to remain asymptotically free and high-energy complete while entailing condensate formation and thus dynamical (gauge) symmetry breaking at low-energies. Such theories would be *technically natural* [64–68] and thus of great interest to model building. In addition, however, the transition to low energies requires a breaking of the Luttinger spinors and the corresponding spin-base symmetry down to Dirac spinors. For this, we suggest an explicit breaking through a Dirac kinetic term which is RG relevant and comes with a dimensionful coupling. Though the scale setting of the latter is not technically natural, the associated sensitivity to a high-energy scale is only powercounting linear as opposed to quadratic in the standard model.

For a minimal technically natural extension of the standard model, it appears worthwhile to aim at the construction of a separate Luttinger fermion sector designed such that it forms a bilinear fermion condensate at low energies with the quantum numbers of the standard model Higgs field. In this case, the electroweak scale would be set by the scale of dimensional transmutation of the Luttinger fermion self-interaction which features the desired logarithmic sensitivity to the UV physics. Because of the strong paramagnetic dominance such a model would inevitably exert a strong influence on the (still logarithmic) running of the electroweak gauge sector possibly at the expense of no asymptotic freedom. A conclusive analysis requires to study the RG interplay of the gauge sector with the renormalizable fermionic self-interactions, cf. [69].

To conclude, the new kind of relativistic fermion degrees of freedom pave the way to unprecedented explorations of new particle physics models in four spacetime dimensions.

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APPENDIX A: RELATIVISTIC ABRIKOSOV ALGEBRA

For the construction of the relativistic kinetic action in Eq. (1), we use the anticommuting Clifford algebra of elements $G_{\mu\nu}$ acting on the Luttinger fermions as it has first been written down for the nonrelativistic case by Abrikosov [25]. For $\{G_{\mu\nu}, G_{\kappa\lambda}\} \sim \mathbb{1}$ to transform as a tensor under Lorentz transformations, the right-hand side must be formed from Lorentz covariant tensors. In absence of any further structure, we have the metric $g_{\mu\nu}$ and the Levi-Civita symbol $\epsilon_{\mu\nu\kappa\lambda}$ at our disposal. The latter is excluded by the symmetry requirement $G_{\mu\nu} = G_{\nu\mu}$ following from the kinetic action. This leaves us with the ansatz,

$$\{G_{\mu\nu}, G_{\kappa\lambda}\} = (a g_{\mu\nu} g_{\kappa\lambda} + b(g_{\mu\kappa} g_{\nu\lambda} + g_{\mu\lambda} g_{\nu\kappa}))\mathbb{1}, \quad \text{(A1)}$$

with constants a, b to be determined. The metric factors have been arranged such that the symmetries of the anticommutator and of $G_{\mu\nu}$ are already implemented. The identity $\mathbb{1}$ in spinor space refers to the $d_\gamma \times d_\gamma$ matrix structure of the $G_{\mu\nu}$.

The constants a, b are determined by the requirements that the Luttinger operator should square to the square of the D'Alembertian,

$$G_{\mu\nu}(i\partial^\mu)(i\partial^\nu)G_{\kappa\lambda}(i\partial^\kappa)(i\partial^\lambda) = (\partial^2)^2 \Rightarrow a + 2b = 2, \quad \text{(A2)}$$

and that the $G_{\mu\nu}$ shall be traceless in order to remove the reducible Klein-Gordon part from the kinetic action,

$$0 = \{G_{\mu\nu}, G^\kappa{}_\kappa\} = g^{\kappa\lambda}\{G_{\mu\nu}, G_{\kappa\lambda}\} \Rightarrow da + 2b = 0, \quad \text{(A3)}$$

where d is the spacetime dimension. Solving these equations leads to Eq. (2). By construction, this Abrikosov algebra is invariant under Lorentz transformations,

$$G_{\mu\nu} \to G_{\kappa\lambda} \Lambda^\kappa{}_\mu \Lambda^\lambda{}_\nu, \qquad \Lambda \in \mathrm{SO}(1, d-1) \quad \text{(A4)}$$

as well as under spin-base transformations (4) which correspond to the similarity transformations of the $G_{\mu\nu}$ matrices. [In principle, we can perform $\mathrm{GL}(d_\gamma, \mathbb{C})$ transformations; however, the U(1) phase and a rescaling $\in \mathbb{R}_+$ does not change the $G_{\mu\nu}$ which is why we consider only $\mathrm{SL}(d_\gamma, \mathbb{C})$ [29,70].] For a given matrix $G_{\mu\nu}$, both transformations change the form of that matrix. However, since all representations are connected by similarity transformations [71], there exists a spin-base transformation $\mathcal{S}_{\mathrm{Lor}}$ for each Lorentz transformation Λ^μ_ν, such that the spin-base transformation undoes the Lorentz transformation,

$$G_{\mu\nu} \to \mathcal{S}_{\mathrm{Lor}} G_{\kappa\lambda} \Lambda^\kappa{}_\mu \Lambda^\lambda{}_\nu \mathcal{S}_{\mathrm{Lor}}^{-1} \equiv G_{\mu\nu}. \quad \text{(A5)}$$

Clearly, the set of all $\mathcal{S}_{\mathrm{Lor}}$ forms an $\mathrm{SO}(1, d-1)$ subgroup of $\mathrm{SL}(d_\gamma, \mathbb{C})$. For an accurate discussion of the global aspects and depending on d, a restriction to the identity component may be necessary; e.g. in $d = 4$, the identity component of SO(1,3) has a universal cover that is isomorphic to $\mathrm{SL}(2, \mathbb{C})$, and therefore it is useful to think of $\mathcal{S}_{\mathrm{Lor}}$ as an $\mathrm{SL}(2, \mathbb{C})$ subgroup embedded into $\mathrm{SL}(32, \mathbb{C})$, cf. below. The details of this embedding also quantify how the Luttinger spinor can be decomposed into $\mathrm{SL}(2, \mathbb{C})$ Weyl spinors as irreducible representations of the Lorentz group. The corresponding transformations of the spinors $\psi \to \mathcal{S}_{\mathrm{Lor}}\psi$ can be viewed as the "Lorentz transformations of Luttinger spinors." This corresponds to the conventional picture, where fields transform under Lorentz transformations, but the $G_{\mu\nu}$ (or γ_μ in the Dirac case) remain fixed.

At this point, we should emphasize the difference between spin-base invariance and typical global symmetries such as the axial U(1) symmetry discussed for the NJL model in the main text. The latter is a typical "Noether" symmetry that acts on the fields and goes along with a corresponding Noether current and charge. By contrast, the spin-base symmetry also acts on the $G_{\mu\nu}$ which carries the spin structure but is not considered as a field in standard QFT. However, the $G_{\mu\nu}$ can be understood as fields $G_{\mu\nu}(x)$ in the case of gravitational interactions, where also the RHS of the Abrikosov algebra contains the metric $g_{\mu\nu}$ as the gravitational field variable. In such a scenario, the spin-base symmetry becomes a gauge theory rather than a global Noether symmetry [28,29].

For the metric convention $g = \mathrm{diag}(1, -1, -1, \ldots)$, the squares of the $G_{\mu\nu}$ can be determined from Eq. (2),

$$G_{0i}^2 = -\frac{1}{2}\frac{d}{d-1}, \quad G_{ij\neq i}^2 = \frac{1}{2}\frac{d}{d-1}, \quad G_{\underline{\mu\mu}}^2 = 1. \quad \text{(A6)}$$

Associating hermiticity properties to $G_{\mu\nu}$, this implies that G_{0i} has to be chosen anti-hermitean and all others hermitean. Spanning the $G_{\mu\nu}$ matrices by a Euclidean Clifford algebra,

$$G_{\mu\nu} = a_{\mu\nu}^A \gamma_A, \qquad \{\gamma_A, \gamma_B\} = 2\delta_{AB}, \quad \text{(A7)}$$

with hermitean $d_\gamma \times d_\gamma$ matrices γ_A, the coefficients a_{0i}^A can be chosen imaginary, and all others real. With $G_{\mu\nu} = G_{\nu\mu}$

we need $d_e = \frac{1}{2}d(d+1) - 1$ linearly independent anticommuting elements to span the space of $G_{\mu\nu}$ matrices.

For illustration, let us give an explicit representation of the $G_{\mu\nu}$ matrices for $d = 3+1$ dimensional spacetime in terms of $d_{\mathrm{e}} = 9$ Dirac matrices:

$$
\begin{aligned}
&G_{0i} = i\sqrt{\frac{2}{3}}\gamma_{A=i}, \quad i = 1,2,3,\\
&G_{12} = \sqrt{\frac{2}{3}}\gamma_4, \qquad G_{23} = \sqrt{\frac{2}{3}}\gamma_5, \qquad G_{31} = \sqrt{\frac{2}{3}}\gamma_6,\\
&G_{00} = \gamma_7, \qquad G_{11} = \frac{1}{3}\gamma_7 + \frac{2\sqrt{2}}{3}\gamma_8,\\
&G_{22} = \frac{1}{3}\gamma_7 - \frac{\sqrt{2}}{3}\gamma_8 + \sqrt{\frac{2}{3}}\gamma_9,\\
&G_{33} = \frac{1}{3}\gamma_7 - \frac{\sqrt{2}}{3}\gamma_8 - \sqrt{\frac{2}{3}}\gamma_9.
\end{aligned} \tag{A8}
$$

This representation can be viewed as an appropriate Wick rotation of the one constructed for $d = 4$ Euclidean dimensions in [11].

So far, it seems that the relativistic Abrikosov algebra could be constructed from the irreducible representation of the Euclidean Dirac algebra containing $d_{\mathrm{e}} = 9$ elements which would be the $d_{\gamma,\mathrm{irr}} = 2^{\lfloor \frac{d_{\mathrm{e}}}{2} \rfloor} = 16$ dimensional representation. However, a real action with a unitary time evolution requires the definition of the conjugate spinor $\bar{\psi} = \psi^\dagger h$, involving a spin metric h. In the Euclidean, $h = \mathbb{1}$ can be chosen, as all G matrices are hermitean. This is not a solution for the relativistic case, since the requirement of the action to be real implies the conditions of Eq. (5), $\{h, G_{0i}\} = 0$, $[h, G_{ij}] = 0$, $[h, G_{\underline{\mu}\underline{\mu}}] = 0$, where we have used that $h = h^\dagger$ following from $\bar{\psi}\psi$ being a real scalar. The nonrelativistic choice $h = \mathbb{1}$ is obviously in contradiction with $\{h, G_{0i}\} = 0$.

Let us for the moment assume that we work in the irreducible representation $d_{\gamma,\mathrm{irr}} = 16$. Using the representation Eq. (A8) as an example, the latter anticommutator can only be fulfilled, if h is a (linear superposition of a) product of an odd number of the remaining matrices $\gamma_{4,\ldots,9}$ as this exhausts all possible elements of the algebra. Then, the resulting h commutes only with G_{ij} and $G_{\underline{\mu}\underline{\mu}}$, if it contains the elements γ_A that G_{ij} and $G_{\underline{\mu}\underline{\mu}}$ are composed from. This implies that h must be a product of all remaining γ_A, e.g., $\gamma_4\gamma_5\gamma_6\gamma_7\gamma_8\gamma_9$. However, this is a product of an even number of γ_A matrices in contradiction with the assumption made before. This proves that there exists no spin metric in the $d_{\gamma,\mathrm{irr}} = 16$ dimensional representation of the γ_A matrices and thus no relativistic theory.

The solution as presented in the main text is to use a reducible representation with the smallest one being $d_\gamma = 32$ for $d = 4$. Note that the attribute "reducible" refers to being able to satisfy the Abrikosov algebra. If we read the algebra together with the conditions for the spin metric of Eq. (5), then the $d_\gamma = 32$ dimensional representation cannot be further reduced to a lower dimensional representation without violating one of the requirements. Still, the reducibility in the aforementioned sense has a consequence: as the $d_\gamma = 32$ dimensional representation goes along with two further anticommuting elements γ_{10} and γ_{11}, there are two linearly independent choices for a spin metric,

$$
h = \gamma_1\gamma_2\gamma_3\gamma_{10}, \quad \text{or} \quad \tilde{h} = \gamma_1\gamma_2\gamma_3\gamma_{11}, \tag{A9}
$$

satisfying all requirements of Eq. (5). Any linear combination $h' = \alpha h + \beta\tilde{h}$ with $\alpha^2 + \beta^2 = 1$ and $\alpha, \beta \in \mathbb{R}$ could equally well serve as spin metric. At this point, the most important aspect is that a spin metric exists, rendering the action relativistically invariant, nonzero, and real.

The reducibility (in the above mentioned sense) of the representation goes along with another advantage. There exists another element that anticommutes with all G matrices as well as with the spin metric. For our choice for the spin metric h, this element corresponds to γ_{10} (for the choice $\tilde{h}$, it would be γ_{11}). Using this element, we can define axial transformations,

$$
\psi \to e^{i\vartheta\gamma_{10}}\psi, \qquad \bar{\psi} \to \bar{\psi}e^{i\vartheta\gamma_{10}}, \tag{A10}
$$

which leave the kinetic term invariant. Incidentally, a mass term of the form

$$
S_m = \int d^4x m^2 \bar{\psi}\psi \tag{A11}
$$

would break this symmetry. The situation is therefore rather similar to the Dirac case with respect to both the existence of an axial symmetry in addition to the standard vector symmetry of phase rotations for the kinetic term, as well as the breaking of the axial symmetry by a mass term. This suggests that a Luttinger fermion can be decomposed into "chiral" components using the projectors

$$
P_{\mathrm{R/L}} = \frac{1}{2}(\mathbb{1} \pm \gamma_{10}), \qquad \psi_{\mathrm{R/L}} = P_{\mathrm{R/L}}\psi. \tag{A12}
$$

The kinetic term then decomposes into separate kinetic terms for the chiral components, whereas the mass term couples these components. This is analogous to the decomposition of Dirac fermions into Weyl components. We leave an analogous exploration of the existence of Majorana-Luttinger fermions or representations in terms of real Clifford algebras [72] for future study.

For the construction of possible interaction terms for the Luttinger fermions, it is useful to classify all linearly independent fermion bilinears of the form $\bar{\psi}\Gamma\psi$ with Γ being a Clifford algebra element. The fact that ψ is a

32-component spinor (together with reality requirements for the action) suggests that there might be $32 \times 32 = 1024$ bilinears. They can explicitly be listed using the Euclidean Dirac algebra. For this, we first note that $\gamma_{11} = \prod_{A=1}^{10} \gamma_A$. The complete set of Clifford algebra elements can thus be written as

$$\Gamma = \{\mathbb{1}, \gamma_1, \gamma_2, \ldots, \gamma_{10}, \gamma_1\gamma_2, \ldots, \gamma_9\gamma_{10}, \gamma_1\gamma_2\gamma_3, \ldots, \gamma_{11}\}, \tag{A13}$$

listed in the form of an increasing length of the γ products. Counting this number of products, yields

$$\text{number of } \Gamma = \sum_{k=0}^{10} \binom{10}{k} = 1024, \tag{A14}$$

in agreement with the preceding expectation. It remains an interesting task to determine how these bilinears can be conveniently grouped into Lorentz tensors.

APPENDIX B: SELF-INTERACTING QUANTUM LUTTINGER FIELDS

We have performed the computation of the one-loop β functions for the purely fermionic theories using the functional RG, since the computational tools for such systems are fairly developed [73–75], and generalize straightforwardly to nonperturbative approximation schemes [76–83] to be explored in the future. We employ the Wetterich equation [84] for a scale-dependent effective action Γ_k,

$$\partial_t \Gamma_k = \frac{1}{2} \mathrm{STr}\left[\partial_t R_k (\Gamma_k^{(2)} + R_k)^{-1}\right], \tag{B1}$$

where R_k denotes a regulator function (specified below) in the Euclidean implementing a decoupling of low-momentum modes in the IR; its derivative $\partial_t R_k$ establishes a Wilsonian momentum-shell integration, see [74,85–87] for reviews for the present setting. The supertrace STr includes a minus sign for all Grassmann-valued field components such as the Luttinger fermions considered here. For our purpose, it is useful to split the Hessian of the action into a field-independent kinetic part and a field-dependent part carrying all interactions, $\Gamma^{(2)} + R_k = \mathcal{P} + \mathcal{F}$ with $\mathcal{P} = \Gamma^{(2)}_{\mathrm{kin}}$ and $\mathcal{F} = \Gamma^{(2)}_{\mathrm{int}}$. As a one-loop exact ansatz, we use, for instance, for the Gross-Neveu-type model of (6)

$$\Gamma_k = \int_x \left[-Z\bar{\psi} G_{\mu\nu}\partial^\mu \partial^\nu \psi + \frac{\bar{\lambda}_0}{2}(\bar{\psi}\psi)^2 + \frac{\bar{\lambda}_t}{2}(\bar{\psi}G_{\mu\nu}\psi)^2\right], \tag{B2}$$

where we have introduced a wave function renormalization Z which—together with the bare couplings—is considered as scale dependent. The RG flows of these quantities can be extracted upon an expansion of the Wetterich equation in powers of $\mathcal{F}$. The linear term contains the flow of the wave function which we parametrize in terms of the anomalous dimension of the fermion fields,

$$\eta = -\partial_t \ln Z. \tag{B3}$$

In fact to one-loop order, we find $\eta = 0$ which is consistent with the perturbative argument, that the only contributing tadpole diagram does not give rise to a nontrivial momentum dependence. For generality, we still keep the η dependence explicit, in order to illustrate how higher-loop resummation effects would affect our results. The one-loop flow of the couplings is contained in the terms of order $\sim\mathcal{F}^2$. Introducing the renormalized (dimensionless) couplings

$$\lambda_{0,t} = \frac{k^{d-4}}{Z^2} \bar{\lambda}_{0,t}, \tag{B4}$$

we obtain the RG flows

$$\partial_t \lambda_{0,t} = (d - 4 + 2\eta)\lambda_{0,t} - 4 v_d l_2^{(d)}(0,\eta) f_{0,t}(N_\mathrm{f}, d, \lambda_0, \lambda_t), \tag{B5}$$

where we have kept the spacetime dimension general and used the abbreviation $v_d^{-1} = 2^{d+1}\pi^{d/2}\Gamma(d/2)$, such that $v_4 = 1/(32\pi^2)$. The threshold function $l_2^{(d)}$ parametrizes the regularized loop-momentum integral also containing the information about the specifics of the regulator. In addition to the scaling terms linear in the couplings, the fluctuation-induced terms are defined in terms of the functions

$$\begin{aligned} f_0 &= (N_\mathrm{f} d_\gamma - 2)\lambda_0^2 - \frac{2d}{d-1}(d_e + 2)\lambda_0\lambda_t, \\ f_t &= -\frac{d}{(d-1)d_e}((d_e - 2)N_\mathrm{f} d_\gamma + 2(d_e^2 - d_e + 2))\lambda_t^2 \\ &\quad + \left(2 - \frac{4}{d_e}\right)\lambda_0\lambda_t - \frac{4}{d(d+2)}\lambda_0^2, \end{aligned} \tag{B6}$$

where we have used the abbreviation $d_e = \frac{1}{2}(d+2)(d-1)$. Note that both channels decouple in the large-N_f limit.

In general, the form of the threshold function depends on the chosen regulator R_k, manifesting the regularization scheme. For $d = 4$, we can prove that the one-loop result is universal and independent of the scheme, as it should be. In order to arrive at an explicit expression, we choose a regulator that preserves spin-base invariance, $R_k(p) = G_{\mu\nu} p^\mu p^\nu r(p^2/k^2)$, where r denotes a regulator shape function specifying the decoupling of IR modes. In fact, in the loop integrals, the shape function occurs in the combination $[p^2(1+r)]$ as can be expected from power-counting. The resulting loop structure can therefore be mapped onto that of bosonic threshold functions. In fact, the threshold functions $l_n^d(\omega,\eta)$ are well tabulated in the literature [74], and the one in our equation above agrees with those. Using the common partially linear regulator shape function $r(y)=(y^{-1}-1)\theta(1-y)$ [88], we arrive at the standard result

$$l_2^{(d)}(0,\eta)=\frac{4}{d}\left(1-\frac{\eta}{d+2}\right), \tag{B7}$$

which in $d=4$ and at one-loop with $\eta=0$ boils down to $l_2^{(d=4)}(0,0)=1$. Specializing to $N_\mathrm{f}=1$, $d_\gamma=32$ and $d=4$, the flows of the couplings yield the expressions given in the main text.

For the NJL-type model, the computation is performed analogously. Also in this case, the NJL coupling generates tensor-type couplings at subleading order in N_f which we ignore in our discussion. The flow equation for the NJL coupling can be brought into the form of Eq. (B5) with a corresponding function $f(N_\mathrm{f},d,\lambda)=d_\gamma N_\mathrm{f}$ such that the flow equation for $d=4$ with $\eta=0$ reads

$$\partial_t\lambda=-\frac{d_\gamma N_\mathrm{f}}{8\pi^2}\lambda^2 \tag{B8}$$

in agreement with the special case for $d_\gamma=32$ given in the main text.

APPENDIX C: GAUGE THEORIES FOR LUTTINGER FERMIONS

Let us recall the action (12) for quantum electrodynamics with Luttinger fermions

$$S=\int d^4x\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\bar{\psi}G_{\mu\nu}D^\mu D^\nu\psi-m^2\bar{\psi}\psi\right]. \tag{C1}$$

The one-loop β function can be derived via the effective action, integrating out the fermions in an electromagnetic background field [53]. Apart from a normalization given below, the one-loop correction to the action is given by the fermion determinant arising from the integration over Grassmann-valued fields obeying Fermi-Dirac statistics:

$$\begin{aligned}\Gamma_{1\ell}[A]&=-i\ln\ \det[-G_{\mu\nu}D^\mu D^\nu-m^2]\\&=-\frac{i}{2}\ln\ \det[-G_{\mu\nu}D^\mu D^\nu G_{\kappa\lambda}D^\kappa D^\lambda+m^4],\end{aligned} \tag{C2}$$

where we used the properties $\gamma_{10}^2=1$ and $\{G_{\mu\nu},\gamma_{10}\}=0$, in order to arrive at the squared differential operator—similar to γ_5 hermiticity in standard QED [53].

We can decompose the product $G_{\mu\nu}D^\mu D^\nu G_{\kappa\lambda}D^\kappa D^\lambda$ into symmetric and antisymmetric parts,

$$\begin{aligned}&G_{\mu\nu}D^\mu D^\nu G_{\kappa\lambda}D^\kappa D^\lambda\\&\quad=\left(\frac{1}{2}\{G_{\mu\nu},G_{\kappa\lambda}\}+\frac{1}{2}[G_{\mu\nu},G_{\kappa\lambda}]\right)D^\mu D^\nu D^\kappa D^\lambda.\end{aligned} \tag{C3}$$

Let us study the two pieces separately, starting with the antisymmetric part,

$$[G_{\mu\nu},G_{\kappa\lambda}]D^\mu D^\nu D^\kappa D^\lambda=\frac{1}{2}[G_{\mu\nu},G_{\kappa\lambda}][D^\mu D^\nu,D^\kappa D^\lambda], \tag{C4}$$

since $G_{\mu\nu}=G_{\nu\mu}$. The commutator between covariant derivatives yields

$$\begin{aligned}[D^\mu D^\nu,D^\kappa D^\lambda]=-ie(&F^{\nu\lambda}D^\mu D^\kappa+F^{\nu\kappa}D^\mu D^\lambda\\&+F^{\mu\lambda}D^\kappa D^\nu+F^{\mu\kappa}D^\lambda D^\nu).\end{aligned} \tag{C5}$$

Here we have used the relation $[D^\mu,D^\nu]=-ieF^{\mu\nu}$ and confined ourselves to a constant electromagnetic field, $F^{\mu\nu}=\text{const}$, such that $D^\kappa F^{\mu\nu}=F^{\mu\nu}D^\kappa$.

Since the product $[G_{\mu\nu},G_{\kappa\lambda}]F^{\nu\lambda}$ is symmetric under the exchange $\mu\leftrightarrow\kappa$, Eq. (C4) becomes

$$[G_{\mu\nu},G_{\kappa\lambda}]D^\mu D^\nu D^\kappa D^\lambda=-ie[G_{\mu\nu},G_{\kappa\lambda}]F^{\nu\lambda}\{D^\mu,D^\kappa\}. \tag{C6}$$

The symmetric part of Eq. (C3) can be rewritten using the Abrikosov algebra (2); with the same assumptions, we obtain

$$\{G_{\mu\nu},G_{\kappa\lambda}\}D^\mu D^\nu D^\kappa D^\lambda=2(D^2)^2+\frac{3de^2}{2(d-1)}F_{\kappa\lambda}F^{\kappa\lambda}, \tag{C7}$$

where we work in general spacetime dimensions d for generality, but will later specialize to $d=4$. Inserting Eqs. (C6) and (C7) into Eq. (C3), the one-loop effective action (C2) becomes

$$\begin{aligned}\Gamma_{1\ell}[A]=-\frac{i}{2}\ln\det\Bigg[&-(D^2)^2-\frac{3e^2d}{4(d-1)}F_{\kappa\lambda}F^{\kappa\lambda}\\&+\frac{ie}{2}[G_{\mu\nu},G_{\kappa\lambda}]F^{\nu\lambda}\{D^\mu,D^\kappa\}+m^4\Bigg].\end{aligned} \tag{C8}$$

In order to keep track of potential issues arising from the tachyonic mass poles or their negative residues, we work in Minkowski space, paying careful attention to contour rotations in the complex momentum plane. First, we use the Schwinger proper time formula for the logarithm,

$$\ln\frac{M}{N}=-\lim_{\delta\to0}\int_0^{\infty+i\delta}\frac{dt}{t}(e^{iMt}-e^{iNt}), \tag{C9}$$

such that Eq. (C8) can be written as

$$\Gamma_{1\ell}[A] = \frac{i}{2}\lim_{\delta\to 0}\int_0^{\infty+i\delta}\frac{dt}{t}\mathrm{Tr}\left\{e^{it[-(D^2)^2-\frac{3e^2d}{4(d-1)}F_{\kappa\lambda}F^{\kappa\lambda}+\frac{ie}{2}[G_{\mu\nu},G_{\kappa\lambda}]F^{\nu\lambda}\{D^\mu,D^\kappa\}+m^4]} - e^{it[-(\partial^2)^2+m^4]}\right\}. \tag{C10}$$

Here we have used $\ln\det(M) = \mathrm{Tr}\ln(M)$ and subtracted the free-field case in order to fix the normalization mentioned above. Since we aim at computing the β function, it suffices to compute $\Gamma_{1\ell}$ to order F^2. For this, we define

$$\begin{aligned} A &\equiv -it(D^2)^2, \\ B &\equiv -\frac{et}{2}[G_{\mu\nu},G_{\kappa\lambda}]F^{\nu\lambda}\{D^\mu,D^\kappa\}, \end{aligned} \tag{C11}$$

and employ the Baker–Campbell–Hausdorff formula for the expansion. We find

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} \xrightarrow{\mathrm{Tr}} e^A\left(1+\frac{1}{2}B^2\right) + \mathcal{O}(F^3), \tag{C12}$$

since $[A,B]^2 \sim \mathcal{O}(F^4)$, $B[A,B] \sim \mathcal{O}(F^3)$, and the traces of the terms $\sim B$ and $[A,B]$ vanish. This reduces our expression for the one-loop effective action to

$$\begin{aligned} \Gamma_{1\ell}[A] = {}& \frac{i}{2}\lim_{\delta\to 0}\int_0^{\infty+i\delta}\frac{dt}{t}e^{it[-\frac{3e^2d}{4(d-1)}F_{\kappa\lambda}F^{\kappa\lambda}+m^4]}\mathrm{Tr}\left\{e^{-it(D^2)^2}\left[\mathbb{1}+\frac{1}{2}\left(\frac{-et}{2}[G_{\mu\nu},G_{\kappa\lambda}]F^{\nu\lambda}\{D^\mu,D^\kappa\}\right)^2\right]\right\} \\ & -\frac{i}{2}\lim_{\delta\to 0}\int_0^{\infty+i\delta}\frac{dt}{t}\mathrm{Tr}\{e^{it[-(\partial^2)^2+m^4]}\} + \mathcal{O}(F^4). \end{aligned} \tag{C13}$$

The functional trace runs over coordinate/momentum space as well as spinor space, $\mathrm{Tr} = \mathrm{Tr}_x\mathrm{Tr}_G$. Only the term $\sim B^2$ in Eq. (C12) is nontrivial in spinor space, yielding

$$\begin{aligned} \mathrm{Tr}_G B^2 &= \frac{e^2t^2}{4}F^{\nu\lambda}\{D^\mu,D^\kappa\}F^{\beta\delta}\{D^\alpha,D^\gamma\}\mathrm{Tr}_G\{[G_{\mu\nu},G_{\kappa\lambda}][G_{\alpha\beta},G_{\gamma\delta}]\} \\ &= \frac{e^2t^2}{4}\frac{8d_\gamma}{(d-1)^2}[2d(2-d)F^{\nu\lambda}F_{\nu\kappa}D_\lambda D^\kappa D^2 - d^2F^{\nu\lambda}F_{\nu\lambda}(D^2)^2], \end{aligned} \tag{C14}$$

where we have again kept terms only up to order $\sim F^2$, used $F = \mathrm{const}$, and d_γ denotes the dimension of the Abrikosov algebra. Plugging these results into Eq. (C13), we get to order F^2:

$$\begin{aligned} \Gamma_{1\ell}[A] = {}& \frac{id_\gamma}{2}\lim_{\delta\to 0}\int_0^{\infty+i\delta}\frac{dt}{t}e^{itm^4}\left[1-i\frac{3e^2td}{4(d-1)}F_{\kappa\lambda}F^{\kappa\lambda}\right]\mathrm{Tr}_x\left\{e^{-it(D^2)^2}\right. \\ & \left.\times\left(1+\frac{e^2t^2}{(d-1)^2}[2d(2-d)F^{\nu\lambda}F_{\nu\kappa}\partial_\lambda\partial^\kappa\partial^2 - d^2F^{\nu\lambda}F_{\nu\lambda}(\partial^2)^2]\right)\right\} + \mathcal{O}(F^4) \\ & -\frac{id_\gamma}{2}\lim_{\delta\to 0}\int_0^{\infty+i\delta}\frac{dt}{t}\mathrm{Tr}_x\{e^{it[-(\partial^2)^2+m^4]}\}. \end{aligned} \tag{C15}$$

In order to use heat-kernel methods, we rewrite the factor $e^{-it(D^2)^2}$ in terms of a Fresnel integral. For this, we use the Gaussian integral:

$$\sqrt{\frac{\alpha}{\pi}}\int_{-\infty}^{+\infty}d\mu\, e^{-\alpha\mu^2-2\alpha\beta\mu} = e^{\alpha\beta^2}, \tag{C16}$$

and implicitly rotate the contour by identifying $\alpha \equiv -i$ and $\beta^2 \equiv (D^2)^2 t$, resulting in

$$e^{-it(D^2)^2} = \sqrt{\frac{-i}{\pi}}\int_{-\infty}^{+\infty}d\mu\, e^{i\mu^2}\, e^{2i\sqrt{t}D^2\mu}. \tag{C17}$$

The details of the contour rotation imply the convention $\sqrt{-i} = e^{-i\frac{\pi}{4}}$. We are then left with the following integrals:

$$\Gamma_{1\ell}[A] = \frac{id_\gamma}{2}\lim_{\delta\to 0}\int_0^{\infty+i\delta}\frac{dt}{t}e^{itm^4}\sqrt{\frac{-i}{\pi}}\int_{-\infty}^{+\infty}d\mu\, e^{i\mu^2}\mathrm{Tr}_x[e^{2i\sqrt{t}D^2\mu} - e^{2i\sqrt{t}\partial^2\mu}]$$
$$+\frac{id_\gamma}{2}\lim_{\delta\to 0}\int_0^{\infty+i\delta}\frac{dt}{t}e^{itm^4}\mathrm{Tr}_x\left[e^{-it(\partial^2)^2}\left(\frac{e^2t^2}{(d-1)^2}[2d(2-d)F^{\nu\lambda}F_{\nu\kappa}\partial_\lambda\partial^\kappa\partial^2 - d^2F^{\nu\lambda}F_{\nu\lambda}(\partial^2)^2]\right.\right.$$
$$\left.\left.-it\frac{3e^2d}{4(d-1)}F_{\kappa\lambda}F^{\kappa\lambda}\right)\right] + \mathcal{O}(F^4). \tag{C18}$$

Here, we use the heat kernel of the scalar Laplacian in a constant magnetic background field B [54],

$$\mathrm{Tr}_x e^{i\lambda D^2} = -\frac{i\Omega}{(4\pi)^2}\frac{1}{\lambda^2}\frac{\lambda eB}{\sin\lambda eB} = -\frac{i\Omega}{(4\pi)^2}\frac{1}{\lambda^2}\left(1+\frac{\lambda^2e^2B^2}{6}+\mathcal{O}(B^4)\right), \tag{C19}$$

where Ω denotes the spacetime volume, and the factor of i arises from rotating the trace in momentum space into the Euclidean domain, $dp_0 = idp_4$. The free-field subtraction cancels precisely the constant term in Eq. (C20). The remaining operators to be traced are all diagonal in momentum space and can be done straightforwardly. For instance, in terms of the magnetic background, the results read in our conventions

$$\mathrm{Tr}_x(e^{-it(\partial^2)^2}F^{\nu\lambda}F_{\nu\kappa}(\partial^2)^2) = i\Omega 2B^2\int\frac{d^dp}{(2\pi)^d}e^{-ip^4t}p^4,$$
$$\mathrm{Tr}_x(e^{-it(\partial^2)^2}F^{\nu\lambda}F_{\nu\kappa}\partial_\lambda\partial^\kappa\partial^2) = i\Omega\int\frac{d^dp}{(2\pi)^d}e^{-ip^4t}p_\lambda p^\kappa p^2F^{\nu\lambda}F_{\nu\kappa} = \frac{i\Omega}{d}2B^2\int\frac{d^dp}{(2\pi)^d}e^{-ip^4t}p^4, \tag{C20}$$

where the momentum integrals are understood to run over Euclideanized momenta. Because of the t contour having a small positive imaginary part, the remaining integral converges, yielding, e.g., $\int\frac{d^4p}{(2\pi)^4}e^{-ip^4t}p^4 = -\frac{1}{32\pi^2t^2}$ in $d=4$.

Collecting the contributions of all terms, we arrive at

$$\Gamma_{1\ell}[A] = -\frac{19}{18\pi^2}\frac{d_\gamma}{32}\Omega e^2B^2\lim_{\delta\to 0}\int_{\frac{i}{\Lambda^4}}^{\infty+i\delta}\frac{dt}{t}e^{itm^4} = \frac{19}{18\pi^2}\frac{d_\gamma}{32}\Omega e^2B^2\left[\gamma+\ln\left(\frac{m^4}{\Lambda^4}\right)+O\left(\frac{m^4}{\Lambda^4}\right)\right], \tag{C21}$$

to leading order $\mathcal{O}(B^2)$ in the field strength. By combining this result with the bare Maxwell Lagrangian using $\Gamma_{1\ell} = \Omega\mathcal{L}_{1\ell}$ for a homogeneous field, we get for the relativistic Luttinger case $d_\gamma = 32$

$$\mathcal{L}_{\mathrm{eff}} = \mathcal{L}_{\mathrm{M}} + \mathcal{L}_{1\ell} = -\frac{1}{2}B^2 + \frac{19}{18\pi^2}e^2B^2\left[\gamma+\ln\left(\frac{m^4}{\Lambda^4}\right)\right]. \tag{C22}$$

Defining the wave function renormalization

$$Z^{-1} = 1-\frac{19}{9\pi^2}e^2\left[\gamma+\ln\left(\frac{\mu^4}{\Lambda^4}\right)\right], \tag{C23}$$

where μ is the RG scale, we introduce the renormalized field and coupling

$$B_{\mathrm{R}}^2 = Z^{-1}B^2, \qquad e_{\mathrm{R}}^2 = Ze^2. \tag{C24}$$

The one-loop β function is then given by

$$\beta_{e^2} \equiv \mu\frac{\partial}{\partial\mu}e_{\mathrm{R}}^2(\mu) = \frac{\partial}{\partial\mu}Ze^2 = \frac{4\cdot 19}{9\pi^2}e_{\mathrm{R}}^4. \tag{C25}$$

By keeping track of which contribution arises from the Laplacian and which from the endomorphisms or the spin-field coupling terms, we can decompose the factor 19 into dia- and paramagnetic contributions as is given in Eq. (14) in the main text.

The generalization to the non-Abelian case is evident. We define the QCD action with one relativistic Luttinger quark by

$$S = \int_x\left[-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \bar\psi^iG_{\mu\nu}(D^\mu)^{ij}(D^\nu)^{jk}\psi^k - m^2\bar\psi^i\psi^i\right], \tag{C26}$$

where $i,j = 1,\dots,N_{\mathrm{c}}$ labels fundamental and $a = 1,\dots,N_{\mathrm{c}}^2-1$ adjoint color indices. A generalization to arbitrary flavor numbers N_{f} is straightforward. The covariant derivative is now given by $D^\mu_{ij} = \partial^\mu - ig\tau^a_{ij}A^{\mu,a}$, with τ^a being the generators of $SU(N_{\mathrm{c}})$, $\mathrm{Tr}_{\mathrm{c}}(\tau^a\tau^b) = \frac{1}{2}\delta^{ab}$.

The computation of the quark contribution to the QCD β function can be mapped to that of the QED case, by using a pseudo-Abelian background field $A^a_\mu = n^a\tilde A_\mu$, where $\tilde A_\mu$ is an Abelian vector potential and n^a is a constant unit vector in color space ($n^a n^a = 1$). The covariant derivative then reduces to $D^\mu_{ij} = \partial^\mu - ig(\tau^a n^a)_{ij}\tilde{A}^\mu$, which implies that the background is covariantly constant, $[D^\kappa, F^{\nu\lambda}] = 0$. The computation of the quark determinant proceeds in complete analogy to the QED case supplemented by the trace over color space. To leading order $\sim F^2$, this trace reduces to $\mathrm{tr_c}((\tau^a n^a)_{ij}(\tau^b n^b)_{jk}) = \frac{1}{2}\delta^{ab}n^a n^b = \frac{1}{2}$, leading to a modification of the quark contribution to the QCD β function by a factor of $\frac{1}{2}$ compared with QED,

$$\beta_{g^2}|_{\mathrm{quark-loop}} \equiv \mu \frac{\partial}{\partial\mu} g_{\mathrm{R}}^2(\mu) = \frac{2\cdot 19}{9\pi^2} g_{\mathrm{R}}^4, \qquad \text{(C27)}$$

for a single quark flavor. A different flavor number is accounted for by a factor of N_{f}. Together with the gluon (and ghost) loops, the final result for the QCD β function with relativistic Luttinger quarks is given in Eq. (15) in the main text.

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