Conformal Killing cosmology: Geometry, dark sector, growth of structures, and a big rip

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(Received 30 May 2024; accepted 20 August 2024; published 11 September 2024)

We introduce Sinyukov-like tensors, a special kind of conformal Killing tensors. In Robertson-Walker space-times, they have the perfect-fluid form and only depend on two constants and the scale factor. They are the candidates for the dark term of the newly proposed conformal Killing gravity, by Harada. In addition to ordinary matter, the Friedmann equations contain a dark term and a Λ term that parametrize the Sinyukov-like tensor. The expression of $H(z)$ is tested on cosmological data based on cosmic chronometers or including baryon acoustic oscillations. There is a large incertitude in Ω_Λ and Ω_{dark} that may become negative, but their sum is close to Ω_Λ of Lambda cold dark matter (ΛCDM). In any case, there is a future singularity, that is, a big rip for all positive Ω. We solve the equation for the evolution, in linear approximation, of the density contrast in a matter-dominated universe. The dark sector and the Λ term give no significative deviation from ΛCDM and general relativity results.

DOI: 10.1103/PhysRevD.110.064041

I. INTRODUCTION

Since the observation of supernovae SnIa by Riess and Perlmutter in 1998–1999, dark energy stands as an unsolved theoretical problem. The reintroduction of the Λ term in the Friedmann equations successfully accounted for a major accordance with observational cosmology, together with the cold dark matter (CDM) assumption. Nevertheless, the puzzle remains, with discrepancies between theory and data. Several modifications of the Standard Model have been proposed, in two mainstreams. The first one modifies the energy-momentum source with new fields [see the review by Copeland [1]]: phantom field [2], quintessence [3], Chaplygin gas [4], and others. The second class modifies geometry [see [5,6]] and contains popular models such as $f(R)$ [7,8], $f(G)$ [9], $f(T)$ [10], mimetic gravity [11], and others. These theories and others in Friedmann-Robertson-Walker (FRW) space-times fit in the class of models of Cotton gravity [12]. A third route is explored in [13], with axion dark matter and dark energy resulting from modified gravity with $f(R) = \exp(-\beta R)$.

In 2023, Harada [14] introduced a modified theory of gravity to explain the present accelerated phase of the Universe without the explicit introduction of dark energy. It is based on the following field equations in 4 dimensions:

$$\begin{aligned}&\nabla_j R_{kl} + \nabla_k R_{jl} + \nabla_l R_{jk}\\&\quad -\frac{1}{3}(g_{kl}\nabla_j R + g_{jl}\nabla_k R + g_{jk}\nabla_l R)\\&= \nabla_j T_{kl} + \nabla_k T_{jl} + \nabla_l T_{jk}\\&\quad -\frac{1}{6}(g_{kl}\nabla_j T + g_{jl}\nabla_k T + g_{jk}\nabla_l T).\end{aligned} \tag{1}$$

They are manifestly third-order in the derivatives of the metric tensor and extend general relativity (GR): a solution of the Einstein equations $R_{kl} - \frac{1}{2}Rg_{kl} = T_{kl}$ is a solutions of the new equations.

In static spherical symmetry Harada obtained the Schwarzschild solution with new terms that are significant at large distances, $-g_{00} = 1/g_{rr} = 1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2 - \frac{\lambda}{5}r^4$. He then derived an equation for the scale factor in RW space-times. In a matter-dominated universe, the explicit solution shows a transition from decelerating to accelerating expansion without introducing dark sources in the equations; Λ results as an integration constant.

Shortly after, we published a parametrization showing that Harada's equations are equivalent to the Einstein equations modified by a supplemental conformal Killing tensor that is also divergence-free [15]:

$$R_{kl} - \frac{1}{2}Rg_{kl} = T_{kl} + K_{kl}, \tag{2}$$

$$\begin{aligned}&\nabla_j K_{kl} + \nabla_k K_{jl} + \nabla_l K_{jk}\\&= \frac{1}{6}(g_{kl}\nabla_j K + g_{jl}\nabla_k K + g_{jk}\nabla_l K).\end{aligned} \tag{3}$$

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For this reason, the theory was named conformal Killing gravity (CKG).

The reformulation makes the extension of GR explicit through the Killing term, that satisfies $\nabla^k K_{kl} = 0$ and poses as a natural candidate for the energy-momentum of the dark sector. In this direction, we proved that a space-time is generalized Robertson-Walker (GRW) if and only if it admits a conformal Killing tensor with the perfect-fluid form with a closed velocity field. Requiring conformal flatness restricts the space-time to RW.

We wrote the two Friedmann equations for the dark pressure and the dark energy density, recovered Harada's solution for the scale factor $a(t)$ in a matter-dominated universe, and discussed a toy model. Both cases led to the conclusion that the dark fluid determines the late time evolution of the scale parameter, with the equation of state $p_D/\mu_D \to -\frac{5}{3}$ (phantom energy).

The cosmological scenarios of CKG were further explored by Harada in [16] for various values of Ω_m in the evolution equation of the Hubble parameter

$$\frac{H(z)}{1+z} = H_0\left[\Omega_m(1+z) + \Omega_r(1+z)^2 + \Omega_k + \frac{\Omega_\Lambda}{(1+z)^2} + \frac{\Omega_D}{(1+z)^4}\right]^{1/2} \tag{4}$$

and compared with ΛCDM. If the effective dark energy $\Omega_D = 1 - \Omega_m - \Omega_\Lambda$ (radiation and curvature are neglected) is present in a moderate amount, CKG holds the potential to resolve the Hubble tension. This remains true if Ω_D is dominant in the total energy budget with $\Lambda = 0$.

Solutions with $T_{kj} = 0$ were obtained by Clément and Noucier [17] in a variety of cases including singularity-free eternal cosmologies and universes evolving symmetrically in finite time from big bang to big crunch.

The static spherical solution was retaken by Barnes [18] to include the case $g_{rr}g_{00} \neq -1$. The important issue of the validity of Birkhoff's theorem is discussed in [19].

Junior *et al.* [20] studied black hole solutions in CKG coupled to nonlinear electrodynamics (NLED) with canonical or phantom-like scalar fields. They found generalizations of the Schwarzschild–Reissner-Nordström–AdS solutions that extend the class of known regular black hole solutions. In [21], they explore black bounce solutions, that extend Bardeen-type and Simpson-Visser geometries, in CKG coupled to NLED and scalar fields.

Solutions of CKG linearly coupled to Maxwell fields were obtained by Barnes [19,22] as powers series; the general closed-form solutions were then found by Clément and Nouicer [17]. While these results were obtained by solving the third-order Eq. (1) in static spherical symmetry, we obtained them in a simpler manner by solving the equivalent second-order Einstein-like equations [23]. Barnes obtained the most general pp-wave solutions in CKG and their plane-wave specialization [24].

In this paper, we investigate conformal Killing gravity in RW space-time. To be specific, the conformal Killing tensor (3) is a perfect fluid and Sinyukov-like. In this case, the geometry is wholly determined by two integration constants that parametrize the dark fluid. Their numerical values are fixed by the knowledge of Ω_Λ and Ω_D in the evolution (4) of $H(z)$, for which data are still limited.

In Sec. II, we prove that a perfect-fluid tensor $K_{ij} = Ag_{ij} + Bu_iu_j$ is a conformal Killing tensor if the velocity is shear-free. The fluid is conserved, $\nabla^i K_{ij} = 0$, if the velocity is also vorticity-free. If the velocity is also acceleration-free, the perfect-fluid tensor is of special type: we name it Sinyukov-like. Conformal Killing gravity in generalized Robertson-Walker space-times is thus provided by Sinyukov tensors, presented in Sec. III. In the RW setting the Ricci tensor is a perfect fluid and the Sinyukov tensor is fixed by the scale parameter and two constants.

In Sec. IV, we write the analogs of the Friedmann equations: the covariant approach is particularly effective for this. We give the formulas for $H(z)$, the deceleration $q(z)$, and the look-back time. In Sec. V, restricting to $\Omega_m + \Omega_\Lambda + \Omega_D = 1$, a preliminary fit shows a qualitative difference if the dataset includes baryon acoustic oscillations (BAO). With exclusion, the result is $\Omega_D > 0$, allowing for an interpretation of dark energy density and the occurrence of a "big rip" future singularity. With inclusion, it is $\Omega_D < 0$: a different future singularity occurs, with $\dot{a} = 0$, and the dark energy density remains positive for most of the universe lifetime. In both cases, Ω_m (matter) is stable and close to the ΛCDM value.

In Sec. VI, we obtain the equation for the evolution of the density contrast δ_m in CKG (Sinyukov) cosmology in matter dominance, with the Λ term. It is solved in the relevant cases: $\Omega_m + \Omega_\Lambda = 1$ (ΛCDM), $\Omega_m + \Omega_D = 1$ (CKG with $\Omega_\Lambda = 0$), and $\Omega_m + \Omega_\Lambda + \Omega_D = 1$ (CKG). In matter dominance, the analytic and numerical (full CKG) solutions do not deviate from ΛCDM. Section VII contains a brief discussion of $\sigma_8(z)$, the root mean square of $\delta_m(z)$, whose values can be inferred from galaxy surveys. The result is that Λ, which is an integration constant in CKG, must be nonzero to produce the observed maximum of $f(z)\sigma_8(z)$. Conclusions summarize the results.

Notation: $i, j, k, \ldots = 0, 1, 2, \ldots$; $\mu, \nu, \ldots = 1, 2, \ldots$ The dot derivative is $\dot{H} = u^k\nabla_k H$, i.e., the time derivative in the comoving frame ($u^0 = 1$, $u^\mu = 0$).

II. PERFECT FLUID CONFORMAL KILLING TENSORS

In this section, we study conformal Killing tensors that are perfect-fluid tensors and the restrictions they pose on the space-time.

In dimension n, a symmetric tensor K_{ij} is a conformal Killing tensor (CKT) if [25]

$$\nabla_i K_{jl} + \nabla_j K_{li} + \nabla_l K_{ij} = \eta_i g_{jl} + \eta_j g_{li} + \eta_l g_{ij}, \quad (5)$$

$$\eta_i = \frac{\nabla_i K + 2\nabla_j K^j{}_i}{n+2}, \quad (6)$$

where η_i is the associated conformal vector, $K = g^{ij}K_{ij}$ is the trace, and $\nabla_j K^j{}_i$ is the divergence of the tensor. The CKT tensor is divergence-free if and only if

$$\eta_i = \frac{\nabla_i K}{n+2}. \quad (7)$$

A perfect-fluid tensor is characterized by a velocity vector field $u_p u^p = -1$ and scalar fields A and $B \neq 0$:

$$K_{jk} = A g_{jk} + B u_j u_k. \quad (8)$$

We recall that a velocity field has the canonical decomposition

$$\nabla_i u_j = H(g_{ij} + u_i u_j) - u_i \dot{u}_j + \omega_{ij} + \sigma_{ij}, \quad (9)$$

where $H = \frac{1}{n-1}\nabla_r u^r$ is the expansion parameter, $\dot{u}_i = u^k \nabla_k u_i$ is the acceleration, $\omega_{ij} = -\omega_{ji}$ is the vorticity, $\sigma_{ij} = \sigma_{ji}$ is the shear, with $g^{ij}\sigma_{ij} = 0$, $\omega_{ij}u^j = \sigma_{ij}u^j = 0$.

The following theorem parallels statements in [26] and [27]. A proof, simpler in our view and in tensor index notation, is given in Appendix A:

Theorem 1. The perfect-fluid tensor (8) is conformal Killing if and only if

(1) the velocity is shear-free,

(2) the expansion parameter H and the scalar B satisfy

$$H = \frac{\dot{B}}{2B}, \quad (10)$$

$$\nabla_j B = -\dot{B} u_j + 2B \dot{u}_j. \quad (11)$$

Then the conformal vector is

$$\eta_j = \nabla_j A + \dot{B} u_j. \quad (12)$$

The scalar field A is unconstrained.

Now we impose that the perfect-fluid CKT is divergence-free. The condition (7) is $(n+2)\eta_j = \nabla_j K$, where $K = nA - B$. At the same time, (12) holds. Then

$$\dot{B} u_j = -\nabla_j \frac{2A + B}{n+2}. \quad (13)$$

The conformal vector becomes $\eta_j = -\frac{1}{2}(n-1)\dot{B}u_j - B\dot{u}_j$ and Eq. (12) gives

$$\nabla_j A = -\frac{1}{2}(n+1)\dot{B}u_j - B\dot{u}_j.$$

We distinguish two cases:

$\dot{B} = 0$: $\eta_j = \nabla_j A$ and $2A + B$ is a number. Since Eq. (11) simplifies to $\nabla_j B = 2B\dot{u}_j$ the acceleration is closed. If also $\omega_{ij} = 0$, then the space-time is static:

$$\nabla_i u_j = -u_i \dot{u}_j, \qquad \nabla_i \dot{u}_j = \nabla_j \dot{u}_i.$$

$\dot{B} \neq 0$: Eq. (13) shows that u_i is hypersurface orthogonal. This implies that the vorticity is zero: $\omega_{ij} = 0$. Then $\nabla_i u_j = H(g_{ij} + u_i u_j) - u_i \dot{u}_j$.

Proposition 2. If $\dot{B} \neq 0$, then either the acceleration $\dot{u}_j$ is not closed or $\dot{u}_j = 0$.

Proof. Suppose that the acceleration is closed: $\nabla_i \dot{u}_j = \nabla_j \dot{u}_i$. Evaluate: $0 = \nabla_i \nabla_j B - \nabla_j \nabla_i B = -u_j \nabla_i \dot{B} + u_i \nabla_j \dot{B} - \dot{B}(u_i \dot{u}_j - \dot{u}_i u_j)$. This implies $\nabla_j \dot{B} = -\ddot{B}u_j - \dot{B}\dot{u}_j$. Next evaluate: $0 = \nabla_i \nabla_j A - \nabla_j \nabla_i A = (n+2)\dot{B}(u_i \dot{u}_j - u_j \dot{u}_i)$, which gives $\dot{B}\dot{u}_j = 0$. ∎

Hereafter, we choose $\dot{B} \neq 0$ and $\dot{u}_j = 0$. This specializes the theory of conserved perfect-fluid CKT tensors to space-times of cosmology.

A GRW space-time has the warped metric

$$ds^2 = -dt^2 + a(t)^2 g^\star_{\mu\nu}(\mathbf{x}) dx^\mu dx^\nu, \quad (14)$$

where $g^\star_{\mu\nu}(\mathbf{x})$ is the Riemannian metric of a spacelike hypersurface and $a(t)$ is the scale factor. A covariant characterization is the existence of a unit timelike vector field, $u_k u^k = -1$, such that [see [28,29]]:

$$\nabla_j u_k = H(g_{jk} + u_j u_k), \quad (15)$$

$$\nabla_j H = -\dot{H} u_j. \quad (16)$$

Condition (16) is equivalent to requiring that u_j is an eigenvector of the Ricci tensor $R_{ij}u^j = \xi u_i$. The eigenvalue is $\xi = (n-1)(H^2 + \dot{H})$.

In cosmology, H is the Hubble parameter and ξ is related to the acceleration:

$$H = \frac{\dot{a}}{a}, \qquad \xi = (n-1)\frac{\ddot{a}}{a}. \quad (17)$$

The Ricci tensor and the scalar curvature in a GRW space-time are

$$R_{kl} = \frac{R - n\xi}{n-1} u_l u_k + \frac{R - \xi}{n-1} g_{kl} - (n-2)E_{kl}, \quad (18)$$

$$R = \frac{R^\star}{a^2} + (n-1)(n-2)H^2 + 2\xi, \quad (19)$$

where $E_{kl} = u^j u^m C_{jklm}$ is the electric part of the Weyl curvature tensor C_{jklm} and $R^\star$ is the curvature of the spacelike hypersurface.

Proposition 3. If $K_{ij} = Ag_{ij} + Bu_i u_j$ is a divergence-free CKT with $\dot{u}_j = 0$, then the space-time is GRW.

Proof. By hypothesis, (15) holds and we only need show Eq. (16). With $H = \dot{B}/(2B)$ and $\nabla_j B = -\dot{B}u_j$, let us evaluate:

$$\begin{aligned}\nabla_j \dot{B} &= \nabla_j (u^k \nabla_k B),\\ &= (\nabla_j u^k)(-\dot{B}u_k) + u^k \nabla_j \nabla_k B,\\ &= u^k \nabla_k \nabla_j B = -u_j \ddot{B},\end{aligned}$$

Therefore, $\nabla_j H = -[\frac{\ddot{B}}{2B} + \frac{(\dot{B})^2}{2B^2}] u_j = -\dot{H}u_j$. ∎

If $\dot{u}_j = 0$, the perfect-fluid divergence-free CKT tensor lives in a GRW space-time and has a special property:

$$\begin{aligned}\nabla_i K_{jl} &= g_{jl}\nabla_i A + \nabla_i (Bu_j u_l)\\ &= -\frac{1}{2}(n+1)\dot{B}u_i g_{jl} - \dot{B}u_i u_j u_l\\ &\quad + BH(2u_i u_j u_l + g_{ij}u_l + g_{il}u_j)\\ &= -(n+1)BHu_i g_{jl} + BHu_j g_{li} + BHu_l g_{ij}. \qquad (20)\end{aligned}$$

This motivates the definition in the next section.

III. SINYUKOV-LIKE TENSORS AND GRW SPACE-TIMES

We consider symmetric tensors satisfying the relation

$$\nabla_j K_{kl} = a_j g_{kl} + b_k g_{jl} + b_l g_{jk} \qquad (21)$$

and call them "Sinyukov-like" tensors [30,31]. When K_{kl} is the Ricci tensor, we recover the characterization of Sinyukov manifolds, investigated, for example, in [32].

Contractions with g^{jk} and g^{kl} give a_j and b_j in terms of $K = g^{ij}K_{ij}$ and the divergence of the tensor. Thus, explicitly

$$\begin{aligned}\nabla_j K_{kl} = &\frac{(n+1)\nabla_j K - 2\nabla^p K_{pj}}{(n+2)(n-1)} g_{kl}\\ &+ \frac{n\nabla^p K_{pk} - \nabla_k K}{(n+2)(n-1)} g_{jl} + \frac{n\nabla^p K_{pl} - \nabla_l K}{(n+2)(n-1)} g_{jk}. \qquad (22)\end{aligned}$$

The cyclic sum of (22) shows that a Sinyukov-like tensor is a CKT.

At the end of Sec. II, we showed that a perfect-fluid conformal Killing tensor $K_{ij} = Ag_{ij} + Bu_i u_j$ that is divergence-free and acceleration-free, i.e.,

$$\nabla_i u_j = H(g_{ij} + u_i u_j), \qquad (23)$$

$$H = \frac{\dot{B}}{2B}, \qquad \nabla_j B = -\dot{B}u_j, \qquad \nabla_j A = -\frac{n+1}{2}\dot{B}u_j \qquad (24)$$

is a Sinyukov-like tensor, and the space-time is a GRW.

Now we easily prove the opposite: in a GRW space-time characterized by the velocity field u_j, a perfect-fluid tensor $Ag_{ij} + Bu_i u_j$ with A and B satisfying (24) is Sinyukov-like and divergence-free.

Proof. The evaluation (20) proves that K_{ij} is Sinyukov-like. The contraction with g^{ij} shows that it is also divergence-free. ∎

We then conclude:

Theorem 4. A space-time is GRW if and only if it admits a divergence-free, acceleration-free, perfect-fluid Sinyukov-like tensor.

Equation (24) can be solved in terms of the scale function $a(t)$. The first one, $H = \dot{a}/a = \dot{B}/(2B)$, has the solution $B(t) = \frac{1}{n-1}Ca^2(t)$, where C is constant. The second and third equations give $A(t) = \frac{1}{2}(n+1)B(t) - \Lambda$, where Λ is another constant. Therefore, the final form of the Sinyukov-like divergence-free tensor is only parametrized by the scale function $a(t)$ and the constants C and Λ:

$$K_{kl} = g_{kl}\left[\frac{1}{2}\frac{n+1}{n-1}Ca^2 - \Lambda\right] + u_k u_l \frac{Ca^2}{n-1}. \qquad (25)$$

Hereafter, we set the space-time dimension $n = 4$ in all the equations.

IV. CONFORMAL KILLING GRAVITY IN RW SPACE-TIMES

A GRW space-time is RW whenever the Weyl tensor is zero. With $E_{kl} = 0$, the Ricci tensor (18) has the perfect-fluid form and the Riemann tensor is [33]

$$\begin{aligned}R_{jklm} = &\frac{1}{6}(R - 2\xi)(g_{km}g_{jl} - g_{kl}g_{jm})\\ &+ \frac{1}{6}(R - 4\xi)(g_{km}u_j u_l - g_{kl}u_j u_m - g_{jm}u_k u_l\\ &+ g_{jl}u_k u_m). \qquad (26)\end{aligned}$$

Consider Eq. (3) of conformal Killing gravity in a RW space-time with the Sinyukov-like perfect-fluid term K_{kl} describing the imposing *dark sector* of gravity.

Equations (18) and (25) give the perfect-fluid energy-momentum tensor of ordinary matter with energy density μ and pressure p:

$$\begin{aligned} T_{kl} &= (p+\mu)u_k u_l + p g_{kl} \\ &= R_{kl} - \frac{1}{2}R g_{kl} - K_{kl} \\ &= -\frac{1}{6}(R + 2\xi + 5Ca^2 - 6\Lambda)g_{kl} \\ &\quad + \frac{1}{3}(R - 4\xi - Ca^2)u_k u_l. \end{aligned} \tag{27}$$

After specifying R with (19) and $\xi = 3(H^2 + \dot{H})$, the Friedmann equations for CKG with ordinary matter are

$$\mu = \frac{R^\star}{2a^2} + 3H^2 + \frac{1}{2}Ca^2 - \Lambda, \tag{28}$$

$$p = -\frac{R^\star}{6a^2} - 3H^2 - 2\dot{H} - \frac{5}{6}Ca^2 + \Lambda. \tag{29}$$

They reduce to the GR equations when $C = 0$ and $\Lambda = 0$.

We obtain

$$\mu + 3p = -6\frac{\ddot{a}}{a} - 2Ca^2 + 2\Lambda. \tag{30}$$

In [15], we showed that (28) and (29) imply Eq. (32) by Harada [14]. The converse was shown by Clément and Noucier [17].

Let ordinary matter be composed of dust with energy density μ_m and zero pressure, and radiation with EoS $p_r = \frac{1}{3}\mu_r$. Then $\mu = \mu_r + \mu_m$ and $p = p_r$. They are conserved independently. Dust: $0 = \nabla_k(\mu_m u^k u_l) = (\dot{\mu}_m + 3H\mu_m)u_l$. Radiation: $0 = \nabla_k(\mu_r + p_r)u^k u_l + \nabla_l p_r = \frac{4}{3}(\dot{\mu}_r + 3H\mu_r)u_l + \frac{1}{3}\nabla_l \mu_r$. The two equations give $\mu_m = \mu_{m0}(\frac{a}{a_0})^{-3}$ and $\mu_r = \mu_{r0}(\frac{a}{a_0})^{-4}$, where $a_0 = a(t_0)$ is the present-time scale.

The Friedmann equation (28) of CKG with radiation, matter, dark energy, Λ term, and curvature term is

$$\mu_{m0}\left(\frac{a}{a_0}\right)^{-3} + \mu_{r0}\left(\frac{a}{a_0}\right)^{-4} = \frac{R^\star}{2a^2} + 3H^2 + \frac{1}{2}Ca^2 - \Lambda.$$

We divide by $3H_0^2$ (the value at scale a_0) and obtain

$$\left(\frac{H}{H_0}\right)^2 = \Omega_r\left(\frac{a}{a_0}\right)^{-4} + \Omega_m\left(\frac{a}{a_0}\right)^{-3} + \Omega_k\left(\frac{a}{a_0}\right)^{-2} + \Omega_\Lambda + \Omega_D\left(\frac{a}{a_0}\right)^2, \tag{31}$$

$$\Omega_m = \frac{\mu_{m0}}{3H_0^2}, \qquad \Omega_r = \frac{\mu_{r0}}{3H_0^2}, \qquad \Omega_k = -\frac{R^\star}{6H_0^2 a_0^2},$$
$$\Omega_\Lambda = \frac{\Lambda}{3H_0^2}, \qquad \Omega_D = -\frac{Ca_0^2}{6H_0^2}. \tag{32}$$

Evidently, the cosmological parameters satisfy

$$\Omega_m + \Omega_r + \Omega_k + \Omega_\Lambda + \Omega_D = 1.$$

Since $H = \dot{a}/a$, Eq. (31) gives the time evolution of the scale function. In terms of the redshift $1 + z = a_0/a$, it becomes Eq. (4) by Harada:

$$\left(\frac{H(z)}{H_0}\right)^2 = \Omega_r(1+z)^4 + \Omega_m(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda + \frac{\Omega_D}{(1+z)^2}. \tag{33}$$

The "lookback time" of a photon emitted at time t_e at scale $a_e = a(t_e)$, and received at present time t_0 at scale factor a_0, is

$$t_L = \int_{t_e}^{t_0} dt = \int_{a_e}^{a_0} \frac{da}{\dot{a}} = \int_0^{z_e} \frac{dz}{(1+z)H(z)}. \tag{34}$$

The age of the Universe, if finite, is the limit $z \to \infty$ of t_L. The future time span of the Universe from now ($z = 0$) is the integral

$$\tau = \int_{z_f}^0 \frac{dz}{(1+z)H(z)}. \tag{35}$$

We obtain a bound for the future time span, valid when all $\Omega_j > 0$, $z_f = -1$:

$$\begin{aligned} \tau H_0 &\le \int_{-1}^0 \frac{dz}{\sqrt{\Omega_\Lambda(1+z)^2 + \Omega_D}} \\ &= \frac{1}{2\sqrt{\Omega_\Lambda}} \log \frac{\sqrt{\Omega_D + \Omega_\Lambda} + \sqrt{\Omega_\Lambda}}{\sqrt{\Omega_D + \Omega_\Lambda} - \sqrt{\Omega_\Lambda}}. \end{aligned} \tag{36}$$

V. CKG PARAMETERS FROM DATA FITS FOR H(Z)

In the following, we consider a flat space submanifold $\Omega_k = 0$, and also $\Omega_r = 0$. The smallness of $\Omega_r \approx 9.16 \times 10^{-5}$ (neutrino corrected) compared to $\Omega_m = 0.31$ (Planck data) determines a large value $1 + z_m = \Omega_m/\Omega_r$ where all but matter contributions are negligible. At decreasing redshifts, the Universe is in the matter-dominated era until Λ and D take over.

With three relevant terms, it is

$$\left(\frac{H(z)}{H_0}\right)^2 = \Omega_m(1+z)^3 + \Omega_\Lambda + \frac{\Omega_D}{(1+z)^2}. \tag{37}$$

If $\Omega_D > 0$, the squared Hubble parameter diverges to $+\infty$ for $z \to -1^-$ (future singularity). If $\Omega_D < 0$, the requirement $H^2(z) \ge 0$ determines a critical value $H^2(z_c) = 0$, where $\dot{a} = 0$ at finite scale $a = a_0/(1 + z_c)$.

The deceleration parameter is $q = -\frac{a\ddot{a}}{\dot{a}^2} = -1 - \frac{\dot{H}}{H^2}$. With $\dot{H} = -\frac{1}{2}\frac{dH^2}{dz}(z+1)$, it is (see Fig. 3)

$$q(z) = \frac{\Omega_m(1+z)^5 - 2\Omega_\Lambda(1+z)^2 - 4\Omega_D}{2\Omega_m(1+z)^5 + 2\Omega_\Lambda(1+z)^2 + 2\Omega_D}. \quad (38)$$

As a preliminary test, we make a best fit for the parameters α_k in

$$H(z) = \sqrt{\alpha_m(1+z)^3 + \alpha_\Lambda + \alpha_D(1+z)^{-2}}$$

with the available dataset (z_j, H_j, σ_j) for cosmic chronometers (CC, 32 points) and CC + BAO (58 points) taken from [34–38] and listed in Appendix B. The fit is made with *Mathematica*, `NonLinearFit` with weights (see Figs. 1 and 2).

	CC	CC	CC + BAO	CC + BAO
	ΛCDM	CKG	ΛCDM	CKG
α_m	$1483_{\pm 173}$	$1586_{\pm 303}$	$1309_{\pm 51}$	$1262_{\pm 63}$
α_Λ	$3163_{\pm 540}$	$1877_{\pm 3119}$	$3567_{\pm 193}$	$4544_{\pm 788}$
α_D	⋯	$1638_{\pm 3919}$	⋯	$-1688_{\pm 1320}$

Next, we evaluate $H_0 = H(0)$, $\Omega_k = \alpha_k/H_0^2$, and related values listed in the table:

	CC	CC	CC + BAO	CC + BAO
	ΛCDM	CKG	ΛCDM	CKG
H_0	68.16	71.42	69.83	64.17
q_0	–0.516	–0.854	–0.549	–0.130
Ω_m	0.323	0.311	0.301	0.306
Ω_Λ	0.677	0.368	0.699	1.103
Ω_D	⋯	+0.321	⋯	–0.410
Age	13.59	13.47	13.92	14.10

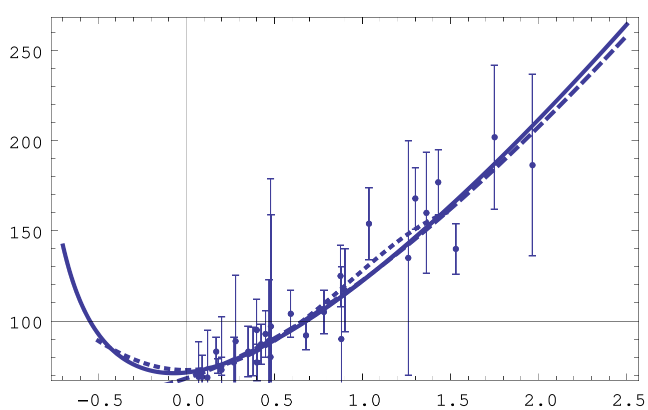

FIG. 1. Best fit for $H(z)$ for ΛCDM (dashed) and CKT (full), with 32 experimental data points with cosmic chronometers, including the recent point $z = 1.26$ [38]. The dotted line is the output of the genetic algorithm with CC data (31 points), Eq. (7) in [55], extrapolated out of $0 < z < 1.4$.

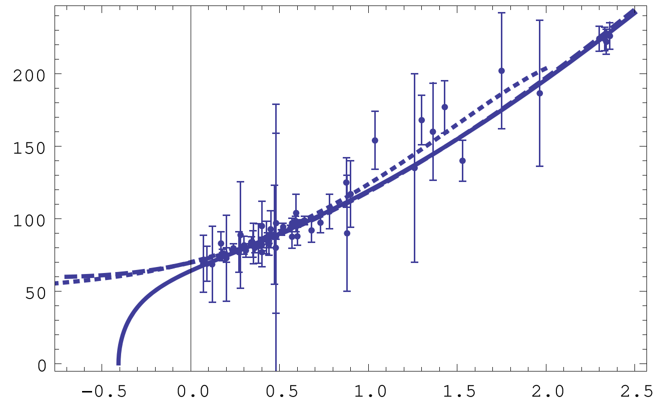

FIG. 2. Best fit for $H(z)$ for ΛCDM (dashed) and CKT (full) and 58 CC + BAO data. The dotted line is the output of the genetic algorithm, Eq. (9) in [55] extrapolated out of $0 < z < 1.4$.

With cosmic chronometer (CC) data, one notes that Ω_m is little changed by the presence of the Sinyukov terms. The ΛCDM value of Ω_Λ splits into the sum of the two dark terms. The ΛCDM and CKG fitting lines almost overlap in the past and deviate for z near zero. The deviation is more marked in $q(z)$ toward the future: driven by the D term, the CKG universe accelerates more than the ΛCDM solution, driven by the Λ term.

With CC + BAO data, the significant feature is $\Omega_D < 0$. This makes $H^2(z)$ vanish at $z_c = -0.407$ which is a limit value ($H^2 < 0$ in the interval $(-1, z_c)$).

Given the great uncertainty of the values α_Λ and α_D, a fit with SnIa and new data with small z are highly desirable.

The future time span of the Universe from now ($z = 0$) is the integral

$$\tau = \int_{z_f}^{0} \frac{dz}{(1+z)H(z)} = \begin{cases} 21.0 \text{ Gy} & \Omega_D > 0, z_f = -1 \\ 15.1 \text{ Gy} & \Omega_D < 0, z_f = -0.407 \end{cases}. \quad (39)$$

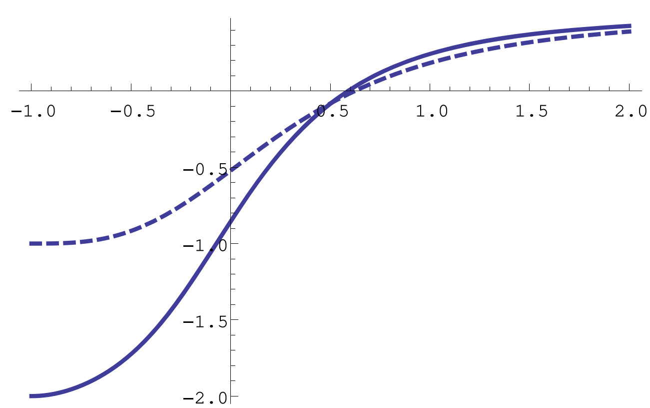

FIG. 3. The deceleration parameter $q(z)$, Eq. (38), for ΛCDM (dashed) and CKT (full) (CC data).

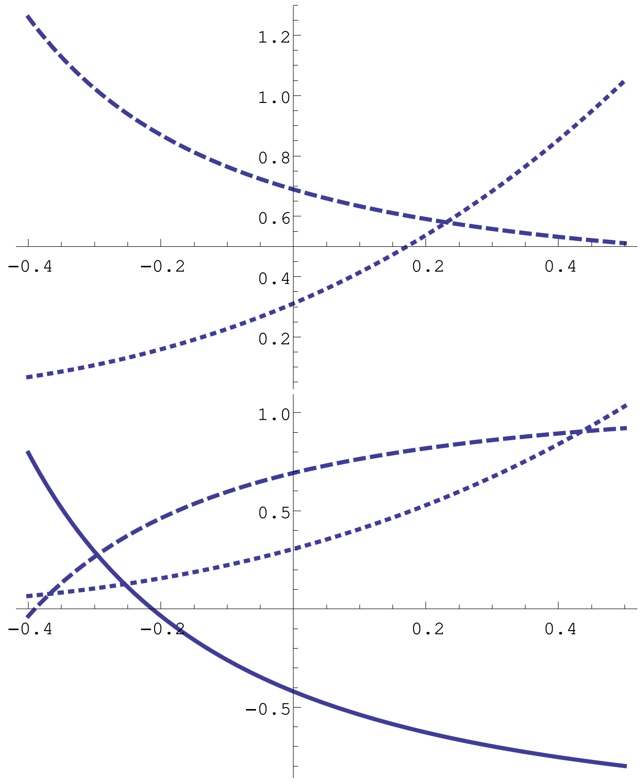

FIG. 4. The dark energy density $\mu_D(z)$ (dashed), the dark pressure $p_D(z)$ (full), and the matter energy density $\mu_m(z)$ (dotted) in units $3H_0^2$, evaluated with CC (top), CC + BAO (bottom). For CC, the pressure is negative and not shown. Note that $\mu_D > 0$ throughout almost the whole range; it turns negative close to the future singularity with CC + BAO data.

The value τ for $\Omega_D > 0$ is only slightly smaller than the estimate $\tau < 21Gy$ with Eq. (36): this signals the ongoing marginality of the matter term. Different from ΛCDM, τ is finite because of the occurrence of future singularities. Future singularities are discussed in [39,40].

$\Omega_D > 0$: Both $H(z)$ and $a(t)$ diverge at $z = -1$. The parameters of the Sinyukov tensor (25) may be interpreted as energy density and pressure of the dark perfect fluid:

$$\mu_D = -\frac{1}{2}Ca^2(t) + \Lambda = 3H_0^2\left[\frac{\Omega_D}{(1+z)^2} + \Omega_\Lambda\right], \quad (40)$$

$$p_D = \frac{5}{6}Ca^2(t) - \Lambda = -3H_0^2\left[\frac{5}{3}\frac{\Omega_D}{(1+z)^2} + \Omega_\Lambda\right], \quad (41)$$

where $\mu_D > 0$ and $p_D < 0$ (see Fig. 4). As both diverge, the future singularity is a "big rip" [41]. $w(z) = p_D/\mu_D$ ranges from -1 (past) to $-5/3$.

$\Omega_D < 0$: $H(z)$ vanishes as a square root at $z(\tau)$ and $a(\tau)$ is finite. Given the values of the CC + BAO fit, for $z > -0.39$, the energy density is positive; for $z > -0.21$, the pressure is negative. At $z(\tau)$, the pressure is finite and positive, while $\mu_D < 0$. This future singularity is not in the scheme in [40].

In [42], the authors discussed a model for the present accelerated phase of the Universe without the explicit introduction of dark energy or new degrees of freedom: it relies on the coupling between dark and ordinary matter through an effective metric. A similar approach is pursued in [43], where an exotic effective fluid arises from a disformal transformation. In this scenario, the past evolution is similar to ΛCDM, but the late dynamics gives rise to a monotonous increase of expansion.

The Hubble tension is a prominent problem in high-precision cosmology. It refers to the fact that the local measurement for H_0 is significantly higher than the value extrapolated from cosmological observations, notably CMB measurements.

The SH0ES team recently found $H_0 = 73.0 \pm 1\ \mathrm{km\,s^{-1}/Mps}$ [44]. On the other hand, CMB data refer to the early Universe, and to extrapolate H_0 requires a cosmological model. In ΛCDM, the value inferred from Planck data is $H_0 = 67.8 \pm 0.5\ \mathrm{km\,s^{-1}/Mps}$ [45]. The value $H_0 = 71.4\ \mathrm{km\,s^{-1}/Mps}$ that is here obtained in CKG with the CC dataset seems to alleviate the tension. A detailed survey of the H_0 tension is in Sec. 2 of [46], that also reports a list of measures of H_0 (Table 1 and Fig. 10).

VI. GROWTH OF PERTURBATIONS IN CKG COSMOLOGY

In this section, we investigate the equations governing the growth of perturbations. We follow the framework of spherical collapse illustrated by Abramo *et al.* [47]. The procedure has been used in several theories of extended gravity, such as mimetic gravity [48], energy momentum-squared gravity [49], time-dependent Λ [50], modified torsion cosmology [51], and generalized Rastall gravity [52].

We set the ordinary matter of the spatially flat FRW Universe ($R^\star = 0$) to be a pressureless dust, in the presence of the Λ and dark terms. Equations (28) and (29) give

$$\mu_m = 3H^2 + \frac{C}{2}a^2 - \Lambda, \quad (42)$$

$$\frac{\ddot{a}}{a} = -\frac{\mu_m}{6} - \frac{C}{3}a^2 + \frac{\Lambda}{3}. \quad (43)$$

Consider a local perturbation of the density that changes the background value μ_m to $\mu_m^c = \mu_m(1+\delta_m)$. In spherical symmetry, the mass $\mu_m a^3$ in a spherical volume with radius a^3 fills a spherical volume with radius a_p^3 with mass density μ_m^c: $\mu_m^c a_p^3 = \mu_m a^3$. We infer

$$a_p = a(1+\delta_m)^{-1/3} \approx a\left(1 - \frac{1}{3}\delta_m\right).$$

The parameter $\delta_m = (\mu_m^c/\mu_m) - 1$ is the "density contrast" [53].

Let the conservation rule for the perturbed sphere be $\dot{\mu}_m^c + 3h\mu_m^c = 0$, where $h = \dot{a}_p/a_p$ is the "local" Hubble parameter. While the background Universe evolves as (43), an analogous evolution for the perturbed cluster is assumed:

$$\frac{\ddot{a}_p}{a_p} = -\frac{\mu_m^c}{6} - \frac{C}{3}a_p^2 + \frac{\Lambda}{3}. \quad (44)$$

In the linear approximation, it is $\dot{a}_p = \dot{a}(1 - \frac{1}{3}\delta_m) - \frac{1}{3}a\dot{\delta}_m$. Another dot-derivative gives $\ddot{a}_p = \ddot{a}(1 - \frac{1}{3}\delta_m) - \frac{2}{3}\dot{a}\dot{\delta}_m - \frac{1}{3}a\ddot{\delta}_m$. We divide by a_p and use the evolution equations:

$$-\frac{\mu_m^c}{6} - \frac{C}{3}a_p^2 = -\frac{\mu_m}{6} - \frac{C}{3}a^2 - \frac{2}{3}H\dot{\delta}_m - \frac{1}{3}\ddot{\delta}_m.$$

We simplify by inserting $\mu_m^c = \mu_m(1+\delta_m)$ and $a_p^2 = a^2(1 - \frac{2}{3}\delta_m)$, using (43) to specify μ_m. Then

$$\ddot{\delta}_m + 2H\dot{\delta}_m - \left[\frac{3}{2}H^2 - \frac{5C}{12}a^2 - \frac{\Lambda}{2}\right]\delta_m = 0. \quad (45)$$

We trade the dot-derivative with the derivative with respect to the scale factor

$$\dot{\delta}_m = \frac{d\delta_m}{da}\dot{a} = \frac{d\delta_m}{da}Ha,$$
$$\ddot{\delta}_m = \frac{d^2\delta_m}{da^2}\dot{a}^2 + \frac{d\delta_m}{da}\ddot{a} = \frac{d^2\delta_m}{da^2}H^2a^2 - \frac{d\delta_m}{da}a\left[\frac{H^2}{2} + \frac{5}{12}Ca^2 - \frac{\Lambda}{2}\right]$$

and obtain

$$\frac{d^2\delta_m}{da^2}a^2 + a\frac{d\delta_m}{da}\left[\frac{3}{2} - \frac{5C}{12H^2}a^2 + \frac{\Lambda}{2H^2}\right] - \left[\frac{3}{2} - \frac{5C}{12H^2}a^2 - \frac{\Lambda}{2H^2}\right]\delta_m = 0.$$

The equation may be written in redshift space $a = \frac{1}{1+z}$ ($a_0 = 1$). It is

$$\frac{d\delta_m}{da} = -\frac{d\delta_m}{dz}(1+z)^2,$$
$$\frac{d^2\delta_m}{da^2} = \frac{d^2\delta_m}{dz^2}(1+z)^4 + 2\frac{d\delta_m}{dz}(1+z)^3,$$
$$\frac{d^2\delta_m}{dz^2}(1+z)^2 + \frac{d\delta_m}{dz}(1+z)\left[\frac{1}{2} + \frac{5C - 6\Lambda(1+z)^2}{12H^2(1+z)^2}\right] - \left[\frac{3}{2} - \frac{5C + 6\Lambda(1+z)^2}{12H^2(1+z)^2}\right]\delta_m = 0.$$

At this point, we employ $C = -6H_0^2\Omega_D$ and $\Lambda = 3H_0^2\Omega_\Lambda$:

$$0 = \frac{d^2\delta_m}{dz^2}(1+z)^2 + \frac{d\delta_m}{dz}(1+z) \times \frac{(H/H_0)^2(1+z)^2 - 5\Omega_D - 3\Omega_\Lambda(1+z)^2}{2(H/H_0)^2(1+z)^2} = -\frac{3(H/H_0)^2(1+z)^2 + 5\Omega_D - 3\Omega_\Lambda(1+z)^2}{2(H/H_0)^2(1+z)^2}\delta_m.$$

And we insert $H(z)/H_0$ given in (37):

$$0 = \frac{d^2\delta_m}{dz^2}(1+z)^2 + \frac{1}{2}\frac{d\delta_m}{dz}(1+z) \times \frac{\Omega_m(1+z)^5 - 2\Omega_\Lambda(1+z)^2 - 4\Omega_D}{\Omega_m(1+z)^5 + \Omega_\Lambda(1+z)^2 + \Omega_D} - \frac{\delta_m}{2}\frac{3\Omega_m(1+z)^5 + 8\Omega_D}{\Omega_m(1+z)^5 + \Omega_\Lambda(1+z)^2 + \Omega_D}. \quad (46)$$

Now we discuss various cases:

(1) When $\Omega_D = 0$ and $\Omega_\Lambda = 0$, we recover the GR evolution of the density contrast:

$$\delta_{m,\mathrm{GR}}(z) = c_1(1+z)^{3/2} + c_2(1+z)^{-1}. \quad (47)$$

The constants c_1 and c_2 are determined by initial conditions at a reference redshift $z_i \gg 1$ (matter dominated era):

$$\delta_{m,\mathrm{GR}}(z_i) = c_1(1+z_i)^{3/2} + c_2(1+z_i)^{-1},$$
$$\delta'_{m,\mathrm{GR}}(z_i) = \frac{3}{2}c_1(1+z_i)^{1/2} - c_2(1+z_i)^{-2}.$$

It is required that the fluctuation is small and the derivative is negative and small (initial growth of structures). This rules out c_1 as unphysical and poses the"adiabatic condition"

$$\delta'_m(z_i) = -\frac{\delta_m(z_i)}{1+z_i} \quad (48)$$

that is used as a criterion to fix the constants in more general conditions.

(2) Equation (46) is now studied for $\Omega_\Lambda \neq 0$ and $\Omega_D = 0$. This is standard ΛCDM [see Peebles [53] and Martel [54]]. In this approximation, we put $\alpha = \Omega_m/\Omega_\Lambda$. The common acceptance is $\alpha \approx 3/7$. Matter dominance means $\Omega_m(1+z)^3 \gg 1 - \Omega_m$, i.e., $z \gg \sqrt[3]{\alpha} = 0.3$.

The equation for δ_m is

$$0 = \frac{d^2\delta_m}{dz^2}(1+z)^2 + \frac{1}{2}\frac{d\delta_m}{dz}(1+z)\frac{\alpha(1+z)^3-2}{\alpha(1+z)^3+1} - \frac{\delta_m}{2}\frac{3\alpha(1+z)^3}{\alpha(1+z)^3+1}. \quad (49)$$

Let us introduce the new variable $x = \frac{1}{\alpha}(1+z)^{-3}$. The differential equation becomes

$$0 = x^2(1+x)\frac{d^2\delta_m}{dx^2} + \frac{x}{6}\frac{d\delta_m}{dx}(10x+7) - \frac{1}{6}\delta_m.$$

It reduces to a hypergeometric equation by appropriate γ in $\delta_m(x) = x^\gamma F(x)$. F solves the equation

$$x^2(1+x)F'' + \frac{x}{6}F'[2x(5+6\gamma)+7+12\gamma] + \frac{F}{6}(6\gamma^2+6\gamma-1+2x\gamma(3\gamma+2)] = 0.$$

The choices $\gamma = \frac{1}{3}$ and $\gamma = -\frac{1}{2}$ give two hypergeometric equations for $G(x) = F(-x)$:

$$x(1-x)G'' + \frac{G'}{6}[-2x(5+6\gamma)+7+12\gamma] - \frac{G}{3}\gamma(3\gamma+2) = 0.$$

For $\gamma = -\frac{1}{2}$, $G(x) = {}_2F_1(\frac{1}{6}, -\frac{1}{2}; \frac{1}{6}; x) = \sqrt{1-x}$. For $\gamma = \frac{1}{3}$, $G(x) = {}_2F_1(1, \frac{1}{3}; \frac{11}{6}; x)$.

The general solution is

$$\delta_m(x) = \kappa_1\sqrt{\frac{1+x}{x}} + \kappa_2 x^{1/3}{}_2F_1\left(1, \frac{1}{3}; \frac{11}{6}; -x\right).$$

It coincides with the case $m = 0$ for the evolution of density perturbations in the Newtonian approximation with $\Lambda = 3\alpha a(t)^{-m}$, by Silveira and Waga [50]. Back to redshift:

$$\delta_m(z) = \kappa_1\sqrt{1+\alpha(1+z)^3} + \kappa_2\frac{1}{1+z}{}_2F_1\left(1, \frac{1}{3}; \frac{11}{6}; -\frac{1}{\alpha(1+z)^3}\right). \quad (50)$$

In the dominant matter regime, $\alpha(1+z)^3$ is large and the hypergeometric function is around unity. The ΛCDM density contrast is given by the above formula with $\kappa_1 = 0$ and becomes GR for large z. For large z_i, it fulfills the adiabatic condition (48).

(3) Equation (46) is studied with $\Omega_\Lambda = 0$ (in CKG, the cosmological constant Λ is an integration constant, not a term of the field equations). Now Ω_D comes into play. Let $\beta = \Omega_m/\Omega_D$.

$$\frac{d^2\delta_m}{dz^2}(1+z)^2 + \frac{1}{2}\frac{d\delta_m}{dz}(1+z)\frac{\beta(1+z)^5-4}{\beta(1+z)^5+1} - \frac{1}{2}\frac{3\beta(1+z)^5+8}{\beta(1+z)^5+1}\delta_m = 0. \quad (51)$$

The equation is solved in the new variable $x = \frac{1}{\beta(1+z)^5}$. $(1+z)\frac{d\delta_m}{dz} = -5x\frac{d\delta_m}{dx}$, and $(1+z)^2\frac{d^2\delta_m}{dz^2} = 30x\frac{d\delta_m}{dx} + 25x^2\frac{d^2\delta_m}{dx^2}$. Equation (51) takes the form

$$x^2(1+x)\frac{d^2\delta_m}{dx^2} + \frac{x}{10}(16x+11)\frac{d\delta_m}{dx} - \frac{8x+3}{50}\delta_m = 0. \quad (52)$$

Now change the dependent variable $\delta_m(x) = x^\gamma F(x)$:

$$x(1+x)\frac{d^2F}{dx^2} + \frac{dF}{dx}\left[2\gamma(1+x) + \frac{16x+11}{10}\right] + \left(\gamma - \frac{1}{5}\right)\left[\frac{1}{x}\left(\gamma + \frac{3}{10}\right) + \left(\gamma + \frac{4}{5}\right)\right]F = 0.$$

The value $\gamma = \frac{1}{5}$ simplifies the equation $x(1+x)F'' + F'(2x + \frac{3}{2}) = 0$, i.e.,

$$\frac{d}{dx}\left[x(1+x)F'(x) + \frac{1}{2}F(x)\right] = 0.$$

Then $2x(1+x)F'(x) + F(x) = f_1$, where f_1 is a constant, with the solution $F(x) = f_2\sqrt{\frac{1+x}{x}} + f_1$. The final result is a simple expression:

$$\delta_m(z) = f_1\frac{1}{1+z} + f_2\frac{\sqrt{1+\beta(1+z)^5}}{1+z}. \quad (53)$$

For large $\beta(1+z)$, we recover the functional form of the GR result (47). The evolution of the matter and dark energy densities are

$$\Omega_m(z) = \frac{\Omega_m(1+z)^3}{(H(z)/H_0)^2}, \qquad \Omega_D(z) = \frac{\Omega_D(1+z)^{-2}}{(H(z)/H_0)^2}.$$

Thus, the condition of matter dominance $\Omega_m(z)/\Omega_D(z) > 1$ is $\beta(1+z)^5 > 1$, i.e., $z > \sqrt[5]{\Omega_D/\Omega_m} - 1$. Since for large z the expression of the density contrast is very similar to the GR expression, we apply the adiabatic condition (48), as usually done in the literature:

$$\frac{d\delta_m(z)}{dz} = -\frac{f_1}{(1+z)^2} + f_2\frac{3\beta(1+z)^5-2}{2(1+z)^2\sqrt{1+\beta(1+z)^5}}.$$

The adiabatic condition (48) at large z_i imposes $f_2 = 0$. The density contrast grows linearly in the scale factor as in GR.

(4) Evolution in CKG with $\Omega_\Lambda \neq 0$, $\Omega_m + \Omega_\Lambda + \Omega_D = 1$. Equation (46) is rewritten in the variable x:

$$x = \frac{1}{\beta(1+z)^5}, \qquad \beta = \frac{\Omega_m}{\Omega_D}, \qquad \eta = \sqrt[5]{\frac{\Omega_\Lambda^5}{\Omega_m^2 \Omega_D^3}},$$

$$0 = 25x^2 \frac{d^2\delta_m}{dx^2} + \frac{5}{2} x \frac{d\delta_m}{dx} \frac{11 + 14\eta x^{3/5} + 16x}{1 + \eta x^{3/5} + x} - \frac{\delta_m}{2} \frac{3 + 8x}{1 + \eta x^{3/5} + x}. \tag{54}$$

For z in the range 10–100 (small x), the leading behaviour of the independent solutions is read in the approximation $0 = 25x^2 \frac{d^2\delta_m}{dx^2} + \frac{55}{2} x \frac{d\delta_m}{dx} - \frac{3\delta_m}{2}$ that gives an expanding and a decreasing asymptotic solution: $\delta_m(x) = \kappa_- x^{-3/10} + \kappa_+ x^{1/5}$.

Starting with the factorization $\delta_m^- = x^{-3/10} F^-(x)$, Eq. (54) becomes

$$25x^2 \frac{d^2F^-}{dx^2}(1 + \eta x^{3/5} + x) + \frac{5}{2} x[5 + 10x + 8\eta x^{3/5}] \frac{dF^-}{dx} - \frac{1}{4} F[3\eta x^{3/5} + 25x] = 0.$$

Now make the ansatz $F^-(x) = \sqrt{1 + A\eta x^{3/5} + Bx}$. Neglecting terms of order $x^{6/5}$ and smaller, it is $x^2 \frac{d^2F^-}{dx^2} = -\frac{3}{25} A\eta x^{3/5}$ and $x \frac{dF^-}{dx} = \frac{3}{10} A\eta x^{3/5} + \frac{B}{2} x$. The equation for F^- simplifies to $3(A-1)\eta x^{3/5} + 25x(B-1) = 0$ that gives $A = B = 1$. One then obtains the following expression, correct up to vanishing terms $x^{6/5}$:

$$\delta_m^-(x) = \sqrt{\frac{1 + \eta x^{3/5} + x}{x^{3/5}}}. \tag{55}$$

Next, by posing $\delta^+(x) = \delta^-(x) F^+(x)$, one obtains the other approximate solution (this procedure is equivalent to exploiting the Wronskian).

$$\delta_m^+(x) = \sqrt{\frac{1 + \eta x^{3/5} + x}{x^{3/5}}} \int_0^x \frac{dx'}{\sqrt{x'}(1 + \eta x'^{3/5} + x')^{3/2}}. \tag{56}$$

Similar expressions, with different exponents, were obtained by Hugo Martel in the study of density perturbations of the Friedmann equations with the Λ term in a linear adiabatic regime [Eqs. (10) and (11) in [54]].

Equation (46) is also solved numerically with NDSolve (*Mathematica* 7). For comparison with other approximations, the initial conditions $\delta_m(z_i)$ and $\delta'_m(z_i)$ are chosen as identical. We put $\delta_m(400) = 0.0001$ and $\delta'_m(400) = -\delta_m(400)/401$.

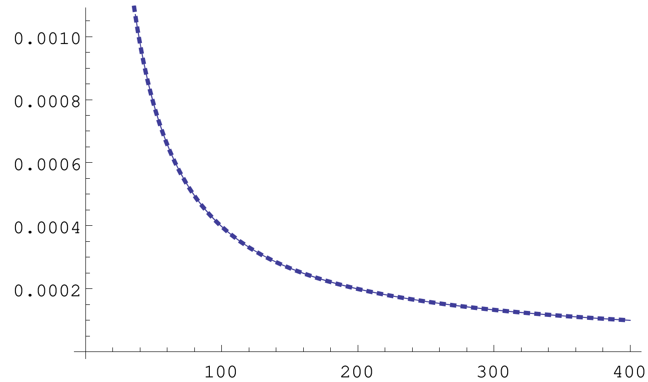

FIG. 5. The density contrast $\delta_m(z)$ with initial conditions $\delta_m(400) = 0.0001$ and $\delta'_m(400) = -\delta_m(400)/401$ for ΛCDM (dotted), and the numerical solution for CKG with CC data (*Mathematica*).

Despite the different analytic expressions and values of Ω_j, the plots in Fig. 5 of CKG gravity with or without cosmological constant do not show significative differences with the expanding mode of GR and ΛCDM in the matter-dominated phase.

VII. THE PARAMETER σ_8

A robust measurable quantity in redshift surveys, that is related to the density contrast $\delta_m(z)$, is the product $f(z)\sigma_8(z)$ [56] [for a simple account of the theory, see [57]]. $\sigma_8(z)$ is the root-mean-square mass fluctuation at the scale $R_8 = 8h^{-1}$ Mpc:

$$\sigma_8^2(z) = 4\pi \int_0^\infty k^2 dk \tilde{W}^2(k) P_{\delta_m}(k, z).$$

In the integral, $P_{\delta_m}(k, z)$ is the mass power spectrum, i.e., the Fourier transform of the correlator $\langle \delta_m(\mathbf{x}) \delta_m(\mathbf{x}') \rangle$ at redshift z. $\tilde{W}(k)$ is the Fourier transform of a window function $W(r)$ that filters the spatial scale; a choice is $W(r) = 3/(4\pi R_8^3)$ if $r < R_8$ and $W(r) = 0$ if $r > R_8$.

f is the growth rate of linear perturbations:

$$f = \frac{d \log \delta_m}{d \log a} = -\frac{1+z}{\delta_m(z)} \frac{d\delta_m}{dz}.$$

In the linear perturbation regime, $\delta_m(z)$ and $\sigma_8(z)$ are proportional [see Eq. (2.21) of [58]]. Then, let us consider the function

$$g(z) = \frac{f(z)\sigma_8(z)}{\sigma_8(0)} = -\frac{1+z}{\delta_m(0)} \frac{d\delta_m(z)}{dz}. \tag{57}$$

The function is plotted for ΛCDM and CKG in Fig. 6. The curve for ΛCDM shows a maximum, while the CKG

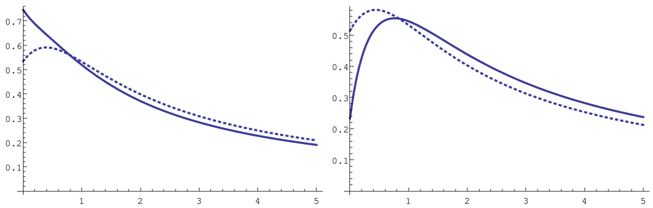

FIG. 6. The function $g(z)$, Eq. (57), for ΛCDM (dashed) and CKT (full). Left panel: CC data. Right panel: CC + BAO data.

curve does not for CC values of parameters. The maximum occurs in a redshift range that is borderline of the dominant matter era, wherein the linear evolution equations for $\delta_m(z)$ were obtained. Therefore, $\delta_m(0)$ is an extrapolated value.

Experimental measures of $f(z)\sigma_8(z) \propto g(z)$ show a weak maximum, as in Fig. 1 of [59].

To speculate about the occurrence of a maximum for $g(z)$ in CKG, let us fix the value $\Omega_M = 0.3$. It is numerically found that the maximum occurs at $z = 0$ for $\Omega_\Lambda = 0.55$ (then $\Omega_D = 0.15$). By increasing Ω_Λ, the maximum shifts to higher z. At $\Omega_\Lambda = 0.7$, the function $g(z)$ is ΛCDM; at higher values, Ω_D becomes negative (see Fig. 7). For the limit case $g'(0) = 0$, with the aid of Eq. (46), one obtains

$$g(0) = \frac{3\Omega_M + 8\Omega_D}{\Omega_M + 4\Omega_\Lambda + 6\Omega_D} = 0.617.$$

An interesting point of the plot is $z^\star \approx 0.85$, where the curves almost meet with the value $g^\star = 0.55$. It is then $f(z^*)\sigma_8(z^*) = 0.55\sigma_8(0)$ independently of the values of $\Omega_D + \Omega_\Lambda = 0.7$.

The tension in $\sigma_8 = \sigma_8(0)$ is the discrepancy between the values measured in the late Universe, that are smaller than the values found in CMB (early Universe). Planck data [45] give $\sigma_8 = 0.8120 \pm 0.0073$.

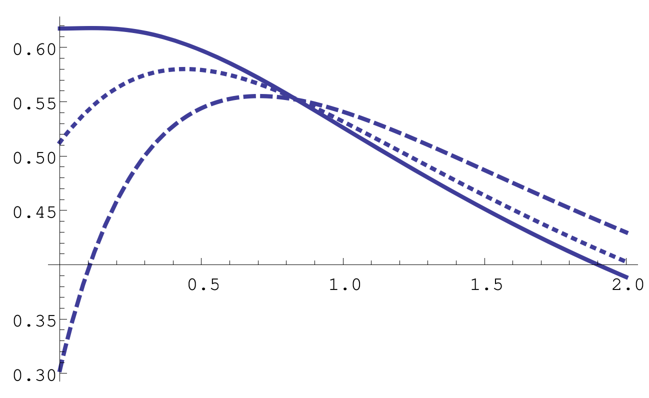

FIG. 7. The function $g(z)$, Eq. (57), with $\Omega_M = 0.3$ in three cases: (1) CKG limit case $\Omega_\Lambda = 0.55$, $\Omega_D = 0.15$ (full line); (2) ΛCDM with $\Omega_\Lambda = 0.7$, $\Omega_D = 0$ (dotted line); (3) CKG with $\Omega_\Lambda = 1$, $\Omega_D = -0.3$ (dashed line).

A complete survey of σ_8 tension is contained in [46], Sec. III.1, together with a wide list of measures of σ_8 (Table 2 and Fig. 21).

The function $g(z)$ and datasets for $f(z)\sigma_8(z)$ allow in principle a determination of σ_8 in CKG. According to the linear theory, Eq. (57), the ratio $f(z)\sigma_8(z)/g(z)$ is constant and equal to σ_8. We use two datasets, Benisty [Table 1 with 47 points in [59]], and Kazantzidis and Perilovaropoulos [Table II with 63 points in [60]], and evaluate $g(z_j)$ with CC or CC + BAO parameters. Different points yield different ratios; the best fitting values σ_8 (omitting error bar analysis of input data) are reported below, in the different cases:

	[59]	[60]
ΛCDM (CC)	0.779	0.768
CKG (CC)	0.727	0.711
ΛCDM (CC + BAO)	0.791	0.780
CKG (CC + BAO)	0.943	0.927

Due to the very uncertain behaviour of $g(z)$ in CKG with CC or CC + BAO datasets, a comparison with experimental data to verifty the σ_8 tension is premature.

VIII. CONCLUSIONS

A space-time is generalized RW if and only if it admits a divergence-free, acceleration-free, perfect-fluid Sinyukov-like tensor. Such tensors are conformal Killing tensors and are candidates for the dark sector of conformal Killing gravity. We study the Friedmann equations for CKG and determine the analytic form of $H(z)$. Neglecting radiation, in spatially flat RW space-time, $H(z)$ is fitted against experimental datasets for CC or CC + BAO to estimate the dark sector parameters Ω_D, Ω_Λ and the matter parameter Ω_m. While the model shows nonrelevant deviation of $H(z)$ in CKG from ΛCDM in the past, the future is driven by the dark term to a future singularity: a big rip if Ω_D and Ω_Λ are positive, an exotic one if $\Omega_D < 0$. Next we solve the equation for the evolution of the density contrast $\delta_m(z)$ in the linear regime. The solution in CKG, for values $\Omega_m + \Omega_\Lambda + \Omega_D = 1$, shows no sensible deviation in the matter-dominated regime from the ΛCDM or GR solutions.

While conformal Killing gravity, as an extension of Einstein's gravity, does not contradict the present view of the past evolution of the Universe, it differs in describing the late accelerated phase. New data for $H(z)$ are needed to solve the sign ambiguity of Ω_D that influences the evolution picture offered by the theory. The discussion of σ_8 clarifies that Λ, as an integration constant in CKG, must be taken as nonzero with $\Omega_\Lambda \geq 0.55$, to comply with present measures of $f(z)\sigma_8(z)$. Fitting with CC data does not yield a large enough value of Ω_Λ. The extension of data other than CC and BAO will be important to better determine the CKG parameters and validate the theory.

APPENDIX A: PROOF OF THEOREM 1

The perfect-fluid tensor $Ag_{jk} + Bu_j u_k$ is a CKT if (5) holds, with conformal vector (6):

$$(n+2)\eta_i = (n+2)\nabla_i A - \nabla_i B + 2\dot{B}u_i + 2B\dot{u}_i + 2Bu_i\nabla_j u^j. \tag{A1}$$

The condition (5) with (A1) is

$$\begin{aligned}0 = &(n+2)[\nabla_i(Bu_j u_l) + \nabla_j(Bu_i u_l) + \nabla_l(Bu_i u_j)]\\ &- g_{jl}[-\nabla_i B + 2\dot{B}u_i + 2B\dot{u}_i + 2Bu_i(\nabla_p u^p)]\\ &- g_{il}[-\nabla_j B + 2\dot{B}u_j + 2B\dot{u}_j + 2Bu_j(\nabla_p u^p)]\\ &- g_{ij}[-\nabla_l B + 2\dot{B}u_l + 2B\dot{u}_l + 2Bu_l(\nabla_p u^p)].\end{aligned} \tag{A2}$$

Contraction with $u^i u^j u^l$ is

$$0 = 3(n+2)u^j u^l u^i \nabla_i(Bu_j u_l) - 3[3\dot{B} + 2B(\nabla_p u^p)],$$

i.e.,

$$\frac{\nabla^p u_p}{n-1} = H = \frac{\dot{B}}{2B}. \tag{A3}$$

Contraction of (A2) with $u^j u^l$ is

$$\begin{aligned}0 &= (n+2)[\nabla_i B - 2\dot{B}u_i - 2B\dot{u}_i] - \nabla_i B + 2\dot{B}u_i + 2B\dot{u}_i\\ &\quad + 2Bu_i(\nabla_p u^p) - 2u_i[-3\dot{B} - 2B(\nabla_p u^p)]\\ &= (n+1)\nabla_i B - 2(n-2)\dot{B}u_i - 2(n+1)B\dot{u}_i\\ &\quad + 6Bu_i(\nabla_p u^p).\end{aligned}$$

We use (A3) and obtain Eq. (11): $\nabla_i B = -\dot{B}u_i + 2B\dot{u}_i$. This and (A3) are inserted in (A2):

$$\begin{aligned}0 = &2B(\dot{u}_i u_j u_l + u_i \dot{u}_j u_l + u_i u_j \dot{u}_l)\\ &+ B[\nabla_i(u_j u_l) + \nabla_j(u_i u_l) + \nabla_l(u_i u_j)]\\ &- (u_j u_l + g_{jl})\dot{B}u_i - (u_i u_l + g_{il})\dot{B}u_j - (u_i u_j + g_{ij})\dot{B}u_l.\end{aligned}$$

Contraction with u^i is

$$\nabla_j u_l + \nabla_l u_j = \frac{\dot{B}}{B}(u_j u_l + g_{jl}) - u_j \dot{u}_l - u_l \dot{u}_j.$$

If we insert the standard decomposition (9) of $\nabla_j u_l$, the equation is satisfied with shear $\sigma_{il} = 0$. The CKT condition (5) is now identically verified. The conformal vector (6) becomes $(n+2)(\eta_i - \nabla_i A) = 3\dot{B}u_i + 2B\nabla_i u^i$, i.e., $\eta_i = \nabla_i A + \dot{B}u_i$.

Let us prove the opposite. Suppose that the perfect fluid tensor (8) has shear-free velocity with expansion $\dot{B}/2B$ and B that satisfies (11).

$\nabla_i K_{jl} = g_{jl}\nabla_i A + (-\dot{B}u_i + 2B\dot{u}_i)u_j u_l + B(u_j \nabla_i u_l + u_l \nabla_i u_j) = g_{jl}\nabla_i A + B(2\dot{u}_i u_j u_l - u_i u_j \dot{u}_l - u_i \dot{u}_j u_l) + \frac{1}{2}\dot{B}(u_j g_{il} + u_l g_{ij}) + \frac{1}{2}B(u_j \omega_{il} + u_l \omega_{ij})$.

In the cyclic sum, many terms cancel: $\nabla_i K_{jl}$ + cyclic $= g_{jl}(\nabla_i A + \dot{B}u_i)$ + cyclic. The CKT condition is satisfied with $\eta_i = \nabla_i A + \dot{B}u_i$. ∎

APPENDIX B: DATASETS FOR H(z)

The left half is the CC dataset, and the right half is the BAO dataset [37,38].

z	H(z)	σ	z	H(z)	σ
0.070	69	19.6	0.24	79.69	2.99
0.090	69	12	0.30	81.7	6.22
0.120	68.6	26.2	0.31	78.18	4.74
0.170	83	8	0.34	83.8	3.66
0.1791	75	4	0.35	82.7	9.1
0.1993	75	5	0.36	79.94	3.38
0.200	72.9	29.6	0.38	81.5	1.9
0.270	77	14	0.40	82.04	2.03
0.280	88.8	36.6	0.43	86.45	3.97
0.3519	83	14	0.44	82.6	7.8
0.3802	83	13.5	0.44	84.81	1.83
0.400	95	17	0.48	87.79	2.03
0.4004	77	10.2	0.51	90.4	1.9
0.4247	87.1	11.2	0.52	94.35	2.64
0.4497	92.8	12.9	0.56	93.34	2.3
0.470	89	34	0.57	87.6	7.8
0.4783	80.9	9	0.57	96.8	3.4
0.480	97	62	0.59	98.48	3.18
0.593	104	13	0.60	87.9	6.1
0.6797	92	8	0.61	97.3	2.1
0.7812	105	12	0.64	98.82	2.98
0.8754	125	17	0.73	97.3	7.0
0.880	90	40	2.30	224	8.6
0.900	117	23	2.33	224	8
1.037	154	20	2.34	222	8.5
1.26	135	65	2.36	226	9.3
1.300	168	17			
1.363	160	33.6			
1.430	177	18			
1.530	140	14			
1.750	202	40			
1.965	186.5	50.4			

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