Test of the conjectured critical black-hole formation–null geodesic correspondence: The case of self-gravitating scalar fields

Shahar Hod
The Ruppin Academic Center, Emeq Hefer 40250, Israel
and The Hadassah Academic College, Jerusalem 91010, Israel

(Received 8 August 2024; accepted 14 August 2024; published 11 September 2024)

It has recently been conjectured [A. Ianniccari, A. J. Iovino, A. Kehagias, D. Perrone, and A. Riotto, Phys. Rev. Lett. **133**, 081401 (2024)] that there exists a correspondence between the critical threshold of black-hole formation and the stability properties of null circular geodesics in the curved spacetime of the collapsing matter configuration. In the present compact paper we provide a nontrivial test of this intriguing conjecture. In particular, using analytical techniques we study the physical and mathematical properties of self-gravitating scalar field configurations that possess marginally stable (degenerate) null circular geodesics. We reveal the interesting fact that the *analytically* calculated critical compactness parameter $\mathcal{C}^{\text{analytical}} \equiv \max_r\{m(r)/r\} = 6/25$, which signals the appearance of the first (marginally stable) null circular geodesic in the curved spacetime of the self-gravitating scalar fields, agrees quite well (to within $\sim 10\%$) with the exact compactness parameter $\mathcal{C}^{\text{numerical}} \equiv \max_t\{\max_r\{m(r)/r\}\} \simeq 0.265$ which is computed *numerically* using fully nonlinear numerical simulations of the gravitational collapse of scalar fields at the threshold of black-hole formation [here $m(r)$ is the gravitational mass contained within a sphere of radius r].

DOI: 10.1103/PhysRevD.110.064036

I. INTRODUCTION

Curved spacetimes of highly compact matter configurations may possess closed light rings (null circular geodesics) [1–3] on which massless particles can orbit the central compact object. These null geodesics are of fundamental physical importance in theoretical and observational studies of highly curved spacetimes (see [1–15] and references therein).

In particular, it has been shown that the optical appearance of a highly compact collapsing star is determined by the physical properties of its external light rings [4,5]. Likewise, the physically important phenomenon of strong gravitational lensing is closely related to the existence of closed null circular geodesics in the curved spacetimes of the central compact objects [6]. In addition, it is well established that the eikonal quasinormal resonance spectra that characterize the relaxation dynamics of highly compact self-gravitating objects (black holes and spatially regular ultracompact objects that possess light rings) are determined by the physical properties of these closed null circular geodesics (see [7–11] and references therein).

Intriguingly, Ianniccari *et al.* [16] have recently provided evidence that null circular geodesics may also be related to the formation of primordial black holes. In particular, it has been demonstrated in the physically important work [16] that there is a correspondence between the critical threshold of primordial black-hole formation and the appearance of the first (marginally stable) null circular geodesic in the curved spacetime of the collapsing matter configuration.

The main goal of the present compact paper is to test, using *analytical* techniques, the validity of the black-hole formation–null geodesic correspondence proposed in the physically interesting work [16]. To this end, we shall analyze the physical and mathematical properties of self-gravitating scalar field configurations that possess light rings. In particular, we shall determine the critical value,

$$\mathcal{C}^{\text{analytical}} \equiv \max_r\left\{\frac{m(r)}{r}\right\}, \tag{1}$$

of the dimensionless compactness parameter which signals the appearance of the first null circular geodesic in the curved spacetime of the self-gravitating field configuration.

Interestingly, below we shall reveal the fact that the *analytically* derived value $\mathcal{C}^{\text{analytical}}$ of the dimensionless compactness parameter, which signals the critical formation of a marginally stable null circular geodesic in the curved spacetime of the self-gravitating scalar configuration, agrees quite well (to within $\sim 10\%$) with the corresponding exact value $\mathcal{C}^{\text{numerical}}$ of the compactness parameter as determined from fully nonlinear *numerical* simulations [17] of the gravitational collapse of scalar fields at the critical threshold of black-hole formation (see also [18,19] and references therein).

II. DESCRIPTION OF THE SYSTEM

We consider self-gravitating scalar field configurations which are characterized by the spherically symmetric curved line element [2,20]

$$ds^2 = -\alpha^2(r,t)dt^2 + a^2(r,t)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (2)$$

The Einstein equations yield the dimensionless functional relations [17,21]

$$\frac{r\alpha'}{\alpha} - \frac{ra'}{a} + 1 - a^2 = 0 \quad (3)$$

and

$$a(r) = \left[1 - \frac{2m(r)}{r}\right]^{-1/2} \quad (4)$$

for the metric functions, where [22]

$$m(r) = \int_0^r 4\pi x^2 \rho(x) dx \quad (5)$$

is the gravitational mass contained within a sphere of radius r.

It is convenient to define the dimensionless compactness function

$$\mathcal{C}(r) = \frac{m(r)}{r}, \quad (6)$$

which is characterized by the physically motivated boundary condition

$$\mathcal{C}(r = 0) = 0 \quad (7)$$

of a regular origin.

III. TESTING THE CONJECTURED CRITICAL BLACK-HOLE FORMATION–NULL GEODESIC CORRESPONDENCE

In the present section we shall test the validity of the physically intriguing correspondence suggested in [16] between the threshold of black-hole formation and the appearance of the first (marginally stable) circular geodesic in the corresponding curved spacetime of the self-gravitating matter configuration.

Following [16] we shall consider a physical situation in which the curved spacetime is temporarily stationary. It is interesting to note that this assumption is particularly suitable for the present model of a self-gravitating scalar field whose critical solution at the threshold of black-hole formation has a discrete self-similar character [17–19]. Thus, the critical solution of the scalar field model at the threshold of black-hole formation is characterized by a discrete set of stationary times $\{t_n\}_{n=1}^{n=\infty}$ for which physically measurable quantities, such as the dimensionless compactness parameter $\mathcal{C}_{\max}(t) \equiv \max_r\{\mathcal{C}(r)\}$, are temporarily stationary. In particular, below we shall use analytical techniques in order to estimate the maximum value,

$$\mathcal{C}^*_{\max} \equiv \max_t\{\max_r\{\mathcal{C}(r,t)\}\}, \quad (8)$$

of the compactness parameter which is defined by the relation

$$\frac{d\mathcal{C}_{\max}}{dt} = 0 \quad \text{for } t \in \{t_n\}_{n=1}^{n=\infty}. \quad (9)$$

The condition for the existence of light ring(s) in the curved spacetime is given by the compact functional relation [16,23–25]

$$\mathcal{F}(r) \equiv \frac{r\alpha'}{\alpha} - 1 = 0 \quad \text{for } r = r_{\rm c}. \quad (10)$$

It is well established [23–25] that spatially regular curved spacetimes generally possess an *even* (or zero) number of null circular geodesics (closed light rings). In particular, the first appearance of a marginally stable light ring in the curved spacetime is characterized by the degenerate functional relation [16,23–25]

$$\mathcal{F}(r_{\rm c}) = \mathcal{F}'(r_{\rm c}) = 0. \quad (11)$$

From Eqs. (10) and (11) one obtains the characteristic relation

$$\alpha'' = 0 \quad \text{for } r = r^*_{\rm c} \quad (12)$$

for the radial location, $r = r^*_{\rm c}$, of the marginally stable (degenerate) null circular geodesic in the curved spacetime.

Taking cognizance of Eqs. (3), (4), (10), and (12) one obtains the two coupled equations

$$ra' - (2 - a^2)a = 0 \quad \text{for } r = r^*_{\rm c} \quad (13)$$

and

$$(a' + ra'')a - ra'^2 + 2a^3a' = 0 \quad \text{for } r = r^*_{\rm c}, \quad (14)$$

which yield the functional relations [see Eqs. (1) and (4)] [26]

$$4\mathcal{C}_{\rm c} + r^*_{\rm c}\mathcal{C}'_{\rm c} = 1 \quad (15)$$

and

$$20\mathcal{C}_{\rm c} - r^{*2}_{\rm c}\mathcal{C}''_{\rm c} = 5 \quad (16)$$

for the marginally stable null circular geodesic. [The subscript c means that the physical quantities are evaluated at the critical radius $r = r_c^*$ of the marginally stable (degenerate) null circular geodesic].

To determine analytically the values of the dimensionless physical quantities $\{\mathcal{C}_c, r_c^*\mathcal{C}_c', r_c^{*2}\mathcal{C}_c''\}$ we shall use the functional expansion

$$\mathcal{C}(r \simeq r_c^*) = \mathcal{C}_c + \mathcal{C}_c' \cdot (r - r_c^*) + \frac{1}{2}\mathcal{C}_c'' \cdot (r - r_c^*)^2 + O\{[(r - r_c^*)/r_c^*]^3\} \tag{17}$$

of the compactness function. Taking cognizance of Eqs. (7), (15), (16), and (17) one finds the dimensionless relations [27]

$$\mathcal{C}_c = \frac{7}{30}; \quad r_c^*\mathcal{C}_c' = \frac{1}{15}; \quad r_c^{*2}\mathcal{C}_c'' = -\frac{1}{3}. \tag{18}$$

The peak (maximum value) of the characteristic compactness function is determined by the gradient relation

$$\mathcal{C}'(r = r_p) = 0, \tag{19}$$

which, using Eqs. (17) and (18), yields the dimensionless relations

$$\frac{r_p - r_c^*}{r_c^*} = \frac{1}{5} \tag{20}$$

and

$$\mathcal{C}^{\text{analytical}} \equiv \mathcal{C}(r = r_p) = \frac{6}{25}. \tag{21}$$

IV. SUMMARY AND DISCUSSION

Motivated by the important results recently presented in [16], which provide compelling evidence for an interesting relation between the critical threshold of black-hole formation and the stability properties of closed light rings (null circular geodesics) in the curved spacetime of the collapsing matter configurations, we have analyzed the physical and mathematical properties of self-gravitating scalar field configurations.

In particular, using analytical techniques we have proved that the critical compactness parameter, which signals the appearance of the first (marginally stable) light ring in the curved spacetime of the field configuration, is given by the compact relation [see Eqs. (1), (6), and (21)]

$$\mathcal{C}^{\text{analytical}} \equiv \max_r\left\{\frac{m(r)}{r}\right\} = \frac{6}{25}. \tag{22}$$

To test the validity of the conjectured critical black-hole formation–null geodesic correspondence suggested in [16], one should compare the *analytically* derived value (22) of the critical compactness parameter with the corresponding exact (*numerically* computed) value $\mathcal{C}_{\text{max}}^{\text{numerical}}$ of the compactness parameter as determined from fully nonlinear numerical simulations of the gravitational collapse of scalar fields at the critical threshold of black-hole formation [17–19]. In particular, from Figs. 3 and 4 of [18] one finds the numerically computed value [see Eq. (8)]

$$\mathcal{C}_{\text{max}}^{\text{numerical}} \simeq 0.265. \tag{23}$$

From Eqs. (22) and (23) one learns that the *analytically* determined value of the critical compactness parameter, whose derivation in the present compact paper is based on the conjectured black-hole formation–null geodesic correspondence of [16], agrees to within ∼10% with the corresponding exact value of the compactness parameter as determined *numerically* [17–19] using fully nonlinear simulations of the collapse of self-gravitating scalar fields at the threshold of black-hole formation.

Thus, our analytical results indicate that, in accord with the physically interesting conjecture made in [16], there may be a nontrivial relation between the critical threshold of black-hole formation and the appearance of the first light ring (marginally stable null circular geodesic) in the curved spacetime of the collapsing matter configuration.

ACKNOWLEDGMENTS

This research is supported by the Carmel Science Foundation. I thank Yael Oren, Arbel M. Ongo, Ayelet B. Lata, and Alona B. Tea for helpful discussions.

[1] J. M. Bardeen, W. H. Press, and S. A. Teukolsky, Astrophys. J. **178**, 347 (1972).
[2] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, New York, 1983).
[3] S. L. Shapiro and S. A. Teukolsky, *Black Holes, White Dwarfs and Neutron Stars: The Physics of Compact Objects*, 1st ed. (Wiley-Interscience, New York, 1983).
[4] M. A. Podurets, Astron. Zh. **41**, 1090 (1964); [Sov. Astron. **8**, 868 (1965)].
[5] W. L. Ames and K. S. Thorne, Astrophys. J. **151**, 659 (1968).
[6] I. Z. Stefanov, S. S. Yazadjiev, and G. G. Gyulchev, Phys. Rev. Lett. **104**, 251103 (2010).
[7] C. J. Goebel, Astrophys. J. **172**, L95 (1972).

[8] B. Mashhoon, Phys. Rev. D **31**, 290 (1985).
[9] S. R. Dolan, Phys. Rev. D **82**, 104003 (2010).
[10] Y. Dećanini, A. Folacci, and B. Raffaelli, Phys. Rev. D **81**, 104039 (2010); **84**, 084035 (2011).
[11] S. Hod, Phys. Rev. D **80**, 064004 (2009); **78**, 084035 (2008); **75**, 064013 (2007); Classical Quantum Gravity **24**, 4235 (2007); Phys. Lett. B **715**, 348 (2012).
[12] S. Hod, Phys. Rev. D **84**, 124030 (2011); **84**, 104024 (2011).
[13] S. Hod, Phys. Lett. B **718**, 1552 (2013); **751**, 177 (2015); Classical Quantum Gravity **33**, 114001 (2016).
[14] J. Novotný, J. Hladík, and Z. Stuchlík, Phys. Rev. D **95**, 043009 (2017).
[15] S. Hod, Phys. Rev. D **97**, 084018 (2018).
[16] A. Ianniccari, A. J. Iovino, A. Kehagias, D. Perrone, and A. Riotto, arXiv:2404.02801.
[17] M. W. Choptuik, Phys. Rev. Lett. **70**, 9 (1993).
[18] S. Hod and T. Piran, Phys. Rev. D **55**, 3485 (1997); M.Sc. thesis, The Hebrew University, Jerusalem, 1995.
[19] S. Hod and T. Piran, Phys. Rev. D **55**, R440 (1997).
[20] We use natural units in which $G = c = \hbar = 1$.
[21] Here a prime $'$ denotes a differentiation with respect to the radial coordinate r.
[22] Here $\rho = -T^t_t$ is the enrgy density of the self-gravitating matter configuration.
[23] S. Hod, Phys. Lett. B **739**, 383 (2014).
[24] P. V. P. Cunha, E. Berti, and C. A. R. Herdeiro, Phys. Rev. Lett. **119**, 251102 (2017).
[25] S. Hod, Phys. Lett. B **776**, 1 (2018).
[26] Here we have used the relations $a' = \mathcal{C}'/(1-2\mathcal{C})^{3/2}$ and $a'' = 3\mathcal{C}'^2/(1-2\mathcal{C})^{5/2} + \mathcal{C}''/(1-2\mathcal{C})^{3/2}$ [see Eqs. (4) and (6)].
[27] Here we assume that the compactness function is broad enough.