Gravitational bremsstrahlung waveform at the fourth post-Minkowskian order and the second post-Newtonian level

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Using the multipolar post-Minkowskian formalism, we compute the frequency-domain waveform generated by the gravitational scattering of two nonspinning bodies at the fourth post-Minkowskian order ($O(G^4)$, or two-loop order), and at the fractional second post-Newtonian accuracy [$O(v^4/c^4)$]. The waveform is decomposed in spin-weighted spherical harmonics and the needed radiative multipoles, $U_{\ell m}(\omega), V_{\ell m}(\omega)$, are explicitly expressed in terms of a small number of master integrals. The basis of master integrals contains both (modified) Bessel functions, and solutions of inhomogeneous Bessel equations with Bessel-function sources. We show how to express the latter in terms of Meijer G functions. The low-frequency expansion of our results is checked against existing classical soft theorems. We also complete our previous results on the $O(G^2)$ bremsstrahlung waveform by computing the $O(G^3)$ spectral densities of radiated energy and momentum, in the rest frame of one body, at the thirtieth order in velocity.

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I. INTRODUCTION

The gravitational-wave (GW) emission from gravitationally interacting binary systems has been the focus of intense theoretical research over many years because of its relevance to the network of interferometric GW detetectors. See [1] for a review. Recently, a renewed theoretical effort, utilizing advances in quantum scattering amplitudes methods, has been aimed at computing the gravitational waveform emitted by the scattering of two massive bodies [2–22].

The current accuracy of these bremsstrahlung waveforms for nonspinning bodies is the one-loop level corresponding to the third post-Minkowskian (3PM) order [$O(G^3)$]. In terms of the classical waveform[1] $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ we have the PM expansion

$$h_{\mu\nu}(x^\lambda) = G h^{\rm lin}_{\mu\nu} + G^2 h^{\rm 2PM\ or\ tree}_{\mu\nu} + G^3 h^{\rm 3PM\ or\ one\text{-}loop}_{\mu\nu} + G^4 h^{\rm 4PM\ or\ two\text{-}loop}_{\mu\nu} + O(G^5), \qquad (1.1)$$

where we indicated the dictionary between the quantum nomenclature (tree, one-loop,...) and the classical PM one.

Anticipating on the ongoing effort of computing the *two-loop* waveform (G^4 contribution to $h_{\mu\nu}$, denoted $h^{G^4}_{\mu\nu}$), the aim of this paper is to present the first five orders, namely $\eta^0+\eta^1+\eta^2+\eta^3+\eta^4$ (with $\eta\equiv\frac{1}{c}$) in the small-velocity expansion of the classical 4PM (two-loop) waveform, in the frequency domain. We recall that each power of $\eta\sim\frac{v}{c}$ is traditionally referred to as being *half* a post-Newtonian (PN) order, so that our present $O(\eta^4)$ accuracy corresponds to a fractional 2PN accuracy. We hope the results presented here will provide useful benchmarks when two-loop waveform results are obtained.

Denoting the complex asymptotic waveform as

$$h_c(T_r,\theta,\phi) = \lim_{R\to\infty} \epsilon^\mu \epsilon^\nu R h_{\mu\nu} = \lim_{R\to\infty}(R(h_+ - i h_\times)), \qquad (1.2)$$

where R denotes the radial distance, and where $\epsilon^\mu = \frac{1}{\sqrt{2}}(e^\mu_\theta - i e^\mu_\phi)$ is a null polarization vector, we present below the value of its $O(G^4)$ frequency-domain contribution at the $\eta^0+\eta^1+\eta^2+\eta^3+\eta^4$ accuracy

$$h^{G^4}_c(\omega,\theta,\phi) = \int_{-\infty}^{+\infty} dT_r e^{i\omega T_r} h^{G^4}_c(T_r,\theta,\phi) = G^4[\eta^0+\eta^1+\eta^2+\eta^3+\eta^4+O(\eta^5)]. \qquad (1.3)$$

Here, we work in a Bondi-like coordinate system T_r, R, θ, ϕ, anchored on the center-of-mass (cm) of the binary system. In particular, $T_r \simeq t - \frac{r}{c} - \frac{2G\mathcal{M}}{c^3}\ln\frac{r}{cb_0}$ denotes a retarded time which contains a logarithmic shift proportional to the total mass-energy of the system, $\mathcal{M}\equiv\frac{E}{c^2}$, and

[1]We use the mostly plus signature, $\eta_{\mu\nu} = {\rm diag}(-1,+1,+1,+1)$.

which depends on the choice of an arbitrary timescale b_0. Denoting the direction of GW emission as $\mathbf{n}(\theta,\phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ we use as null polarization vector

$$\boldsymbol{\epsilon} = \bar{\mathbf{m}} = \frac{1}{\sqrt{2}}\left(\partial_\theta \mathbf{n}(\theta,\phi) - \frac{i}{\sin\theta}\partial_\phi \mathbf{n}(\theta,\phi)\right). \tag{1.4}$$

The method we use here to compute $h_c^{G^4}$ is the (PN-matched) multipolar post-Minkowskian (MPM) formalism Refs. [23–31]. This formalism (which is a vast generalization of the classic quadrupole-radiation formula of Einstein [32]), has allowed one to reach a high PN accuracy [33,34] on the time-domain waveform, $h_c(T_r,\theta,\phi)$, expressed as a PM-expanded, *retarded functional*[2] of some multipole moments of the material source. The latter source multipole moments are essentially defined as a regularized version of the irreducible multipole moments of the total (matter + gravitational), conserved stress-energy tensor, $\tau^{\mu\nu} = |g|(T^{\mu\nu}_{\rm mat} + T^{\mu\nu}_{\rm grav})$.

The expression of the time-domain waveform $h_c(T_r,\theta,\phi)$ as a retarded functional of the source multipoles is not the end of the story. Indeed, one must still: (i) evaluate the source multipole moments for the considered case of a binary system moving along hyperboliclike trajectories; and (ii) compute the Fourier transform of the time-domain waveform. As in our previous works [21,35] we shall use for these computations the explicit quasi-Keplerian representation of the (PN-expanded) general-relativistic solution for scattering motions [36–38].

For brevity, we shall not repeat here the minireviews of the structure of the MPM formalism, and of the quasi-Keplerian representation, recently presented in [21,35] (see Secs. II and IV of [35], and Secs. II and III of [21]). We will just review the notation used for the multipolar decomposition of the waveform $h_c(T_r,\theta,\phi)$.

We start by recalling that it is technically convenient to factor out of the complex waveform the overall factor $\frac{4G}{c^4} \equiv 4G\eta^4$ which appears in the classic (leading PN order) Einstein quadrupole formula

$$\epsilon^i\epsilon^j h^{\rm LO}_{ij}(t,r,\mathbf{n}) \approx \frac{4G}{c^4}\frac{\frac{1}{2}\epsilon^i\epsilon^j I^{(2)}_{ij}(t-r/c)}{r}, \tag{1.5}$$

where $I_{ij}(t) \approx \int d^3x \frac{T^{00}}{c^2}(x^i x^j - \frac{1}{3}\mathbf{x}^2\delta^{ij})$ is the quadrupole moment of the mass-energy distribution, and where the superscript $^{(2)}$ denotes a second time derivative.

When writing $h_c(T_r,\theta,\phi) = \lim_{R\to\infty}\epsilon^\mu\epsilon^\nu R h_{\mu\nu}$ in the factorized form

$$h_c(T_r,\theta,\phi) \equiv 4G\eta^4 W(T_r,\theta,\phi), \tag{1.6}$$

the rescaled waveform $W(T_r,\theta,\phi)$ has, at leading PN order, the *Newtonian-level* value $\frac{1}{2}\epsilon^i\epsilon^j I^{(2)}_{ij}(T_r)$. The MPM formalism obtains $W(T_r,\theta,\phi)$ as a multipolar series labeled by the total angular momentum ℓ, combined with a double series in G and in $\eta = \frac{1}{c}$. Namely, one has, say after taking the Fourier transform over T_r, $W(\omega,\theta,\phi) \equiv \int dT_r e^{i\omega T_r} W(T_r,\theta,\phi)$,

$$\begin{aligned} W_{\rm MPM}(\omega,\theta,\phi) &= U_2 + \eta(V_2+U_3) + \eta^2(V_3+U_4) \\ &\quad + \eta^3(V_4+U_5) + \eta^4(V_5+U_6) \\ &\quad + O(\eta^5), \end{aligned} \tag{1.7}$$

where we use the standard definitions and normalizations of the various 2^ℓ multipole contributions, see, e.g., [1] or, for a recent review, our previous work [35].

Each even-parity, $U_\ell(\omega)$, or, odd-parity, $V_\ell(\omega)$, *radiative* multipole starts (for $\omega \neq 0$) at PM order G and at Newtonian order η^0. The various factors η^n in Eq. (1.7) indicate that, e.g., the Newtonian-level $G^1\eta^0$ term in U_ℓ contributes at order $G^1\eta^{\ell-2}$ to W and the $G^1\eta^0$ term in V_ℓ contributes at order $G^1\eta^{\ell-1}$ to W.

Here, we are considering the nonstationary part of the waveform, corresponding to nonzero frequencies ω. For instance, at leading PN order, this is equivalent, modulo a factor $-i\omega$, to considering the Fourier transform of $\frac{1}{2}\epsilon^i\epsilon^j I^{(3)}_{ij}(T_r)$. The latter involves the third derivative of the quadrupole moment, which vanishes for free motions, and therefore contains a factor G^1. More precisely,

$$\begin{aligned} U_\ell(\omega,\theta,\phi) &= U_\ell^G(\eta^0+\eta^2+\eta^4+\cdots) \\ &\quad + U_\ell^{G^2}(\eta^0+\eta^2+\eta^3+\eta^4+\cdots) \\ &\quad + U_\ell^{G^3}(\eta^0+\eta^2+\eta^3+\eta^4+\cdots) \\ &\quad + \cdots \\ V_\ell(\omega,\theta,\phi) &= V_\ell^G(\eta^0+\eta^2+\eta^4+\cdots) \\ &\quad + V_\ell^{G^2}(\eta^0+\eta^2+\eta^3+\eta^4+\cdots) \\ &\quad + V_\ell^{G^3}(\eta^0+\eta^2+\eta^3+\eta^4+\cdots) \\ &\quad + \cdots. \end{aligned} \tag{1.8}$$

Recent works [21,35] have displayed the building blocks $U_\ell^G(\omega,\theta,\phi)$, $U_\ell^{G^2}(\omega,\theta,\phi)$ and $V_\ell^G(\omega,\theta,\phi)$, $V_\ell^{G^2}(\omega,\theta,\phi)$ needed to reach the overall accuracy[3] $G^3\eta^5$ for the even-in-ϕ part of the waveform $h_c(\omega,\theta,\phi) \equiv 4G\eta^4 W(\omega,\theta,\phi)$. [The odd-in-$\phi$ part of the waveform was given with the

[2]This functional comprises various linear and nonlinear tail effects, as well as instantaneouslike nonlinear contributions, see [1] for a detailed review, and [35] for a streamlined presentation.

[3]When computing the accuracy of the waveform $h_c(\omega,\theta,\phi)$, we count the overall power of G, but we discount the overall factor η^4 to focus on the fractional PN accuracy.

reduced accuracy $G^3\eta^3$.] This corresponds to the accuracy $U_2 \sim (G+G^2)(\eta^0+\eta^2+\eta^3+\eta^4+\eta^5)$ for $\ell = 2^+$, the accuracy $V_2 \sim U_3 \sim (G+G^2)(\eta^0+\eta^2+\eta^3)$ for $\ell = 2^-$ and $\ell = 3^+$, the accuracy $V_3 \sim U_4 \sim (G+G^2)(\eta^0+\eta^2+\eta^3)$ for $\ell = 3^-$ and $\ell = 4^+$, the accuracy $V_4 \sim U_5 \sim (G+G^2)(\eta^0+\eta^2)$ for $\ell = 4^-$ and $\ell = 5^+$ (not given in our previous work), and the accuracy $V_5 \sim U_6 \sim (G+G^2)(\eta^0)$ for $\ell = 4^-$ and $\ell = 5^+$.

Here, our aim is to reach the overall accuracy $G^4\eta^4$ (4PM and 2PN) on the c^4-rescaled waveform $\eta^{-4}h_c(\omega,\theta,\phi) \equiv 4GW(\omega,\theta,\phi)$. This will be obtained by having the following accuracies on the $U_\ell^{G^3}(\omega,\theta,\phi)$ and $V_\ell^{G^3}(\omega,\theta,\phi)$:

$$\begin{aligned} &U_2^{G^3} \sim G^3(\eta^0+\eta^2+\eta^3+\eta^4),\\ &V_2^{G^3} \sim U_3^{G^3} \sim G^3(\eta^0+\eta^2+\eta^3),\\ &V_3^{G^3} \sim U_4^{G^3} \sim G^3(\eta^0+\eta^2),\\ &V_4^{G^3} \sim U_5^{G^3} \sim V_5^{G^3} \sim U_6^{G^3} \sim G^3\eta^0. \end{aligned} \tag{1.9}$$

In the following, we focus on the structure of, and result for, the even-parity radiative quadrupole U_2, which requires the highest fractional PN accuracy.

II. STRUCTURE OF THE 4PM QUADRUPOLAR WAVEFORM AT 2PN FRACTIONAL ACCURACY

The even-parity, quadrupolar contribution U_2 to the rescaled waveform $W(T_r,\theta,\phi) = \frac{c^4}{4G}h_c(T_r,\theta,\phi)$ reads

$$U_2(T_r,\theta,\phi) = \frac{1}{2}\bar{m}^{ij}U_{ij}, \tag{2.1}$$

where the radiative quadrupole (measured at future null infinity) is given, in the MPM formalism, by a retarded functional of the form

$$\begin{aligned} U_{ij}(T_r) = I_{ij}^{(2)}(T_r) &+ \frac{2G\mathcal{M}}{c^3}\int_{-\infty}^{T_r} dT'_r\left[\ln\left(\frac{T_r-T'_r}{2b_0}\right)+\frac{11}{12}\right]I_{ij}^{(4)}(T'_r)\\ &+ O\left(\frac{G^2}{c^5}\right). \end{aligned} \tag{2.2}$$

Here, the (time-dependent part of the) source multipole $I_{ij}(t)$ has to be evaluated at the $(G+G^2+G^3)(\eta^0+\eta^2+\eta^4)$ accuracy (i.e., the 2PN accuracy) in the first term on the right-hand side (rhs), while only its Newtonian-level $[O(\eta^0)]$, but 2PM-accurate $[O(G+G^2)]$, value is needed in the second, tail, term on the rhs. We recall that $\mathcal{M} = \frac{E_{\rm cm}}{c^2}$, and that b_0 denotes the arbitrary timescale entering the MPM formalism (both in the definition of the log contribution to the retarded time, and in the *partie finie* regularization entering the nonlinear MPM iteration scheme). For illustration, the beginning of the PN expansion of the (time-domain) MPM source quadrupole moment reads [26]

$$\begin{aligned} I_{ij}(t) = &\int d^3x\sigma(t,\mathbf{x})\hat{x}^{ij} + \frac{1}{14c^2}\frac{d^2}{dt^2}\int d^3x\sigma(t,\mathbf{x})\mathbf{x}^2\hat{x}^{ij}\\ &- \frac{20}{21c^2}\frac{d}{dt}\int d^3x\sigma_k(t,\mathbf{x})\hat{x}^{ijk} + O\left(\frac{1}{c^4}\right). \end{aligned} \tag{2.3}$$

Here, $\sigma = \frac{T^{00}_{\rm mat}+T^{kk}_{\rm mat}}{c^2}$, $\sigma_k = \frac{T^{0k}_{\rm mat}}{c}$, $\hat{x}^{ij\cdots}$ denotes the symmetric-trace-free (STF) projection of $x^ix^j\cdots$, and the coordinates x^i refer to a source-rooted harmonic coordinate system. As in Refs. [21,35], the Cartesian components of all multipoles, and the associated emission angles θ,ϕ, are defined with respect to a cm-frame spatial basis e_x, e_y, e_z where e_x, e_y are anchored on the classical averaged cm-frame momenta. More precisely, the time axis $\bar{e}_0$ of the cm frame is

$$\bar{e}_0^\mu = \frac{\bar{p}_1^\mu+\bar{p}_2^\mu}{|\bar{p}_1^\mu+\bar{p}_2^\mu|}. \tag{2.4}$$

The vector e_y lies in the spatial direction of $\bar{p}_1$ (i.e., the bisector between the incoming and the outgoing spatial momentum of the first particle in the cm frame),

$$\bar{p}_1 = E_1\bar{e}_0 + \bar{P}_{\rm cm}e_y, \qquad \bar{p}_2 = E_2\bar{e}_0 - \bar{P}_{\rm cm}e_y, \tag{2.5}$$

where $E_a = \sqrt{m_a^2+P_{\rm cm}^2}$, and $P_{\rm cm}$ is the magnitude of the spatial part of the incoming momentum, while $\bar{P}_{\rm cm} = P_{\rm cm}\cos\frac{1}{2}\chi$. The axis vector e_x is in the plane of motion and orthogonal to e_y (and oriented from particle 2 toward particle 1). It is also the direction of the eikonal vectorial impact parameter, $b_{\rm eik}e_x$. Here $b_{\rm eik}$ is the eikonal-type impact parameter, linked to the usual, cm, incoming-momenta impact parameter b by $b_{\rm eik} = \frac{b}{\cos\frac{1}{2}\chi}$. The last spatial axis vector e_z is orthogonal to the plane of motion (and such that e_x, e_y, e_z is positively oriented). All vectors and tensors are decomposed in the frame $\bar{e}_0$, e_x, e_y, e_z and the angles θ,ϕ are accordingly defined, so that

$$\begin{aligned} \mathbf{n}(\theta,\phi) &= \sin\theta\cos\phi e_x + \sin\theta\sin\phi e_y + \cos\theta e_z,\\ k &= \omega(\bar{e}_0+\mathbf{n}(\theta,\phi)),\\ \bar{m} &= \frac{1}{\sqrt{2}}\left[\partial_\theta\mathbf{n}(\theta,\phi) - \frac{i}{\sin\theta}\partial_\phi\mathbf{n}(\theta,\phi)\right]. \end{aligned} \tag{2.6}$$

Here $\mathbf{n}$ is the spatial unit vector that characterizes the direction of the gravitational wave vector $k = (\omega,\mathbf{k})$.

The explicit computation (at the accuracy needed here) of the Fourier-transform of the first contribution to U_{ij} then proceeds as follows: (i) one inserts the $T^{\mu\nu}_{\rm mat}$ describing two point masses at 2PN accuracy (see [39,40]); (ii) one inserts the explicit 2PN-accurate quasi-Keplerian representation of the relative hyperbolic motion, $x^i(t) = x_1^i(t) - x_2^i(t)$, namely

$$\mathbf{x}(t) = r(t)(\cos\varphi(t)e_x + \sin\varphi(t)e_y), \tag{2.7}$$

where the polar coordinates of the relative motion are obtained as functions of time by eliminating the (hyperbolictype) "eccentric anomaly" variable v between equations of the type

$$\begin{aligned} \bar{n}t &= e_t \sinh(v) - v + O(\eta^4), \\ r &= \bar{a}_r(e_r\cosh(v) - 1) + O(\eta^4), \\ \varphi &= 2K\arctan\left(\sqrt{\frac{e_\varphi+1}{e_\varphi-1}}\tanh\left(\frac{v}{2}\right)\right) + O(\eta^4). \end{aligned} \tag{2.8}$$

Here, the quasi-Keplerian quantities $\bar{n}, e_t, e_r, e_\varphi, K$ are (PN-expanded) functions of the c.m. energy, E, and angular momentum, J, of the binary system (see Refs. [36–38]). [E and J are then reexpressed in terms of the incoming Lorentz factor γ and the (incoming-momenta) impact parameter b.] The quasi-Keplerian representation Eq. (2.8) incorporates (in the conservative case) a time symmetry around $t = 0$, corresponding to the closest approach between the two bodies. The asymptotic logarithmic drift of the two worldlines is embodied in the v parametrization involving hyperbolic functions.

Finally, the Fourier transform is conveniently computed by considering $I_{ij}^{(3)}(\omega) = -i\omega I_{ij}^{(2)}(\omega)$ and by taking the *large eccentricity* expansion of the Fourier integrals generated by the quasi-Keplerian representation, which have the form

$$\int dv e^{i\frac{\omega}{\bar{n}}(e_t\sinh(v)-v)}(F_0(v) + \eta^2 F_2(v) + \cdots). \tag{2.9}$$

The eccentricity e_t (when expressed in terms of the cm energy E and the cm angular momentum J) is of the form

$$e_t(E,J,G) \sim \sqrt{1 + \frac{p_\infty^2 J^2}{(Gm_1m_2)^2}}\left(1 + O\left(\frac{p_\infty^2}{c^2}\right) + O(G^2)\right), \tag{2.10}$$

where we parametrize the total cm energy by the variable $p_\infty \equiv c\sqrt{\gamma^2 - 1}$ (having the dimension of a velocity), with γ the relative Lorentz factor between the incoming particles. In view of the expression (2.10), the large-eccentricity expansion corresponds to a PM expansion in powers of G.

In previous (one-loop-level) works [21,35], we only had to consider the first two terms in the large-eccentricity expansion of the multipole moments. The novelty of the present study is to take into account the third term, $O(\frac{1}{e_t^3}) = O(G^3)$, in the large-eccentricity expansion of the frequency-domain multipoles, as given by the quasi-Keplerian representation (2.8). As discussed next, the Fourier integrals appearing at order $O(\frac{1}{e_t^3}) = O(G^3)$ introduce new transcendental functions which go beyond the ones introduced at orders $O(G + G^2)$. [At orders $O(G + G^2)$ the frequency-domain waveform $W(\omega)$ was fully expressible in terms of the two Bessel K functions $K_0(\frac{\omega b}{p_\infty})$ and $K_1(\frac{\omega b}{p_\infty})$, together with $\exp(-\frac{\omega b}{p_\infty})$. See Appendix D for the order G^1 and Refs. [21,35] for the order G^2.]

Finally, the frequency-domain radiative quadrupole moment of a scattering binary is obtained as a double expansion in G and η of the form

$$\begin{aligned} U_2(\omega,\theta,\phi) &= U_2^{G^1}(\omega,\theta,\phi) + U_2^{G^2}(\omega,\theta,\phi) \\ &\quad + U_2^{G^3}(\omega,\theta,\phi) + O(G^4), \end{aligned} \tag{2.11}$$

where each PM contribution $O(G^n)$, for $n = 1, 2, 3$, has a small-velocity (PN) expansion of the form (at our present accuracy)

$$\begin{aligned} U_2^{G^n}(\omega,\theta,\phi) &= U_2^{G^n\eta^0} + U_2^{G^n\eta^2} + U_2^{G^n\eta^3} \\ &\quad + U_2^{G^n\eta^4} + O(G^n\eta^5). \end{aligned} \tag{2.12}$$

The small-velocity expansion is taken at a fixed value of the following dimensionless frequency parameter

$$u \equiv \frac{\omega b}{p_\infty}. \tag{2.13}$$

Here, $b = \frac{J}{P_{\rm cm}}$ is the impact parameter.

When $U_2(\omega)$ (with dimensions $[MV^2T]$) is expressed in terms of the dimensionless variables u, θ, ϕ the velocity expansions of its successive PM contributions have the structure (where we factor out an overall symmetric mass ratio $\nu \equiv \frac{m_1m_2}{M^2}$, with $M \equiv m_1 + m_2$)

$$\begin{aligned} U_2^{G^1}(u,\theta,\phi) &= \nu\frac{GM^2}{p_\infty}\left[1 + \frac{p_\infty^2}{c^2} + \frac{p_\infty^4}{c^4} + O\left(\frac{p_\infty^6}{c^6}\right)\right] \\ U_2^{G^2}(u,\theta,\phi) &= \nu\frac{GM^2}{p_\infty}\left(\frac{GM}{bp_\infty^2}\right)\left[1 + \frac{p_\infty^2}{c^2} + \frac{p_\infty^3}{c^3} \right. \\ &\quad \left. + \frac{p_\infty^4}{c^4} + O\left(\frac{p_\infty^5}{c^5}\right)\right] \\ U_2^{G^3}(u,\theta,\phi) &= \nu\frac{GM^2}{p_\infty}\left(\frac{GM}{bp_\infty^2}\right)^2\left[1 + \frac{p_\infty^2}{c^2} + \frac{p_\infty^3}{c^3} \right. \\ &\quad \left. + \frac{p_\infty^4}{c^4} + O\left(\frac{p_\infty^5}{c^5}\right)\right]. \end{aligned} \tag{2.14}$$

Here, the dimensionless factor $(\frac{GM}{bp_\infty^2})$ keying the PM expansion is of order G but is of Newtonian order η^0

(remember that p_∞ is a velocity). This factor is of order of the scattering angle χ.

The explicit values of $U_2^{G^1}(u,\theta,\phi)$ and $U_2^{G^2}(u,\theta,\phi)$ have been given in Refs. [21,35] (and their ancillary files) to a rather high PN accuracy. Here, we consider only the 3PM contribution $U_2^{G^3}(u,\theta,\phi)$ to $W(u,\theta,\phi)$ (corresponding to the 4PM waveform) at the fractional PN accuracy indicated above, i.e., $1+\frac{p_\infty^2}{c^2}+\frac{p_\infty^3}{c^3}+\frac{p_\infty^4}{c^4}+O(\frac{p_\infty^5}{c^5})$. The explicit values of the complementary higher multipolar contributions V_2, U_3 (at 1.5PN accuracy), V_3, U_4 (at 1PN accuracy), and V_4, U_5, V_5, U_6 (at the leading Newtonian order) are given in the ancillary files of the arxiv submission of this work.

In order to clarify the u dependence of the multipolar waveforms, it is convenient to factor out the angular dependence which is simply given by appropriate spin-weighted spherical harmonics (SWSH). For example, for $\ell=2^+$, we have

$$U_2(u,\theta,\phi)=U_{22}(u)Y_{\bar{2};22}(\theta,\phi)+U_{20}(u)Y_{\bar{2};20}(\theta,\phi)+U_{2\bar{2}}(u)Y_{\bar{2};2\bar{2}}(\theta,\phi), \quad (2.15)$$

where, as stated above, each term $U_{2m}(u)$ (for $m=2,0,-2$) has both a G-expansion and an η-expansion and we used the notation $Y_{\bar{s}=-s;\ell m}(\theta,\phi)$ for the SWSH.

III. THE 1.5PN, TAIL CONTRIBUTION TO THE 4PM WAVEFORM

It is convenient to discuss first the $\frac{p_\infty^3}{c^3}$ (1.5PN) contribution to $U_2^{G^3}(u,\theta,\phi)$. This contribution only comes from the nonlocal-in-time (tail) term in Eq. (2.2) because the PN expansion of the local-in-time term $I_{ij}^{(2)}(t)$ reads $\sim\eta^0+\eta^2+\eta^4+\eta^5$. In Fourier-space the nonlocal tail kernel diagonalizes and becomes a factor (involving Euler's constant γ_E and the sign of the frequency)

$$f_{U_2}^{\rm tail}(\omega)=\frac{2G\mathcal{M}|\omega|}{c^3}\left[\frac{\pi}{2}+i\,{\rm sign}(\omega)\left(\ln(2|\omega|b_0)+\gamma_E-\frac{11}{12}\right)\right]. \quad (3.1)$$

The tail factor (3.1) involves the timescale b_0 introduced in the definition of the retarded time.

As this factor is $O(G)$, the tail contribution to the $O(G^3)$ quadrupole U_2 is obtained by multiplying $U_2^{G^2\eta^0}$ by $f_{U_2}^{\rm tail}(\omega)$. On the other hand, we found in Ref. [35] that $U_2^{G^2\eta^0}$ differed from the leading-order term $U_2^{G^1\eta^0}$ only by a factor

$$f^{\frac{G^2\eta^0}{G^1\eta^0}}(\omega)=\frac{\pi}{2}\left(\frac{GM}{bp_\infty^2}\right)u. \quad (3.2)$$

Finally, the 1.5PN, tail contribution to $U_2^{G^3}(u,\theta,\phi)$ is simply

$$U_2^{G^3\eta^3}(u,\theta,\phi)=f_{U_2}^{\rm tail}(\omega)f^{\frac{G^2\eta^0}{G^1\eta^0}}(\omega)U_2^{G^1\eta^0}(u,\theta,\phi). \quad (3.3)$$

Here the leading-order term $U_2^{G^1\eta^0}$ (written here for simplicity in the equatorial plane, $\theta=\frac{\pi}{2}$) is

$$U_2^{G^1\eta^0}\left(u,\frac{\pi}{2},\phi\right)=-\frac{GM^2\nu}{2p_\infty}[(2iu\sin(2\phi)+\cos(2\phi)+1)K_0(u)+(2i\sin(2\phi)+2u\cos(2\phi))K_1(u)]. \quad (3.4)$$

See the ancillary file of Ref. [35] for the corresponding result outside of the equatorial plane, given in terms of SWSHs.

Similarly to the tail contribution to U_2 shown above, working at the 2PN level of accuracy we have to include the tail contributions to V_2 and U_3. We find

$$V_2^{G^3\eta^3}(u,\theta,\phi)=f_{V_2}^{\rm tail}(\omega)f^{\frac{G^2\eta^0}{G^1\eta^0}}(\omega)V_2^{G^1\eta^0}(u,\theta,\phi),$$
$$U_3^{G^3\eta^3}(u,\theta,\phi)=f_{U_3}^{\rm tail}(\omega)f^{\frac{G^2\eta^0}{G^1\eta^0}}(\omega)U_3^{G^1\eta^0}(u,\theta,\phi), \quad (3.5)$$

where

$$f_{V_2}^{\rm tail}(\omega)=\frac{2G\mathcal{M}|\omega|}{c^3}\left[\frac{\pi}{2}+i\,{\rm sign}(\omega)\left(\ln(2|\omega|b_0)+\gamma_E-\frac{7}{6}\right)\right],$$
$$f_{U_3}^{\rm tail}(\omega)=\frac{2G\mathcal{M}|\omega|}{c^3}\left[\frac{\pi}{2}+i\,{\rm sign}(\omega)\left(\ln(2|\omega|b_0)+\gamma_E-\frac{97}{60}\right)\right]. \quad (3.6)$$

IV. MASTER INTEGRALS FOR THE NEWTONIAN, 1PN AND 2PN CONTRIBUTIONS TO THE 4PM WAVEFORM

We now come to the instantaneous contributions to the waveform. We recall that at order G^1 the SWSH components of the quadrupolar waveform $U_2(u)$ involve (at any velocity order) linear combinations of the modified Bessel functions $K_0(u)$ and $K_1(u)$ with coefficients given by polynomials in u. By contrast, at order G^2 there are two types of contributions: some involve linear combinations of $K_0(u)$ and $K_1(u)$, while others (which start at order $G^2\eta^2$) involve the new transcendental function $\frac{e^{-u}}{u}$, multiplied by a polynomial in u.

The origin of these results is the following. First, we recall that the news function $\dot{h}_c\sim\sum(\dot{U}_\ell+\dot{V}_\ell)$ vanishes when evaluated along free motions. It is therefore expressible (when viewed from a PM perspective) as the product of

G with some integral expression involving the PM expansion of the relative motion (in the cm frame). The latter cm-frame PM expansion is a direct reflection of the covariant PM expansion of two worldlines: $z_A^\mu = b_A^\mu + \tau_A u_A^\mu + G z_A^{\mu 1\mathrm{PM}} + O(G^2)$. The $O(G^1)$ value of $W(\omega)$ only depends on the insertion in the latter expression of the free-motion solution $z_A^\mu = b_A^\mu + \tau_A u_A^\mu + O(G)$. This is easily seen to generate (in the cm frame, and at any velocity order) Fourier integrals of the generic form (n denoting a positive integer)

$$I_1(n) = \int dT \frac{e^{iuT}}{(1+T^2)^{n+\frac{1}{2}}}. \tag{4.1}$$

By contrast, the $O(G^2)$ value of $W(\omega)$ depends on the insertion of the $G z_A^{\mu 1\mathrm{PM}}$ correction to incoming free motions (projected on the cm-frame relative motion), as well as on $O(\frac{GM}{r})$ fractional corrections entering the source multipole moments. The 1PM correction to the worldline contains three types of terms (see, e.g., [41]): terms rational in τ_A, algebraic terms involving $D(\tau_A) = \sqrt{b^2 + p_\infty^2 \tau_A^2}$, and transcendental terms involving $\ln(p_\infty \tau_A + D(\tau_A))$. The rational terms yield contributions of the type $I_1(n)$. The algebraic terms involving $D(\tau_A) = \sqrt{b^2 + p_\infty^2 \tau_A^2}$, together with the $\propto \frac{GM}{r}$ fractional corrections to the multipole moments (which, in the PN expansion, start at order $G^2\eta^2$) add an extra factor $\frac{1}{(1+T^2)^{\frac{1}{2}}}$ in the first Fourier integrals above, leading to contributions of the generic form

$$I_2(n) = \int dT \frac{e^{iuT}}{(1+T^2)^{n+1}}. \tag{4.2}$$

Finally, the transcendental terms yield contributions of the new type

$$I_3(n) = \int dT \frac{e^{iuT}}{(1+T^2)^{n+\frac{1}{2}}} \mathrm{arcsinh} T. \tag{4.3}$$

The first type of integral, $I_1(n)$, is expressible in terms of $K_n(u)$ functions. The second type, $I_2(n)$, which can be easily computed as a Cauchy integral, is expressible in terms of e^{-u}. The third type, $I_3(n)$, is expressible in terms of the *first* derivative of a Bessel function $K_\nu(u)$ with respect to the order ν, when taking the limit $\nu \to n$ after differentiation. Then, known identities concerning $\partial_\nu K_\nu(u)$ at integer values of ν [42,43], allow one to reexpress $I_3(n)$ in terms of ordinary $K_n(u)$ functions. See also Appendix A for a direct proof.

When going to the two-loop level (G^3 in W) one gets more complicated Fourier integrals defining higher transcendental functions of u. Here, we discuss the integrals that appeared at the limited PN accuracy of our present study. [All the integrals discussed here correspond to some diagrams of exchanged gravitons between the two particle worldlines. In our approach the exchange of gravitons is encoded in the quasi-Keplerian solution, and it is the G expansion of the corresponding multipole moments which generate the corresponding integrals.] In order to clarify the structure of the resulting waveform, and the possibility of expressing the waveform in terms of a finite basis of *master integrals*, we took a systematic approach to, and a more systematic notation for, the appearing integrals. Namely, let us denote the simpler integrals appearing in Eqs. (4.1) and (4.2) as

$$Q_\alpha(u) \equiv \int dT \frac{e^{iuT}}{(1+T^2)^\alpha}, \tag{4.4}$$

for a generic index α (possibly complex, if analytic continuations in α helps its discussion). We generally assume here that the argument u is (strictly) positive.

For generic values of α the special functions $Q_\alpha(u)$ defined by Eq. (4.4) can be expressed in terms of modified Bessel K functions, namely

$$Q_\alpha(u) = \frac{2^{\frac{3}{2}-\alpha}\sqrt{\pi} u^{-\frac{1}{2}+\alpha}}{\Gamma(\alpha)} \mathrm{BesselK}\left(-\frac{1}{2}+\alpha, u\right). \tag{4.5}$$

[We recall that Bessel functions of half-integer orders are expressible in terms of exponential functions.] The integrals appearing in the G^3 level of the rescaled waveform W (at our considered PN accuracy) are all directly expressed, without using any integration by parts (IBP), either in terms of integrals of the type (4.4), or of the following more complicated integrals:

$$\begin{aligned} Q_\alpha^{\mathrm{at}}(u) &= \int dT \frac{e^{iuT}}{(1+T^2)^\alpha} \arctan(T), \\ Q_\alpha^{\mathrm{as}}(u) &= \int dT \frac{e^{iuT}}{(1+T^2)^\alpha} \mathrm{arcsinh}(T), \\ Q_\alpha^{\mathrm{as2}}(u) &= \int dT \frac{e^{iuT}}{(1+T^2)^\alpha} \mathrm{arcsinh}^2(T). \end{aligned} \tag{4.6}$$

Here, the index α takes (for the relevant integrals) half-integer values except for the second integral (Q_α^{as}), where it takes integer values.

Note that all those integrals are of the generic form

$$I_\alpha^f(u) \equiv \int dT \frac{e^{iuT}}{(1+T^2)^\alpha} f(T), \tag{4.7}$$

with some function $f(T)$. In addition, the functions $f(T)$ entering the first two types of integrals in (4.6) have the special property that the derivative of the extra factor $f(T)$ satisfies

$$f'(T) = \frac{1}{(1+T^2)^\beta}, \tag{4.8}$$

with $\beta = 1$ for the arctan integral and $\beta = 1/2$ for the arcsinh integral.

One can derive some simple *ladder relations* linking various generic $I^f_\alpha(u)$ integrals. Introducing the first-order differential operator

$$\hat{T} = \frac{1}{i}\frac{d}{du}, \tag{4.9}$$

it is easy to see that[4]

$$(\hat{T}^2 + 1) I^f_\alpha(u) = I^f_{\alpha-1}(u). \tag{4.10}$$

Using the integration by parts (IBP) identities

$$0 \sim \frac{d}{dT}\left(\frac{e^{iuT}}{(1+T^2)^\alpha} f(T)\right),$$
$$0 \sim \frac{d}{dT}\left(\frac{T e^{iuT}}{(1+T^2)^\alpha} f(T)\right), \tag{4.11}$$

we get

$$0 = iu I^f_\alpha - 2\alpha \hat{T} I^f_{\alpha+1} + I^{f'}_\alpha,$$
$$0 = (1 + iu\hat{T} - 2\alpha) I^f_\alpha + 2\alpha I^f_{\alpha+1} + \hat{T} I^{f'}_\alpha. \tag{4.12}$$

In the case where the derivative $f'(T)$ satisfies the relation (4.8) we get the following more informative result

$$0 = iu I^f_\alpha - 2\alpha \hat{T} I^f_{\alpha+1} + Q_{\alpha+\beta},$$
$$0 = (1 + iu\hat{T} - 2\alpha) I^f_\alpha + 2\alpha I^f_{\alpha+1} + \hat{T} Q_{\alpha+\beta}. \tag{4.13}$$

The latter result yields ladder relations expressing I^f_α in terms of first-order u-derivatives of any one of its two neighbors $I^f_{\alpha+1}$ or $I^f_{\alpha-1}$, and of extra source terms involving the known, simpler integrals $Q_{\alpha+\beta}$, and their first-order u-derivative.

If we combine the general ladder relations (4.12) we get a second-order inhomogeneous differential equation for I^f_α, namely

$$\left(\hat{T}^2 + 1 - \frac{2(\alpha-1)}{iu}\hat{T}\right) I^f_\alpha(u) = -\frac{1}{iu}(\hat{T}^2+1) I^{f'}_\alpha = -\frac{1}{iu} I^{f'}_{\alpha-1}. \tag{4.14}$$

In the particular case where $f'(T) = \frac{1}{(1+T^2)^\beta}$, this second-order inhomogeneous differential equation simplifies to

$$\left(\hat{T}^2 + 1 - \frac{2(\alpha-1)}{iu}\hat{T}\right) I^f_\alpha(u) = -\frac{1}{iu} Q_{\alpha+\beta-1}. \tag{4.15}$$

Especially useful is the second ladder relation in (4.13). Indeed, it can be solved for $I^f_{\alpha+1}$ when $\alpha \neq 0$, namely

$$I^f_{\alpha+1} = \frac{1}{2\alpha}(2\alpha - 1 - iu\hat{T}) I^f_\alpha - \frac{1}{2\alpha}\hat{T} Q_{\alpha+\beta}. \tag{4.16}$$

When $\alpha \neq 0$, one can express, by using this relation recursively, the infinite sequence of integrals $I^f_{\alpha+n}$, $n = 0, 1, 2, 3, \ldots$ in terms of the single integral I^f_α, and of derivatives of the simpler $Q_{\alpha+\beta}$'s. And when $\alpha = 0$, one can express the infinite sequence of integrals I^f_n in terms of only two master integrals I^f_0 and I^f_1 and the $Q_{\alpha+\beta}$'s. Moreover, when $\alpha = 0$, one can use the first ladder relation (4.13) to relate I^f_0 to Q_β, namely

$$I^f_0(u) = \frac{i}{u} Q_\beta(u). \tag{4.17}$$

In other words, in all cases *only one new master integral* is needed: I^f_α if $\alpha \neq 0$, and I^f_1 if $\alpha = 0$.

Equations (4.13) and (4.15) can be cast in the following compact form

$$\begin{pmatrix} I^f_{\alpha+1} \\ \frac{d}{du} I^f_{\alpha+1} \end{pmatrix} = \mathcal{A}(\alpha) \begin{pmatrix} I^f_\alpha \\ \frac{d}{du} I^f_\alpha \end{pmatrix} + \mathcal{B}(\alpha), \tag{4.18}$$

where $\mathcal{A}(\alpha)$ is a 2×2 matrix with elements

$$\mathcal{A}_{11}(\alpha) = 1 - \frac{1}{2\alpha},$$
$$\mathcal{A}_{12}(\alpha) = \mathcal{A}_{21}(\alpha) = -\frac{u}{2\alpha},$$
$$\mathcal{A}_{22(\alpha)} = 0, \tag{4.19}$$

and where the column matrix $\mathcal{B}(\alpha)$ reads

$$\mathcal{B}_1(\alpha) = \frac{i}{2\alpha}\frac{d}{du} Q_{\alpha+\beta}(u),$$
$$\mathcal{B}_2(\alpha) = \frac{i}{2\alpha}\left(\frac{d^2}{du^2} Q_{\alpha+\beta}(u) + Q_{\alpha+\beta-1}(u)\right). \tag{4.20}$$

Recalling the definition (4.4), the last term $\mathcal{B}_2(\alpha)$ can also be rewritten as

$$\mathcal{B}_2(\alpha) = \frac{i}{2\alpha} Q_{\alpha+\beta}(u), \tag{4.21}$$

[4]Here and below we assume that the real part of α is large enough for all the considered integrals to be convergent. Limiting cases must be treated separately (e.g., using analytic continuation in α).

so that

$$\mathcal{B}_1(\alpha) = \frac{d}{du}\mathcal{B}_2(\alpha). \tag{4.22}$$

The first two sequences entering the two-loop waveform are: (i) $Q^{\rm at}_{n+\frac{1}{2}}(u)$ (with $\beta = 1$) which can be reduced to $Q^{\rm at}_{\frac{1}{2}}(u)$ and combination of derivatives of $Q_{n+\frac{1}{2}}(u)$, further reducible to K_0 and K_1; and (ii) $Q^{\rm as}_n(u)$ (with $\beta = \frac{1}{2}$) which can be reduced to $Q^{\rm as}_1(u)$, its derivative, and to the $Q_{n+\frac{1}{2}}(u)$'s, i.e., again to K_0 and K_1.

The third sequence $Q^{\rm as2}_{n+\frac{1}{2}}(u)$ is more complicated because $f'(T)$ is not of the form $\frac{1}{(1+T^2)^\beta}$. However, it is of the form

$$\frac{d}{dT}\mathrm{arcsinh}^2(T) = 2\frac{\mathrm{arcsinh}(T)}{\sqrt{1+T^2}}. \tag{4.23}$$

In that case we get ladder relations with source terms of the form of the already reduced $Q^{\rm as}_n(u)$'s, modulo a single new master integral $Q^{\rm as2}_{\frac{1}{2}}(u)$. We have shown here how to express the waveform in terms of a small set of master integrals, namely $Q^{\rm at}_{\frac{1}{2}}(u)$, $Q^{\rm as}_1(u)$, and $Q^{\rm as2}_{\frac{1}{2}}(u)$. We can then consider that the latter functions are defined by their integral representations. We can also give alternative representations of these master integrals. In particular, we can use the inhomogeneous second-order differential equation they satisfy. The left-hand side (lhs) of Eq. (4.14) features an operator equivalent to the Bessel operator (if one replaces $I^f_\alpha(u) \to u^{-\frac{1}{2}+\alpha}F(u)$, and $\alpha \to \nu + \frac{1}{2}$), say

$$\mathcal{L}_\nu(F(u)) = \left(u^2\frac{d^2}{du^2} + u\frac{d}{du} - (u^2+\nu^2)\right)F(u) = 0, \tag{4.24}$$

with solutions $I_\nu(u)$, and $K_\nu(u)$. This allows one to solve an inhomogeneous equation of the type

$$\mathcal{L}_\nu(F(u)) = S(u), \tag{4.25}$$

with the boundary condition $F(u) \to 0$ as $u \to +\infty$, by using a Green's function bilinear in $I_\nu(u)$, and $K_\nu(u)$ (or, equivalently, by using Lagrange's method of varying constants). At the end, the solution can be expressed in terms of indefinite integrals $\int du I_\nu(u)S(u)$ and $\int du K_\nu(u)S(u)$, with source terms $S(u)$ given by modified Bessel functions.

We find that our first two master integrals satisfy the equations

$$\mathcal{L}_0\left(Q^{\rm at}_{\frac{1}{2}}(u)\right) = -2iuK_0(u). \tag{4.26}$$

and

$$\frac{d^2}{du^2}Q^{\rm as}_1(u) - Q^{\rm as}_1(u) = -\frac{2i}{u}K_0(u). \tag{4.27}$$

The differential operator appearing in the latter equation is equivalent to the Bessel operator $\mathcal{L}_{\frac{1}{2}}$ applied to $u^{-1/2}Q^{\rm as}_1(u)$, that is Eq. (4.27) becomes

$$\mathcal{L}_{\frac{1}{2}}(u^{-1/2}Q^{\rm as}_1(u)) = -2iu^{1/2}K_0(u). \tag{4.28}$$

This approach also applies to $Q^{\rm as2}_{\frac{1}{2}}(u)$. Indeed, the rhs of Eq. (4.14) features [when $\alpha = \frac{1}{2}$, taking into account Eq. (4.23)] the function $Q^{\rm as}_{\alpha-\frac{1}{2}} = Q^{\rm as}_0$. However, the first ladder relation (4.13) allows one, when $\alpha = 0$, to express $Q^{\rm as}_0(u)$ in terms of $K_0(u)$. We thereby obtain the following inhomogeneous Bessel equation for $Q^{\rm as2}_{\frac{1}{2}}(u)$

$$\mathcal{L}_0 Q^{\rm as2}_{\frac{1}{2}}(u) = 4K_0(u). \tag{4.29}$$

One can easily recognize in the latter inhomogeneous Bessel equation the result of differentiating twice the Bessel equation $\mathcal{L}_\nu K_\nu(u)$ with respect to the order ν, before setting $\nu \to 0$. One can then show (see Appendix A for details) that $Q^{\rm as2}_{\frac{1}{2}}(u)$ can be expressed in terms of the second ν-derivative of $K_\nu(u)$ at $\nu = 0$, through

$$Q^{\rm as2}_{\frac{1}{2}}(u) = -\frac{\pi^2}{2}K_0(u) + 2\frac{d^2}{d\nu^2}K_\nu(u)|_{\nu=0}. \tag{4.30}$$

We can then use the literature on second ν-derivatives of $K_\nu(u)$ (see Refs. [42,43]) to further express $Q^{\rm as2}_{\frac{1}{2}}(u)$ in terms of Meijer G functions. We also show in the Appendices how to express the two other master integrals in terms of Meijer G functions.

The main purpose of the present section was to indicate the structures (together with simple IBP's) allowing one to reduce the computation of the 4PM waveform to a small number of master integrals. We leave to the future the issue of finding the most useful (from a practical point of view) master integrals. For instance, it may happen that the reduction of $Q^{\rm as2}_{\frac{1}{2}}(u)$ to Meijer G functions is less useful than considering simply that the function $Q^{\rm as2}_{\frac{1}{2}}(u)$ is defined by a specific integral (which can be numerically evaluated when needed).

More details of the explicit reduction process we used are given in Appendix A.

V. EXPLICIT EXPRESSION FOR THE NEWTONIAN, 1PN AND 2PN CONTRIBUTIONS TO THE 4PM WAVEFORM

Considering, for simplicity, only the $\ell m = 22$ SWSH contribution to $U_2^{G^3}$, at PN orders η^0, η^2, and η^4 we have four different types of contributions: arctan ($p_{22}^{Q^{\rm at}}$), arcsinh ($p_{22}^{Q^{\rm as}}$), arcsinh^2 ($p_{22}^{Q^{\rm as2}}$), together with the simpler type involving Q_α integrals (p_{22}^{Q}). For example, we formally write, at each considered PN order,

$$U_{22}^{G^3\eta^n} = \frac{\nu G^3 M^4}{b^2 p_\infty^{5-n}} \left(p_{Q^{\rm at}}^{22\eta^n} + p_{Q^{\rm as}}^{22\eta^n} + p_{Q^{\rm as2}}^{22\eta^n} + p_{Q}^{22\eta^n} \right). \tag{5.1}$$

All the contributions $U_{2m}^{G^3\eta^n}$ are given in the companion ancillary file. Explicitly, upon reduction to master integrals, $U_{22}^{G^3\eta^0}$ reads

$$U_{22}^{G^3\eta^0} = \frac{\nu G^3 M^4 \sqrt{5\pi}}{5 b^2 p_\infty^5} \left[2(u^2+3u+2)K_0(u) + (2u^2+5u+4)K_1(u) - u^2(u+1)\frac{d}{du}Q_{\frac{1}{2}}^{\rm as2}(u) + u^2\left(u+\frac{1}{2}\right)Q_{\frac{1}{2}}^{\rm as2}(u) \right],$$

$$\begin{aligned} U_{22}^{G^3\eta^2} = \frac{\nu G^3 M^4 \sqrt{5\pi}}{u^2 b^2 p_\infty^3} \Bigg[& \left(\left(\frac{12\nu}{35} - \frac{19}{105}\right)u^5 + \left(\frac{10\nu}{7} - \frac{51}{14}\right)u^4 + \left(\frac{18\nu}{7} + \frac{337}{105}\right)u^3 + \left(\frac{44\nu}{35} + \frac{228}{35}\right)u^2 \right) K_0(u) \\ & + \left(\left(\frac{12\nu}{35} - \frac{19}{105}\right)u^5 + \left(\frac{44\nu}{35} - \frac{373}{105}\right)u^4 + \left(\frac{79\nu}{35} + \frac{61}{105}\right)u^3 + \left(\frac{44\nu}{35} - \frac{2}{7}\right)u^2 \right) K_1(u) \\ & + \left(\left(\frac{19}{210} - \frac{6\nu}{35}\right)u^6 + \left(\frac{269}{420} - \frac{13\nu}{35}\right)u^5 + \left(\frac{19}{70} - \frac{11\nu}{35}\right)u^4 \right) \frac{d}{du}Q_{\frac{1}{2}}^{\rm as2}(u) \\ & + \left(\left(\frac{6\nu}{35} - \frac{19}{210}\right)u^6 + \left(\frac{2\nu}{7} - \frac{25}{42}\right)u^5 + \left(\frac{11\nu}{70} - \frac{17}{35}\right)u^4 \right) Q_{\frac{1}{2}}^{\rm as2}(u) + \left(\frac{6iu^4}{5} + \frac{6iu^3}{5} + \frac{6iu^2}{5}\right) Q_1^{\rm as}(u) \\ & + \left(-\frac{6iu^4}{5} - \frac{6iu^3}{5}\right)\frac{d}{du}Q_1^{\rm as}(u) + \left(\frac{12iu^3}{5} + \frac{6iu^2}{5}\right)Q_{\frac{1}{2}}^{\rm at}(u) + \left(-\frac{12iu^3}{5} - \frac{12iu^2}{5}\right)\frac{d}{du}Q_{\frac{1}{2}}^{\rm at}(u) \Bigg], \end{aligned} \tag{5.2}$$

$$\begin{aligned} U_{22}^{G^3\eta^4} = \frac{\nu G^3 M^4 \sqrt{5\pi}}{u^2 b^2 p_\infty} \Bigg[& \Bigg(\left(\frac{82\nu^2}{945} - \frac{247\nu}{945} + \frac{71}{945}\right)u^6 + \left(\frac{262\nu^2}{945} - \frac{3433\nu}{945} + \frac{4309}{1890}\right)u^5 \\ & + \left(\frac{136\nu^2}{189} + \frac{4867\nu}{945} - \frac{12101}{945}\right)u^4 + \left(\frac{256\nu^2}{315} + \frac{6361\nu}{630} - \frac{1903}{180}\right)u^3 + \left(\frac{32\nu^2}{105} + \frac{787\nu}{105} + \frac{1138}{105}\right)u^2 \Bigg) K_0(u) \\ & + \Bigg(\left(\frac{82\nu^2}{945} - \frac{247\nu}{945} + \frac{71}{945}\right)u^6 + \left(\frac{221\nu^2}{945} - \frac{6619\nu}{1890} + \frac{2119}{945}\right)u^5 + \left(\frac{596\nu^2}{945} + \frac{7157\nu}{1890} - \frac{45053}{3780}\right)u^4 \\ & + \left(\frac{232\nu^2}{315} + \frac{9641\nu}{1260} - \frac{2092}{315}\right)u^3 + \left(\frac{32\nu^2}{105} + \frac{589\nu}{105} - \frac{379}{30}\right)u^2 \Bigg) K_1(u) \\ & + \Bigg(\left(-\frac{41\nu^2}{945} + \frac{247\nu}{1890} - \frac{71}{1890}\right)u^7 + \left(-\frac{7\nu^2}{135} + \frac{1157\nu}{1890} - \frac{1073}{3780}\right)u^6 + \left(-\frac{2\nu^2}{21} + \frac{37\nu}{30} - \frac{19}{84}\right)u^5 \\ & + \left(-\frac{8\nu^2}{105} + \frac{43\nu}{84} + \frac{913}{840}\right)u^4 \Bigg) \frac{d}{du}Q_{\frac{1}{2}}^{\rm as2}(u) \\ & + \Bigg(\left(\frac{41\nu^2}{945} - \frac{247\nu}{1890} + \frac{71}{1890}\right)u^7 + \left(\frac{19\nu^2}{630} - \frac{689\nu}{1260} + \frac{167}{630}\right)u^6 + \left(\frac{16\nu^2}{315} - \frac{1117\nu}{1260} - \frac{79}{504}\right)u^5 \\ & + \left(\frac{4\nu^2}{105} - \frac{677\nu}{840} + \frac{109}{210}\right)u^4 \Bigg) Q_{\frac{1}{2}}^{\rm as2}(u) \\ & + \left(\left(\frac{39i\nu}{35} - \frac{61i}{70}\right)u^5 + \left(\frac{54i\nu}{35} - \frac{67i}{35}\right)u^4 + \left(\frac{54i\nu}{35} - \frac{67i}{35}\right)u^3 + \left(\frac{9i\nu}{7} - \frac{51i}{70}\right)u^2 \right) Q_1^{\rm as}(u) \\ & + \left(\left(\frac{61i}{70} - \frac{39i\nu}{35}\right)u^5 + \left(\frac{67i}{35} - \frac{54i\nu}{35}\right)u^4 + \left(\frac{51i}{70} - \frac{9i\nu}{7}\right)u^3 \right) \frac{d}{du}Q_1^{\rm as}(u) \\ & + \left(\left(\frac{114i\nu}{35} - \frac{143i}{35}\right)u^4 + \left(\frac{99i\nu}{35} + \frac{109i}{70}\right)u^3 + \left(\frac{9i\nu}{7} - \frac{51i}{70}\right)u^2 \right) Q_{\frac{1}{2}}^{\rm at}(u) \\ & + \left(\left(\frac{143i}{35} - \frac{114i\nu}{35}\right)u^4 + \left(\frac{17i}{35} - \frac{156i\nu}{35}\right)u^3 + \left(-\frac{18i\nu}{7} - \frac{243i}{35}\right)u^2 \right) \frac{d}{du}Q_{\frac{1}{2}}^{\rm at}(u) \Bigg], \end{aligned} \tag{5.3}$$

Reconstructing the original quantity $U_2^{G^3}$ is then straightforward

$$\begin{aligned} U_2^{G^3} &= (U_{22}^{G^3\eta^0} + U_{22}^{G^3\eta^2}\eta^2 + U_{22}^{G^3\eta^4}\eta^4) Y_{\bar{2};22}(\theta,\phi) \\ &\quad + (U_{20}^{G^3\eta^0} + U_{20}^{G^3\eta^2}\eta^2 + U_{20}^{G^3\eta^4}\eta^4) Y_{\bar{2};20}(\theta,\phi) \\ &\quad + (U_{2\bar{2}}^{G^3\eta^0} + U_{2\bar{2}}^{G^3\eta^2}\eta^2 + U_{2\bar{2}}^{G^3\eta^4}\eta^4) Y_{\bar{2};2\bar{2}}(\theta,\phi), \end{aligned} \tag{5.4}$$

where the notation $\bar{2} = -2$ has been used.

For instance, the Newtonian term $U_2^{G^3\eta^0}$ evaluated at $\theta = \frac{\pi}{2}$ is given by

$$\begin{aligned} U_2^{G^3\eta^0} = \frac{\nu G^3 M^4}{b^2 p_\infty^5}\Bigg[&\left(\left(\frac{u^2}{2}+1\right)\cos(2\phi)+\frac{3}{2}iu\sin(2\phi)\right)K_0(u) \\ &+\left(\left(\frac{iu^2}{2}+i\right)\sin(2\phi)+\frac{5}{4}u\cos(2\phi)+\frac{u}{4}\right)K_1(u) \\ &+\left(-\frac{1}{4}u^3\cos(2\phi)-\frac{1}{4}iu^2\sin(2\phi)\right)\frac{d}{du}Q_{\frac{1}{2}}^{\mathrm{as}2}(u) \\ &+\left(\frac{1}{4}iu^3\sin(2\phi)+\frac{1}{8}u^2\cos(2\phi)+\frac{u^2}{8}\right)Q_{\frac{1}{2}}^{\mathrm{as}2}(u)\Bigg], \end{aligned} \tag{5.5}$$

with [see Eq. (A30)]

$$\begin{aligned} Q_{\frac{1}{2}}^{\mathrm{as}2}(u) &= -\frac{\pi^2}{2}K_0(u) + 2\frac{d^2}{d\nu^2}K_\nu(u)\Big|_{\nu=0}, \\ \frac{d}{du}Q_{\frac{1}{2}}^{\mathrm{as}2}(u) &= \frac{\pi^2}{2}K_1(u) - 2\frac{d^2}{d\nu^2}K_\nu(u)\Big|_{\nu=1}. \end{aligned} \tag{5.6}$$

VI. SOFT EXPANSION OF THE 4PM WAVEFORM

The first three terms of the low frequency ("soft") expansion of the complex waveform [in its rescaled version $W(\omega,\mathbf{n}) = \frac{c^4}{4G}h_c(\omega,\mathbf{n})$] are universal and have the structure

$$W(\omega) \overset{\omega\to 0^+}{=} \frac{\mathcal{A}}{\omega} + \mathcal{B}\ln\omega + \mathcal{C}\omega(\ln\omega)^2 + \cdots \tag{6.1}$$

The leading-order term $\frac{\mathcal{A}}{\omega}$ was derived in the classic work of Weinberg [44]. The sub-leading soft terms $\mathcal{B}\ln\omega$ and $\mathcal{C}\omega(\ln\omega)^2$ were derived in Refs. [45–47], and have been re-written in a simplified way in Appendix B of Ref. [35] in the case of the conservative $2\to 2$ scattering, in the cm frame. [Radiative contributions to the scattering angle start at order $O(\frac{G^3}{c^5})$ while the present study is limited to the $O(\frac{G^3}{c^4})$ accuracy.] We have found that the results obtained from these universal soft theorems agree with the direct low-frequency expansion of our explicit frequency-domain waveform presented above, which read (limiting our computations to the equatorial plane, and including in the waveform only even-in-ϕ terms, for simplicity)

$$\begin{aligned} \mathcal{A}^{G^3} &= \frac{iM^4G^3\nu}{b^3p_\infty^4}\Bigg\{\sin(2\phi) + \eta^2p_\infty^2\left[\left(2\nu-\frac{1}{2}\right)\sin(2\phi) + \left(\frac{9\nu}{2}-\frac{3}{2}\right)\sin(4\phi)\right] \\ &\quad + \eta^4p_\infty^4\left[\left(\frac{11\nu^2}{16}+\frac{105\nu}{16}-\frac{251}{16}\right)\sin(2\phi) + \left(\frac{21\nu^2}{4}+\frac{45\nu}{4}-\frac{9}{2}\right)\sin(4\phi) + \left(\frac{95\nu^2}{16}-\frac{95\nu}{16}+\frac{19}{16}\right)\sin(6\phi)\right]\Bigg\}, \\ \mathcal{B}^{G^3} &= \frac{M^4G^3\nu}{b^2p_\infty^5}\Bigg\{-\cos(2\phi) + \eta^2p_\infty^2[(-2\nu-2)\cos(2\phi) + (1-3\nu)\cos(4\phi)] \\ &\quad + \eta^4p_\infty^4\left[\left(-\frac{11\nu^2}{16}-\frac{57\nu}{16}+\frac{65}{16}\right)\cos(2\phi) + \left(-\frac{7\nu^2}{2}-3\nu+\frac{3}{2}\right)\cos(4\phi) + \left(-\frac{45\nu^2}{16}+\frac{45\nu}{16}-\frac{9}{16}\right)\cos(6\phi)\right]\Bigg\}, \\ \mathcal{C}^{G^3} &= \frac{iM^4G^3\nu}{2bp_\infty^6}\Bigg\{\sin(2\phi) + \eta^2p_\infty^2\left(\left(2\nu-\frac{3}{2}\right)\sin(2\phi) + \left(\frac{3\nu}{2}-\frac{1}{2}\right)\sin(4\phi)\right) \\ &\quad + \eta^4p_\infty^4\left(\left(\frac{11\nu^2}{16}-\frac{55\nu}{16}-\frac{83}{16}\right)\sin(2\phi) + \left(\frac{7\nu^2}{4}-\frac{15\nu}{4}+1\right)\sin(4\phi) + \left(\frac{15\nu^2}{16}-\frac{15\nu}{16}+\frac{3}{16}\right)\sin(6\phi)\right)\Bigg\}. \end{aligned} \tag{6.2}$$

The latter 2PN-accurate expansions were obtained by deriving the low-frequency expansion ($u\to 0$) of our master integrals $Q_{\frac{1}{2}}^{\mathrm{at}}(u)$, $Q_1^{\mathrm{as}}(u)$ and $Q_{\frac{1}{2}}^{\mathrm{as}2}(u)$ from their corresponding Meijer G representations, see Eqs. (A15) and (B6)–(B9) below.

VII. CONCLUDING REMARKS

As a benchmark in the current effort to increase the PM accuracy in solving the gravitational bremsstrahlung problem, we have provided here the $O(G^4)$ (two-loop) asymptotic waveform, at 2PN accuracy $c^4h_c \sim O(G^4\eta^4)$, computing the multipolar contributions U_2, V_2, U_3, V_3, U_4, V_4, U_5, V_5, and U_6 at the needed accuracy level. The resulting expressions in Fourier space have been found to be more involved than at the previously treated $O(G^3)$ (one-loop) case. They involve a finite number of new master integrals of the form displayed in Eqs. (4.6). We have explicitly discussed the IBP reduction of these integrals, and shown how they can also be expressed either as integrals of integrands bilinear in Bessel functions (of integer or half-integer, orders), or, more explicitly, in terms of MeijerG functions. See Appendices below for details. The low-frequency limit of our 2PN-accurate $O(G^4)$ waveform was checked against soft theorems.

Finally, as a side study of our previous results on the $O(G^2)$ bremsstrahlung waveform, we computed the $O(G^3)$ spectral densities of radiated energy and momentum (in the rest frame of particle A) at the thirtieth order in velocity. These results are given here as additional benchmarks in the study of the gravitational bremsstrahlung problem in the post-Minkowskian approach.

Our results are explicitly given in electronic form in the ancillary files of the arxiv submission of this work.

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APPENDIX A: EVALUATION AND REDUCTION OF THE INTEGRALS ENTERING THE $O(G^4)$ WAVEFORM

Let us discuss how to evaluate (and reduce) the needed integrals (4.6), which are of the general form

$$I^f_\alpha(u) = \int dT \frac{e^{iuT}}{(1+T^2)^\alpha} f(T), \tag{A1}$$

with $f(T) = [\arctan(T), \mathrm{arcsinh}(T), \mathrm{arcsinh}^2(T)]$. The first two are purely imaginary (for symmetry reasons) while the third one is real. Let us discuss the first two types, having the special property that in both cases

$$f'(T) = \frac{1}{(1+T^2)^\beta}, \tag{A2}$$

with $\beta = 1$ for the arctan integral and $\beta = 1/2$ for the arcsinh integral. Denoting

$$\hat{T} = \frac{1}{i}\frac{d}{du}, \tag{A3}$$

we have

$$(\hat{T}^2 + 1) I^f_\alpha(u) = I^f_{\alpha-1}(u). \tag{A4}$$

Moreover, the IBP relations, Eqs. (4.11), imply ($f(T)$ carrying the index β)

$$\begin{aligned} 0 &= iuI^f_\alpha - 2\alpha\hat{T}I^f_{\alpha+1} + Q_{\alpha+\beta}, \\ 0 &= (1 + iu\hat{T} - 2\alpha)I^f_\alpha + 2\alpha I^f_{\alpha+1} + \hat{T}Q_{\alpha+\beta}, \end{aligned} \tag{A5}$$

leading to the following recurrence relations

$$\begin{aligned} I^f_{\alpha-1} &= \frac{2(\alpha-1)}{iu}\hat{T}I^f_\alpha - \frac{1}{iu}Q_{\alpha+\beta-1}, \\ I^f_{\alpha+1} &= \frac{1}{2\alpha}(2\alpha - 1 - iu\hat{T})I^f_\alpha - \frac{1}{2\alpha}\hat{T}Q_{\alpha+\beta}. \end{aligned} \tag{A6}$$

Therefore, as explained in the text, one can reduce the first two types of integrals to those with the lowest possible values of α, namely $\alpha = \frac{1}{2}$ for $f(T) = \arctan(T)$, and $\alpha = 1$ for $f(T) = \mathrm{arcsinh}(T)$. Some details follow.

1. Integrals of the type $Q^{\rm at}_\alpha(u)$

Let us consider the case $f(T) = \arctan(T)$, i.e., $I^f_\alpha = Q^{\rm at}_\alpha$, $\beta = 1$. The first equation of Eqs. (4.13) for the first few values of $\alpha = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$ gives

$$\begin{aligned} R_{\frac{1}{2}} \quad 0 &= iuQ^{\rm at}_{\frac{1}{2}}(u) + i\frac{d}{du}Q^{\rm at}_{\frac{3}{2}}(u) + 2uK_1(u), \\ R_{\frac{3}{2}} \quad 0 &= iuQ^{\rm at}_{\frac{3}{2}}(u) + 3i\frac{d}{du}Q^{\rm at}_{\frac{5}{2}}(u) + \frac{2}{3}u^2K_0(u) \\ &\quad + \frac{4}{3}uK_1(u), \\ R_{\frac{5}{2}} \quad 0 &= iuQ^{\rm at}_{\frac{5}{2}}(u) + 5i\frac{d}{du}Q^{\rm at}_{\frac{7}{2}}(u) + \frac{8}{15}u^2K_0(u) \\ &\quad + \frac{2u}{15}(u^2+8)K_1(u), \end{aligned} \tag{A7}$$

while the second one gives

$$\begin{aligned} S_{\frac{1}{2}} \quad 0 &= u\frac{d}{du}Q^{\rm at}_{\frac{1}{2}}(u) + 2iuK_0(u) + Q^{\rm at}_{\frac{3}{2}}(u), \\ S_{\frac{3}{2}} \quad 0 &= -2Q^{\rm at}_{\frac{3}{2}}(u) + u\frac{d}{du}Q^{\rm at}_{\frac{3}{2}}(u) + \frac{2}{3}iu^2K_1(u) \\ &\quad + 3Q^{\rm at}_{\frac{5}{2}}(u), \\ S_{\frac{5}{2}} \quad 0 &= -4Q^{\rm at}_{\frac{5}{2}}(u) + u\frac{d}{du}Q^{\rm at}_{\frac{5}{2}}(u) + \frac{2iu^3}{15}K_0(u) \\ &\quad + \frac{4iu^2}{15}K_1(u) + 5Q^{\rm at}_{\frac{7}{2}}(u). \end{aligned} \tag{A8}$$

Solving the relation $S_{\frac{1}{2}}$ for $Q^{\rm at}_{\frac{3}{2}}(u)$ and substituting it into the relation $R_{\frac{1}{2}}$ gives the following second order differential equation for $Q^{\rm at}_{\frac{1}{2}}(u)$

$$-iu\frac{d^2}{du^2}Q^{\rm at}_{\frac{1}{2}}(u)-i\frac{d}{du}Q^{\rm at}_{\frac{1}{2}}(u)+iuQ^{\rm at}_{\frac{1}{2}}(u)+2K_0(u)=0. \tag{A9}$$

The latter is a real equation since $Q^{\rm at}_{\frac{1}{2}}(u)=i\tilde{Q}^{\rm at}_{\frac{1}{2}}(u)$, with $\tilde{Q}^{\rm at}_{\frac{1}{2}}(u)$ a real function of u, so that

$$u\frac{d^2}{du^2}\tilde{Q}^{\rm at}_{\frac{1}{2}}(u)+\frac{d}{du}\tilde{Q}^{\rm at}_{\frac{1}{2}}(u)-u\tilde{Q}^{\rm at}_{\frac{1}{2}}(u)=-2K_0(u). \tag{A10}$$

Recalling Bessel's modified differential equation of order n

$$\mathcal{L}_n(f(u))=\left(u^2\frac{d^2}{du^2}+u\frac{d}{du}-(u^2+n^2)\right)f(u)=0, \tag{A11}$$

with solutions $f_n(u)=I_n(u),K_n(u)$, the above equation reads

$$\mathcal{L}_0\left(\tilde{Q}^{\rm at}_{\frac{1}{2}}(u)\right)=-2uK_0(u). \tag{A12}$$

The solution for $\tilde{Q}^{\rm at}_{\frac{1}{2}}(u)$ is thus given by

$$\begin{aligned}\tilde{Q}^{\rm at}_{\frac{1}{2}}(u)&=c_1K_0(u)+c_2I_0(u)-2I_0(u)\int_0^u K_0^2(x)dx\\&\quad+2K_0(u)\int_0^u I_0(x)K_0(x)dx,\end{aligned} \tag{A13}$$

namely $\tilde{Q}^{\rm at}_{\frac{1}{2}}(u)$ is expressed as some kind of *iterated Bessel function*. The integration constants c_1 and c_2 corresponding to the solutions of the associated homogeneous equation are chosen in order to ensure that $\tilde{Q}^{\rm at}_{\frac{1}{2}}(u)$ satisfies the needed limits as $u\to 0$ and $u\to\infty$. This implies $c_1=0$ and $c_2=\frac{\pi^2}{2}$. The final solution can be expressed in terms of Meijer G functions as follows

$$\begin{aligned}\tilde{Q}^{\rm at}_{\frac{1}{2}}(u)&=\frac{\pi^2}{2}I_0(u)-2I_0(u)\frac{\sqrt{\pi}u}{4}G^{3,1}_{2,4}\left(u^2\middle|\begin{matrix}\frac{1}{2},\frac{1}{2}\\0,0,0,-\frac{1}{2}\end{matrix}\right)\\&\quad+2K_0(u)\frac{u}{4\sqrt{\pi}}G^{2,2}_{2,4}\left(u^2\middle|\begin{matrix}\frac{1}{2},\frac{1}{2}\\0,0,-\frac{1}{2},0\end{matrix}\right).\end{aligned} \tag{A14}$$

For small values of u we find

$$\begin{aligned}\tilde{Q}^{\rm at}_{\frac{1}{2}}(u)&=u\left(2+\frac{5u^2}{18}+\frac{89u^4}{7200}\right)\ln\left(\frac{1}{2}ue^{\gamma_E}\right)+\frac{\pi^2}{2}\left(1+\frac{u^2}{4}+\frac{u^4}{64}\right)\\&\quad-u\left(4+\frac{37}{54}u^2+\frac{7393}{216000}u^4\right)+O(u^6),\end{aligned} \tag{A15}$$

whereas for large values of u

$$\tilde{Q}^{\rm at}_{\frac{1}{2}}(u)\sim\frac{\sqrt{2\pi}}{4u^{3/2}}e^{-u}\left[1+\left(2u-\frac{1}{4}\right)\ln\left(8ue^{\gamma_E}\right)\right]. \tag{A16}$$

Notice that this solution gives $\tilde{Q}^{\rm at}_{\frac{1}{2}}(0)=\frac{\pi^2}{2}$, as it can be checked directly by using the method of regions. Considering complex values of α, one can distinguish two regions: region 1 ($uT\ll 1$), and region 2 ($uT\gg 1$). One then formally power-expand the integrand in each region. In region 1 one gets zero, whereas in region 2, one gets $\pi^2/2$ upon changing to the variable $x=uT$ and then taking $\alpha\to 1/2$. Beware that one cannot first take the limit $\alpha\to 1/2$, and then take the limit $u\to 0$ in the integrand because of the arising of a logarithmic divergence.

Using the recurrence relations above, we then find

$$\begin{aligned}Q^{\rm at}_{\frac{3}{2}}(u)&=-u\frac{d}{du}Q^{\rm at}_{\frac{1}{2}}(u)-2iK_0(u)u,\\
Q^{\rm at}_{\frac{5}{2}}(u)&=-\frac{2}{3}u\frac{d}{du}Q^{\rm at}_{\frac{1}{2}}(u)-\frac{4}{3}iK_0(u)u+\frac{1}{3}Q^{\rm at}_{\frac{1}{2}}(u)u^2\\&\quad-\frac{8}{9}iK_1(u)u^2,\\
Q^{\rm at}_{\frac{7}{2}}(u)&=\left(-\frac{16}{15}u-\frac{46}{225}u^3\right)iK_0(u)-\frac{64}{75}iK_1(u)u^2\\&\quad+\left(-\frac{8}{15}u-\frac{1}{15}u^3\right)\frac{d}{du}Q^{\rm at}_{\frac{1}{2}}(u)+\frac{4}{15}Q^{\rm at}_{\frac{1}{2}}(u)u^2,\\
Q^{\rm at}_{\frac{9}{2}}(u)&=\left(-\frac{176}{735}u^3-\frac{32}{35}u\right)iK_0(u)\\&\quad+\left(-\frac{192}{245}u^2-\frac{352}{11025}u^4\right)iK_1(u)\\&\quad+\left(-\frac{8}{105}u^3-\frac{16}{35}u\right)\frac{d}{du}Q^{\rm at}_{\frac{1}{2}}(u)\\&\quad+\left(\frac{8}{35}u^2+\frac{1}{105}u^4\right)Q^{\rm at}_{\frac{1}{2}}(u),\\
Q^{\rm at}_{\frac{11}{2}}(u)&=\left(-\frac{688}{2835}u^3-\frac{256}{315}u-\frac{1126}{297675}u^5\right)iK_0(u)\\&\quad+\left(-\frac{2048}{2835}u^2-\frac{2752}{59535}u^4\right)iK_1(u)\\&\quad+\left(-\frac{8}{105}u^3-\frac{128}{315}u-\frac{1}{945}u^5\right)\frac{d}{du}Q^{\rm at}_{\frac{1}{2}}(u)\\&\quad+\left(\frac{4}{315}u^4+\frac{64}{315}u^2\right)Q^{\rm at}_{\frac{1}{2}}(u).\end{aligned} \tag{A17}$$

2. Integrals of the type $Q^{\rm as}_\alpha(u)$

Let us consider now the case $f(T)=\operatorname{arcsinh}(T)$, i.e., $I^f_\alpha=Q^{\rm as}_\alpha$, $\beta=1/2$. Using the recurrence relations (4.13) we find

$$Q_2^{\rm as}(u) = \frac{1}{2}Q_1^{\rm as}(u) - \frac{u}{2}\frac{d}{du}Q_1^{\rm as}(u) - iuK_0(u)$$
$$Q_3^{\rm as}(u) = -\frac{3}{8}u\frac{d}{du}Q_1^{\rm as}(u) + \left(\frac{3}{8}+\frac{u^2}{8}\right)Q_1^{\rm as}(u) - \frac{3iu}{4}K_0(u) - \frac{5iu^2}{12}K_1(u)$$
$$Q_4^{\rm as}(u) = +\left(\frac{5}{16}+\frac{u^2}{8}\right)Q_1^{\rm as}(u) + \left(-\frac{5}{16}u - \frac{1}{48}u^3\right)\frac{d}{du}Q_1^{\rm as}(u) + \left(-\frac{5iu}{8} - \frac{11}{120}iu^3\right)K_0(u) - \frac{161}{360}iK_1(u)u^2$$
$$Q_5^{\rm as}(u) = \left(\frac{15}{128}u^2 + \frac{1}{384}u^4 + \frac{35}{128}\right)Q_1^{\rm as}(u) + \left(-\frac{5}{192}u^3 - \frac{35}{128}u\right)\frac{d}{du}Q_1^{\rm as}(u) + \left(-\frac{35}{64}iu - \frac{587}{5040}iu^3\right)K_0(u) + \left(-\frac{969}{2240}iu^2 - \frac{31}{2240}iu^4\right)K_1(u)$$
$$Q_6^{\rm as}(u) = +\left(\frac{1}{256}u^4 + \frac{7}{64}u^2 + \frac{63}{256}\right)Q_1^{\rm as}(u) + \left(-\frac{7}{256}u^3 - \frac{63}{256}u - \frac{1}{3840}u^5\right)\frac{d}{du}Q_1^{\rm as}(u) + \left(-\frac{63}{128}iu - \frac{24883}{201600}iu^3 - \frac{193}{120960}iu^5\right)K_0(u) + \left(-\frac{82841}{201600}iu^2 - \frac{187}{8640}iu^4\right)K_1(u), \tag{A18}$$

where $Q_1^{\rm as}(u) = i\tilde{Q}_1^{\rm as}(u)$ satisfies the following (real) differential equation

$$\frac{d^2}{du^2}\tilde{Q}_1^{\rm as}(u) - \tilde{Q}_1^{\rm as}(u) = -\frac{2}{u}K_0(u). \tag{A19}$$

The solution reads

$$\tilde{Q}_1^{\rm as}(u) = \frac{\pi^2}{2}e^{-u} + \sqrt{\pi}e^u G^{3,0}_{2,3}\left(2u\middle|\begin{matrix}\frac{1}{2},1\\0,0,0\end{matrix}\right) - \frac{e^{-u}}{\sqrt{\pi}}G^{3,1}_{2,3}\left(2u\middle|\begin{matrix}\frac{1}{2},1\\0,0,0\end{matrix}\right), \tag{A20}$$

which vanishes both at $u = 0$ and $u \to \infty$.

3. Integrals of the type $Q_\alpha^{\rm as2}(u)$

Finally, let us consider the case $f(T) = \mathrm{arcsinh}^2(T)$, i.e., $I_\alpha^f = Q_\alpha^{\rm as2}$, $\beta = 1/2$. The basic integral with $\alpha = \frac{1}{2}$ is

$$Q_{\frac{1}{2}}^{\rm as2}(u) = \int_{-\infty}^{\infty} dT \frac{e^{iuT}}{\sqrt{1+T^2}}\mathrm{arcsinh}^2 T, \tag{A21}$$

which can be computed as follows. Consider the identity ([43], 10.32.7)

$$\cos\left(\frac{\nu}{2}\pi\right)K_\nu(x) = \int_0^\infty \cos(x\sinh t)\cosh(\nu t)dt = \frac{1}{2}\int_{-\infty}^\infty e^{ix\sinh t}\cosh(\nu t)dt. \tag{A22}$$

Differentiating both sides with respect to ν one gets

$$-\frac{\pi}{2}\sin\left(\frac{\nu}{2}\pi\right)K_\nu(x) + \cos\left(\frac{\nu}{2}\pi\right)\frac{\partial}{\partial\nu}K_\nu(x) = \frac{1}{2}\int_{-\infty}^\infty e^{ix\sinh t}t\sinh(\nu t)dt. \tag{A23}$$

Changing the variable as

$$T = \sinh(t), \quad dT = \cosh t dt, \quad dt = \frac{dT}{\sqrt{1+T^2}} \tag{A24}$$

one gets

$$-\frac{\pi}{2}\sin\left(\frac{\nu}{2}\pi\right)K_\nu(x) + \cos\left(\frac{\nu}{2}\pi\right)\frac{\partial}{\partial\nu}K_\nu(x) = \frac{1}{2}\int_{-\infty}^\infty e^{ixT}\mathrm{arcsinh}(T)\sinh(\nu\,\mathrm{arcsinh}(T))\frac{dT}{\sqrt{1+T^2}}. \tag{A25}$$

This equation can be used to reduce the integrals

$$I_3(n) = Q^{\rm as}_{n+\frac{1}{2}}(u) = \int dT \frac{e^{iuT}}{(1+T^2)^{n+\frac{1}{2}}}\mathrm{arcsinh}T \tag{A26}$$

to $K_n(u)$ Bessel functions [and thereby to $K_0(u)$ and $K_1(u)$] and exponential functions. Indeed, taking $\nu = 1$ in (A25) yields

$$\int_{-\infty}^\infty dT e^{ixT}\frac{T}{\sqrt{1+T^2}}\mathrm{arcsinh}(T) = -\pi K_1(x) = +\pi K_0'(x). \tag{A27}$$

Integrating both sides over x then yields

$$Q^{\rm as}_{\frac{1}{2}}(x) = i\pi K_0(x), \tag{A28}$$

where both sides vanish as $x \to +\infty$.

Further differentiating both sides of (A25) with respect to ν and evaluating for $\nu = 0$ one finds

$$-\frac{\pi^2}{4}K_0(x)+\frac{d^2}{d\nu^2}K_\nu(x)\Big|_{\nu=0} =\frac{1}{2}\int_{-\infty}^{\infty}dT\frac{e^{ixT}}{\sqrt{1+T^2}}\text{arcsinh}^2T, \tag{A29}$$

implying

$$Q^{\rm as2}_{\frac{1}{2}}(u)=-\frac{\pi^2}{2}K_0(u)+2\frac{d^2}{d\nu^2}K_\nu(u)\Big|_{\nu=0}, \tag{A30}$$

where the second term is given in Eq. (B2) below.

The simplest way to get the integrals corresponding to higher values of α consists in changing the variable as

$$T=\frac{v_0t}{b},\qquad dT=\frac{v_0}{b}dt, \tag{A31}$$

differentiating then with respect to b leading to

$$Q^{\rm as2}_{\frac{3}{2}}(u)=-u\frac{dQ^{\rm as2}_{\frac{1}{2}}(u)}{du}-2\frac{d\tilde{Q}^{\rm as}_1(u)}{du}, \tag{A32}$$

where $\tilde{Q}^{\rm as}_1(u)$ is given by Eq. (A20). Similarly we get

$$\begin{aligned}
Q^{\rm as2}_{\frac{5}{2}}(u)&=-\frac{2}{3}uK_1(u)+\frac{1}{3}u^2Q^{\rm as2}_{\frac{1}{2}}(u)-\frac{2}{3}u\frac{dQ^{\rm as2}_{\frac{1}{2}}(u)}{du}-\frac{4}{3}\frac{d\tilde{Q}^{\rm as}_1(u)}{du}+u\tilde{Q}^{\rm as}_1(u),\\
Q^{\rm as2}_{\frac{7}{2}}(u)&=-\frac{3}{10}u^2K_0(u)-\frac{2}{3}uK_1(u)+\left(-\frac{8}{15}u-\frac{1}{15}u^3\right)\frac{dQ^{\rm as2}_{\frac{1}{2}}(u)}{du}+\frac{4}{15}u^2Q^{\rm as2}_{\frac{1}{2}}(u)-\left(\frac{u^2}{4}+\frac{16}{15}\right)\frac{d\tilde{Q}^{\rm as}_1(u)}{du}+\frac{11}{12}u\tilde{Q}^{\rm as}_1(u),\\
Q^{\rm as2}_{\frac{9}{2}}(u)&=-\frac{13}{36}u^2K_0(u)+\left(-\frac{29}{420}u^3-\frac{28}{45}u\right)K_1(u)+\left(-\frac{16}{35}u-\frac{8}{105}u^3\right)\frac{dQ^{\rm as2}_{\frac{1}{2}}(u)}{du}+\left(\frac{8}{35}u^2+\frac{1}{105}u^4\right)Q^{\rm as2}_{\frac{1}{2}}(u)\\
&\quad-\left(\frac{7}{24}u^2+\frac{32}{35}\right)\frac{d\tilde{Q}^{\rm as}_1(u)}{du}+\left(\frac{1}{24}u^3+\frac{33}{40}u\right)\tilde{Q}^{\rm as}_1(u),\\
Q^{\rm as2}_{\frac{11}{2}}(u)&=\left(-\frac{65u^4}{6048}-\frac{33647u^2}{90720}\right)K_0(u)+\left(-\frac{3109u^3}{30240}-\frac{328u}{567}\right)K_1(u)+\left(-\frac{u^5}{945}-\frac{8u^3}{105}-\frac{128u}{315}\right)\frac{dQ^{\rm as2}_{\frac{1}{2}}(u)}{du}\\
&\quad+\left(\frac{4}{315}u^4+\frac{64}{315}u^2\right)Q^{\rm as2}_{\frac{1}{2}}(u)-\left(\frac{u^4}{192}+\frac{283u^2}{960}+\frac{256}{315}\right)\frac{d\tilde{Q}^{\rm as}_1(u)}{du}+\left(\frac{17u^3}{288}+\frac{5053u}{6720}\right)\tilde{Q}^{\rm as}_1(u),\\
Q^{\rm as2}_{\frac{13}{2}}(u)&=\left(-\frac{109u^4}{5940}-\frac{521129u^2}{1425600}\right)K_0(u)+\left(-\frac{281u^5}{221760}-\frac{1188767u^3}{9979200}-\frac{7664u}{14175}\right)K_1(u)\\
&\quad+\left(-\frac{2u^5}{1155}-\frac{256u^3}{3465}-\frac{256u}{693}\right)\frac{dQ^{\rm as2}_{\frac{1}{2}}(u)}{du}+\left(\frac{u^6}{10395}+\frac{16u^4}{1155}+\frac{128u^2}{693}\right)Q^{\rm as2}_{\frac{1}{2}}(u)\\
&\quad-\left(\frac{5u^4}{576}+\frac{1289u^2}{4480}+\frac{512}{693}\right)\frac{d\tilde{Q}^{\rm as}_1(u)}{du}+\left(\frac{u^5}{1920}+\frac{383u^3}{5760}+\frac{5597u}{8064}\right)\tilde{Q}^{\rm as}_1(u).
\end{aligned} \tag{A33}$$

The remaining relations needed at 2PN can be found in the ancillary file.

APPENDIX B: DERIVATIVES OF BESSELK FUNCTIONS WITH RESPECT TO THE ORDER: EXPLICIT EXPRESSIONS

The first-order derivatives of the BesselK functions with respect to the order are given by

$$\begin{aligned}
\frac{\partial}{\partial\nu}K_\nu(u)\Big|_{\nu=0}&=0,\\
\frac{\partial}{\partial\nu}K_\nu(u)\Big|_{\nu=1}&=\frac{1}{u}K_0(u).
\end{aligned} \tag{B1}$$

The second-order derivatives are instead much more complicated (see, e.g., Ref. [42]) and involve MeijerG functions. We find

$$\begin{aligned}
\frac{\partial^2}{\partial\nu^2}K_\nu(u)\Big|_{\nu=0}&=A_{K_0}(u)K_0(u)+A_{I_0}(u)I_0(u),\\
\frac{\partial^2}{\partial\nu^2}K_\nu(u)\Big|_{\nu=1}&=B_{K_0}(u)K_0(u)+B_{K_1}(u)K_1(u)\\
&\quad+B_{I_0}(u)I_0(u)+B_{I_1}(u)I_1(u),
\end{aligned} \tag{B2}$$

where (using the traditional notation for the MeijerG functions)

$$
\begin{aligned}
A_{K_0}(u) = &-i\pi \ln\left(\frac{i}{2} u e^{\gamma_E}\right) \\
&- \frac{i\pi}{4} u^2 {}_3F_4\left(1, 1, \frac{3}{2}; 2, 2, 2, 2; u^2\right) \\
&- \frac{\sqrt{\pi}}{2} G^{3,5}_{3,1}\left(-u^2 \middle| \begin{matrix} \frac{1}{2}, -\frac{1}{2}, 1 \\ 0, 0, 0, 0, -\frac{1}{2} \end{matrix}\right), \\
A_{I_0}(u) = &-\frac{\pi^{3/2}}{2} G^{4,0}_{4,5}\left(-u^2 \middle| \begin{matrix} 0, \frac{1}{2}, \frac{1}{2}, 1 \\ 0, 0, 0, 0, \frac{1}{2} \end{matrix}\right) \\
&+ \frac{i\pi^{3/2}}{2} G^{3,5}_{3,1}\left(-u^2 \middle| \begin{matrix} \frac{1}{2}, -\frac{1}{2}, 1 \\ 0, 0, 0, 0, -\frac{1}{2} \end{matrix}\right),
\end{aligned} \tag{B3}
$$

and

$$
\begin{aligned}
B_{K_0}(u) = &-\frac{i\pi}{u} - \frac{i\pi u}{2} {}_3F_4\left(1, 1, \frac{3}{2}; 2, 2, 2, 2; u^2\right) \\
&- \frac{3i\pi u^3}{64} {}_3F_4\left(2, 2, \frac{5}{2}; 3, 3, 3, 3; u^2\right) \\
&- \frac{\sqrt{\pi}}{u} G^{2,1}_{2,4}\left(-u^2 \middle| \begin{matrix} \frac{1}{2}, -\frac{1}{2} \\ 0, 0, 0, -\frac{1}{2} \end{matrix}\right), \\
B_{K_1}(u) = &\frac{i\pi}{2}\left[3i\pi + 2\ln\left(\frac{u}{2} e^{\gamma_E}\right)\right] \\
&+ \frac{i\pi u^2}{4} {}_3F_4\left(1, 1, \frac{3}{2}; 2, 2, 2, 2; u^2\right) \\
&- \frac{\sqrt{\pi}}{2} G^{3,1}_{2,5}\left(-u^2 \middle| \begin{matrix} \frac{1}{2}, -\frac{1}{2}, 1 \\ 0, 0, 0, 0, -\frac{1}{2} \end{matrix}\right), \\
B_{I_0}(u) = &-\frac{\pi^{3/2}}{u}\left[G^{3,0}_{3,4}\left(-u^2 \middle| \begin{matrix} 0, \frac{1}{2}, \frac{1}{2} \\ 0, 0, 0, \frac{1}{2} \end{matrix}\right)\right. \\
&\left. + iG^{2,1}_{2,4}\left(-u^2 \middle| \begin{matrix} \frac{1}{2}, -\frac{1}{2} \\ 0, 0, 0, -\frac{1}{2} \end{matrix}\right)\right], \\
B_{I_1}(u) = &\frac{\pi^{3/2}}{u}\left[G^{4,0}_{4,5}\left(-u^2 \middle| \begin{matrix} 0, \frac{1}{2}, \frac{1}{2}, 1 \\ 0, 0, 0, 0, \frac{1}{2} \end{matrix}\right)\right. \\
&\left. + iG^{3,1}_{3,5}\left(-u^2 \middle| \begin{matrix} \frac{1}{2}, -\frac{1}{2}, 1 \\ 0, 0, 0, 0, -\frac{1}{2} \end{matrix}\right)\right].
\end{aligned} \tag{B4}
$$

The following relation holds

$$
\frac{d}{du}\left(\frac{\partial^2}{\partial\nu^2} K_\nu(u)\bigg|_{\nu=0}\right) = -\frac{\partial^2}{\partial\nu^2} K_\nu(u)\bigg|_{\nu=1}. \tag{B5}
$$

Finally, using the results of this section, we can write the limiting expressions of the master integral for $u \to 0$

$$
Q^{\rm as}_1(u) = iu[c^{\rm as}_1 + c^{\rm as}_{\ln} \ln(u) + c^{\rm as}_{\ln^2} \ln^2 u] + O(u^2), \tag{B6}
$$

with

$$
\begin{aligned}
c^{\rm as}_1 &= 2 - 2\gamma_E + \gamma_E^2 - \frac{\pi^2}{6} + 2\ln 2 - 2\gamma_E \ln(2) + \ln^2 2, \\
c^{\rm as}_{\ln} &= -2 + 2\gamma_E - 2\gamma_E \ln(2), \\
c^{\rm as}_{\ln^2} &= 1.
\end{aligned} \tag{B7}
$$

Similarly,

$$
\begin{aligned}
Q^{\rm at}_{\frac{1}{2}}(u) = &\, i\frac{\pi^2}{2} + 2iu[-2 + \gamma_E - \ln 2 + \ln(u)] \\
&+ O(u^2),
\end{aligned} \tag{B8}
$$

and

$$
\begin{aligned}
Q^{\rm as2}_{\frac{1}{2}}(u) = &-\frac{2}{3}\log^3(u) + \frac{1}{6}(12\log(2) - 12\gamma_E)\log^2(u) \\
&+ \frac{1}{6}\left(\pi^2 - 12\log^2(2) + 12\gamma_E(\log(4) - \gamma_E)\right)\log(u) \\
&+ \frac{1}{6}(-8\zeta(3) \\
&- \left((\gamma_E - \log(2))((\log(4) - 2\gamma_E)^2 - \pi^2)\right)) \\
&+ O(u^2).
\end{aligned} \tag{B9}
$$

APPENDIX C: SOME PROPERTIES OF THE MEIJER-*G* FUNCTION: A BRIEF REMINDER

The Meijer-*G* function is usually defined by the following Mellin-Barnes integral representation, see e.g., Eq. (16.17.1) of [43]

$$
\begin{aligned}
&G^{m,n}_{p,q}\left(z \middle| \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right) \\
&= \frac{1}{2\pi i}\int_L \frac{\prod_{\ell=1}^m \Gamma(b_\ell - s)\prod_{\ell=1}^n \Gamma(1 - a_\ell + s)}{\prod_{\ell=m}^{q-1}\Gamma(1 - b_{\ell+1} + s)\prod_{\ell=n}^{p-1}\Gamma(a_{\ell+1} - s)} z^s ds,
\end{aligned} \tag{C1}
$$

where the integration path L separates the poles of the factors $\Gamma(b_\ell - s)$ from those of the factors $\Gamma(1 - a_\ell + s)$. Here, m and n are integers such that $0 \le m \le q$ and $0 \le n \le p$, and none of $a_k - b_j$ is a positive integer when $1 \le k \le n$ and $1 \le j \le m$.

Let us shortly recall some properties of the Meijer-*G* function (see, e.g., Sec. 8.2 of Ref. [48]).

(i) Reduction formulas

$$
\begin{aligned}
&G^{m,n}_{p,q}\left(z \middle| \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_{q-1}, a_1 \end{matrix}\right) \\
&= G^{m,n-1}_{p-1,q-1}\left(z \middle| \begin{matrix} a_2, \ldots, a_p \\ b_1, \ldots, b_{q-1} \end{matrix}\right),
\end{aligned} \tag{C2}
$$

and

$$G^{m,n}_{p,q}\left(z\middle|\begin{matrix}a_1,\ldots,a_{p-1},b_1\\ b_1,\ldots,b_q\end{matrix}\right)=G^{m-1,n}_{p-1,q-1}\left(z\middle|\begin{matrix}a_1,\ldots,a_{p-1}\\ b_2,\ldots,b_q\end{matrix}\right). \quad \text{(C3)}$$

(ii) Derivative formulas

$$\frac{d}{dz}\left[z^{1-a_1}G^{m,n}_{p,q}\left(z\middle|\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)\right]=z^{-a_1}G^{m,n}_{p,q}\left(z\middle|\begin{matrix}a_1-1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right),\quad n\geq 1, \quad \text{(C4)}$$

and

$$\frac{d}{dz}\left[z^{1-a_p}G^{m,n}_{p,q}\left(z\middle|\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)\right]=-z^{-a_p}G^{m,n}_{p,q}\left(z\middle|\begin{matrix}a_1,\ldots,a_p-1\\ b_1,\ldots,b_q\end{matrix}\right),\quad n\leq p-1. \quad \text{(C5)}$$

(iii) Translation formulas in the parameters

$$z^\alpha G^{m,n}_{p,q}\left(z\middle|\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)=G^{m,n}_{p,q}\left(z\middle|\begin{matrix}a_1+\alpha,\ldots,a_p+\alpha\\ b_1+\alpha,\ldots,b_q+\alpha\end{matrix}\right). \quad \text{(C6)}$$

(iv) Relation with the generalized hypergeometric function ${}_pF_q$

$${}_pF_q\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\middle|-x\right)=\frac{\prod_{\ell=1}^q\Gamma(b_\ell)}{\prod_{\ell=1}^p\Gamma(a_\ell)}G^{1,p}_{p,q+1}\left(x\middle|\begin{matrix}1-a_1,\ldots,1-a_p\\ 0,1-b_1,\ldots,1-b_q\end{matrix}\right). \quad \text{(C7)}$$

APPENDIX D: HIGH PN ACCURACY SPECTRAL DENSITIES OF ENERGY AND MOMENTUM LOSSES AT $O(G^3)$ IN THE REST FRAME OF PARTICLE A

As a side project of our current works on gravitational wave emission from scattering motions (notably [49]) we give in this appendix the $O(p_\infty^{30})$-accurate values of the spectral densities of energy and momentum losses at $O(G^3)$, computed in the rest frame of particle A. Let

$$h^{\mathrm{A\,TT}}_{ij}=\frac{f_{ij}(u^{\mathrm{A}}_{\mathrm{ret}},\theta^{\mathrm{A}},\phi^{\mathrm{A}})}{R_{\mathrm{A}}}+O\left(\frac{1}{R_{\mathrm{A}}^2}\right). \quad \text{(D1)}$$

denote the bremsstrahlung waveform recorded in the rest frame of particle A (expressed in terms of the retarded time and emission angle associated with the A rest frame, [49]). We consider the A-frame Fourier transform of $f^{\mathrm{A\,TT}}_{ij}$

$$\hat f_{ij}(\omega^{\mathrm{A}},\theta^{\mathrm{A}},\phi^{\mathrm{A}})=\int_{-\infty}^{+\infty}du^{\mathrm{A}}_{\mathrm{ret}}e^{i\omega^{\mathrm{A}}u^{\mathrm{A}}_{\mathrm{ret}}}f_{ij}(u^{\mathrm{A}}_{\mathrm{ret}},\theta^{\mathrm{A}},\phi^{\mathrm{A}}). \quad \text{(D2)}$$

It is then convenient to work with a dimensionless rescaled version of the frequency ω_A that we shall denote as u_A (it should not be confused with the retarded time $u^{\mathrm{A}}_{\mathrm{ret}}$). We also define the dimensionless time variable T_A associated with u_A, namely

$$u_A\equiv\frac{\omega_A b}{p_\infty}=\frac{\omega_A b}{\gamma v},\qquad T_A\equiv\frac{\gamma v}{b}u^{\mathrm{A}}_{\mathrm{ret}}. \quad \text{(D3)}$$

We then get

$$\hat f_{ij}(\omega_{\mathrm{A}},\theta^{\mathrm{A}},\phi^{\mathrm{A}})=\frac{b}{\gamma v}\int dT_{\mathrm{A}}e^{i\omega^{\mathrm{A}}\frac{b}{\gamma v}T_{\mathrm{A}}}f_{ij}(T_{\mathrm{A}},\theta^{\mathrm{A}},\phi^{\mathrm{A}})=\frac{b}{\gamma v}\int dT_{\mathrm{A}}e^{iu_{\mathrm{A}}T_{\mathrm{A}}}f_{ij}(T_{\mathrm{A}},\theta^{\mathrm{A}},\phi^{\mathrm{A}}). \quad \text{(D4)}$$

Hereafter, we often simplify the notation by deleting the A labels, i.e., by writing

$$u_{\mathrm{A}}=u,\Omega_{\mathrm{A}}=(\theta_{\mathrm{A}},\phi_{\mathrm{A}})=\Omega. \quad \text{(D5)}$$

Defining

$$\mathfrak{F}(\omega_{\mathrm{A}},\Omega^{\mathrm{A}})=(-i\omega_A\hat f_{ij}(\omega_{\mathrm{A}},\Omega^{\mathrm{A}}))(-i\omega_A\hat f_{ij}(\omega_{\mathrm{A}},\Omega^{\mathrm{A}}))^* \quad \text{(D6)}$$

the spectral density of radiated A-frame energy reads

$$\frac{dE_A}{d\omega_A}(\omega_A)=\frac{1}{2\pi}\frac{1}{32\pi G}\int d\Omega_{\mathrm{A}}\mathfrak{F}(\omega_{\mathrm{A}},\Omega^{\mathrm{A}}), \quad \text{(D7)}$$

while the spectral density of radiated A-frame momentum reads

$$\frac{d\mathbf{P}_A}{d\omega_A}(\omega_A)=\frac{1}{2\pi}\frac{1}{32\pi G}\int d\Omega_{\mathrm{A}}\mathfrak{F}(\omega_{\mathrm{A}},\Omega^{\mathrm{A}})\mathbf{n}_A. \quad \text{(D8)}$$

Here we define the A-frame polar angles (θ_A,ϕ_A) such that the direction of gravitational wave emission reads

$$\mathbf{n}_A(\theta_A,\phi_A)=\cos\theta_A e^A_x+\sin\theta_A\cos\phi_A e^A_y+\sin\theta_A\sin\phi_A e^A_z, \quad \text{(D9)}$$

where (following [50]) e^A_x is the direction of motion of body B in the A-frame, e^A_y is in the direction of the impact parameter, while e^A_z is orthogonal to the plane of motion.

We found that the real integrand $\mathfrak{F}(\omega_{\rm A}, \Omega^{\rm A})$ depends on the angle ϕ_A only as $\sim c_0 + c_2\cos(2\phi_A) + c_4\cos(4\phi_A) + \cdots$. This implies that, after integrating over angles, the linear momentum spectral density $\frac{d\mathbf{P}_A}{d\omega_A}(\omega_A)$ is entirely directed along the axis e_x^A.

The corresponding integrated losses are

$$E_A^{\rm rad} = \int_{-\infty}^{+\infty} d\omega_A \frac{dE_A}{d\omega_A}(\omega_A), \tag{D10}$$

and

$$P_A^{x\rm rad} = \int_{-\infty}^{+\infty} d\omega_A \frac{dP_A^x}{d\omega_A}(\omega_A). \tag{D11}$$

Starting from the time-domain expression of the A-frame waveform (first computed in Ref. [50]) we computed the Fourier transform $\hat{f}_{ij}(\omega_{\rm A}, \theta^{\rm A}, \phi^{\rm A})$ and then the angle-integrated spectral energy loss $\frac{dE_A}{d\omega_A}(\omega_A)$ to the thirtieth order in the relative velocity $v = p_\infty/\gamma$. It is convenient to express the result in terms of a rescaled flux $\mathcal{F}(u)$ defined so that the *one-sided* integrand, $2d\omega_A \frac{dE_A}{d\omega_A}(\omega_A)$, of the total energy loss (D10) reads

$$2d\omega_A \frac{dE_A}{d\omega_A}(\omega_A)\Big|_{\omega_A = \frac{\gamma v}{b}u} = \frac{\gamma v}{b}\frac{G^3 m_1^2 m_2^2}{b^3} du \mathcal{F}(u), \tag{D12}$$

Here, the extra factor 2 on the lhs is added because one must integrate both sides only on the positive real axis.

The velocity expansion of the dimensionless spectral density $\mathcal{F}(u)$ contains only even powers of v,

$$\mathcal{F}(u) = \sum_{k=0}^{15} \mathcal{F}_{2k}(u) v^{2k}, \tag{D13}$$

with expansion coefficients which are bilinear in $K_0(u)$ and $K_1(u)$, namely

$$\mathcal{F}_{2k}(u) = A_{00}^{(2k)}(u)K_0^2(u) + A_{01}^{(2k)}(u)K_0(u)K_1(u) + A_{11}^{(2k)}(u)K_1^2(u), \tag{D14}$$

where $A_{ij}^{(k)}(u)$ are polynomials in u whose orders increase with k. More precisely, $A_{00}^{(0)}(u), A_{11}^{(0)}(u) = P_4(u)$, $A_{01}^{(0)}(u) = P_3(u)$; $A_{00}^{(2)}(u), A_{11}^{(2)}(u) = P_6(u)$, $A_{01}^{(2)}(u) = P_5(u)$, etc. The values of these polynomials are given in Tables I–III below. They are also given in the ancillary file to this work. We recall that the argument $u = u_A$ used here is defined as $\frac{\omega_A b}{\gamma v}$.

As a check on our results, we have computed the total A-frame radiated energy and found it to agree with the exact value of $E_A^{\rm rad} = \frac{G^3 m_A^2 m_B^2}{b^3}\mathcal{E}$ (first computed in [51,52]), namely

TABLE I. Coefficients of the PN-expansion of the A-frame radiated energy spectrum at 3PM, $\frac{dE_A^{\rm rad}}{du} = \frac{1}{\pi}\frac{G^3 m_1^2 m_2^2}{b^2}\frac{\gamma v}{b}\sum_{k=0}^{15}\mathcal{F}_{2k}(u)v^{2k}$, with $\mathcal{F}_{2k}(u) = A_{00}^{(2k)}(u)K_0^2(u) + A_{01}^{(2k)}(u)K_0(u)K_1(u) + A_{11}^{(2k)}(u)K_1^2(u)$.

0	$A_{00}^{(0)}$	$\frac{32}{5}(\frac{u^2}{3} + u^4)$
	$A_{01}^{(0)}$	$\frac{96}{5}u^3$
	$A_{11}^{(0)}$	$\frac{32}{5}(u^2 + u^4)$
2	$A_{00}^{(2)}$	$\frac{16u^6}{21} + \frac{16u^4}{105} - \frac{160u^2}{21}$
	$A_{01}^{(2)}$	$\frac{288u^3}{35} - \frac{736u^5}{105}$
	$A_{11}^{(2)}$	$\frac{16u^6}{21} - \frac{104u^4}{35} + \frac{160u^2}{7}$
4	$A_{00}^{(4)}$	$-\frac{992}{315}u^2 - \frac{592}{105}u^4 + \frac{9752}{2835}u^6 + \frac{32}{567}u^8$
	$A_{01}^{(4)}$	$\frac{2144}{315}u^3 - \frac{25792}{2835}u^5 - \frac{32}{27}u^7$
	$A_{11}^{(4)}$	$\frac{352}{9}u^2 + \frac{22808}{2835}u^4 + \frac{8152}{2835}u^6 + \frac{32}{567}u^8$
6	$A_{00}^{(6)}$	$\frac{23648}{3465}u^2 - \frac{35824}{3465}u^4 + \frac{280094}{31185}u^6 + \frac{15857}{31185}u^8 + \frac{4}{1485}u^{10}$
	$A_{01}^{(6)}$	$-\frac{3616}{385}u^3 - \frac{569656}{31185}u^5 - \frac{160163}{31185}u^7 - \frac{208}{2079}u^9$
	$A_{11}^{(6)}$	$\frac{19168}{385}u^2 + \frac{516416}{31185}u^4 + \frac{206834}{31185}u^6 + \frac{14339}{31185}u^8 + \frac{4}{1485}u^{10}$
8	$A_{00}^{(8)}$	$\frac{885344}{45045}u^2 - \frac{2395024}{225225}u^4 + \frac{3230374}{160875}u^6 + \frac{7295669}{3378375}u^8 + \frac{132532}{3378375}u^{10} + \frac{128}{1447875}u^{12}$
	$A_{01}^{(8)}$	$-\frac{2715296}{75075}u^3 - \frac{13389016}{375375}u^5 - \frac{50489827}{3378375}u^7 - \frac{764656}{1126125}u^9 - \frac{1472}{289575}u^{11}$
	$A_{11}^{(8)}$	$\frac{236576}{4095}u^2 + \frac{4481456}{160875}u^4 + \frac{9462994}{675675}u^6 + \frac{95519}{51975}u^8 + \frac{33844}{921375}u^{10} + \frac{128}{1447875}u^{12}$
10	$A_{00}^{(10)}$	$\frac{1541216}{45045}u^2 - \frac{32624}{6825}u^4 + \frac{31139132}{779625}u^6 + \frac{67172618}{10135125}u^8 + \frac{6949772}{30405375}u^{10} + \frac{1264}{675675}u^{12} + \frac{64}{30405375}u^{14}$
	$A_{01}^{(10)}$	$-\frac{15965344}{225225}u^3 - \frac{651864016}{10135125}u^5 - \frac{23864242}{675675}u^7 - \frac{11806268}{4343625}u^9 - \frac{209312}{4343625}u^{11} - \frac{5248}{30405375}u^{13}$
	$A_{11}^{(10)}$	$\frac{222304}{3465}u^2 + \frac{39704584}{921375}u^4 + \frac{38610844}{1447875}u^6 + \frac{4944236}{921375}u^8 + \frac{249748}{1216215}u^{10} + \frac{464}{259875}u^{12} + \frac{64}{30405375}u^{14}$

TABLE II. See I.

12	$A_{00}^{(12)}$	$\frac{38234848}{765765}u^2+\frac{664179056}{80405325}u^4+\frac{609846477308}{8442559125}u^6+\frac{3586085674}{216475875}u^8+\frac{13409764}{15049125}u^{10}+\frac{1070874916}{75983032125}u^{12}+\frac{4604272}{75983032125}u^{14}+\frac{64}{1688511825}u^{16}$
	$A_{01}^{(12)}$	$-\frac{691906016}{6185025}u^3-\frac{904026194128}{8442559125}u^5-\frac{15650879038}{216475875}u^7-\frac{5320118264}{649427625}u^9-\frac{18801541964}{75983032125}u^{11}-\frac{32693504}{15196606425}u^{13}-\frac{5056}{1206079875}u^{15}$
	$A_{11}^{(12)}$	$\frac{53204192}{765765}u^2+\frac{527378010872}{8442559125}u^4+\frac{390447480532}{8442559125}u^6+\frac{108397714028}{8442559125}u^8+\frac{3457791644}{4469590125}u^{10}+\frac{991326004}{75983032125}u^{12}+\frac{4446448}{75983032125}u^{14}+\frac{64}{1688511825}u^{16}$
14	$A_{00}^{(14)}$	$\frac{967086496}{14549535}u^2+\frac{44553349744}{1527701175}u^4+\frac{497489574458}{4113041625}u^6+\frac{5743861651901}{160408623375}u^8+\frac{52138451363}{19249034805}u^{10}+\frac{3173164499}{47140493400}u^{12}+\frac{55198609}{97053957000}u^{14}+\frac{455251}{320817246750}u^{16}+\frac{2}{3749811975}u^{18}$
	$A_{01}^{(14)}$	$-\frac{241339616992}{1527701175}u^3-\frac{2966753846488}{17823180375}u^5-\frac{109459029181}{822608325}u^7-\frac{9833194123268}{481225870125}u^9-\frac{1055131768589}{1154942088300}u^{11}-\frac{1547366591}{113229616500}u^{13}-\frac{63589199}{962451740250}u^{15}-\frac{3152}{41247931725}u^{17}$
	$A_{11}^{(14)}$	$\frac{63368224}{855855}u^2+\frac{10005565904}{116491375}u^4+\frac{1324684450874}{17823180375}u^6+\frac{95116552793}{3564636075}u^8+\frac{193685786059}{84922212375}u^{10}+\frac{1403038676257}{23098841766000}u^{12}+\frac{1238993491}{2309884176600}u^{14}+\frac{362519}{262486838250}u^{16}+\frac{2}{3749811975}u^{18}$
16	$A_{00}^{(16)}$	$\frac{1216661536}{14549535}u^2+\frac{267653884592}{4583103525}u^4+\frac{54952443734834}{288735522075}u^6+\frac{20058843936491}{288735522075}u^8+\frac{4280998872419}{618718975875}u^{10}+\frac{25253447517269}{103944787947000}u^{12}+\frac{337766531953}{103944787947000}u^{14}+\frac{1260100943}{77958590960250}u^{16}+\frac{2937982}{116937886440375}u^{18}+\frac{64}{10630716949125}u^{20}$
	$A_{01}^{(16)}$	$-\frac{955109600096}{4583103525}u^3-\frac{5432756646472}{22210424775}u^5-\frac{7243357382909}{32081724675}u^7-\frac{3929659712072}{88388425125}u^9-\frac{140892100269227}{51972393973500}u^{11}-\frac{149172517973}{2474875903500}u^{13}-\frac{30142913}{59555837250}u^{15}-\frac{13319576}{8995222033875}u^{17}-\frac{256}{236238154425}u^{19}$
	$A_{11}^{(16)}$	$\frac{59755616}{765765}u^2+\frac{32681704037984}{288735522075}u^4+\frac{244034023054}{2170943775}u^6+\frac{72104443743799}{1443677610375}u^8+\frac{10514507163607}{1856156927625}u^{10}+\frac{8908446852703}{41577915178800}u^{12}+\frac{12489841249}{4157791517880}u^{14}+\frac{3609992339}{233875772880750}u^{16}+\frac{2874974}{116937886440375}u^{18}+\frac{64}{10630716949125}u^{20}$
18	$A_{00}^{(18)}$	$\frac{4840422944}{47805615}u^2+\frac{10152535152304}{105411381075}u^4+\frac{9454139678145704}{33204585038625}u^6+\frac{152535831520652}{1229799445875}u^8+\frac{81453937212064}{5242829216625}u^{10}+\frac{2799488660851}{3881055394125}u^{12}+\frac{163614684321623}{11953650613905000}u^{14}+\frac{374166171979}{3448168446318750}u^{16}+\frac{22895301998}{67239284703215625}u^{18}+\frac{23391824}{67239284703215625}u^{20}+\frac{32}{574694741053125}u^{22}$
	$A_{01}^{(18)}$	$-\frac{27673801976032}{105411381075}u^3-\frac{114027510015271744}{33204585038625}u^5-\frac{11935974246697772}{33204585038625}u^7-\frac{8691906334862104}{99613755115875}u^9-\frac{2053081414878682}{298841265347625}u^{11}-\frac{12737341722449}{60988013336250}u^{13}-\frac{117364295407867}{44826189802143750}u^{15}-\frac{128906265112}{9605612100459375}u^{17}-\frac{1699134176}{67239284703215625}u^{19}-\frac{2624}{213458046676875}u^{21}$
	$A_{11}^{(18)}$	$\frac{27301225312}{334639305}u^2+\frac{4786820572883816}{33204585038625}u^4+\frac{21948751897528}{135528918525}u^6+\frac{572935348006816}{6640917007725}u^8+\frac{148132470366512}{11953650613905}u^{10}+\frac{37225425774913}{59768253069525}u^{12}+\frac{742786600096999}{59768253069525000}u^{14}+\frac{391765471967}{3842244840183750}u^{16}+\frac{4411424734}{13447856940643125}u^{18}+\frac{22980416}{67239284703215625}u^{20}+\frac{32}{574694741053125}u^{22}$
20	$A_{00}^{(20)}$	$\frac{199549245664}{1673196525}u^2+\frac{9764857623088}{68207364225}u^4+\frac{1641965775544509392}{4017754789673625}u^6+\frac{49012919505720124}{236338517039625}u^8+\frac{1903731817581247496}{60266321845104375}u^{10}+\frac{334904293968193186}{180798965535313125}u^{12}+\frac{25930496940068909}{556304509339425000}u^{14}+\frac{17132205851197}{32556836530968750}u^{16}+\frac{107724646980226}{40679767245445453125}u^{18}+\frac{224663483288}{40679767245445453125}u^{20}+\frac{1345072}{347690318337140625}u^{22}+\frac{64}{149010136430203125}u^{24}$
	$A_{01}^{(20)}$	$-\frac{370945963204832}{1159525191825}u^3-\frac{1866351240994389472}{4017754789673625}u^5-\frac{349790886360 28}{64309800555}u^7-\frac{9524631180898201616}{60266321845104375}u^9-\frac{2793320184131759948}{180798965535313125}u^{11}-\frac{40697934644030207}{66962579827893750}u^{13}-\frac{283210361675487461}{27119844830296968750}u^{15}-\frac{3264882343848404}{40679767245445453125}u^{17}-\frac{3618884910856}{13559922415148484375}u^{19}-\frac{510977344}{1506658046127609375}u^{21}-\frac{10816}{94824632273765625}u^{23}$
	$A_{11}^{(20)}$	$\frac{6167532064}{72747675}u^2+\frac{717617592078933608}{4017754789673625}u^4+\frac{901260951613981888}{4017754789673625}u^6+\frac{2810388801812460928}{20088773948368125}u^8+\frac{488570775885582 16}{1986801819069375}u^{10}+\frac{567056362248643441}{361597931070626250}u^{12}+\frac{1505165417688550327}{36159793107062625000}u^{14}+\frac{2832028348273429}{5811395320777921875}u^{16}+\frac{102403591879046}{40679767245445453125}u^{18}+\frac{16757096072}{3129212865034265625}u^{20}+\frac{3975952}{104307095501142187 5}u^{22}+\frac{64}{149010136430203125}u^{24}$

$$
\begin{aligned}
\mathcal{E} &= \int_0^{+\infty} du \mathcal{F}(u)\\
&= \frac{37\pi v}{15}+\frac{2393\pi v^3}{840}+\frac{61703\pi v^5}{10080}+\frac{3131839\pi v^7}{354816}\\
&\quad+\frac{513183289\pi v^9}{46126080}+\frac{60697345\pi v^{11}}{4612608}+\frac{588430385\pi v^{13}}{39207168}\\
&\quad+\frac{12755740946147\pi v^{15}}{762814660608}+\frac{27966105533111\pi v^{17}}{1525629321216}\\
&\quad+O(v^{18}),
\end{aligned} \tag{D15}
$$

where we have shown only the first few terms of the exact result (the first fifteen terms where previously computed by us in [53]).

Finally, the only nonvanishing component $d\omega_A \frac{dP^x_A}{d\omega_A}(\omega_A)$ of the spectral momentum loss is found not to carry any new information with respect to the energy spectrum. Indeed, we found that it is proportional to it and given by [when normalized as in (D12)]

$$
\mathcal{F}_P^x(u) = \sqrt{\frac{\gamma-1}{\gamma+1}}\mathcal{F}(u). \tag{D16}
$$

The corresponding integrated radiated momentum agrees with, e.g., Eq. (5.13) of Ref. [53].

TABLE III. See I.

22	$A_{00}^{(22)}$	$\frac{138121194272}{1003917915}u^2+\frac{230926708047248}{1159525191825}u^4+\frac{6831720750861217138}{12053264369020875}u^6+\frac{3964026888009660151}{12053264369020875}u^8+\frac{10734813118303152454}{180798965535313125}u^{10}+\frac{1845038794817382917}{433917517284751500}u^{12}+\frac{36287858835600971}{267850319311575000}u^{14}+\frac{110285140133743067}{54239689660593937500}u^{16}+\frac{19034122266002267}{1301752551854254500000}u^{18}+\frac{12004414078169}{244078603472672718750}u^{20}+\frac{2889194543}{4067976724544545 3125}u^{22}+\frac{38765938}{1098353715627027234375}u^{24}+\frac{8}{2852866793836434375}u^{26}$
	$A_{01}^{(22)}$	$-\frac{23201056084832}{61027641675}u^3-\frac{7345590336598137608}{12053264369020875}u^5-\frac{634470727221493499}{803550957934725}u^7-\frac{48543744264598270504}{180798965535313125}u^9-\frac{34326059254774142599}{1084793793211878750}u^{11}-\frac{73690227510294553}{475786751408718750}u^{13}-\frac{267301770040765541}{7748527094370562500}u^{15}-\frac{5388608317539481}{147926426347074375 00}u^{17}-\frac{13489885200283}{7396321317353718750}u^{19}-\frac{29596144306}{7178782455078609375}u^{21}-\frac{806517454}{219670743125405446875}u^{23}-\frac{4192}{4754777989727390625}u^{25}$
	$A_{11}^{(22)}$	$\frac{88029018016}{1003917915}u^2+\frac{2607781020372568672}{12053264369020875}u^4+\frac{3626025590836924766}{12053264369020875}u^6+\frac{1443756504976386533}{6696257982789375}u^8+\frac{3500727809438547682}{77485270943705625}u^{10}+\frac{1533336522008968 3411}{4339175172847515000}u^{12}+\frac{87904585329932533}{737954961368625000}u^{14}+\frac{568236625586326}{305862911619890625}u^{16}+\frac{10423951630297}{759039388836300000}u^{18}+\frac{1918269573533}{40679767245445453125}u^{20}+\frac{76010981173}{1098353715627027234375}u^{22}+\frac{38283302}{1098353715627027234375}u^{24}+\frac{8}{2852866793836434375}u^{26}$
24	$A_{00}^{(24)}$	$\frac{22731815702432}{145568097675}u^2+\frac{115603050920726128}{437140997318025}u^4+\frac{451179259109528549 80534}{59073048672571308375}u^6+\frac{29498256429730204303753}{59073048672571308375}u^8+\frac{9279817153199196353 4586}{88609573008856962 5625}u^{10}+\frac{55837067899921955 13491}{625479338886049147500}u^{12}+\frac{33657015207573984 03943}{96664988736934868 25000}u^{14}+\frac{3574705737283178077}{5392063266259044375 00}u^{16}+\frac{4052792619327763738 01}{637988925663770130 4500000}u^{18}+\frac{52353104538342463}{170889890802795570656250}u^{20}+\frac{85801830424603}{1196229235619568994 59375}u^{22}+\frac{3990890959582}{5383031560288060475671875}u^{24}+\frac{131576072}{489366505480732770515625}u^{26}+\frac{512}{326244336987155180 34375}u^{28}$
	$A_{01}^{(24)}$	$-\frac{193661419297856992}{437140997318025}u^3-\frac{46043637666221020423544}{59073048672571308375}u^5-\frac{14539140817652527995 47}{1312734414946029075}u^7-\frac{38364719058288069642 8896}{886095730088569625625}u^9-\frac{31965613855937305026 6541}{5316574380531417753750}u^{11}-\frac{4725791040087774061 8083}{13291435951328544384375}u^{13}-\frac{2623283196897156857 1821}{265828719026570887687500}u^{15}-\frac{980956610410665101 63}{7249874155270115118 7500}u^{17}-\frac{5581010221842893798}{5981146178097844972 96875}u^{19}-\frac{19070642631431098}{598114617809784497296875}u^{21}-\frac{2300004655006}{4523555933015176870 3125}u^{23}-\frac{2287755488}{6990950078296182435 9375}u^{25}-\frac{14464}{2509571822978116771875}u^{27}$
	$A_{11}^{(24)}$	$\frac{175359843808}{1940907969}u^2+\frac{15194805030800967293776}{59073048672571308375}u^4+\frac{23200489852455602449538}{59073048672571308375}u^6+\frac{10459835589057082288667}{32818360373650726875}u^8+\frac{2073387769308374704 56826}{265828719026570887 6875}u^{10}+\frac{6740868007239141331783}{92462163139676830 5000}u^{12}+\frac{1603948907326717598 27483}{531657438053141775375000}u^{14}+\frac{265009247773036462667}{443047865044284812 81250}u^{16}+\frac{112925406438133572 4451}{191396677699131039135 00000}u^{18}+\frac{173903961494652086}{598114617809784497296875}u^{20}+\frac{3726162538260727}{5383031560288060475671875}u^{22}+\frac{780704894114}{10766063120576120951 34375}u^{24}+\frac{130169672}{489366505480732770515625}u^{26}+\frac{512}{326244336987155180 34375}u^{28}$
26	$A_{00}^{(26)}$	$\frac{789497272234592}{4512611027925}u^2+\frac{4596440091268254832}{13551370916858775}u^4+\frac{756880375909572252 4636}{7536067937653129875}u^6+\frac{1487763100593570913 51718}{20347383431663450 6625}u^8+\frac{48637507979688620220412}{277464319522683418125}u^{10}+\frac{693878834263823254 0448}{398100980184719686875}u^{12}+\frac{891099779139418716613927}{1098758705309826335775000}u^{14}+\frac{130035949810116642 62176}{686724190818641459859375}u^{16}+\frac{15961661288759457239 6437}{692217984345190591538250000}u^{18}+\frac{599618301729406061621}{403793824201361178397312500}u^{20}+\frac{151438036020571013 6}{302845368151020883797984375}u^{22}+\frac{109817796696688}{1304118810219707155110937 5}u^{24}+\frac{1586655202096}{247782573941744359471078125}u^{26}+\frac{430053856}{247782573941744359471078125}u^{28}+\frac{256}{336356435215037592042187 5}u^{30}$
	$A_{01}^{(26)}$	$-\frac{6886863104740471648}{13551370916858775}u^3-\frac{34831414760907777662 56}{3569716391519903625}u^5-\frac{16164736259909734889558}{107091491745597108 75}u^7-\frac{20614165481495780467348}{308293688358537131 25}u^9-\frac{484600587577407183 9112}{451050371637859743 75}u^{11}-\frac{108470408870665347241271}{1445735138565560968 1250}u^{13}-\frac{17301771086412169696966 1}{686724190818641459859375}u^{15}-\frac{3729662124594768702 24091}{8652724804314882394 2281250}u^{17}-\frac{936661169949530201 77391}{242276294520816707038387 5000}u^{19}-\frac{1083110701838558012}{593814447354942909407 8125}u^{21}-\frac{12029661170367721 24}{272560831335918795418185 9375}u^{23}-\frac{6674225083336}{1304118810219707155110937 5}u^{25}-\frac{6724566656}{275313971046382621634531 25}u^{27}-\frac{52736}{16337312567587540184906 25}u^{29}$
	$A_{11}^{(26)}$	$\frac{14441703673952}{155607276825}u^2+\frac{20418835843447440844616}{67824611438878168875}u^4+\frac{10198962256215684473 1004}{203473834316634506625}u^6+\frac{1068404789529093262 09996}{2347775011345782768 75}u^8+\frac{117138008778282039044 8852}{9156322544248552798125}u^{10}+\frac{45244603486308867471913}{3231643250911253928750}u^{12}+\frac{380735404206575144921 7887}{549379352654913167887 5000}u^{14}+\frac{617403067392999225892}{3657111075365546235 9375}u^{16}+\frac{102693293378253579322 2439}{4845525890416334140767750000}u^{18}+\frac{14093441917289474603 8}{10094845605034029459932 8125}u^{20}+\frac{130389055930867653 38}{272560831335918795418185 9375}u^{22}+\frac{31136983799756}{381203959910375937647812 5}u^{24}+\frac{518868897952}{825941913139147864903593 75}u^{26}+\frac{1278192416}{7433477218252330784132343 75}u^{28}+\frac{256}{336356435215037592042187 5}u^{30}$
28	$A_{00}^{(28)}$	$\frac{875145464391392}{4512611027925}u^2+\frac{2869288439758786 9328}{67756854584293875}u^4+\frac{16917452906724359911 20668}{130804607774979325 6875}u^6+\frac{9506814253980196831066946}{9156322544248552798125}u^8+\frac{771368753053530837249 5164}{27468967632745658394375}u^{10}+\frac{52813960863680648444 6231}{164813805796473950366 25}u^{12}+\frac{1934589889361000596 8287}{11098572780907336725000}u^{14}+\frac{10120163748982104082 04}{208098239642012563593 75}u^{16}+\frac{85833877292121815579}{1180252317724110130500 00}u^{18}+\frac{433303058674684267239 67}{726828883562450121115162 5000}u^{20}+\frac{117150021962742087743}{436097330137470072669097 5000}u^{22}+\frac{232004047450754431}{355514127829459298371546 8750}u^{24}+\frac{790577954597104}{973431540485424269350664 0625}u^{26}+\frac{214402978984}{461099150756253601271367 1875}u^{28}+\frac{453495712}{471739900389090222839167 96875}u^{30}+\frac{128}{396420084360580019192578125}u^{32}$
	$A_{01}^{(28)}$	$-\frac{38995906600648639904}{67756854584293875}u^3-\frac{15689421867390450880 15888}{130804607774979325 6875}u^5-\frac{21126687563468915957 9914}{105245086715500606875}u^7-\frac{27353804323694142943330456}{27468967632745658394375}u^9-\frac{300651468146746645093 4134}{16481380579647395036625}u^{11}-\frac{405770107769554206386 9821}{274689676327456583943750}u^{13}-\frac{134275512321977171029519}{228908063606213819953125}u^{15}-\frac{35061636405583062746 0663}{28842416014382941314093750}u^{17}-\frac{33002725933057025882 0161}{2422762945208167070383875000}u^{19}-\frac{912791336175170588417}{10902433253436751816727 43750}u^{21}-\frac{22925754568163347780 9}{817682494007756386254557 81250}u^{23}-\frac{337355954747939048}{681402078339796988545464843 75}u^{25}-\frac{138183098407624}{322769405529377520889957031 25}u^{27}-\frac{24349412096}{1572466334630300742797226562 5}u^{29}-\frac{12416}{792840168721160038385156 25}u^{31}$
	$A_{11}^{(28)}$	$\frac{13842435028768}{145568097675}u^2+\frac{45482271062818460146 2632}{130804607774979325 6875}u^4+\frac{114907509147511814020 8476}{183126450884971055 9625}u^6+\frac{296285246971349422431 56}{46955500226915655375}u^8+\frac{165670626993468202240 53628}{824069028982369751 83125}u^{10}+\frac{417608713557307151056939}{16481380579647395036625}u^{12}+\frac{80759027223797168945 97419}{549379352654913167887 5000}u^{14}+\frac{29425088012532655789}{686381000318482218750}u^{16}+\frac{32068917185139009311 93881}{4845525890416334140767750000}u^{18}+\frac{17732972435528737759}{319204604111747967112 5000}u^{20}+\frac{151596732490556699963}{594678177460186462730587 5000}u^{22}+\frac{103985251398356231}{165522772066347446610234 3750}u^{24}+\frac{646767703118128}{8176824940077563862545578125}u^{26}+\frac{1869579635032}{40884124700387819312727890625}u^{28}+\frac{34600736}{362876846453146325260898 4375}u^{30}+\frac{128}{396420084360580019192578125}u^{32}$

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