Near-horizon properties of trajectories with finite force relevant for Bañados-Silk-West effect

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According to the Bañados-Silk-West (BSW) effect, two particles moving toward a black hole can collide near the horizon with an unbounded energy in the center-of-mass frame. This requires one of the particles to have fine-tuned parameters in such a way that the time component of generalized momentum is zero $X = 0$. Thus the existence of such trajectories is a necessary condition for the BSW effect. However, it is insufficient since the forward-in-time condition requires $X > 0$ outside the horizon. We examine this condition for different types of particles and horizons and find configurations for which the BSW effect is possible. In doing so, we take into account a finite force of unspecified nature exerted on particles. It includes relationships between numbers characterizing the rate with which four-velocity, acceleration, and metric functions change near the horizon. For some aforementioned relations, parameters of a system control the sign of X; in other cases they are required for X to be a real quantity. In the simplest case of free particles the BSW effect for the Kerr or Kerr-Newman black hole is impossible if a fine-tuned particle has a negative energy, so in this sense a combination of the Penrose process and the BSW effect is forbidden.

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I. INTRODUCTION

If two particles collide in the vicinity of a black hole, the energy E_{cm} in the center-of-mass frame can grow unbounded, provided one of particles has fine-tuned parameters [1]. This is the Bañados-Silk-West (BSW) effect. Originally, this was observed for free particles in the background of a rotating black hole. However, the counterpart of it exists also for static charged black holes [2] as well as for a combination of electric charge and rotation [3]. Instead of considering the electromagnetic force, one can scrutinize the effect of a force as such, not specifying its nature. This helps to understand the nature of the BSW effect, evaluate the role of gravitation radiation in the context of the BSW effect [4] (at least qualitatively, to the extent that it can be modeled by a force), take into account the influence of the surrounding medium, etc. For nonextremal and extremal black holes, this was done in [5] and [6]. A general approach was developed in [7].

The key ingredient of the BSW effect is the existence of a fine-tuned trajectory. It is selected by the condition on the time component of generalized momentum $X = 0$ on the horizon. [It is more direct than using the time component of the four-velocity u^t that would lead to a more complicated classification; see Eq. (15) in [7].] Meanwhile, there is also another condition that, to the best of our knowledge, was overlooked or remained underappreciated. It consists in the requirement $dX/dr > 0$ in the immediate vicinity of the horizon to satisfy the forward-in-time condition for a fine-tuned particle $X > 0$ outside the horizon. For a free moving particle, this is an obvious constraint on its parameters but even in this case it leads to meaningful consequences restricting possible scenarios of the BSW effect that were not noticed before (see below). The situation becomes nontrivial when a force acts on a particle since in some situations the sign of X outside the horizon cannot be chosen arbitrarily and is determined by dynamics, so one is led to the analysis of equations of motion. But even if this sign can be imposed by hand, other constraints may exist (for example, from the requirement that X should be a real

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quantity). In other words, nontrivial interplay between kinematics and dynamics can occur in the vicinity of the horizon as a factor selecting scenarios appropriate for the BSW effect.

For example, for the Reissner-Nordsröm metric $X = E - \frac{qQ}{r}$, where E is the Killing energy, q being particle charge, and Q a black hole mass. If we choose parameters in such a way that $E = qQ/r_h$ on a horizon $r = r_h$, the fine-tuned (critical) trajectory does exist for $qQ > 0$. If $qQ < 0$, it still exists for $E < 0$. However, this does not save the matter since in the immediate vicinity of the horizon $X < 0$ outside and thus the BSW effect cannot occur. In principle, similar restrictions should be valid for neutral particles in the rotating black hole background that impose constraints on the behavior of acceleration and other particle parameters on the horizon.

The main goal of our work is to find conditions which have to be satisfied by the external force to have $dX/dr > 0$ in the vicinity of the horizon and analyze for which types of particles this is in principle possible. To this end, in Sec. II we develop a general approach and for particles moving in the equatorial plane we relate the rate of change dX/dr with the external force. In Sec. III we analyze all possible distinct cases which give different expressions for dX/dr and summarize which conditions are required to produce the desired sign for dX/dr. In Sec. V we apply the conditions developed in Sec. III to different types of particles and analyze for which types this sign is controlled by an external force and how to keep it positive by the action of an external force.

II. GENERAL SETUP

We investigate the motion of particles in the background of a rotating black hole which is described in the generalized Boyer-Lindquist coordinates (t, r, θ, φ) by the metric

$$ds^2 = -N^2 dt^2 + g_{\varphi\varphi}(dt - \omega d\varphi)^2 + \frac{dr^2}{A} + g_{\theta\theta} d\theta^2, \tag{1}$$

where all metric coefficients do not depend on t and φ. The horizon is located at $r = r_h$ where $A(r_h) = N(r_h) = 0$. Near the horizon, we use a general expansion for the functions N^2, A, and ω:

$$N^2 = \kappa_p v^p + o(v^p), \quad A = A_q v^q + o(v^q), \tag{2}$$

$$\omega = \omega_H + \omega_k v^k + o(v^k), \tag{3}$$

where q, p, and k are numbers that characterize the rate of change of the metric functions near the horizon, and $v = r - r_h$.

If a particle is freely moving, the space-time symmetries with respect to ∂_t and ∂_φ impose conservation of the corresponding components of the four-momentum: $mu_t = -E$, $mu_\varphi = L$. Here, E is energy, L is angular momentum, m is particle mass, and u^μ is the four-velocity. We assume the symmetry with respect to the equatorial plane and restrict ourselves to equatorial motion. Then, it follows from equations of motion that the four-velocity of a free-falling particle can be written in the following form:

$$u^\mu = \left(\frac{\mathcal{X}}{N^2}, \sigma \frac{\sqrt{A}}{N} P, 0, \frac{\mathcal{L}}{g_{\varphi\varphi}} + \frac{\omega \mathcal{X}}{N^2}\right), \tag{4}$$

where $\sigma = \pm 1$, $\mathcal{X} = \epsilon - \omega\mathcal{L}$, $\epsilon = E/m$, $\mathcal{L} = L/m$ and P is given by

$$P = \sqrt{\mathcal{X}^2 - N^2\left(1 + \frac{\mathcal{L}^2}{g_{\varphi\varphi}}\right)}. \tag{5}$$

The forward-in-time condition states that $u^t > 0$, where

$$\mathcal{X} > 0 \tag{6}$$

outside the horizon. On the horizon itself $\mathcal{X} = 0$ is admissible, and the corresponding particle (trajectory) is fine-tuned. Additionally, we make an assumption that a force which acts on the particle does not depend on t and φ. Then, the expression (4) retains its form but with E and $\mathcal{L}$ that can depend on r. In doing so, we take advantage of some equations already derived in previous works, especially in [5]. (The case when a force can depend on t and φ is also of interest since it can be relevant for a description of dissipation effects or gravitational radiation but this case is more complicated and is beyond the scope of the present paper.)

For a description of motion near the horizon, one can introduce the tetrad attached to an observer. To this end, it is convenient to use the so-called zero angular momentum observer (ZAMO) [8]. More precisely, we consider such an observer for which $r =$ const and call it OZAMO ("orbital" ZAMO) since both such an observer and fine-tuned particle do not cross the horizon. As a result, components of acceleration in this frame remain finite. Then, we rely on Eq. (114) from [5]:

$$\frac{d\mathcal{X}}{d\tau} = N a_o^{(t)} - \frac{d\omega}{d\tau}\mathcal{L}. \tag{7}$$

According to Eq. (111) of [5],

$$a_o^{(t)} = \frac{N}{\mathcal{X}}\left[\frac{\mathcal{L}}{\sqrt{g_\phi}} a_o^{(\varphi)} + \frac{u^r}{\sqrt{A}} a_0^{(r)}\right]. \tag{8}$$

Then, (7) is equivalent to

$$\frac{d\mathcal{X}}{d\tau} = \frac{N^2}{\mathcal{X}}\left[\frac{\mathcal{L}}{\sqrt{g_\phi}} a_o^{(\varphi)} + \frac{u^r}{\sqrt{A}} a_0^{(r)}\right] - \frac{d\omega}{d\tau}\mathcal{L}. \tag{9}$$

Due to the independence of t and φ, we have

$$\frac{d\mathcal{X}}{d\tau} = \frac{d\mathcal{X}}{dr} u^r, \quad \frac{d\omega}{d\tau} = \frac{d\omega}{dr} u^r, \tag{10}$$

where

$$\frac{d\mathcal{X}}{dr} = \frac{N^2}{\mathcal{X}}\left[\frac{a_0^{(r)}}{\sqrt{A}} + \frac{\mathcal{L}}{\sqrt{g_\phi} u^r} a_o^{(\varphi)}\right] - \mathcal{L}\omega'(r). \tag{11}$$

Using (4), we can also rewrite it in the form

$$\frac{d\mathcal{X}}{dr} = \frac{N^2}{\mathcal{X}\sqrt{A}}\left[a_0^{(r)} + \sigma \frac{\mathcal{L}N}{\sqrt{g_\phi}P} a_o^{(\varphi)}\right] - \mathcal{L}\omega'(r). \tag{12}$$

Now, we are interested in the question: when does the fine-tuned particle become bad? By "bad" we imply that $\frac{d\mathcal{X}}{dr}(r_h) < 0$. Then, it is impossible to have $\mathcal{X}(r_h) = 0$ since in a small vicinity of the horizon in the outer region we would have $\mathcal{X} < 0$ in contradiction with the forward-in-time condition (6).

A. Free particle: No Penrose effect for a fine-tuned one

Let us consider a particularly important case when a particle is free. We assume $\omega' < 0$ that is typical of asymptotically flat metrics such as the Kerr and Kerr-Newman ones. We obtain that for a "bad" particle $\mathcal{L} < 0$. In principle, a particle can have $E < 0$ due to the ergoregion and we can achieve $\mathcal{X} = 0$ on the horizon due to $\mathcal{L} < 0$. However, in a small neighborhood of the horizon, (6) will be violated. This means that near the horizon $\mathcal{L} > 0$ for any fine-tuned particle. In turn, this means that a fine-tuned particle must have $E > 0$. As a fine-tuned particle is an essential ingredient of the BSW effect, now this effect is prohibited for fine-tuned particles with $E < 0$.

In the Penrose process particle 0 decays to two fragments 1 and 2, where particle 1 has a negative energy while particle 2 escapes to infinity thus giving the energy gain [9]. It follows from what is said above that for a fine-tuned particle the Penrose process is impossible.

III. GENERAL CASE

Now, let us consider the most general case when some force acts on a particle. Our aim is to find such conditions that near the horizon $\frac{d\mathcal{X}}{dr} > 0$. To this end, we assume that near the horizon expansions hold

$$N^2 = \kappa_p v^p + o(v^p), \qquad A = A_q v^q + o(v^q), \tag{13}$$

$$\omega = \omega_H + \omega_k v^k + o(v^k). \tag{14}$$

Additionally, we assume that near the horizon parameters of a fine-tuned particle behave in such a way that

$$\mathcal{X} = X_s v^s + o(v^s), \quad \mathcal{L} = L_H + L_b v^b + o(v^b), \tag{15}$$

so $\mathcal{X}$ obeys the condition $\mathcal{X}(r_h) = 0$.

Acceleration near the horizon reads

$$a_o^{(r)} = \left(a_o^{(r)}\right)_{n_1} v^{n_1} + o(v^{n_1}), \tag{16}$$

$$a_o^{(\varphi)} = \left(a_o^{(\varphi)}\right)_{n_2} v^{n_2} + o(v^{n_2}). \tag{17}$$

For the four-velocity, using its normalization, one finds

$$u^r = -\frac{\sqrt{A}}{N}\sqrt{\mathcal{X}^2 - N^2\left(1 + \frac{\mathcal{L}^2}{g_\varphi}\right)}, \tag{18}$$

where the minus sign is chosen because the particle is considered to be infalling. Near-horizon behavior of this quantity was analyzed in [7]. Generally, this behavior may be described by the quantity c, such that $u^r \approx (u^r)_c v^c$, where c depends on quantity s. If $0 \le s \le p/2$, then $c = \frac{q-p}{2} + s$. The case $s > p/2$ is impossible, because the quantity u^r would become complex. However, if coefficients in expansion for X are chosen in such a way that they satisfy the condition

$$\mathcal{X}^2 = N^2\left(1 + \frac{\mathcal{L}^2}{g_\varphi}\right) \text{ plus } v^{2c+p-q} \text{ terms}, \tag{19}$$

then $u^r \approx (u^r)_c v^c$ where c may be any value, higher then $q/2$.

Now let us analyze an expression (12) for $\frac{d\mathcal{X}}{dr}$.

Substituting near-horizon expansions, one obtains

$$\frac{d\mathcal{X}}{dr} = sX_s v^{s-1} = \frac{\kappa_p}{X_s}\frac{\left(a_0^{(r)}\right)_{n_1}}{\sqrt{A_q}} v^{n_1+p-\frac{q}{2}-s} + \frac{\kappa_p}{X_s}\frac{L_H\left(a_o^{(\phi)}\right)_{n_2}}{\sqrt{g_{\phi H}}(u^r)_c} v^{n_2+p-c-s} - L_H k\omega_k v^{k-1}. \tag{20}$$

Thus we have three terms having the orders $n_1 + p - \frac{q}{2} - s$, $n_2 + p - c - s$, and $k - 1$. We have to analyze all the cases when: only one from these terms is dominant, two terms are of the same order, and all three terms are of the same order. To cover all these situations, let us analyze them case by case.

A. First term is dominant

For this to be true, one has to have

$$n_2 > n_1 + c - \frac{q}{2}, \tag{21}$$

$$k > n_1 + p - \frac{q}{2} + 1 - s. \tag{22}$$

In this case, (20) becomes

$$sX_s v^{s-1} = \frac{\kappa_p}{X_s} \frac{\left(a_0^{(r)}\right)_{n_1}}{\sqrt{A_q}} v^{n_1+p-\frac{q}{2}-s}. \tag{23}$$

From this equation, it follows that s has to satisfy the condition

$$n_1 = 2s + \frac{q}{2} - p - 1. \tag{24}$$

Substituting this into (21) and (22), one obtains that these conditions become

$$n_2 > 2s - p - 1 + c, \tag{25}$$

$$k > s. \tag{26}$$

If it is so, then

$$sX_s = \frac{\kappa_p}{X_s} \frac{\left(a_0^{(r)}\right)_{n_1}}{\sqrt{A_q}} \Rightarrow (X_s)^2 = \frac{\kappa_p}{s} \frac{\left(a_0^{(r)}\right)_{n_1}}{\sqrt{A_q}}. \tag{27}$$

One sees that in this case we can define only $(X_s)^2$, and thus a force does not control the sign of the X_s. However, for X_s to be real, one has to require $(a_0^{(r)})_{n_1} > 0$.

B. Second term is dominant

For this to be true, one has to have

$$n_1 > n_2 + \frac{q}{2} - c, \tag{28}$$

$$k > n_2 + p - c + 1 - s. \tag{29}$$

In this case, (20) becomes

$$sX_s v^{s-1} = \frac{\kappa_p}{X_s} \frac{L_H \left(a_o^{(\phi)}\right)_{n_2}}{\sqrt{g_{\phi H}}(u^r)_c} v^{n_2+p-c-s}. \tag{30}$$

From this equation it follows that s has to satisfy the condition

$$n_2 = 2s + c - p - 1. \tag{31}$$

Substituting this into (28) and (29), one gets

$$n_1 > 2s + \frac{q}{2} - p - 1, \tag{32}$$

$$k > s. \tag{33}$$

If it is so, then

$$sX_s = \frac{\kappa_p}{X_s} \frac{L_H \left(a_o^{(\phi)}\right)_{n_2}}{\sqrt{g_{\phi H}}(u^r)_c} \Rightarrow (X_s)^2 = \frac{\kappa_p}{s} \frac{L_H \left(a_o^{(\phi)}\right)_{n_2}}{\sqrt{g_{\phi H}}(u^r)_c}. \tag{34}$$

One sees that in this case we can define only $(X_s)^2$, and thus a force does not control the sign of the X_s. However, for X_s to be real, one has to require $\frac{L_H (a_o^{(\phi)})_{n_2}}{(u^r)_c} > 0$. As for infalling particles $(u^r)_c < 0$, signs of angular momentum and angular components of acceleration have to be different.

C. Third term is dominant

For this to be true, one has to have

$$n_1 > k - 1 + \frac{q}{2} - p + s, \tag{35}$$

$$n_2 > k - 1 + c - p + s. \tag{36}$$

In this case, (20) becomes

$$sX_s v^{s-1} = -L_H k \omega_k v^{k-1}. \tag{37}$$

From this equation it follows that s has to satisfy the condition

$$s = k. \tag{38}$$

Substituting this into (35) and (36), one gets

$$n_1 > 2s + \frac{q}{2} - p - 1, \tag{39}$$

$$n_2 > 2s + c - p - 1. \tag{40}$$

If it is so, then

$$X_s = -L_H \omega_k. \tag{41}$$

X_s will be positive if $L_H \omega_k < 0$ only. This condition is the same as for the case of freely moving particles.

D. First and second terms are dominant

For this to be true, one has to have

$$n_2 = n_1 + c - \frac{q}{2}, \tag{42}$$

$$k > n_1 + p - \frac{q}{2} + 1 - s. \tag{43}$$

In this case, (20) becomes

$$sX_s v^{s-1} = \frac{\kappa_p}{X_s}\left[\frac{(a_0^{(r)})_{n_1}}{\sqrt{A_q}} + \frac{L_H (a_o^{(\phi)})_{n_2}}{\sqrt{g_{\phi H}}(u^r)_c}\right] v^{n_1+p-\frac{q}{2}-s}. \tag{44}$$

From this equation it follows that s has to satisfy the condition

$$n_1 = 2s + \frac{q}{2} - p - 1. \tag{45}$$

Substituting this into (42) and (43), one gets

$$n_2 = 2s + c - p - 1, \tag{46}$$

$$k > s. \tag{47}$$

If it is so, then

$$sX_s = \frac{\kappa_p}{X_s}\left[\frac{(a_0^{(r)})_{n_1}}{\sqrt{A_q}} + \frac{L_H (a_o^{(\phi)})_{n_2}}{\sqrt{g_{\phi H}}(u^r)_c}\right] \Rightarrow (X_s)^2 = \frac{\kappa_p}{s}\left[\frac{(a_0^{(r)})_{n_1}}{\sqrt{A_q}} + \frac{L_H (a_o^{(\phi)})_{n_2}}{\sqrt{g_{\phi H}}(u^r)_c}\right]. \tag{48}$$

One sees that in this case we can define only $(X_s)^2$, and thus we cannot control the sign of the X_s. However, for X_s to be real, one has to require $\frac{(a_0^{(r)})_{n_1}}{\sqrt{A_q}} + \frac{L_H (a_o^{(\phi)})_{n_2}}{\sqrt{g_{\phi H}}(u^r)_c} > 0$.

E. All three terms are of the same order

Now let us analyze this case. We postpone the analysis of two additional cases when the first and third terms and second and third terms are dominant. Motivation for this is that these cases could be obtained from the case when all three terms are of the same order after taking several parameters to zero. For this to be true, one has to have

$$n_2 = n_1 + c - \frac{q}{2}, \tag{49}$$

$$n_1 = k + \frac{q}{2} - p + s - 1. \tag{50}$$

In this case, (20) becomes

$$sX_s v^{s-1} = \left(\frac{\kappa_p}{X_s}\left[\frac{(a_0^{(r)})_{n_1}}{\sqrt{A_q}} + \frac{L_H (a_o^{(\phi)})_{n_2}}{\sqrt{g_{\phi H}}(u^r)_c}\right] - L_H k \omega_k\right) v^{n_1+p-\frac{q}{2}-s}. \tag{51}$$

From this equation it follows that s has to satisfy the condition

$$n_1 = 2s + \frac{q}{2} - p - 1. \tag{52}$$

Substituting this into (49) and (50), one gets

$$n_2 = 2s + c - p - 1, \tag{53}$$

$$k = s. \tag{54}$$

If it is so, then

$$sX_s = \frac{\kappa_p}{X_s}\left[\frac{(a_0^{(r)})_{n_1}}{\sqrt{A_q}} + \frac{L_H (a_o^{(\phi)})_{n_2}}{\sqrt{g_{\phi H}}(u^r)_c}\right] - L_H k \omega_k. \tag{55}$$

Multiplying by X_s, one obtains

$$s(X_s)^2 + kL_H\omega_k X_s - \kappa_p\left[\frac{(a_0^{(r)})_{n_1}}{\sqrt{A_q}} + \frac{L_H (a_o^{(\phi)})_{n_2}}{\sqrt{g_{\phi H}}(u^r)_c}\right] = 0. \tag{56}$$

Solving this equation, we find

$$X_s = -\frac{kL_H\omega_k}{2s} \pm \frac{\sqrt{D}}{2s}, \tag{57}$$

where

$$D = (kL_H\omega_k)^2 + 4s\kappa_p\left[\frac{(a_0^{(r)})_{n_1}}{\sqrt{A_q}} + \frac{L_H (a_o^{(\phi)})_{n_2}}{\sqrt{g_{\phi H}}(u^r)_c}\right]. \tag{58}$$

To analyze what signs these roots can have, let us introduce

$$a = \frac{kL_H\omega_k}{2s}, \quad b = \frac{\kappa_p}{s}\left[\frac{(a_0^{(r)})_{n_1}}{\sqrt{A_q}} + \frac{L_H (a_o^{(\phi)})_{n_2}}{\sqrt{g_{\phi H}}(u^r)_c}\right], \tag{59}$$

thus the solution under discussion becomes

$$X_s = -a \pm \sqrt{a^2 + b}. \tag{60}$$

If $b > 0$, then, independently on the sign of a, one of the roots is positive. But if $b < 0$, then one of the roots is positive only if $a < 0$. These conditions mean that if

$$\frac{\left(a_0^{(r)}\right)_{n_1}}{\sqrt{A_q}} + \frac{L_H\left(a_o^{(\phi)}\right)_{n_2}}{\sqrt{g_{\phi H}}(u^r)_c} > 0, \tag{61}$$

then positive X_s exists independently on $L_H\omega_k$. However, if

$$\frac{\left(a_0^{(r)}\right)_{n_1}}{\sqrt{A_q}} + \frac{L_H\left(a_o^{(\phi)}\right)_{n_2}}{\sqrt{g_{\phi H}}(u^r)_c} < 0, \tag{62}$$

then positive X_s exists only if $L_H\omega_k < 0$. Additionally, we have to mention that if

$$\frac{\left(a_0^{(r)}\right)_{n_1}}{\sqrt{A_q}} + \frac{L_H\left(a_o^{(\phi)}\right)_{n_2}}{\sqrt{g_{\phi H}}(u^r)_c} = 0, \tag{63}$$

then there is only one root

$$X_s = -2a. \tag{64}$$

This root is positive if $L_H\omega_k < 0$ only.

F. First and third terms are dominant

For this to be true, one has to have

$$n_1 = k + \frac{q}{2} - p + s - 1, \tag{65}$$

$$n_2 > k + c - 1 - p + s. \tag{66}$$

In this case, (20) becomes

$$sX_s v^{s-1} = \left(\frac{\kappa_p}{X_s}\frac{\left(a_0^{(r)}\right)_{n_1}}{\sqrt{A_q}} - L_H k\omega_k\right) v^{n_1+p-\frac{q}{2}-s}. \tag{67}$$

From this equation it follows that s has to satisfy condition

$$n_1 = 2s + \frac{q}{2} - p - 1. \tag{68}$$

Substituting this into (65) and (66), one gets

$$n_2 > 2s + c - p - 1, \tag{69}$$

$$k = s. \tag{70}$$

If it is so, then

$$sX_s = \frac{\kappa_p}{X_s}\frac{\left(a_0^{(r)}\right)_{n_1}}{\sqrt{A_q}} - L_H k\omega_k. \tag{71}$$

Multiplying by X_s (only if $X_s \neq 0$), one obtains

$$s(X_s)^2 + kL_H\omega_k X_s - \kappa_p\frac{\left(a_0^{(r)}\right)_{n_1}}{\sqrt{A_q}} = 0. \tag{72}$$

The solution of this equation is

$$X_s = -a \pm \sqrt{a^2 + b},$$

where

$$a = \frac{kL_H\omega_k}{2s}, \qquad b = \frac{\kappa_p}{s}\frac{\left(a_0^{(r)}\right)_{n_1}}{\sqrt{A_q}}. \tag{73}$$

One sees that this solution is similar to the case when all three roots are of the same order, but with a slight change of coefficient b. Following the lines, we see that if

$$\frac{\left(a_0^{(r)}\right)_{n_1}}{\sqrt{A_q}} > 0, \tag{74}$$

then positive X_s exists independently on $L_H\omega_k$. However, if

$$\frac{\left(a_0^{(r)}\right)_{n_1}}{\sqrt{A_q}} < 0, \tag{75}$$

then positive X_s exists only if $L_H\omega_k < 0$. Additionally, we have to mention that if

$$\frac{\left(a_0^{(r)}\right)_{n_1}}{\sqrt{A_q}} = 0, \tag{76}$$

then there is only one root

$$X_s = -2a. \tag{77}$$

This root is positive only if $L_H\omega_k < 0$.

G. Second and third terms are dominant

For this to be true, one has to have

$$n_2 = k + c - 1 - p + s, \tag{78}$$

$$n_1 > k + \frac{q}{2} - p + s - 1. \tag{79}$$

In this case, (20) becomes

$$sX_s v^{s-1} = \left(\frac{\kappa_p}{X_s}\frac{L_H\left(a_o^{(\phi)}\right)_{n_2}}{\sqrt{g_{\phi H}}(u^r)_c} - L_H k\omega_k\right) v^{n_2+p-c-s}. \tag{80}$$

From this equation it follows that s has to satisfy the condition

$$n_2 = 2s + c - p - 1. \tag{81}$$

Substituting this into (78) and (79), one gets

$$n_1 > 2s + \frac{q}{2} - p - 1, \tag{82}$$

$$k = s. \tag{83}$$

If it is so, then

$$sX_s = \frac{\kappa_p}{X_s} \frac{L_H (a_o^{(\phi)})_{n_2}}{\sqrt{g_{\phi H}} (u^r)_c} - L_H k \omega_k. \tag{84}$$

Multiplying by X_s (only if $X_s \neq 0$), one obtains

$$s(X_s)^2 + kL_H \omega_k X_s - \kappa_p \frac{L_H (a_o^{(\phi)})_{n_2}}{\sqrt{g_{\phi H}} (u^r)_c} = 0. \tag{85}$$

The solution of this equation is

$$X_s = -a \pm \sqrt{a^2 + b},$$

where

$$a = \frac{kL_H \omega_k}{2s}, \qquad b = \frac{\kappa_p}{s} \frac{L_H (a_o^{(\phi)})_{n_2}}{\sqrt{g_{\phi H}} (u^r)_c}. \tag{86}$$

One sees that this solution is similar to the case when all three roots are of the same order, but with a slight change of coefficient b. Following the same analysis as in the previous case, we see that if

$$\frac{L_H (a_o^{(\phi)})_{n_2}}{\sqrt{g_{\phi H}} (u^r)_c} > 0, \tag{87}$$

then positive X_s exists independently on $L_H \omega_k$. However, if

$$\frac{L_H (a_o^{(\phi)})_{n_2}}{\sqrt{g_{\phi H}} (u^r)_c} < 0, \tag{88}$$

then positive X_s exists only if $L_H \omega_k < 0$. Additionally, we have to mention that if

$$\frac{L_H (a_o^{(\phi)})_{n_2}}{\sqrt{g_{\phi H}} (u^r)_c} = 0, \tag{89}$$

then there is only one root

$$X_s = -2a. \tag{90}$$

This root is positive only if $L_H \omega_k < 0$.

We summarize all the possible cases in Table I.

IV. PARTICULAR CASES

A. Radial acceleration

Let us also derive general relations in the case when acceleration has only the radial component $a_o^{(r)} \neq 0$, whereas $a_o^{(\phi)} = 0$. Behavior of $\frac{d\mathcal{X}}{dr}$ in this case can be found using the results we previously obtained. Indeed, if $a_o^{(\phi)} = 0$ then (20) becomes

$$sX_s v^{s-1} = \frac{\kappa_p}{X_s} \frac{(a_0^{(r)})_{n_1}}{\sqrt{A_q}} v^{n_1 + p - \frac{q}{2} - s} - L_H k \omega_k v^{k-1}. \tag{91}$$

Thus, we have only two terms, which we have to compare. If $n_1 < n_1^* \equiv k + \frac{q}{2} + s - p - 1$, then the first term is dominant. This case was already analyzed and is presented by the first line in Table I. If $n_1 = n_1^*$, then both terms are of the same order and this case corresponds to the fifth line in Table I. If $n_1 > n_1^* = k + \frac{q}{2} + s - p - 1$, then the second term is dominant and this case corresponds to the third line in Table I.

B. Angular acceleration

In the opposite case $a_o^{(r)} = 0$, $a_o^{(\phi)} \neq 0$. Then, (20) becomes

$$sX_s v^{s-1} = \frac{\kappa_p}{X_s} \frac{L_H (a_o^{(\phi)})_{n_2}}{\sqrt{g_{\phi H}} (u^r)_c} v^{n_2 + p - c - s} - L_H k \omega_k v^{k-1}.$$

If $n_2 < n_2^* \equiv k + c - 1 - p + s$, the first terms is dominant and this corresponds to the second line in Table I. If $n_2 = n_2^*$, both terms are of the same order and this corresponds to the sixth line in Table I. If $n_2 > n_2^*$, the second term is dominant and this corresponds to the third line in Table I.

The case of pure angular acceleration was already considered in Sec. V C of [5] for a particular case of extremal black holes. It corresponds to $p = q = 2$, $b = 1 = k = n_2$, so $n_2^* = 0$ and $n_2 > n_2^*$ (where n_2^* is the value of the degree n_2 from Ref. [5]).

C. Static space-time

For static space-times there is no ω and (20) becomes

$$sX_s v^{s-1} = \frac{\kappa_p}{X_s} \frac{(a_0^{(r)})_{n_1}}{\sqrt{A_q}} v^{n_1 + p - \frac{q}{2} - s} + \frac{\kappa_p}{X_s} \frac{L_H (a_o^{(\phi)})_{n_2}}{\sqrt{g_{\phi H}} (u^r)_c} v^{n_2 + p - c - s}. \tag{92}$$

As one can see, the terms on the right-hand side are the same as the first and second terms in (20). Then, the results

TABLE I. Table showing which conditions have to hold for different terms to be dominant in the expression for $\frac{dX}{dr}$ and under which conditions this quantity is positive. Here, the first term means the term with the radial acceleration in Eq. (12), the second one corresponds to coupling between the angular acceleration and angular momentum there, and the third term does not contain acceleration and arises entirely due to the angular momentum. We denoted $b = \frac{\kappa_p}{s}\left[\frac{(a_0^{(r)})_{n_1}}{\sqrt{A_q}} + \frac{L_H (a_o^{(\phi)})_{n_2}}{\sqrt{g_{\phi H}}(u^r)_c}\right]$.

Dominant terms	Restrictions on n_1, n_2, k	Conditions for $X_s > 0$	Conditions for real X_s
First	$n_1 = 2s + \frac{q}{2} - p - 1$ $n_2 > 2s + c - p - 1$ $k > s$	⋯	$(a_0^{(r)})_{n_1} > 0$
Second	$n_1 > 2s + \frac{q}{2} - p - 1$ $n_2 = 2s + c - p - 1$ $k > s$	⋯	$\frac{L_H (a_o^{(\phi)})_{n_2}}{(u^r)_c} > 0$
Third	$n_1 > 2s + \frac{q}{2} - p - 1$ $n_2 > 2s + c - p - 1$ $k = s$	$L_H \omega_k < 0$	⋯
First and second	$n_1 = 2s + \frac{q}{2} - p - 1$ $n_2 = 2s + c - p - 1$ $k > s$	⋯	$\frac{(a_0^{(r)})_{n_1}}{\sqrt{A_q}} + \frac{L_H (a_o^{(\phi)})_{n_2}}{\sqrt{g_{\phi H}}(u^r)_c} > 0$
First and third	$n_1 = 2s + \frac{q}{2} - p - 1$ $n_2 > 2s + c - p - 1$ $k = s$	$\frac{(a_0^{(r)})_{n_1}}{\sqrt{A_q}} > 0$ or if $\frac{(a_0^{(r)})_{n_1}}{\sqrt{A_q}} \leq 0$ and $L_H \omega_k < 0$	⋯
Second and third	$n_1 > 2s + \frac{q}{2} - p - 1$ $n_2 = 2s + c - p - 1$ $k = s$	$\frac{L_H (a_o^{(\phi)})_{n_2}}{\sqrt{g_{\phi H}}(u^r)_c} > 0$ or if $\frac{L_H (a_o^{(\phi)})_{n_2}}{\sqrt{g_{\phi H}}(u^r)_c} \leq 0$ and $L_H \omega_k < 0$	⋯
First, second, and third	$n_1 = 2s + \frac{q}{2} - p - 1$ $n_2 = 2s + c - p - 1$ $k = s$	$b > 0$ or if $b \leq 0$ and $L_H \omega_k < 0$	⋯

of Sec. III D apply. According to Table I, for all these cases a force does not control the sign of X_s. In [10] the particular case of the Schwarzschild black hole with a radial force acting on a particle was considered. There, the sign of acceleration was chosen by hand to ensure the existence of critical trajectories but the question about the sign of dX/dr was not posed. Now, we are making the second step describing this issue.

V. DIFFERENT TYPES OF PARTICLES

Now let us consider different types of particles and how the possibility of having positive dX/dr is related to the type of a particle. For this we have to review main properties of different types of particles. As was said in Sec. III, they differ by the relation between numbers s and c. Namely, for subcritical particles $0 < s < \frac{p}{2}$, $c = s + \frac{q-p}{2}$, for critical $s = \frac{p}{2}$, $c = \frac{q}{2}$, for ultracritical $s = \frac{p}{2}$, $c > \frac{q}{2}$ (see Table 1 in [7]). Now let us apply this to the analysis of different cases to elucidate when a force controls the sign of dX/dr. As follows from Table I, this is possible only in cases when either only the third term is dominant; or if first and second, first, and third or second and third terms are dominant; or if all three terms are of the same order. Thus we have to consider which conditions we obtain for each of these cases for different types of particles. Also note that one can additionally add the requirement of finiteness of external forces (namely, requiring $n_{1,2} \geq 0$), but extensive analysis of the conditions required to produce a finite force was already done in [7] (see Tables IV–VI in there), so we will not repeat them here. Only in the next section we will provide this analysis for different types of horizons and show that it correlates with the results in [7].

A. Third term is dominant

The third term is dominant if $n_1 > 2s + \frac{q}{2} - p - 1$, $n_1 > 2s + c - p - 1$, and $k = s$.

For subcritical particles $c = s + \frac{q-p}{2}$, thus conditions for $n_{1,2}$ become

$$n_1 > 2s + \frac{q}{2} - p - 1, \tag{93}$$

$$n_2 > 2s + \frac{q}{2} - p - 1 + \left(s - \frac{p}{2}\right). \tag{94}$$

As for subcritical particles $s < \frac{p}{2}$, the lower bound for n_2 has to be smaller then the lower bound for n_1. However, the conditions themselves do not tell which number is greater, so there may exist any relation between them. However, it is obvious that, as for subcritical particles $0 < s$, independently on the exact value of s holds $n_1 > \frac{q}{2} - p - 1$, $n_2 > \frac{q}{2} - \frac{3p}{2} - 1$. Condition $k = s$ gives us for subcritical particles $0 < k = s < \frac{p}{2}$. Thus, summarizing, for subcritical particles n_1 and n_2 are unrelated, but $n_1 > \frac{q}{2} - p - 1$, $n_2 > \frac{q}{2} - \frac{3p}{2} - 1$. Additionally $0 < k = s < \frac{p}{2}$.

For critical particles $s = \frac{p}{2}$ and $c = \frac{q}{2}$. Substituting this into conditions on $n_{1,2}$, one gets

$$n_1 > \frac{q}{2} - 1, \tag{95}$$

$$n_2 > \frac{q}{2} - 1. \tag{96}$$

One sees that the lower bounds for both $n_{1,2}$ are the same, but conditions do not restrict which number has to be greater. Summarizing, for critical particles, n_1 and n_2 are unrelated but $n_{1,2} > \frac{q}{2} - 1$. Additionally $k = \frac{p}{2}$.

For ultracritical particles $s = \frac{p}{2}$, $c > \frac{q}{2}$. Conditions for $n_{1,2}$ give

$$n_1 > \frac{q}{2} - 1, \tag{97}$$

$$n_2 > c - 1. \tag{98}$$

In this case, the lower bound for n_1 is smaller, but the relation between these numbers is not restricted, so that for ultracritical particles n_1 and n_2 are unrelated, but $n_1 > \frac{q}{2} - 1$, $n_2 > c - 1$. Additionally, $k = \frac{p}{2}$.

B. First and third terms are dominant

The first and second terms are dominant if $n_1 = 2s + \frac{q}{2} - p - 1$, $n_2 > 2s + c - p - 1$, $k = s$.

For subcritical particles $c = s + \frac{q-p}{2}$, thus conditions for $n_{1,2}$ become

$$n_1 = 2s + \frac{q}{2} - p - 1, \tag{99}$$

$$n_2 > 2s + \frac{q}{2} - p - 1 + \left(s - \frac{p}{2}\right). \tag{100}$$

From these conditions we see that $n_2 > n_1 + (s - \frac{p}{2})$. As $s < \frac{p}{2}$, we see that from this condition does not follow what is greater: n_1 or n_2, but for them has to follow $n_1 > \frac{q}{2} - p - 1$, $n_2 > \frac{q}{2} - \frac{3p}{2} - 1$. Thus for subcritical particles n_1 and n_2 are unrelated, but $n_1 > \frac{q}{2} - p - 1$, $n_2 > \frac{q}{2} - \frac{3p}{2} - 1$. Additionally $0 < k = s < \frac{p}{2}$.

For critical particles $s = \frac{p}{2}$ and $c = \frac{q}{2}$ and conditions for $n_{1,2}$ become

$$n_1 = \frac{q}{2} - 1, \tag{101}$$

$$n_2 > \frac{q}{2} - 1. \tag{102}$$

Thus we see, that for them holds $n_2 > n_1$. So, for critical particles $n_2 > n_1 = \frac{q}{2} - 1$. Additionally, $k = p/2$.

For ultracritical $s = \frac{p}{2}$, $c > \frac{q}{2}$. Thus conditions for $n_{1,2}$ become

$$n_1 = \frac{q}{2} - 1, \tag{103}$$

$$n_2 > c - 1. \tag{104}$$

As $c > \frac{q}{2}$, we see that $n_2 > n_1$. Thus, for ultracritical particles $n_2 > n_1 = \frac{q}{2} - 1$. Additionally, $k = p/2$.

C. Second and third terms are dominant

The second and third terms are dominant if $n_1 > 2s + \frac{q}{2} - p - 1$, $n_2 = 2s + c - p - 1$, $k = s$.

For subcritical particles the conditions for $n_{1,2}$ become

$$n_1 > 2s + \frac{q}{2} - p - 1, \tag{105}$$

$$n_2 = 2s + \frac{q}{2} - p - 1 + \left(s - \frac{p}{2}\right). \tag{106}$$

Combining these conditions we have $n_1 > n_2 + (\frac{p}{2} - s)$. As $s < \frac{p}{2}$ we see that $n_1 > n_2$. Thus, for subcritical particles $n_1 > n_2$ and $n_1 > \frac{q}{2} - p - 1$, $n_2 > \frac{q}{2} - \frac{3p}{2} - 1$. Additionally, $0 < k = s < \frac{p}{2}$.

For critical particles conditions for $n_{1,2}$ become

$$n_1 > \frac{q}{2} - 1, \tag{107}$$

$$n_2 = \frac{q}{2} - 1. \tag{108}$$

One can see that $n_1 > n_2$. Thus summarizing, for critical particles $n_1 > n_2$ and $n_1 > \frac{q}{2} - 1$, $n_2 = \frac{q}{2} - 1$. Additionally, $k = s = \frac{p}{2}$.

For ultracritical particles $s = \frac{p}{2}$ and $c > \frac{q}{2}$, so the conditions for $n_{1,2}$ become

TABLE II. Table showing how n_1 and n_2 are related for different types of particles and different dominant terms.

	Subcritical	Critical	Ultracritical
Third term	n_1 and n_2 are not related	n_1 and n_2 are not related	n_1 and n_2 are not related
First and third term	n_1 and n_2 are not related	$n_2 > n_1$	$n_2 > n_1$
Second and third term	$n_1 > n_2$	$n_1 > n_2$	n_1 and n_2 are not related
First, second and third term	$n_1 > n_2$	$n_1 = n_2$	$n_2 > n_1$

$$n_1 > \frac{q}{2} - 1, \tag{109}$$

$$n_2 = c - 1. \tag{110}$$

As $c > \frac{q}{2}$, $n_2 > \frac{q}{2} - 1$ and we see that n_1 and n_2 are unrelated. Thus, for ultracritical particles n_1 and n_2 are unrelated, but $n_{1,2} > \frac{q}{2} - 1$. Additionally $k = \frac{p}{2}$.

D. All three terms are of the same order

All three terms are of the same order if $n_1 = 2s + \frac{q}{2} - p - 1$, $n_2 = 2s + c - p - 1$, $k = s$.

For subcritical particles the conditions for $n_{1,2}$ become

$$n_1 = 2s + \frac{q}{2} - p - 1, \tag{111}$$

$$n_2 = 2s + \frac{q}{2} - p - 1 + \left(s - \frac{p}{2}\right) = n_1 + \left(s - \frac{p}{2}\right). \tag{112}$$

As $s < \frac{p}{2}$, we see that $n_2 < n_1$. Thus summarizing, for subcritical particles $n_1 > n_2$ and $n_1 > \frac{q}{2} - p - 1$, $n_2 > \frac{q}{2} - \frac{3p}{2} - 1$. Additionally, $0 < k = s < \frac{p}{2}$.

For critical particles,

$$n_1 = \frac{q}{2} - 1 = n_2, \tag{113}$$

so that for critical particles $n_1 = n_2 = \frac{q}{2} - 1$. Additionally, $k = s = \frac{p}{2}$.

For ultracritical particles $s = \frac{p}{2}$, $c > \frac{q}{2}$. Then, the conditions for $n_{1,2}$ become

$$n_1 = \frac{q}{2} - 1, \tag{114}$$

$$n_2 = c - 1. \tag{115}$$

As $c > \frac{q}{2}$, we have $n_2 > n_1$. Thus we have that for ultracritical particles $n_2 > n_1 = \frac{q}{2} - 1$. Additionally, $k = s = \frac{p}{2}$.

VI. ANALYSIS FOR DIFFERENT TYPES OF HORIZONS

We summarize all these cases in Table II. Now let us investigate for what types of horizons and what types of particles constraints arise that control the sign of X_s.

A. Nonextremal horizon

For nonextremal horizons $q = p = 1$. We will show that for all types of particles for such horizons a finite force does not control the sign of X_s. As we have shown in the previous section, in all cases when a force controls the sign of X_s, condition $k = s$ has to hold. As for subcritical particles $0 < s < p/2$, we see that k has to be less than 1, which is forbidden as we consider only integers in Taylor expansions of metric functions. For critical and ultracritical particles the situation is familiar: $k = s = p/2 < 1$.

B. Extremal horizon

For extremal horizons $q = 2$, $p \geq 2$. In this case it is in principle possible to have $k = s$. Note that the third term may always be dominant if $k = s$. If the first and third are dominant, then condition

$$n_1 = 2s + \frac{q}{2} - p - 1 \tag{116}$$

should be fulfilled.

For subcritical particles this quantity is negative (because $q = 2$ and $s < p/2$); that means that the force has to diverge. But for critical and ultracritical particles $n_1 = 0$ and force is finite.

The second and third terms are dominant if

$$n_2 = 2s + c - p - 1. \tag{117}$$

For subcritical particles

$$c = s + \frac{q-p}{2}, \quad n_2 = 2s + \frac{q}{2} - p - 1 + \left(s - \frac{p}{2}\right). \tag{118}$$

We see, that as $q = 2$ and $s < p/2$ for them, then $n_2 < 0$. However, for critical and subcritical particles $c \geq \frac{q}{2}$ and n_2 is non-negative.

If all three terms are of the same order, all previous conditions have to hold. To conclude, we see that in the case when a force is absent (or if it is quite small near the horizon, so that the third term is dominant) one can control the sign of X_s for any type of particle. However, if the force is high enough, then it becomes possible only for critical and ultracritical particles.

C. Ultra-extremal horizon

For ultra-extremal horizons ($p \geq 2$, $q \geq 3$) there is no special restriction and all types of particles may have controllable X_s. Thus we see that if the force is small enough (or absent) such that the third term is dominant, then we can control the sign of dX/dr for all types of particles if the horizon is extremal or ultra-extremal. If a force becomes higher in such a way that it starts to change the dynamics of a system (corresponding conditions for this to happen are given in the last three lines in Table I), we can control the sign of dX/dr for subcritical particles only if the horizon is ultra-extremal, and for critical and ultracritical only if the horizon is extremal or ultra-extremal. Note that in our previous work where we analyzed for which fine-tuned particles and which types of horizons it is possible to have finite forces, we obtained the same set of combinations of types of particles and types of horizons (see Table VIII in [7]), except for the case when a force is absent [as was mentioned after Eq. (60) in [7]; such cases were not analyzed there].

VII. CONCLUSIONS

In this work we scrutinized self-consistent dynamics of fine-tuned particles in the vicinity of the horizon. As such a type of particle is a necessary ingredient of the BSW effect; self-consistent solutions of equations of motion give us the condition when this effect does exist. In doing so, we have analyzed which constraints the presence of a force imposes on the sign of X for fine-tuned particles. We have established general relations between kinematic properties of a particle (energy, angular momentum) and dynamic ones (forces in different directions). This has a crucial consequence for the existence (or nonexistence) of the BSW effect. As on the horizon itself $X = 0$ for fine-tuned particles, near the horizon we examined how the sign of dX/dr (or the first nonvanishing derivative on the horizon) is related to the acceleration and properties of the metric. Term-by-term analysis has shown that the sign of dX/dr is controlled by force only if several conditions hold: if the rate of change of the metric coefficient ω is the same as the rate of change of X near the horizon ($k = s$) and if the forces satisfy the conditions listed in Table I. In addition, there is the case of freely moving particles (or particles, for which a force near the horizon is negligibly small), for which the relation $L_H \omega_k < 0$ is required. In some cases, there are no constraints on sign but there are constraints that are necessary for X to be a real quantity.

We analyzed which conditions hold for different types of particles. We have found that for all types of particles one is led in general to control the sign of dX/dr, but not for all types of horizons. For example, we have shown that for nonextremal horizons there are no restrictions that control the sign of dX/dr; for extremal it happens only for critical and ultracritical particles (if the force is in some sense high enough) or for all types of particles (if the force is absent or small enough), and for ultracritical it is irrelevant.

Our results show that there is a nontrivial interplay between kinematics and dynamics in determining under which configurations the BSW effect is possible when particles experience the action of a force.

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