Charged dilatonic black holes in dilaton-massive gravity

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In this paper, we focus on massive Einstein-dilaton gravity including the coupling of dilaton scalar field to massive graviton terms, and then derive static and spherically symmetric solutions of charged dilatonic black holes in four dimensional spacetime. We find that the dilatonic black hole could possess different horizon structures for some suitably parameters. Then, we also investigate the thermodynamic properties of charged dilatonic black holes where $f(r)$ approaches $+\infty$ and $-\infty$, respectively.

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I. INTRODUCTION

Despite Einstein's general relativity (GR) successfully explaining many observations, alternative theories are being sought due to the cosmological constant problem [1], and the origin of acceleration of our Universe based on the supernova data [2,3] and cosmic microwave background (CMB) radiation [4,5]. An alternative theory to GR is dilaton gravity, which arises from the low-energy limit of string theory. In this theory, Einstein's gravity is reinstated alongside a scalar dilaton field through nonminimal coupling with other fields such as axion and gauge fields [6]. The dilaton field is essential in string theory when coupled with gravity and other gauge fields. Many attempts have been made to investigate this theory. For instance, Refs. [7–13] found that the dilaton field alters the causal structure of black holes, leading to curvature singularities at finite radii. The dilaton potential can be seen as a generalization of the cosmological constant and can also modify the asymptotic behavior of the solutions. References [14,15] investigated black hole solutions in (A) dS spacetimes by combining three Liouville-type dilaton potentials. Additionally, scalar-tensor generalizations of GR have been investigated, incorporating various curvature corrections to the standard Einstein-Hilbert Lagrangian coupled with the dilaton scalar field [16–18]. A specific model, Einstein-dilaton-Gauss-Bonnet (EdGB) gravity, was extensively studied in Refs. [19,20]. It was found that the scalar dilaton acts as secondary hair, with its charge expressed in terms of the black hole mass. Later, EdGB gravity was used to study black holes in various dimensions [21–23], rotating black holes [24,25], wormholes [26], and rapidly rotating neutron stars [27].

From the perspective of modern particle physics [28,29], the gravity field is described by a unique theory involving a spin-2 graviton. Massive gravity is a straightforward modification achieved by giving mass to the graviton. Dating back to 1939, Fierz and Pauli [30] formulated a linear theory of massive gravity. However, this theory consistently faces the Boulware-Deser (BD) ghost issue at the nonlinear level [31,32]. Notice that the authors [33] of eliminated the BD ghost by introducing higher-order interaction terms into the Lagrangian. Subsequently, the ghost-free massive theory known as de Rham-Gabadadze-Tolley (dRGT) massive gravity was developed and discussed in Refs. [34,35]. In dRGT massive gravity, (charged) black hole solutions and their thermodynamics in asymptotically AdS spacetime were investigated [36–44]. It was found that the coefficients in the potential associated with the graviton mass play a role similar to that of charge in thermodynamic phase space. Other black hole solutions were also studied in massive gravity [45–53]. Meanwhile, some charged dilatonic black hole solutions have been discovered [54–57]. Recently, quasidilaton massive gravity, a scalar extension of dRGT massive gravity with a shift symmetry, has also been studied [58–62]. Building upon these studies, we aim to extend our investigation by considering the nonminimal coupling of the dilaton field to

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the graviton, and derive analytical solutions for charged dilatonic black holes in massive dilaton gravity.

The work is organized as follows. In Sec. II, we will present the static and spherically symmetric charged black hole solutions in four-dimensional massive Einstein-dilaton gravity and investigate the solution structures of charged dilatonic black holes. In Sec. III, we will discuss the thermodynamics of theses charged dilatonic black holes in dilaton-massive gravity. Finally, we close the work with discussions and conclusions in Sec. IV.

II. SOLUTIONS OF CHARGED DILATONIC BLACK HOLES

The action for massive gravity with a nonminimal coupling of dilaton field φ in four dimensional spacetime is given by

$$I=\frac{1}{16\pi}\int d^4x\sqrt{-g}\Big[\mathcal{R}-2(\nabla\varphi)^2-e^{-2\alpha\varphi}F_{\mu\nu}F^{\mu\nu}+m_0^2\sum_{i=1}^{4}c_ie^{\alpha_i\varphi}\mathcal{U}_i(g,h)\Big], \tag{1}$$

where $\varphi=\varphi(r)$ is the dilaton scalar field. The last term in the action denotes the general form of nonminimal coupling between the scalar field and the massive graviton with coupling constants α_i. Here m_0 is the mass of graviton, and c_i are the number of dimensionless coupling coefficients. Moreover, $\mathcal{U}_i$ are symmetric polynomials of the eigenvalues of the 4×4 matrix $K^\mu{}_\nu=\sqrt{g^{\mu\alpha}h_{\alpha\nu}}$ in which h is a fixed rank-2 symmetric tensor, satisfying the following recursion relation [34]

$$\begin{aligned}
&\mathcal{U}_1=[K]=K^\mu{}_\mu,\\
&\mathcal{U}_2=[K]^2-[K^2],\\
&\mathcal{U}_3=[K]^3-3[K][K^2]+2[K^3],\\
&\mathcal{U}_4=[K]^4-6[K^2][K]^2+8[K^3][K]+3[K^2]^2-6[K^4].
\end{aligned} \tag{2}$$

Varying the action with respect to the field variables $g_{\mu\nu}$ and φ, the equations of motion are obtained as

$$G_{\mu\nu}=\mathcal{R}_{\mu\nu}-\frac{1}{2}\mathcal{R}g_{\mu\nu}=2e^{-2\alpha\varphi}\Big[F_{\mu\eta}F^{\eta}_{\nu}-\frac{1}{4}g_{\mu\nu}F_{\eta\tau}F^{\eta\tau}\Big]+2\partial_\mu\varphi\partial_\nu\varphi-(\nabla\varphi)^2g_{\mu\nu}+m_0^2\chi_{\mu\nu}, \tag{3}$$

$$\nabla^2\varphi=-\frac{1}{4}\Big[2\alpha e^{-2\alpha\varphi}F_{\eta\tau}F^{\eta\tau}+m_0^2\sum_{i=1}^{4}\frac{\partial\tilde{c}_i}{\partial\varphi}\mathcal{U}_i\Big], \tag{4}$$

$$\nabla_\mu(e^{-2\alpha\varphi}F^{\mu\nu})=0, \tag{5}$$

where

$$\tilde{c}_i=c_ie^{\alpha_i\varphi}, \tag{6}$$

$$\begin{aligned}
\chi_{\mu\nu}=&\frac{\tilde{c}_1}{2}(\mathcal{U}_1g_{\mu\nu}-K_{\mu\nu})+\frac{\tilde{c}_2}{2}(\mathcal{U}_2g_{\mu\nu}-2\mathcal{U}_1K_{\mu\nu}+2K^2_{\mu\nu})\\
&+\frac{\tilde{c}_3}{2}(\mathcal{U}_3g_{\mu\nu}-3\mathcal{U}_2K_{\mu\nu}+6\mathcal{U}_1K^2_{\mu\nu}-6K^3_{\mu\nu})\\
&+\frac{\tilde{c}_4}{2}(\mathcal{U}_4g_{\mu\nu}-4\mathcal{U}_3K_{\mu\nu}+12\mathcal{U}_2K^2_{\mu\nu}-24\mathcal{U}_1K^3_{\mu\nu}+24K^4_{\mu\nu}).
\end{aligned} \tag{7}$$

Now we introduce the static and spherical symmetry metric ansatz

$$ds^2=-f(r)dt^2+\frac{dr^2}{f(r)}+r^2R(r)^2d\Omega^2, \tag{8}$$

in which $f(r)$ and $R(r)$ are functions of r and $d\Omega^2=d\theta^2+\sin^2\theta d\phi^2$ is the line element for the two dimensional spherical subspace with the constant curvature. The fiducial metric $h_{\mu\nu}$ in the action (1) serves as a Lagrange multiplier to eliminate the BD ghost [36]. Choosing an appropriate form simplifies the calculation. Reference [36] noted that, unlike the dynamical physical metric $g_{\mu\nu}$, the reference metric $h_{\mu\nu}$ is typically fixed and assumed to be non-dynamical in the massive theory. In this work, we will follow [39,40] by choosing the fiducial metric to be

$$h_{\mu\nu}=\mathrm{diag}(0,0,c_0^2,c_0^2\sin^2\theta). \tag{9}$$

From the ansatz (9), the interaction potential in Eq. (2) changes into

$$\mathcal{U}_1=\frac{2c_0}{rR},\qquad \mathcal{U}_2=\frac{2c_0^2}{r^2R^2},\qquad \mathcal{U}_3=\mathcal{U}_4=0. \tag{10}$$

From integrating the Maxwell equation (5), the electromagnetic field tensor can be given

$$F_{tr}=\frac{Qe^{2\alpha\varphi}}{r^2R^2}, \tag{11}$$

where Q is an integration constant related to the electric charge of the black hole. Then, $\chi^\mu{}_\nu$ from Eq. (7) becomes

$$\begin{aligned}
&\chi^1{}_1=\chi^2{}_2=\frac{c_0(c_1rRe^{\alpha_1\varphi}+c_0c_2e^{\alpha_2\varphi})}{r^2R^2},\\
&\chi^3{}_3=\chi^4{}_4=\frac{c_0c_1e^{\alpha_1\varphi}}{2rR},
\end{aligned} \tag{12}$$

and the corresponding components of the equation of motion (3) can be simplified to

$$
\begin{aligned}
G^1{}_1 &= \frac{-1 + rRf'(R + rR') + f[R^2 + r^2R'^2 + 2rR(3R' + rR'')]}{r^2R^2} \\
&= -f\varphi'^2 + m_0^2\chi^1{}_1 - \frac{Q^2e^{2\alpha\varphi}}{r^4R^4},
\end{aligned} \tag{13}
$$

$$
G^2{}_2 = \frac{-1 + rRf'(R + rR') + f(R + rR')^2}{r^2R^2} = f\varphi'^2 + m_0^2\chi^2{}_2 - \frac{Q^2e^{2\alpha\varphi}}{r^4R^4}, \tag{14}
$$

$$
\begin{aligned}
G^3{}_3 = G^4{}_4 &= \frac{1}{2rR}[2(2f + rf')R' + R(2f' + rf'') + 2rfR''] \\
&= -f\varphi'^2 + m_0^2\chi^3{}_3 + \frac{Q^2e^{2\alpha\varphi}}{r^4R^4}.
\end{aligned} \tag{15}
$$

Here the prime $'$ denotes differentiation with respect to the radial coordinate r.

Based on Eqs. (12)–(14), we obtain

$$
2R'(r) + r[R(r)\varphi'(r)^2 + R''(r)] = 0. \tag{16}
$$

We assume that the dilaton field can be expressed as

$$
R(r) = e^{\beta\varphi(r)}, \tag{17}
$$

where β is a constant. In fact, the similar assumption (17) has been extensively used to look for the charged dilaton black hole solutions [41,42] in the Maxwell-dilaton gravity. By solving Eq. (16), we obtain the dilaton field as

$$
\varphi(r) = -\frac{\beta}{1+\beta^2}\ln\left(\frac{ar}{1+br}\right), \tag{18}
$$

where a and b are integration constants, and we set $a = 1$ and $b = 0$.

Considering the scalar field equation (4) and substitute the metric ansatz (8), the scalar field equation becomes

$$
c_0^2e^{\alpha_2\varphi(r)}m_0^2\alpha_2c_2 - \frac{2e^{2\alpha\varphi(r)}Q^2\alpha}{r^2R(r)^2} + rR(r)c_0e^{\alpha_1\varphi(r)}m_0^2\alpha_1c_1 + 2\frac{d}{dr}[r^2R(r)^2f(r)\varphi'(r)] = 0. \tag{19}
$$

According to Eq. (19), along with the assumption $R(r)$ (17) and the dilaton field $\varphi(r)$ (18), we can solve for $f(r)$ as

$$
f(r) = -2mr^{1-\frac{2}{1+\beta^2}} + \frac{Q^2\alpha(1+\beta^2)^2r^{2-\frac{2(2+\alpha\beta)}{1+\beta^2}}}{\beta(1+2\alpha\beta-\beta^2)} + \frac{c_0m_0^2(1+\beta^2)^2c_1\alpha_1r^{\frac{1+2\beta^2-\beta\alpha_1}{1+\beta^2}}}{2\beta(2+\beta^2-\beta\alpha_1)} + \frac{c_0^2m_0^2(1+\beta^2)^2c_2\alpha_2r^{\frac{\beta(2\beta-\alpha_2)}{1+\beta^2}}}{2\beta(1+\beta^2-\beta\alpha_2)}, \tag{20}
$$

where m is an integration constant related to the mass of the black hole.

On the other hand, we further consider the $G^1{}_1$ component of the gravitational field equation, Eq. (13) can be expressed as

$$
\begin{aligned}
&e^{2\alpha\varphi(r)}Q^2 + r^2R(r)^2[r^2f(r)R'(r)^2 - 1 - c_0^2e^{\alpha_2\varphi(r)}m_0^2c_2] + r^2R(r)^4[f(r) + rf'(r) + r^2f(r)\varphi'(r)^2] \\
&+ r^3R(r)^3\{rf'(r)R'(r) + 2f(r)[3R'(r) + rR''(r)] - c_0e^{\alpha_1\varphi(r)}m_0^2c_1\} = 0.
\end{aligned} \tag{21}
$$

Substituting Eq. (20) into Eq. (21), the following parameter constraints can be obtained

$$
\beta = \alpha, \qquad \alpha_1 = \frac{1}{\alpha}, \qquad \alpha_2 = 2\alpha, \qquad \alpha = m_0\sqrt{\frac{c_0c_1}{2+2c_0m_0^2c_1}}. \tag{22}
$$

It should be noted that for a real α, it is required that $m_0^2c_0c_1 > 0$ or $m_0^2c_0c_1 < -1$, thereby excluding the range $-1 < m_0^2c_0c_1 < 0$.

Through (20) and (22), we can obtain the final solution as

$$
f(r) = -2mr^{-\frac{1}{3}-\frac{4}{6+9c_0m_0^2c_1}} + \frac{Q^2(2+3c_0m_0^2c_1)r^{-\frac{4}{3}-\frac{4}{6+9c_0m_0^2c_1}}}{2(1+c_0m_0^2c_1)} + \frac{1}{2}(2+3c_0m_0^2c_1)r^{\frac{2c_0m_0^2c_1}{2+3c_0m_0^2c_1}} + \frac{c_0^2m_0^2(2+3c_0m_0^2c_1)^2c_2}{2(1+c_0m_0^2c_1)(2+c_0m_0^2c_1)}. \tag{23}
$$

Our results reduce to the Schwarzschild case when $c_1 = c_2 = Q = 0$, and reduce to the Reissner-Nordström case when $c_1 = c_2 = 0$.

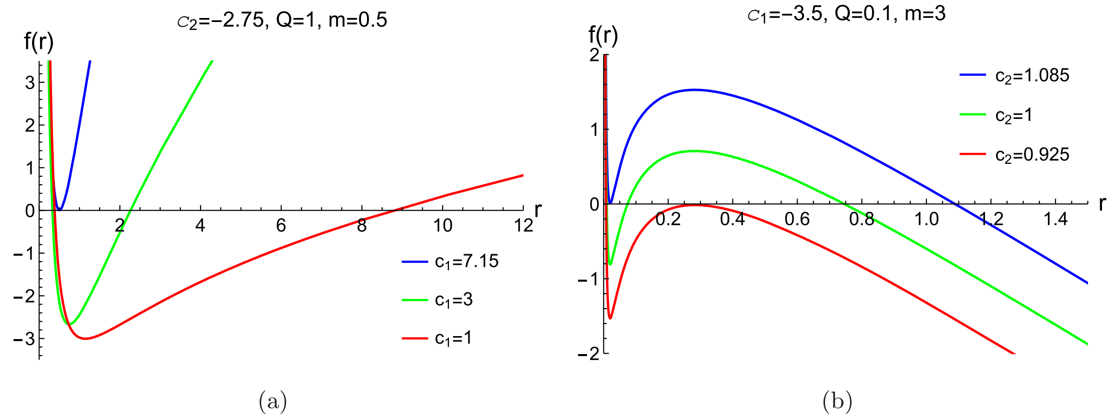


FIG. 1. The function $f(r)$ versus the radius r for different values of c_1 and c_2.

In order to comprehend the behavior of the metric function deeply, we would like to give graphical dependence of the function $f(r)$, and we set $m_0 = c_0 = 1$ for simplify in following discussions. Then, Eq. (23) becomes

$$f(r) = -2mr^{-\frac{2+c_1}{2+3c_1}} + \frac{Q^2(2+3c_1)r^{-\frac{4(1+c_1)}{2+3c_1}}}{2(1+c_1)} + \frac{1}{2}(2+3c_1)r^{\frac{2c_1}{2+3c_1}} + \frac{(2+3c_1)^2 c_2}{2(1+c_1)(2+c_1)}. \tag{24}$$

Evidently, the parameter c_1 affects the asymptotic behavior of $f(r)$. Whereas, $-1 < c_1 < 0$ is excluded.

Now, we calculate the Ricci and Kretschmann scalars to check the spacetime singularities

$$\mathcal{R} = \frac{2 - rR[4f'(R + rR') + rRf''] - 2f[R^2 + r^2R'^2 + 2rR(3R' + rR'')]}{r^2R^2}, \tag{25}$$

$$\mathcal{R}^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu\rho\sigma} = \frac{2r^2R^2f'^2(R + rR')^2 + 4(f(R + rR')^2 - 1)^2 + r^4R^4f''^2}{r^4R^4} + \frac{2r^2R^2[4fR' + f'(R + rR') + 2rfR''^2]^2}{r^4R^4}, \tag{26}$$

where function R has been shown in Eq. (17). One can find that both the Ricci and Kretschmann scalars remain nonsingular at the horizons, indicating that these points are merely coordinate singularities, typical for black holes. Considering the leading terms of asymptotical behaviors of metric at the origin, we obtain

$$\lim_{r\to 0}\mathcal{R} \sim r^{-\frac{2}{3}(5+\frac{2}{2+3c_1})}, \quad \lim_{r\to 0}\mathcal{R}^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu\rho\sigma} \sim r^{-\frac{4}{3}(5+\frac{2}{2+3c_1})}. \tag{27}$$

For $c_1 < -1$ or $c_1 > 0$, the Ricci and Kretschmann scalars diverge at the origin $r = 0$ but finite for $r > 0$. This suggests that the origin $r = 0$ is an essential and physical singularity in the spacetime.

In order to study the asymptotic behavior of the solutions, we expand the metric function $f(r)$ for $r \to \infty$ limit. The figures for metric function $f(r)$ versus radius r are plotted in Fig. 1. If taking $c_1 < -1$ or $c_1 > 0$, we have

$$\lim_{r\to\infty} f(r) = \frac{1}{2}(2+3c_1)r^{\frac{2c_1}{2+3c_1}}. \tag{28}$$

In case of $c_1 > 0$, the function $f(r)$ approaches $+\infty$ as $r \to \infty$. As shown in Fig. 1(a), there are the inner horizon and event horizon when $c_2 = -2.75$, $Q = 1$, $m = 0.5$, and $c_1 = 1$. A naked singularity may appear with the increasing of c_1.

In case of $c_1 < -1$, $f(r)$ approaches $-\infty$ as $r \to \infty$. Figure 1(b) shows a complicated black hole spacetime with $c_1 = -3.5$, $Q = 0.1$, $m = 3$, and $c_2 = 1$, where three horizons emerge: the inner and outer event horizons, and the cosmological horizon. As c_2 increases further to a certain value, the inner and outer event horizons coincide. As c_2 decreases further to a certain value, an extremal black hole known as the Nariai black hole may form, exhibiting a coincidence of the event and cosmological horizons. This indicates that, with the nonminimal coupling to the dilaton field, c_2 plays a crucial role in the behavior of the solution $f(r)$.

III. THERMODYNAMICS OF CHARGED DILATONIC BLACK HOLES

The parameter c_1 determines the asymptotic behavior of $f(r)$, as concluded in the previous section. For $c_1 > 0$, $f(r)$ approaches $+\infty$. And for $c_1 < -1$, $f(r)$ approaches $-\infty$. Now we plan to investigate the thermodynamics of theses charged dilatonic black holes in dilaton-massive gravity.

A. Black holes with $f(r) \to +\infty$

The black hole mass can be expressed in terms of the mass parameter m as mentioned before. Considering that the asymptotic behavior of the metric function (24) is unusual, we will use the Brown and York quasilocal formalism to obtain the quasilocal mass M [63,64]. For the metric [Eq. (2.7) in Ref. [65] or Eq. (3.10) in Ref. [66]]

$$ds^2 = -X(\rho)^2 dt^2 + \frac{d\rho^2}{Y(\rho)^2} + \rho^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{29}$$

provided that the matter field does not contain derivatives of the metric, we can get the quasilocal black hole mass M by using the following relation [Eq. (2.8) in Ref. [65] or Eq. (3.11) in Ref. [66]]

$$\mathcal{M}(\rho) = \rho X(\rho)[Y_0(\rho) - Y(\rho)], \tag{30}$$

with a background metric function $Y_0(\rho)$ which establishes the zero of the mass, and taking the limit $\rho \to \infty$

$$M = \lim_{\rho\to\infty} \mathcal{M}(\rho). \tag{31}$$

Now, we write the metric (8) in the form of Eq. (29) by considering the transformation $\rho = rR(r)$. Thus, one can show that

$$dr^2 = \frac{(1+\beta^2)^2}{R(r)^2} d\rho^2. \tag{32}$$

In our case, we find

$$X(\rho)^2 = f(r), \qquad Y(\rho)^2 = \frac{R(r)^2}{(1+\beta^2)^2} f(r), \tag{33}$$

with $r = r(\rho)$. Substituting these quantities into Eq. (30), we obtain

$$\mathcal{M} = \frac{rR(r)^2}{1+\beta^2} \{[f(r)f_0(r)]^{\frac{1}{2}} - f(r)\}, \tag{34}$$

in which

$$f_0(r) = \frac{Q^2(2+3c_1)r^{-\frac{4(1+c_1)}{2+3c_1}}}{2(1+c_1)} + \frac{1}{2}(2+3c_1)r^{\frac{2c_1}{2+3c_1}} + \frac{(2+3c_1)^2 c_2}{2(1+c_1)(2+c_1)}, \tag{35}$$

where $f_0(r)$ is the metric function $f(r)$ evaluated for the value $m = 0$ of the integration constant. Now, Eq. (34) can be rewritten as

$$\begin{aligned}\mathcal{M} &= \frac{2+2c_1}{2+3c_1} r^{\frac{2+c_1}{2+3c_1}} \left\{ \left\{ \left[-2mr^{-\frac{2+c_1}{2+3c_1}} + f_0(r) \right] f_0(r) \right\}^{\frac{1}{2}} + 2mr^{-\frac{2+c_1}{2+3c_1}} - f_0(r) \right\} \\ &= \frac{2+2c_1}{2+3c_1} r^{\frac{2+c_1}{2+3c_1}} \left\{ f_0(r) \left(1 + \frac{1}{2}\left[\frac{-2mr^{-\frac{2+c_1}{2+3c_1}}}{f_0(r)}\right] - \frac{1}{8}\left[\frac{-2mr^{-\frac{2+c_1}{2+3c_1}}}{f_0(r)}\right]^2 + \mathcal{O}\left[\frac{-2mr^{-\frac{2+c_1}{2+3c_1}}}{f_0(r)}\right]^3 \right) + 2mr^{-\frac{2+c_1}{2+3c_1}} - f_0(r) \right\}. \end{aligned} \tag{36}$$

which leads to the black hole mass

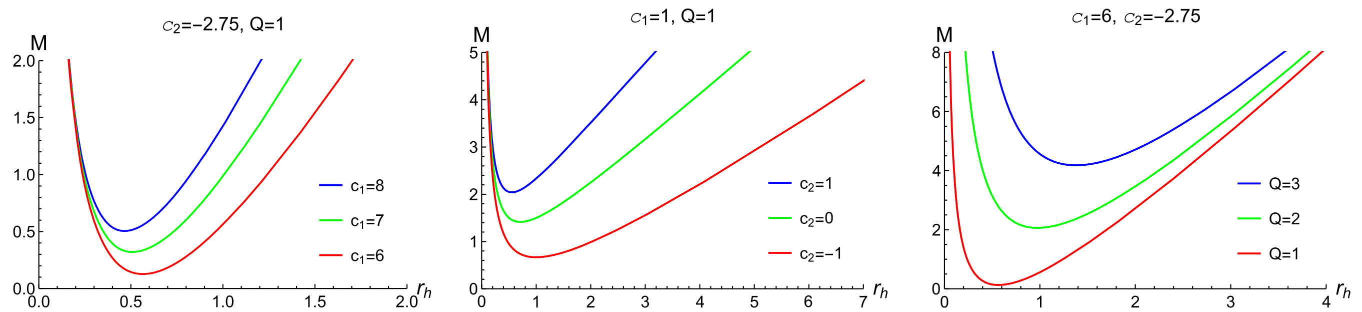
$$M = \lim_{r\to\infty} \mathcal{M} = \lim_{r\to\infty} \frac{2+2c_1}{2+3c_1} \left\{ m - \frac{m^2}{2}\left[\frac{r^{-\frac{2+c_1}{2+3c_1}}}{f_0(r)}\right] - \mathcal{O}\left[\frac{r^{-\frac{2+c_1}{2+3c_1}}}{f_0(r)}\right]^2 \right\}. \tag{37}$$

Note that $f(r)$ approaches $+\infty$ as $r \to \infty$, which requires $c_1 > 0$. Obviously, it is shown that $\lim_{r\to\infty}\left[\frac{r^{-\frac{2+c_1}{2+3c_1}}}{f_0(r)}\right]$ and its higher powers are equal to zero. As a result, we have

$$M = \frac{2m(1+c_1)}{2+3c_1}. \tag{38}$$

Thus, according to the definition of horizon $f(r_h) = 0$, the mass of charged dilatonic black hole is given by

$$M = \frac{1}{2}\left[\frac{Q^2}{r_h} + (1+c_1)r_h + \frac{(2+3c_1)c_2 r_h^{\frac{1}{3}+\frac{4}{6+9c_1}}}{2+c_1}\right]. \tag{39}$$



(a) (b) (c)

FIG. 2. The mass M of black hole versus the horizon radius r_h.

The quasilocal mass M as a function of black hole radius is plotted in Fig. 2(a) for various values of c_1, Fig. 2(b) for various values of c_2, and Fig. 2(c) for various values of Q. In all cases, there exists a minimum mass $M_{\rm min}$, and there are two black holes with the same mass, distinguished by their sizes (a smaller one and a larger one). The quasilocal mass M is always positive. It can be observed that the minimum mass $M_{\rm min}$ increases with c_1, c_2, Q, according to the case.

To develop thermodynamics of charged dilatonic black hole, we need to calculate the Hawking temperature of the black hole geometrically associated with the black hole horizon. In terms of the surface gravity κ corresponding to the null killing vector $(\frac{\partial}{\partial t})^a$ at the horizon, the temperature can be written as

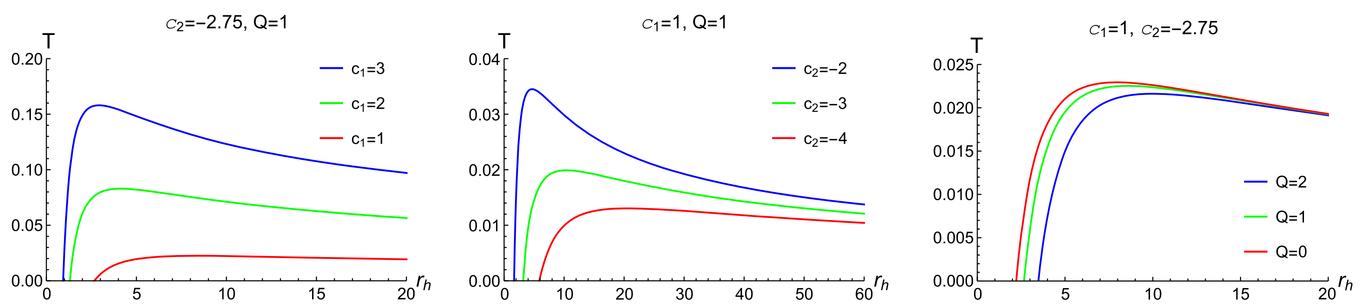
$$T = \frac{\kappa}{2\pi} = \frac{1}{4\pi}\frac{\partial f(r)}{\partial r}\bigg|_{r=r_h} = \frac{(2+3c_1)\left[-Q^2 + r_h\left(r_h + c_1 r_h + c_2 r_h^{\frac{1}{3}+\frac{4}{6+9c_1}}\right)\right] r_h^{-\frac{7}{3}-\frac{4}{6+9c_1}}}{8\pi(1+c_1)}. \tag{40}$$

The temperature of the black hole as a function of radius is depicted in Fig. 3(a) for various values of c_1, in Fig. 3(b) for various values of c_2, and in Fig. 3(c) for various values of Q. In all scenarios, a maximum positive temperature $T_{\rm max}$ is observed for the black hole, indicating the existence of two black holes with identical temperatures but different sizes (a smaller one and a larger one). Moreover, $T_{\rm max}$ increases with c_1, c_2, and Q, depending on the case. And there exists a minimum radius $r_{\rm min}$ where the temperature is 0.

The entropy of charged dilatonic black hole is given by

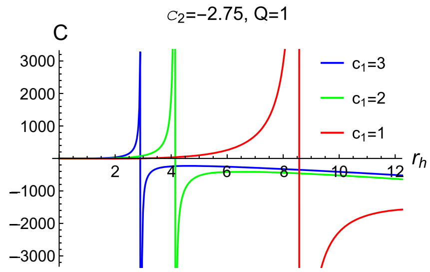
$$S = \pi r_h^2 R(r_h)^2 = \pi r_h^{\frac{4}{3}(1+\frac{1}{2+3c_1})}. \tag{41}$$

References [37–43] have pointed out that the graviton mass does not significantly affect the form of the entropy, and

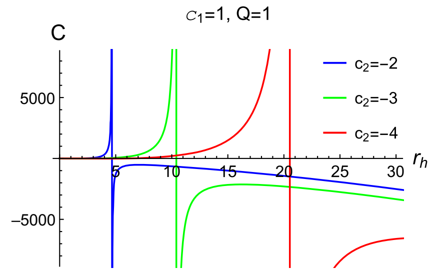


(a) (b) (c)

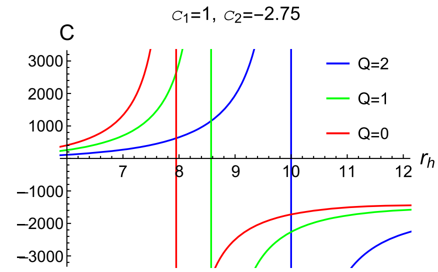
FIG. 3. The temperature T versus the horizon radius r_h. (a) The extreme points of each $T-r_h$ curves $(r_{cri}, T_{\rm max})$ are (2.92518, 0.157982), (4.12896, 0.0829192), (8.56845, 0.0225279) for $c_1 = 3, 2, 1$; (b) The extreme points of each $T-r_h$ curves $(r_{cri}, T_{\rm max})$ are (4.66247, 0.034502), (10.3954, 0.0198907), (20.5486, 0.0130369) for $c_2 = -2, -3, -4$; (c) The extreme points of each $T-r_h$ curves $(r_{cri}, T_{\rm max})$ are (9.98137, 0.0216182), (8.56845, 0.0225279), (7.95021, 0.0229384) for $Q = 2, 1, 0$.



(a)



(b)



(c)

FIG. 4. The heat capacity C versus the horizon radius r_h.

contributes only as a correction for the horizon radius. Then,the potential of charge is

$$U = \frac{Q}{r_h}. \tag{42}$$

It is a matter of calculation to show that the intensive parameters T and U, conjugate to the black hole entropy and charge, satisfy the following relations:

$$\left(\frac{\partial M}{\partial S}\right)_Q = T, \qquad \left(\frac{\partial M}{\partial Q}\right)_S = U. \tag{43}$$

Then, we find that the thermodynamic quantities satisfy the first law of black hole thermodynamics

$$dM = TdS + UdQ. \tag{44}$$

To assess the thermal stability, we compute the heat capacity

$$C_Q = T\left(\frac{\partial S}{\partial T}\right)_Q = T\left(\frac{\partial S/\partial r_h}{\partial T/\partial r_h}\right)_Q, \tag{45}$$

which leads to

$$C_Q = \frac{4\pi(1+c_1)\left[Q^2 - r_h\left(r_h + c_1 r_h + c_2 r_h^{\frac{1}{3}+\frac{4}{6+9c_1}}\right)\right] r_h^{\frac{4}{3}(1+\frac{1}{2+3c_1})}}{2r_h^2 - Q^2(6+7c_1) + r_h\left[c_1(3+c_1)r_h + (2+3c_1)c_2 r_h^{\frac{1}{3}+\frac{4}{6+9c_1}}\right]}. \tag{46}$$

One can find that the nonmonotonic nature of temperature allows us to conclude that the heat capacity exhibits discontinuities as illustrated in Fig. 4, which shows that black holes with relatively smaller horizon radii ($r_{\min} < r_h < r_{cri}$) are stable thermodynamic systems, while a domain of instability exists for larger radii above the point of discontinuity ($r_h > r_{cri}$). In other words, the figures clearly illustrate the stable and unstable regions based on the discontinuities observed in the heat capacity as a function of the horizon radius.

B. Black holes with $f(r) \to -\infty$

When $f(r)$ is for pure dS space, horizons happen at $f(r) = 0$. The biggest root is the cosmological horizon at $r = r_c$, and the next is the black hole event horizon at $r = r_+$. The equations $f(r_+) = 0$ and $f(r_c) = 0$ are rearranged to br as follows [67–69]:

$$M = \frac{2m(1+c_1)}{2+3c_1} = \frac{1}{2}\left[\frac{Q^2}{r_+} + (1+c_1)r_+ + \frac{(2+3c_1)c_2 r_+^{\frac{1}{3}+\frac{4}{6+9c_1}}}{2+c_1}\right], \tag{47}$$

$$M = \frac{2m(1+c_1)}{2+3c_1} = \frac{1}{2}\left[\frac{Q^2}{r_c} + (1+c_1)r_c + \frac{(2+3c_1)c_2 r_c^{\frac{1}{3}+\frac{4}{6+9c_1}}}{2+c_1}\right], \tag{48}$$

from which one can derive

$$M = \frac{[Q^2 + r_c^2(1+c_1)]r_+^{\frac{4(1+c_1)}{2+3c_1}} - [Q^2 + r_+^2(1+c_1)]r_c^{\frac{4(1+c_1)}{2+3c_1}}}{2r_c r_+^{\frac{4(1+c_1)}{2+3c_1}} - 2r_+ r_c^{\frac{4(1+c_1)}{2+3c_1}}}, \tag{49}$$

by eliminating the c_2, and

$$c_2 = \frac{(r_c - r_+)(2+c_1)[Q^2 - r_c r_+(1+c_1)]}{(2+3c_1)r_c r_+(r_c^{\frac{2+c_1}{2+3c_1}} - r_+^{\frac{2+c_1}{2+3c_1}})}, \tag{50}$$

by eliminating the M.

The surface gravities of black hole horizon and the cosmological horizon are

$$\begin{aligned} T_+ &= \frac{\kappa_+}{2\pi} = \frac{1}{4\pi}\frac{\partial f(r)}{\partial r}\bigg|_{r=r_+} \\ &= \frac{(2+3c_1)\left[-Q^2 + r_+\left(r_+ + c_1 r_+ + c_2 r_+^{\frac{1}{3}+\frac{4}{6+9c_1}}\right)\right] r_+^{-\frac{7}{3}-\frac{4}{6+9c_1}}}{8\pi(1+c_1)}, \end{aligned} \tag{51}$$

$$\begin{aligned} T_c &= \frac{\kappa_c}{2\pi} = -\frac{1}{4\pi}\frac{\partial f(r)}{\partial r}\bigg|_{r=r_c} \\ &= -\frac{(2+3c_1)\left[-Q^2 + r_c\left(r_c + c_1 r_c + c_2 r_c^{\frac{1}{3}+\frac{4}{6+9c_1}}\right)\right] r_c^{-\frac{7}{3}-\frac{4}{6+9c_1}}}{8\pi(1+c_1)}. \end{aligned} \tag{52}$$

Considering the connection between the black hole horizon and the cosmological horizon, we can derive the effective thermodynamic quantities and corresponding first law of black hole thermodynamics [70,71]

$$dM = T_{\rm eff} dS + U_{\rm eff} dQ - P_{\rm eff} dV, \tag{53}$$

where the effective temperature $T_{\rm eff}$, effective electric potential $U_{\rm eff}$, and the effective pressure $P_{\rm eff}$ are denoted as

$$T_{\rm eff} = \left(\frac{\partial M}{\partial S}\right)_{Q,V} = \frac{\left(\frac{\partial M}{\partial x}\right)_{r_c}\left(\frac{\partial V}{\partial r_c}\right)_x - \left(\frac{\partial V}{\partial x}\right)_{r_c}\left(\frac{\partial M}{\partial r_c}\right)_x}{\left(\frac{\partial S}{\partial x}\right)_{r_c}\left(\frac{\partial V}{\partial r_c}\right)_x - \left(\frac{\partial V}{\partial x}\right)_{r_c}\left(\frac{\partial S}{\partial r_c}\right)_x}, \tag{54}$$

$$U_{\rm eff} = \left(\frac{\partial M}{\partial Q}\right)_{S,V}, \tag{55}$$

$$P_{\rm eff} = -\left(\frac{\partial M}{\partial V}\right)_{Q,S} = -\frac{\left(\frac{\partial M}{\partial x}\right)_{r_c}\left(\frac{\partial S}{\partial r_c}\right)_x - \left(\frac{\partial S}{\partial x}\right)_{r_c}\left(\frac{\partial M}{\partial r_c}\right)_x}{\left(\frac{\partial V}{\partial x}\right)_{r_c}\left(\frac{\partial S}{\partial r_c}\right)_x - \left(\frac{\partial S}{\partial x}\right)_{r_c}\left(\frac{\partial V}{\partial r_c}\right)_x}. \tag{56}$$

From Eq. (49), one know that

$$M = \frac{[Q^2 + r_c^2(1+c_1)]x^{\frac{4(1+c_1)}{2+3c_1}} - [Q^2 + r_c^2 x^2(1+c_1)]}{2[x^{\frac{4(1+c_1)}{2+3c_1}} - x]r_c}, \tag{57}$$

where $x = r_+/r_c$. Here the thermodynamic volume is that between the black hole horizon and the cosmological horizon, namely

$$\begin{aligned} V = V_c - V_+ &= \frac{4}{3}\pi r_c^3 R(r_c)^3 - \frac{4}{3}\pi r_+^3 R(r_+)^3 \\ &= \frac{4}{3}\pi(r_c^{2+\frac{2}{2+3c_1}} - r_+^{2+\frac{2}{2+3c_1}}) \\ &= \frac{4}{3}\pi r_c^{2+\frac{2}{2+3c_1}}(1 - x^{2+\frac{2}{2+3c_1}}). \end{aligned} \tag{58}$$

The total entropy can be written as

$$\begin{aligned} S = S_+ + S_c + S_{ex} &= \pi r_+^2 R(r_+)^2 + \pi r_c^2 R(r_c)^2 + S_{ex} \\ &= \pi r_c^{\frac{4}{3}(1+\frac{1}{2+3c_1})}[1 + x^{\frac{4}{3}(1+\frac{1}{2+3c_1})} + F(x)]. \end{aligned} \tag{59}$$

Here the undefined function $F(x)$ represents the extra contribution from the correlations of the two horizons. Then, how to determine the function $F(x)$?

In general, the temperatures of the black hole horizon and the cosmological horizon differ, so the globally effective temperature $T_{\rm eff}$ cannot be compared to them. However, in special cases like the lukewarm case, the temperatures of the two horizons are equal. We conjecture that in this special case, the effective temperature should also be the same. Based on this consideration, we can determine the function $F(x)$.

Substituting Eqs. (57)–(59) into Eq. (54), one can obtain

$$T_{\mathrm{eff}}=\left\{r_c^{-\frac{6+7c_1}{2+3c_1}}\left\{Q^2\left[-4x^{\frac{4(1+c_1)}{2+3c_1}}(1+c_1)-4x^{\frac{12+13c_1}{2+3c_1}}(1+c_1)+x^{\frac{10(1+c_1)}{2+3c_1}}(2+c_1)+x^{\frac{6+7c_1}{2+3c_1}}(2+c_1)+x(2+3c_1)\right.\right.\right.$$
$$\left.+x^{\frac{14(1+c_1)}{2+3c_1}}(2+3c_1)\right]+r_c^2\left[2x^{\frac{8+10c_1}{2+3c_1}}c_1(1+c_1)+2x^{\frac{12+13c_1}{2+3c_1}}c_1(1+c_1)+x^{\frac{6+7c_1}{2+3c_1}}(2+3c_1+c_1^2)+x^{\frac{2(7+8c_1)}{2+3c_1}}(2+3c_1+c_1^2)\right.$$
$$\left.\left.\left.-x^3(2+5c_1+3c_1^2)-x^{\frac{14(1+c_1)}{2+3c_1}}(2+5c_1+3c_1^2)\right]\right\}\right\}\Big/\left\{2\pi x\left[x-x^{\frac{4(1+c_1)}{2+3c_1}}\right]^2\left[-4x^{\frac{2}{2+3c_1}}(x+x^{\frac{c_1}{2+3c_1}})(1+c_1)\right.\right.$$
$$\left.\left.-4x^{1+\frac{2}{2+3c_1}}(1+c_1)F(x)+(x^{2+\frac{2}{2+3c_1}}-1)(2+3c_1)F'(x)\right]\right\}, \tag{60}$$

substituting Eq. (57) into Eq. (55), one can obtain

$$U_{\mathrm{eff}}=\frac{Q\left[x^{\frac{4(1+c_1)}{2+3c_1}}-1\right]}{r_c\left[x^{\frac{4(1+c_1)}{2+3c_1}}-x\right]}, \tag{61}$$

substituting Eqs. (57)–(59) into Eq. (56), one can obtain

$$P_{\mathrm{eff}}=\left\{r_c^{-\frac{8+9c_1}{2+3c_1}}\left\{-12(1+c_1)\left[\left[2+3c_1-4x^{\frac{2+c_1}{2+3c_1}}(1+c_1)\right]\left[Q^2+r_c^2x^2(1+c_1)-x^{\frac{4(1+c_1)}{2+3c_1}}[Q^2+r_c^2(1+c_1)]\right]\right.\right.\right.$$
$$\left.-2\left[x-x^{\frac{4(1+c_1)}{2+3c_1}}\right]\left[r_c^2x(1+c_1)(2+3c_1)-2x^{\frac{2+c_1}{2+3c_1}}(1+c_1)[Q^2+r_c^2(1+c_1)]\right]\right]$$
$$\times\left[1+x^{\frac{4(1+c_1)}{2+3c_1}}+F(x)\right]+\left[x-x^{\frac{4(1+c_1)}{2+3c_1}}\right](2+3c_1)\left[r_c^2\left[x^{\frac{4(1+c_1)}{2+3c_1}}-x^2\right](1+c_1)-Q^2\left[x^{\frac{4(1+c_1)}{2+3c_1}}-1\right]\right]$$
$$\left.\left.\times\left[12x^{\frac{2+c_1}{2+3c_1}}(1+c_1)+3(2+3c_1)F'(x)\right]\right\}\right\}\Big/\left\{48\pi\left[x-x^{\frac{4(1+c_1)}{2+3c_1}}\right]^2(1+c_1)\left[4x^{\frac{2}{2+3c_1}}\left(x+x^{\frac{c_1}{2+3c_1}}\right)(1+c_1)\right.\right.$$
$$\left.\left.+4x^{1+\frac{2}{2+3c_1}}(1+c_1)F(x)-\left(x^{2+\frac{2}{2+3c_1}}-1\right)(2+3c_1)F'(x)\right]\right\}. \tag{62}$$

As is well known, in the lukewarm case, we have $T_+=T_c$ [72,73], there is

$$Q^2=\frac{r_c^2x(1+c_1)\left[-x^{\frac{2+c_1}{2+3c_1}}(2+c_1)+x^{\frac{6+7c_1}{2+3c_1}}(2+c_1)+x(2+3c_1)-x^{\frac{6+5c_1}{2+3c_1}}(2+3c_1)\right]}{2+3c_1-4x^{\frac{2+c_1}{2+3c_1}}(1+c_1)+4x^{\frac{6+7c_1}{2+3c_1}}(1+c_1)-x^{\frac{8(1+c_1)}{2+3c_1}}(2+3c_1)}. \tag{63}$$

Using this, we can determine the effective temperature of a lukewarm black hole

$$T_{\mathrm{eff}}=r_c^{-\frac{6+7c_1}{2+3c_1}}\left\{\left[r_c^2x(1+c_1)\left[-4x^{\frac{4(1+c_1)}{2+3c_1}}(1+c_1)-4x^{\frac{12+13c_1}{2+3c_1}}(1+c_1)+x^{\frac{10(1+c_1)}{2+3c_1}}(2+c_1)+x^{\frac{6+7c_1}{2+3c_1}}(2+c_1)+x(2+3c_1)\right.\right.\right.$$
$$\left.+x^{\frac{14(1+c_1)}{2+3c_1}}(2+3c_1)\right]\left[-x^{\frac{2+c_1}{2+3c_1}}(2+c_1)+x^{\frac{6+7c_1}{2+3c_1}}(2+c_1)+x(2+3c_1)-x^{\frac{6+5c_1}{2+3c_1}}(2+3c_1)\right]\Big]\Big/\left[2+3c_1-4x^{\frac{2+c_1}{2+3c_1}}(1+c_1)\right.$$
$$\left.+4x^{\frac{6+7c_1}{2+3c_1}}(1+c_1)-x^{\frac{8(1+c_1)}{2+3c_1}}(2+3c_1)\right]+r_c^2\left[2x^{\frac{8+10c_1}{2+3c_1}}c_1(1+c_1)+2x^{\frac{12+13c_1}{2+3c_1}}c_1(1+c_1)+x^{\frac{6+7c_1}{2+3c_1}}(2+3c_1+c_1^2)\right.$$
$$\left.\left.+x^{\frac{2(7+8c_1)}{2+3c_1}}(2+3c_1+c_1^2)-x^3(2+5c_1+3c_1^2)-x^{\frac{14(1+c_1)}{2+3c_1}}(2+5c_1+3c_1^2)\right]\right\}\Big/\left\{2\pi x\left[x-x^{\frac{4(1+c_1)}{2+3c_1}}\right]^2\right.$$
$$\left.\times\left[-4x^{\frac{2}{2+3c_1}}\left(x+x^{\frac{c_1}{2+3c_1}}\right)(1+c_1)-4x^{1+\frac{2}{2+3c_1}}(1+c_1)F(x)+\left(-1+x^{2+\frac{2}{2+3c_1}}\right)(2+3c_1)F'(x)\right]\right\}. \tag{64}$$

We also note that for the lukewarm black hole, the temperature is

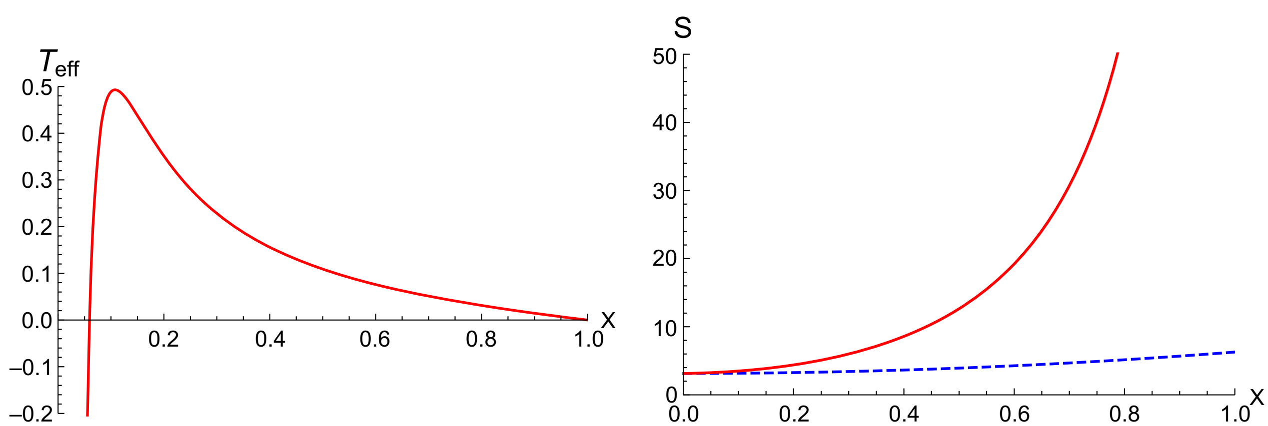
$$T_+ = T_c = -\left\{ r_c^{-\frac{2+c_1}{2+3c_1}}(2+3c_1)\left[2c_1 - 2x^{\frac{8(1+c_1)}{2+3c_1}}c_1 - 4x^2(1+c_1) + 4x^{\frac{2(2+c_1)}{2+3c_1}}(1+c_1) + 2x(2+c_1) - 2x^{\frac{6+5c_1}{2+3c_1}}(2+c_1) - 2x^{\frac{2+c_1}{2+3c_1}}(2+3c_1) + x^{\frac{6+7c_1}{2+3c_1}}(4+6c_1)\right]\right\} \Big/ \left\{8\pi\left(x^{\frac{2+c_1}{2+3c_1}}-1\right)\left[-2-3c_1+4x^{\frac{2+c_1}{2+3c_1}}(1+c_1) - 4x^{\frac{6+7c_1}{2+3c_1}}(1+c_1) + x^{\frac{8(1+c_1)}{2+3c_1}}(2+3c_1)\right]\right\}. \quad (65)$$

Equating temperatures from Eqs. (64) and (65). When $c_1 = -3.5$, we obtain the analytic solution to this equation, which is

$$\begin{aligned} F(x) = \frac{1}{50}(1-x^{\frac{30}{17}})^{\frac{2}{3}}\Bigg[&50D_1 - \frac{1}{1-x^{\frac{3}{17}}-x^{\frac{30}{17}}+x^{\frac{33}{17}}}(1-x^{\frac{30}{17}})^{\frac{1}{3}}(80 - 18x^{\frac{2}{17}} - 97x^{\frac{3}{17}} + 3x^{\frac{5}{17}} + 3x^{\frac{6}{17}} + 3x^{\frac{8}{17}} + 3x^{\frac{9}{17}} + 3x^{\frac{11}{17}} + 3x^{\frac{12}{17}} \\ &+ 3x^{\frac{14}{17}} + 3x^{\frac{15}{17}} + 3x + 3x^{\frac{18}{17}} + 3x^{\frac{20}{17}} + 3x^{\frac{21}{17}} - 47x^{\frac{23}{17}} + 3x^{\frac{24}{17}} + 3x^{\frac{26}{17}} + 3x^{\frac{27}{17}} + 3x^{\frac{29}{17}} - 27x^{\frac{30}{17}} + 21x^{\frac{32}{17}}) \\ &- 18x^{\frac{2}{17}}{}_2F_1\left(\frac{1}{15},\frac{2}{3},\frac{16}{15},x^{\frac{30}{17}}\right) - 17x^{\frac{3}{17}}{}_2F_1\left(\frac{1}{10},\frac{2}{3},\frac{11}{10},x^{\frac{30}{17}}\right) - 15x^{\frac{5}{17}}{}_2F_1\left(\frac{1}{6},\frac{2}{3},\frac{7}{6},x^{\frac{30}{17}}\right) - 14x^{\frac{6}{17}}{}_2F_1\left(\frac{1}{5},\frac{2}{3},\frac{6}{5},x^{\frac{30}{17}}\right) \\ &- 12x^{\frac{8}{17}}{}_2F_1\left(\frac{4}{15},\frac{2}{3},\frac{19}{15},x^{\frac{30}{17}}\right) - 11x^{\frac{9}{17}}{}_2F_1\left(\frac{3}{10},\frac{2}{3},\frac{13}{10},x^{\frac{30}{17}}\right) - 9x^{\frac{11}{17}}{}_2F_1\left(\frac{11}{30},\frac{2}{3},\frac{41}{30},x^{\frac{30}{17}}\right) - 8x^{\frac{12}{17}}{}_2F_1\left(\frac{2}{5},\frac{2}{3},\frac{7}{5},x^{\frac{30}{17}}\right) \\ &- 6x^{\frac{14}{17}}{}_2F_1\left(\frac{7}{15},\frac{2}{3},\frac{22}{15},x^{\frac{30}{17}}\right) - 5x^{\frac{15}{17}}{}_2F_1\left(\frac{1}{2},\frac{2}{3},\frac{3}{2},x^{\frac{30}{17}}\right) - 3x\,{}_2F_1\left(\frac{17}{30},\frac{2}{3},\frac{47}{30},x^{\frac{30}{17}}\right) - 2x^{\frac{18}{17}}{}_2F_1\left(\frac{3}{5},\frac{2}{3},\frac{8}{5},x^{\frac{30}{17}}\right) \\ &+ x^{\frac{21}{17}}{}_2F_1\left(\frac{2}{3},\frac{7}{10},\frac{17}{10},x^{\frac{30}{17}}\right) - \frac{81}{23}x^{\frac{23}{17}}{}_2F_1\left(\frac{2}{3},\frac{23}{30},\frac{53}{30},x^{\frac{30}{17}}\right) + 4x^{\frac{24}{17}}{}_2F_1\left(\frac{2}{3},\frac{4}{5},\frac{9}{5},x^{\frac{30}{17}}\right) - \frac{72}{13}x^{\frac{26}{17}}{}_2F_1\left(\frac{2}{3},\frac{13}{15},\frac{28}{15},x^{\frac{30}{17}}\right) \\ &+ 7x^{\frac{27}{17}}{}_2F_1\left(\frac{2}{3},\frac{9}{10},\frac{19}{10},x^{\frac{30}{17}}\right) - \frac{189}{29}x^{\frac{29}{17}}{}_2F_1\left(\frac{2}{3},\frac{29}{30},\frac{59}{30},x^{\frac{30}{17}}\right)\Bigg]. \end{aligned}$$

We require $D_1 = 1.6$.

In Fig. 5, we depict the total entropy S, effective temperature $T_{\rm eff}$ as functions of x. It is shown that $T_{\rm eff}$ tends to zero as $x \to 1$, namely the charged Nariai limit. Although this result does not agree with that of Bousso and Hawking [74], it is consistent with the entropy. As is depicted in Fig. 5(b), the entropy will diverge as $x \to 1$. Besides, one can see that the entropy is monotonically increasing with the increase of x, while $T_{\rm eff}$ first increases and then decreases. According to the general definition of heat capacity, $C = \frac{\partial M}{\partial T} = T\frac{\partial S}{\partial T}$, these black holes can be thermodynamically stable only in the region of x with the positive temperature and positive slope. This is unexpected. It means that when the black hole horizon and the cosmological horizon are too far apart (small x) or too close together (large x), the black hole cannot be thermodynamically stable.



(a) (b)

FIG. 5. $T_{\rm eff}$ and total entropy S with respect to x. In (a), $T_{\rm eff}$ has a maximum at $x = 0.107593$. In (b), the dashed (blue) curve represents the sum of the two horizon entropies and the solid (red) curve depicts the result in Eq. (59). We set $r_c = 1$ and $Q = 0.2$.

IV. CONCLUSIONS AND DISCUSSIONS

Considering the nonminimal coupling between the graviton and dilaton field, we discussed the charged massive Einstein-dilaton gravity. According to the gravitational, dilaton, and Maxwell field equations obtained by varying the action, we derived the static spherically symmetric solutions of charged dilatonic black holes in four-dimensional spacetime. It should be noted that for a real α, it is required that $m_0^2 c_0 c_1 > 0$ or $m_0^2 c_0 c_1 < -1$, thereby excluding the range $-1 < m_0^2 c_0 c_1 < 0$. Our results reduce to the Schwarzschild case when $c_1 = c_2 = Q = 0$, and reduce to the Reissner-Nordström case when $c_1 = c_2 = 0$.

Later, we have analyzed the singularity of the solution, we give the Ricci and Kretschmann scalars, which suggest that the horizons of black holes are just singularity of coordinate as it should be. What is more, both scalars exhibit the same asymptotic behavior at the origin. When $m_0 = c_0 = 1$, the range $-1 < c_1 < 0$ is excluded. For $c_1 < -1$ or $c_1 > 0$, it can be deduced that there is an essential point located at the origin, and the metric function $f(r)$ tends to infinity as $r \to \infty$. We also show that the black hole solutions can provide one horizon, two horizons (the inner and outer event horizons), three horizons (the inner and outer event horizons, and the cosmological horizon), extreme (Nariai) and naked singularity black holes for the suitably fixed parameters. With dilaton field, the parameter c_1 affects the behavior of the metric function.

Finally, we analyzed the thermodynamics of black holes where $f(r)$ approaches $+\infty$ and $-\infty$, respectively. In the $f(r) \to +\infty$ case, we studied the mass, temperature, and entropy of these charged dilatonic black holes, and checked the first law of black hole thermodynamics. The analysis of the mass suggest that there could exists a minimum of the mass function of black hole horizon. For the temperature of these black holes, we found that it has a maximum positive value $T_{\max}$, distinguishing two sizes of black holes (a smaller one and a larger one) with the same temperature. Moreover, the maximal points of the temperature function correspond to the discontinuous points of the heat capacity. The domain of smaller black hole radii ($r_{\min} < r_h < r_{cri}$) demonstrates stability, while black holes with relatively large horizon radii ($r_h > r_{cri}$) are unstable. In the $f(r) \to -\infty$ case, we have presented the entropy. It is not only the sum of the entropies of black hole horizon and the cosmological horizon, but also with an extra term from the correlation between the two horizons. This idea has twofold advantages. First, without the additional term in the total entropy, the effective temperature is not the same as that of the black hole horizon and the cosmological horizon in the lukewarm case. Second, the method of effective first law of thermodynamics lacks physical explanation or motivation. While taking advantage of the method, we obtain the corrected entropy of the black hole, which may make the method more acceptable.

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