One-loop QCD amplitudes in the Feynman-diagram gauge

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Scattering amplitudes for the massless QCD process, $q\bar{q} \to q'\bar{q}'$, are calculated in the one-loop order in the Feynman-Diagram (FD) gauge, where gluons are quantized on the light cone with opposite direction of the three-momenta. We find non-decoupling of the Faddeev-Popov ghosts and nonconventional UV singularities in dimensional regularization. The known QCD amplitudes with asymptotic freedom are reproduced only after summing propagator and vertex corrections. By quantizing gluons in the Feynman gauge on the FD gauge background, we obtain the one-loop improved FD gauge amplitudes.

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I. INTRODUCTION

Reference [1] proposed a new form of the gauge boson propagator for massless gauge theories like QED and QCD,

$$iG^{\rm FD}_{\mu\nu}(q) = \frac{i}{q^2 + i0}\left(-g_{\mu\nu} + \frac{q_\mu n_\nu(q) + n_\mu(q) q_\nu}{n(q)\cdot q}\right), \qquad (1)$$

where $n^\mu(q)$ is defined as

$$n^\mu(q) = (\mathrm{sgn}(q^0), -q^i/|\vec{q}|). \qquad (2)$$

We use the notation $A^\mu = (A^0, \vec{A}) = (A^0, A^i)$ to separate time and space components of a four-vector. $n^\mu(q)$ is light cone, i.e., $n^\mu(q) n_\mu(q) = 0$. Note that the propagator (1) is not Lorentz covariant.

Using the propagator (1) for the photon and the gluon, it has been shown in Ref. [1] that we can obtain helicity amplitudes that are free from subtle gauge cancellation among interfering Feynman diagrams. This method was later extended [2] to the electroweak theory, where massive gauge bosons are combined with associated Nambu-Goldstone modes forming five-dimensional propagators. It has been found in Refs. [1,2] that the absence of subtle cancellation among interfering Feynman diagrams and the collinear properties of individual diagram are common in the massless [1] and in the massive [2] gauge theories. Because of these common properties,[1] Eq. (1) is named "Feynman-Diagram (FD) gauge" in Ref. [2].

It has later been shown in Ref. [4] that the propagator (1), as well as its generalization to massive gauge bosons [2], can be derived from the gauge fixing term similar to that in the light-cone gauge [7].

In this paper, we study radiative corrections for massless gauge theories in the FD gauge. The rest of this paper is organized as follows. In Sec. II, we show the relevant Feynman rules in the FD gauge for loop calculation. Section III gives details of the one-loop scattering amplitudes for a massless quark scattering process, $q\bar{q} \to q'\bar{q}'$, in the FD gauge. Section IV shows that by quantizing gluons in the Feynman gauge on the FD gauge background, we can obtain one-loop corrected FD gauge amplitudes. Section V summarizes our finding, and some technical details of the loop integrals are given in Appendices A and B.

II. FEYNMAN RULES IN THE FD GAUGE

We work in QCD with massless quarks. The Lagrangian takes the form

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[1]The propagator was called "parton shower gauge" in Ref. [1], because the magnitude of individual Feynman diagram agrees with parton splitting amplitudes [3] in the collinear limit. It was later renamed as Feynman-Diagram gauge in Refs. [2,4] because the term "parton shower gauge" was used in Refs. [5,6] for a specific light-cone gauge.

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \sum_q i\bar{q}_i\gamma^\mu(\partial_\mu\delta_{ij} + igA^a_\mu(T^a)_{ij})q_j + \mathcal{L}_{\rm GF} + \mathcal{L}_{\rm FP}. \tag{3}$$

In this section, we give the forms of the gauge fixing term $\mathcal{L}_{\rm GF}$ and the Faddeev-Popov (FP) ghost term $\mathcal{L}_{\rm FP}$ corresponding to the FD gauge propagator (1).

Following Ref. [4], we consider a gauge fixing

$$\mathcal{L}_{\rm GF} = -\frac{1}{2\xi}(F^a[A])^2, \tag{4}$$

with

$$F^a[A] = \hat{n}^\mu(\partial)A^a_\mu, \tag{5}$$

and the gauge parameter ξ. Here, $\hat{n}^\mu(\partial)$ is a differential operator that may be Lorentz non-covariant and even nonlocal, which was not manifestly written in Ref. [4].

The kinetic term for the gluon is

$$\mathcal{L}_K = \frac{1}{2}A^{a\mu}\left(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu - \frac{1}{\xi}\overleftarrow{\hat{n}}_\mu\hat{n}_\nu\right)A^{a\nu}, \tag{6}$$

with $\overleftarrow{\hat{n}}_\mu = -\hat{n}_\mu$. The equation of motion (EOM) of A, with the source term $J^a_\mu A^{a\mu}$ added, is then

$$\left(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu + \frac{1}{\xi}\hat{n}_\mu\hat{n}_\nu\right)A^{a\nu} = -J^a_\mu. \tag{7}$$

In moving to the momentum space, we set the momentum space representation of $\hat{n}^\mu(\partial)$ as $-in^\mu(q)$ and the EOM (7) gives the gluon propagator,

$$i\delta^{ab}G^{\rm FD}_{\mu\nu}(q)|_\xi = \frac{i\delta^{ab}}{q^2+i0}\left(-g_{\mu\nu} + \frac{q_\mu n_\nu(q) + n_\mu(q)q_\nu}{n(q)\cdot q} - \frac{\xi q^2 q_\mu q_\nu}{(n(q)\cdot q)^2}\right). \tag{8}$$

Hereafter, we set $\xi \to 0$ and obtain

$$i\delta^{ab}G^{\rm FD}_{\mu\nu}(q) = \frac{i\delta^{ab}}{q^2+i0}\left(-g_{\mu\nu} + \frac{q_\mu n_\nu(q) + n_\mu(q)q_\nu}{n(q)\cdot q}\right) \equiv \frac{i\delta^{ab}}{q^2+i0}P^{\rm FD}_{\mu\nu}(q). \tag{9}$$

Equation (9) gives the gluon propagator (1) in the FD gauge. Note that (9) explicitly breaks Lorentz invariance, while keeping space rotational invariance for the light-cone vector of (2).

To calculate loop corrections in the FD gauge, we also need to determine the Lagrangian for the FP ghosts $(c^a, \bar{c}^a)$ associated with the gauge fixing (4) and (5). In the coordinate space, the Lagrangian for the FD ghosts is [8]

$$\begin{aligned}\mathcal{L}_{\rm FP} &= i\bar{c}^a\frac{\delta F^a[A]}{\delta A^b_\mu}(D_\mu c)^b \\ &= i\bar{c}^a\hat{n}^\mu(D_\mu c)^a \\ &= i\bar{c}^a\hat{n}^\mu\partial_\mu c^a - igf^{abc}\bar{c}^a\hat{n}^\mu A^b_\mu c^c,\end{aligned} \tag{10}$$

where $(D_\mu c)^a = \partial_\mu c^a - gf^{abc}A^b_\mu c^c$ is the covariant derivative of the ghost c. The propagator and $c\bar{c}A_\mu$ coupling of the FP ghosts are then given as

$$\langle c^a(q)\bar{c}^b(-q)\rangle = i\delta^{ab}G_{\rm FP}(q) = -\frac{\delta^{ab}}{n(q)\cdot q} = -\frac{\delta^{ab}}{|q^0| + |\vec{q}|}, \tag{11}$$

$$i\Gamma(\bar{c}^a(-p)A^{b\mu}(p-q)c^c(q)) = -igf^{abc}n^\mu(p), \tag{12}$$

respectively. Note that unlike in the light-cone gauge in which n^μ is common for all gluons, the FP ghosts don't decouple from the amplitudes.

III. FOUR-QUARK SCATTERING AMPLITUDES IN THE FD GAUGE

To discuss loop corrections in the FD gauge, we use the massless quark scattering $q\bar{q} \to q'\bar{q}'$ $(q \neq q')$ and calculate the amplitudes with one-loop corrections by gluons, as shown in Fig. 1.

The tree-level amplitudes Fig. 1(a) of the process

$$q_i(p_1) + \bar{q}_j(p_2) \to g^a(q) \to q'_l(p_3) + \bar{q}'_m(p_4), \tag{13}$$

with color indices for quarks (i, j, l, m) and gluon (a), are

$$i\mathcal{M}^{(a)} = -ig^2\bar{v}(p_2)(T^a)_{ji}\gamma^\mu u(p_1)\frac{P^{\rm FD}_{\mu\nu}(q)}{q^2+i0}\bar{u}(p_3)(T^a)_{lm}\gamma^\nu v(p_4). \tag{14}$$

Here, $iP^{\rm FD}_{\mu\nu}(q)/(q^2+i0)$ is the gluon propagator in the FD gauge (9).

Relations $\bar{v}(p_2)\not{q}u(p_1) = \bar{u}(p_3)\not{q}v(p_4) = 0$ follow from the EOM of the quarks. As a consequence, the $n_\mu q_\nu + q_\mu n_\nu$ parts of $P^{\rm FD}_{\mu\nu}$ do not contribute to the amplitude (14), as expected from the gauge independence of the on-shell scattering amplitudes.

We now evaluate the one-loop corrections to the amplitude by gluons. Before showing explicit calculations, we review the structure of the loop corrections. As shown in Fig. 1, they consist of the corrections by gluon self-energies (b, c), quark-quark-gluon vertex corrections (d, e) with wave function corrections of quarks (f), and four-quark box corrections (g, h).

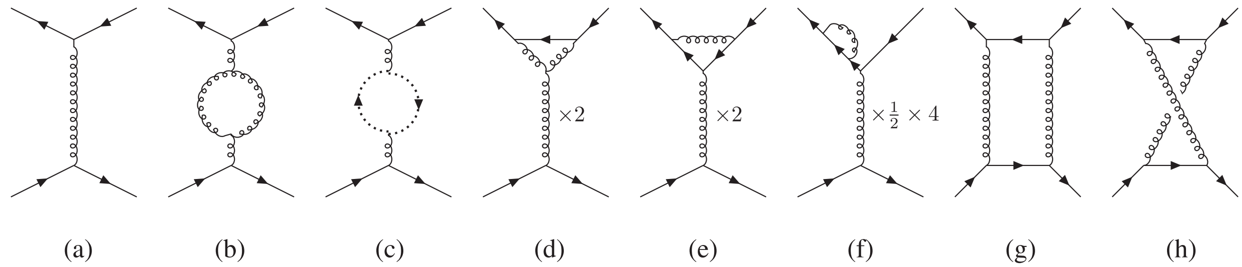

FIG. 1. Feynman diagrams contributing to $q\bar{q} \to q'\bar{q}'$ at the tree (a) and the one-loop order $(b-h)$.

The gluon loop corrections to the amplitude (14) are then

$$i\mathcal{M}(\mathrm{corr}) = i\mathcal{M}^{(b+c)} + i\mathcal{M}^{(d+e+f)} + i\mathcal{M}^{(g+h)}. \tag{15}$$

In terms of the gluon self-energy $i\Pi^{\mu\nu}(q) = i\Pi^{(b)\mu\nu}(q) + i\Pi^{(c)\mu\nu}(q)$, $i\mathcal{M}^{(b+c)}$ is expressed as

$$i\mathcal{M}^{(b+c)} = ig^2 \bar{v}(p_2)(T^a)_{ji}\gamma^\mu u(p_1)\frac{P^{\mathrm{FD}}_{\mu\lambda}(q)}{q^2+i0}\Pi^{\lambda\rho}(q)\frac{P^{\mathrm{FD}}_{\rho\nu}(q)}{q^2+i0}\bar{u}(p_3)(T^a)_{lm}\gamma^\nu v(p_4). \tag{16}$$

Substituting the explicit form of $P^{\mathrm{FD}}_{\mu\nu}$ (9) and EOMs for quarks, the correction is expressed as

$$\begin{aligned} i\mathcal{M}^{(b+c)} = -ig^2 \bar{v}(p_2)(T^a)_{ji}\gamma^\mu u(p_1)\Bigg[&-\frac{1}{q^4}\Pi_{\mu\nu}(q) + \frac{n_\mu(q)}{q^2(n(q)\cdot q)}(q^\lambda \Pi_{\lambda\nu}(q))\frac{1}{q^2} + \frac{1}{q^2}(\Pi_{\mu\rho}(q)q^\rho)\frac{n_\nu(q)}{q^2(n(q)\cdot q)} \\ &- \frac{n_\mu(q)}{q^2(n(q)\cdot q)}(q^\lambda q^\rho \Pi_{\lambda\rho}(q))\frac{n_\nu(q)}{q^2(n(q)\cdot q)}\Bigg]\bar{u}(p_3)(T^a)_{lm}\gamma^\nu v(p_4). \end{aligned} \tag{17}$$

Furthermore, by using the explicit form of $n^\mu(q)$, we have

$$\begin{aligned} i\mathcal{M}^{(b+c)} &= -ig^2 \bar{v}(p_2)(T^a)_{ji}\gamma^\mu u(p_1)\frac{1}{q^4}\bar{u}(p_3)(T^a)_{lm}\gamma^\nu v(p_4) \\ &\quad\times\left[-\Pi_{\mu\nu}(q) + \frac{\mathrm{sgn}(q^0)}{|\vec{q}|}(t_\mu q^\rho \Pi_{\rho\nu}(q) + \Pi_{\mu\sigma}(q)q^\sigma t_\nu) - t_\mu t_\nu \frac{1}{|\vec{q}|^2}q^\rho q^\sigma \Pi_{\rho\sigma}(q)\right], \end{aligned} \tag{18}$$

where $t^\mu = (1,0,0,0)$ is a constant vector. In Eq. (18), the following relation from the quark EOM,

$$\begin{aligned} \frac{n_\mu(q)}{n(q)\cdot q}\bar{v}(p_2)\gamma^\mu u(p_1) &= \frac{1}{|q^0|+|\vec{q}|}\bar{v}(p_2)\left[\mathrm{sgn}(q^0)\gamma^0 + q^i\gamma^i\frac{1}{|\vec{q}|}\right]u(p_1) \\ &= \frac{1}{|q^0|+|\vec{q}|}\bar{v}(p_2)\left[\mathrm{sgn}(q^0)\gamma^0 + q^0\gamma^0\frac{1}{|\vec{q}|}\right]u(p_1) \\ &= \frac{\mathrm{sgn}(q^0)}{|\vec{q}|}\bar{v}(p_2)\gamma^0 u(p_1), \end{aligned} \tag{19}$$

and a similar relation for $\bar{u}(p_3)\gamma^\nu v(p_4)$ are used. As we will see later, contrary to the case in the covariant gauges, $q^\mu \Pi_{\mu\nu}(q)$ in the FD gauge does not vanish in general.

Similarly, $i\mathcal{M}^{(d+e+f)}$ is expressed in terms of the qqg vertex functions $i\Gamma^\mu$ and $i\Gamma^\nu$, which is the sum of the 1PI vertex corrections $(d+e)$ and quark wave function corrections (f), as

$$\begin{aligned} i\mathcal{M}^{(d+e+f)} &= \bar{v}(p_2)(T^a)_{ji} i\Gamma^\mu(-q,p_1,p_2)u(p_1)\frac{iP^{\mathrm{FD}}_{\mu\nu}(q)}{q^2}(-ig)\bar{u}(p_3)(T^a)_{lm}\gamma^\nu v(p_4) \\ &\quad + (-ig)\bar{v}(p_2)(T^a)_{ji}\gamma^\mu u(p_1)\frac{iP^{\mathrm{FD}}_{\mu\nu}(q)}{q^2}\bar{u}(p_3)(T^a)_{lm} i\Gamma^\nu(q,-p_4,-p_3)v(p_4). \end{aligned} \tag{20}$$

By using (19) again, we have

$$i\mathcal{M}^{(d+e+f)} = ig^2\bar{v}(p_2)(T^a)_{ji}\Gamma^\mu(-q,p_1,p_2)u(p_1)\frac{1}{q^2}\left(-g_{\mu\nu}+q_\mu\frac{\mathrm{sgn}(q^0)}{|\vec{q}|}t_\nu\right)\bar{u}(p_3)(T^a)_{lm}\gamma^\nu v(p_4)$$
$$+ ig^2\bar{v}(p_2)(T^a)_{ji}\gamma^\mu u(p_1)\frac{1}{q^2}\left(-g_{\mu\nu}+t_\mu\frac{\mathrm{sgn}(q^0)}{|\vec{q}|}q_\nu\right)\bar{u}(p_3)(T^a)_{lm}\Gamma^\nu(q,-p_4,-p_3)v(p_4). \quad (21)$$

A. UV-divergent parts of the corrections

We now evaluate the UV divergence of each part of the gluon loop correction (15) in the FD gauge.

First, gluon self-energy by gluon loop (b) and by FP ghost loop (c) are

$$i\delta^{ab}\Pi^{(b)}_{\mu\nu}(q) = -\frac{1}{2}f^{acd}f^{bcd}g^2\int\frac{d^Dk}{(2\pi)^D}[(-k+q)_\rho g_{\mu\lambda}+(2k+q)_\mu g_{\lambda\rho}+(-2q-k)_\lambda g_{\mu\rho}]$$
$$\times[(k-q)_\tau g_{\nu\sigma}+(-2k-q)_\nu g_{\sigma\tau}+(2q+k)_\sigma g_{\nu\tau}]\frac{P^{\mathrm{FD}\lambda\sigma}(k)}{k^2}\frac{P^{\mathrm{FD}\rho\tau}(k+q)}{(k+q)^2}, \quad (22)$$

$$i\delta^{ab}\Pi^{(c)}_{\mu\nu}(q) = f^{cad}f^{dbc}g^2\int\frac{d^Dk}{(2\pi)^D}\frac{n_\mu(k)n_\nu(k+q)}{(n(k)\cdot k)(n(k+q)\cdot(k+q))}, \quad (23)$$

respectively. Here, $f^{acd}f^{bcd} = -f^{cad}f^{dbc} = C_A\delta^{ab}$ with $C_A = N_c = 3$. Since we use the dimensional regularization ($D = 4-2\epsilon$), all tadpole contributions with massless fields vanish and are not shown.

Here, we comment on the singularity of the FD gauge propagators (9). As in the covariant gauges, the pole from $1/q^2$ at $q^2 = 0$ should be shifted by the replacement $1/q^2 \to 1/(q^2+i0)$. There is also a singularity from $1/n(q)\cdot q = 1/(|q^0|+|\vec{q}|)$. However, this singularity occurs only at a point $q^\mu = 0$ in the D-dimensional phase space and does not need the $+i0$ prescription.

For calculation, we split the FD gauge gluon propagators $iP^{\mathrm{FD}}_{\mu\nu}(q)/q^2$ in Eq. (22) into two parts, $-ig_{\mu\nu}/q^2$ (“g,” Feynman gauge propagator) and $i(n_\mu(q)q_\nu + q_\mu n_\nu(q))/(n(q)\cdot q)q^2$ (“n”). Equation (22) is then divided as

$$\Pi^{(b)} = \Pi^{(b,gg)} + \Pi^{(b,gn)} + \Pi^{(b,nn)}. \quad (24)$$

The (gg) part, $\Pi^{(b,gg)}_{\mu\nu}(q)$, is the self-energy in the Feynman gauge. As is well known, its UV singular term is [8];

$$i\Pi^{(b,gg)}_{\mu\nu}(q)|_{\mathrm{div}} = -\frac{i}{2}\frac{C_Ag^2}{(4\pi)^2\epsilon}\left(-\frac{19}{6}q^2g_{\mu\nu}+\frac{11}{3}q_\mu q_\nu\right)$$
$$= -\frac{i}{2}\frac{C_Ag^2}{(4\pi)^2\epsilon}\left[\frac{1}{2}(q^0)^2+\frac{19}{6}|\vec{q}|^2, -\frac{11}{3}q^0q^j, \frac{19}{6}q^2\delta^{ij}+\frac{11}{3}q^iq^j\right]. \quad (25)$$

In the second line, we show $i\Pi_{00}$, $i\Pi_{0j}$, and $i\Pi_{ij}$ for later convenience.

We next evaluate $i\Pi^{(b,gn)}_{\mu\nu}$. It contains loop integrals with a factor of $n(k)\cdot k = |k^0|+|\vec{k}|$ in the denominator, such as

$$\int\frac{d^Dk}{(2\pi)^D}\frac{1}{(|k^0|+|\vec{k}|)(k+q)^2}\left(\mathrm{sgn}(k^0), -\frac{k^j}{|\vec{k}|}\right). \quad (26)$$

Here, the momentum integration is to be understood as $d^Dk = d(k^0)d^{D-1}\vec{k}$, namely in $(D-1)$-dimensional space and one-dimensional time.

Since $|k^0|+|\vec{k}|$ is not a polynomial of the loop momentum k^μ, Feynman's formula to combine the denominator of Eq. (26) into the form $((k')^2-C)^n$ does not work. Fortunately, by dimension counting, we find that all the UV divergences of the integrals like Eq. (26) are polynomials of the components of the external momentum q^μ. We therefore differentiate the integrands in (22) and (23) by q to the second order and perform integration of the resulting formulas at $q^\mu = 0$. Details of the integrations are given in Appendix A.

By using the techniques outlined in Appendix A, the (gn) part of the integral (22) is found to give the following UV divergence,

$$i\Pi^{(b,gn)}_{[00,0j,ij]}(q)\Big|_{\rm div} = -\frac{i}{2}\frac{C_A g^2}{(4\pi)^2\epsilon}\left[\frac{20i}{3\pi}(q^0)^2 + \left(2-\frac{76i}{9\pi}\right)|\vec{q}|^2, -\left(2-\frac{16i}{9\pi}\right)q^0q^j,\right.$$
$$\left.\left(2+\frac{20i}{9\pi}\right)(q^0)^2\delta^{ij} + \left(-2-\frac{148i}{45\pi}\right)|\vec{q}|^2\delta^{ij} + \left(2+\frac{64i}{45\pi}\right)q^iq^j\right]. \tag{27}$$

Equation (27) has terms with an extra factor of i/π compared to conventional contributions in the Feynman gauge part (25). They arise from the UV singular integrals with the $1/(n\cdot k)$ factor, which has no on-shell pole.

The (nn) part of the gluon self energy $i\Pi^{(b,nn)}_{\mu\nu}$ and the FP ghost contribution $i\Pi^{(c)}_{\mu\nu}$ are evaluated in the same manner. We find

$$i\Pi^{(b,nn)}_{[00,0j,ij]}(q)\Big|_{\rm div} = -\frac{i}{2}\frac{C_A g^2}{(4\pi)^2\epsilon}\left[-\frac{1}{2}(q^0)^2 + \left(\frac{13}{6}-\frac{8i}{\pi}\right)|\vec{q}|^2, -\left(\frac{1}{3}-\frac{8i}{3\pi}\right)q^0q^j, \left(-\frac{1}{2}+\frac{8i}{3\pi}\right)q^2\delta^{ij} + \left(-1+\frac{8i}{3\pi}\right)q^iq^j\right], \tag{28}$$

and

$$i\Pi^{(c)}_{[00,0j,ij]}(q)\Big|_{\rm div} = -\frac{i}{2}\frac{C_A g^2}{(4\pi)^2\epsilon}\left[-\frac{20i}{3\pi}(q^0)^2 + \frac{4i}{9\pi}|\vec{q}|^2, \frac{8i}{9\pi}q^0q^j, \frac{4i}{9\pi}(q^0)^2\delta^{ij} + \frac{28i}{45\pi}|\vec{q}|^2\delta^{ij} + \frac{56i}{45\pi}q^iq^j\right], \tag{29}$$

respectively. In contrast to the light-cone gauge [7], where n^μ is a constant vector, the FP ghost contribution $i\Pi^{(c)}$ does not vanish. Because the ghost loop in the FD gauge has no on-shell pole, there is no term without a factor of i/π in (29).

Summing Eqs. (25) and (27)–(29), the gluon self energy in the FD gauge is

$$i\Pi^{{\rm FD}(b+c)}_{[00,0j,ij]}(q)|_{\rm div} = -\frac{iC_A g^2}{(4\pi)^2\epsilon}\left[\left(\frac{11}{3}-\frac{8i}{\pi}\right)|\vec{q}|^2, -\left(3-\frac{8i}{3\pi}\right)q^0q^j, \left(\frac{7}{3}+\frac{8i}{3\pi}\right)(q^2\delta^{ij}+q^iq^j)\right], \tag{30}$$

or, equivalently,

$$i\Pi^{{\rm FD}(b+c)}_{\mu\nu}(q)|_{\rm div} = -\frac{iC_A g^2}{(4\pi)^2\epsilon}\left[\left(\frac{7}{3}+\frac{8i}{3\pi}\right)(-q^2 g_{\mu\nu}+q_\mu q_\nu) + \left(\frac{2}{3}-\frac{16i}{3\pi}\right)(q^0(q_\mu t_\nu + t_\mu q_\nu) - 2q^2 t_\mu t_\nu)\right]. \tag{31}$$

We observe that $q^\mu \Pi^{\rm FD}_{\mu\nu}(q)|_{\rm div} \neq 0$, but $q^\mu q^\nu \Pi^{\rm FD}_{\mu\nu}(q)|_{\rm div} = 0$. In fact, $q^\mu q^\nu \Pi^{\rm FD}_{\mu\nu}(q) = 0$ also holds for the UV finite part.

By substituting the self-energy (31) into the $(b+c)$ diagram correction to the amplitude (18), we find

$$i\mathcal{M}^{(b+c)}|_{\rm div} = i\frac{C_A g^4}{(4\pi)^2\epsilon}\bar{v}(p_2)(T^a)_{ji}\gamma^\mu u(p_1)\frac{1}{q^2}\bar{u}(p_3)(T^a)_{lm}\gamma^\nu v(p_4)$$
$$\times\left[\left(-\frac{7}{3}-\frac{8i}{3\pi}\right)(-g_{\mu\nu}+t_\mu t_\nu) + \left(\frac{11}{3}-\frac{8i}{\pi}\right)t_\mu t_\nu + \left(-\frac{4}{3}+\frac{32i}{3\pi}\right)\frac{|q^0|}{|\vec{q}|}t_\mu t_\nu\right]. \tag{32}$$

Next, we calculate the UV-divergent parts of the vertex corrections (d, e) to the $q_i(p_1)\bar{q}_j(p_2) \to g^a(q)$ vertex, as well as the wave function correction (f) of external quarks.

First, the (g, g, q) loop contribution (d) is

$$\bar{v}(p_2)(T^a)_{ji}i\Gamma^{(d)\mu}(-q,p_1,p_2)u(p_1) = ig^3 f^{acd}(T^cT^d)_{ji}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2(k+q)^2(k+p_2)^2}$$
$$\times[(-k+q)^\rho g^{\mu\lambda} + (2k+q)^\mu g^{\lambda\rho} - (k+2q)^\lambda g^{\mu\rho}]$$
$$\times P^{\rm FD}_{\sigma\lambda}(k)P^{\rm FD}_{\rho\tau}(k+q)\bar{v}(p_2)\gamma^\sigma(-\not{k}-\not{p}_2)\gamma^\tau u(p_1). \tag{33}$$

Here, $if^{acd}(T^cT^d)_{ji} = -\frac{1}{2}C_A(T^a)_{ji}$. By dimension counting, the UV-divergent part of (33) should be independent of the external momenta (q, p_1, p_2).

Again, we split the gluon propagators in Eq. (33) into "g" and "n" parts. The Feynman gauge (gg) part is

$$i\Gamma^{(d,gg)\mu}|_{\rm div} = i\frac{C_A g^3}{(4\pi)^2\epsilon}\left(-\frac{3}{2}\gamma^\mu\right). \tag{34}$$

The other parts, (gn) and (nn), are

$$i\Gamma^{(d,gn)[0,i]}|_{\rm div} = i\frac{C_A g^3}{(4\pi)^2\epsilon}\left[\left(\frac{3}{2}-\frac{2i}{\pi}\right)\gamma^0, \left(\frac{3}{2}+\frac{2i}{3\pi}\right)\gamma^i\right], \tag{35}$$

and

$$i\Gamma^{(d,nn)[0,i]}|_{\rm div} = i\frac{C_A g^3}{(4\pi)^2\epsilon}\left[\left(\frac{1}{2}-\frac{2i}{\pi}\right)\gamma^0, \left(-\frac{1}{6}+\frac{2i}{3\pi}\right)\gamma^i\right], \tag{36}$$

respectively. Their summation then gives

$$i\Gamma^{(d)[0,i]}|_{\rm div} = i\frac{C_A g^3}{(4\pi)^2\epsilon}\left[\left(\frac{1}{2}-\frac{4i}{\pi}\right)\gamma^0, \left(-\frac{1}{6}+\frac{4i}{3\pi}\right)\gamma^i\right]. \tag{37}$$

The (q,q,g) loop contribution (e) is given by

$$\begin{aligned}&\bar{v}(p_2)(T^a)_{ji} i\Gamma^{(e)\mu}(-q,p_1,p_2)u(p_1)\\ &\quad = g^3(T^cT^aT^c)_{ji}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2(k+p_1)^2(k-p_2)^2}\\ &\qquad\times P^{\rm FD}_{\nu\rho}(k)\bar{v}(p_2)\gamma^\nu(\not{k}-\not{p}_2)\gamma^\mu(\not{k}+\not{p}_1)\gamma^\rho u(p_1). \end{aligned}\tag{38}$$

Here, $(T^cT^aT^c)_{ji} = (C_F-\frac{1}{2}C_A)(T^a)_{ji}$ with $C_F = (N_c^2-1)/(2N_c) = 4/3$. After splitting the gluon propagator into "g" and "n" parts, we have

$$i\Gamma^{(e,g)\mu}|_{\rm div} = i\frac{g^3}{(4\pi)^2\epsilon}(C_A-2C_F)\left[\frac{1}{2}\gamma^\mu\right], \tag{39a}$$

$$i\Gamma^{(e,n)\mu}|_{\rm div} = i\frac{g^3}{(4\pi)^2\epsilon}(C_A-2C_F)[-\gamma^\mu]. \tag{39b}$$

Note that the "n" part (39b) is Lorentz covariant, unlike the cases of the corrections (b,c,d).

We further include the contribution from the quark wave function correction (f) to the vertex $i\Gamma^{\rm FD}$. The quark self-energy in the FD gauge is

$$i\Sigma_q^{\rm FD}(p_i) = g^2C_F\int\frac{d^Dk}{(2\pi)^D}\frac{\gamma^\nu(\not{k}+\not{p}_i)\gamma^\rho}{k^2(k+p_i)^2}P^{\rm FD}_{\nu\rho}(k). \tag{40}$$

Its UV divergence is, after splitting $P^{\rm FD}_{\mu\nu}$ into $O(g_{\mu\nu})$ and $O(nk)$ terms,

$$(i\Sigma_q^{(g)}(p_i)|_{\rm div}, i\Sigma_q^{(n)}(p_i)|_{\rm div}) = i\frac{g^2C_F}{(4\pi)^2\epsilon}(\not{p}_i, -2\not{p}_i). \tag{41}$$

Then

$$i\Gamma^{(f,g)\mu}|_{\rm div} = i\frac{g^3}{(4\pi)^2\epsilon}C_F[\gamma^\mu], \tag{42}$$

$$i\Gamma^{(f,n)\mu}|_{\rm div} = i\frac{g^3}{(4\pi)^2\epsilon}C_F[-2\gamma^\mu]. \tag{43}$$

They exactly cancel the $O(C_F)$ contributions of the (q,q,g) vertex correction $i\Gamma^{(e)\mu}|_{\rm div}$ ((39a) and (39b)). In total, the UV-divergent qqg vertex correction in the FD gauge is

$$i\Gamma^{(d+e+f)[0,i]}|_{\rm div} = i\frac{g^3C_A}{(4\pi)^2\epsilon}\left[-\frac{4i}{\pi}\gamma^0, \left(-\frac{2}{3}+\frac{4i}{3\pi}\right)\gamma^i\right]. \tag{44}$$

The correction to the amplitude by $i\Gamma^{(d+e+f)}$ for the initial qqg vertex is, by using Eq. (21),

$$\begin{aligned} i\mathcal{M}^{(d+e+f)}_{\rm init}\Big|_{\rm div} &= i\frac{C_Ag^4}{(4\pi)^2\epsilon}\bar{v}(p_2)(T^a)_{ji}\gamma^\mu u(p_1)\frac{1}{q^2}\bar{u}(p_3)(T^a)_{lm}\gamma^\nu v(p_4)\\ &\quad\times\left[\left(-\frac{2}{3}+\frac{4i}{3\pi}\right)(-g_{\mu\nu}+t_\mu t_\nu)+\left(\frac{2}{3}-\frac{16i}{3\pi}\right)\frac{|q^0|}{|\vec{q}|}t_\mu t_\nu+\frac{4i}{\pi}t_\mu t_\nu\right]. \end{aligned}\tag{45}$$

The final $q'q'g$ vertex correction $i\mathcal{M}^{(d+e+f)}_{\rm fin}|_{\rm div}$ is identical to Eq. (45).

Finally, the box corrections $\Delta\mathcal{M}^{(g,h)}$ are, as in the covariant gauges, UV finite.

In total, UV-divergent part of the gluon loop corrections to the amplitude is

$$\begin{aligned} i\mathcal{M}^{\rm FD}({\rm corr})|_{\rm div} &= i\frac{C_Ag^4}{(4\pi)^2\epsilon}\bar{v}(p_2)(T^a)_{ji}\gamma^\mu u(p_1)\frac{1}{q^2}\bar{u}(p_3)(T^a)_{lm}\gamma^\nu v(p_4)\left[-\frac{11}{3}(-g_{\mu\nu}+t_\mu t_\nu)+\frac{11}{3}t_\mu t_\nu\right]\\ &= i\mathcal{M}^{(a)}\times\left(\frac{11}{3}\frac{C_Ag^2}{(4\pi)^2\epsilon}\right). \end{aligned}\tag{46}$$

This result is identical to the one in the covariant gauges and consistent with the beta function [9,10] $\beta(g) = -\frac{11}{3}C_A g^3/(4\pi)^2$ of the gauge coupling g. This result gives an evidence that the FD gauge fixing (5) with the gauge vector (2) in the momentum space gives a consistent procedure for gauge fixing.

B. Transverse and longitudinal contributions

We have seen that loop corrections in the FD gauge have unconventional UV divergences whose coefficients differ from the conventional ones by a factor of $O(i/\pi)$. For better understanding of this type of the loop contributions, we examine the contributions of the transverse and longitudinal parts of the off-shell gluons separately in this subsection.

The FD gauge polarization tensor $P^{\mathrm{FD}\mu\nu}(k)$ is decomposed into the transverse part $P_T^{\mu\nu}$ and the longitudinal part $P_L^{\mu\nu}$, as [1]

$$P^{\mathrm{FD}\mu\nu}(k) = P_T^{\mu\nu}(k) + P_L^{\mu\nu}(k) = \delta_i^\mu\delta_j^\nu\left(\delta^{ij} - \frac{k^i k^j}{|\vec{k}|^2}\right) + k^2\frac{n^\mu(k)n^\nu(k)}{(n(k)\cdot k)^2}. \tag{47}$$

This equation can be verified by using the explicit form of $n^\mu(k)$ (2). The gluon propagator is then decomposed as

$$iG^{\mathrm{FD}\mu\nu}(k) = i\frac{P_T^{\mu\nu}(k)}{k^2 + i0} + i\frac{n^\mu(k)n^\nu(k)}{(n(k)\cdot k)^2}. \tag{48}$$

Since $1/(n(k)\cdot k) = 1/(|k^0| + |\vec{k}|)$ diverges only at a point $k^\mu = 0$, the longitudinal part of the propagator does not correspond to physical states.

In this subsection, we separate the UV-divergent one-loop gluon corrections to the $q\bar{q} \to g(p) \to q'\bar{q}'$ amplitude into transverse (T) and longitudinal (L) internal gluons. For simplicity, we work in the center-of-mass frame of $q\bar{q}$, where $q^\mu = (Q, \vec{0})$ $(Q > 0)$.[2] Note that, in this case, we have $n^\mu(q) = (1, \vec{n})$, where $\vec{n} = -\vec{q}/|\vec{q}|$ is a unit 3D vector whose direction is not determined in the $|\vec{q}| \to 0$ limit. We will find, nevertheless, that this ambiguity of $n^\mu(q)$ does not affect the amplitude (15).

We start from the gluon self-energy. The transverse-transverse (TT), transverse-longitudinal (TL), and longitudinal-longitudinal (LL) parts are, respectively,

$$i\Pi^{(bTT)}_{[00,ij]}(q)\Big|_{\mathrm{div}} = C_A g^2\frac{i}{(4\pi)^2\epsilon}Q^2\left[0, \frac{1}{3}\delta^{ij}\right], \tag{49a}$$

$$i\Pi^{(bTL)}_{[00,ij]}(q)\Big|_{\mathrm{div}} = C_A g^2\frac{i}{(4\pi)^2\epsilon}Q^2\left[0, -\frac{8}{3}\delta^{ij}\right], \tag{49b}$$

$$i\Pi^{(bLL)}_{[00,ij]}(q)\Big|_{\mathrm{div}} = C_A g^2\frac{i}{(4\pi)^2\epsilon}Q^2\left[-\frac{10i}{3\pi}, -\frac{22i}{9\pi}\delta^{ij}\right], \tag{49c}$$

while $i\Pi_{0j}(q) = 0$ by space rotational invariance. It is seen that the unconventional $O(i/\pi)$ term in $i\Pi^{\mathrm{FD}}$ arises from the (LL) part (49c), where the intermediate propagators (two longitudinal gluons) have no cuts. The FP ghost contribution for $q^\mu = (Q, \vec{0})$ is, from Eq. (29),

$$i\Pi^{(c)}_{[00,ij]}(q)\Big|_{\mathrm{div}} = C_A g^2\frac{i}{(4\pi)^2\epsilon}Q^2\left[\frac{10i}{3\pi}, -\frac{2i}{9\pi}\delta^{ij}\right]. \tag{50}$$

In total, the gluon self-energy is

$$i\Pi^{(b+c)}_{[00,ij]}(q)\Big|_{\mathrm{div}} = C_A g^2\frac{i}{(4\pi)^2\epsilon}Q^2\left[0, \left(-\frac{7}{3} - \frac{8i}{3\pi}\right)\delta^{ij}\right]. \tag{51}$$

This result is consistent with the result (30) for general q^μ, as it must be. Since $q^\mu = (Q, \vec{0})$ here, $q^\lambda\Pi^{(b+c)}_{\mu\lambda}(q) = 0$ holds, and the $n(q)$-dependent contributions in the correction (17) to the scattering amplitudes vanish.

The vertex correction (d) is, as for the gluon self-energy (b), decomposed into (TT), (TL), and (LL) parts as

$$i\Gamma^{(dTT)\mu}\Big|_{\mathrm{div}} = i\frac{C_A g^3}{(4\pi)^2\epsilon}\left(-\frac{1}{2}\gamma^\mu\right), \tag{52a}$$

$$i\Gamma^{(dTL)[0,i]}\Big|_{\mathrm{div}} = i\frac{C_A g^3}{(4\pi)^2\epsilon}\left[0, \frac{2}{3}\gamma^i\right], \tag{52b}$$

$$i\Gamma^{(dLL)[0,i]}\Big|_{\mathrm{div}} = i\frac{C_A g^3}{(4\pi)^2\epsilon}\left[\left(1 - \frac{4i}{\pi}\right)\gamma^0, \left(-\frac{1}{3} + \frac{4i}{3\pi}\right)\gamma^i\right]. \tag{52c}$$

The unconventional $O(i/\pi)$ term appears only in the (LL) part with two unphysical propagators, as in the case of the gluon self-energy correction (b) given in Eqs. (49a)–(49c).

The vertex correction (e) is decomposed by separating the gluon propagator in the loop, as

$$i\Gamma^{(eT)[0,i]}\Big|_{\mathrm{div}} = i(C_A - 2C_F)\frac{g^3}{(4\pi)^2\epsilon}\left[\frac{1}{2}\gamma^0, -\frac{1}{6}\gamma^i\right], \tag{53a}$$

$$i\Gamma^{(eL)[0,i]}\Big|_{\mathrm{div}} = i(C_A - 2C_F)\frac{g^3}{(4\pi)^2\epsilon}\left[-\gamma^0, -\frac{1}{3}\gamma^i\right], \tag{53b}$$

giving

$$i\Gamma^{(e)\mu}\Big|_{\mathrm{div}} = i(C_A - 2C_F)\frac{g^3}{(4\pi)^2\epsilon}\left(-\frac{1}{2}\gamma^\mu\right). \tag{54}$$

There are no $O(i/\pi)$ terms in Eqs. (53a) and (53b). Likewise, the quark self-energy is decomposed as

$$i\Sigma_q^{(T)}(p)\Big|_{\mathrm{div}} = i\frac{g^2 C_F}{(4\pi)^2\epsilon}\left[p^0\gamma^0 + \frac{1}{3}p^i\gamma^i\right], \tag{55a}$$

$$i\Sigma_q^{(L)}(p)\Big|_{\mathrm{div}} = i\frac{g^2 C_F}{(4\pi)^2\epsilon}\left[-2p^0\gamma^0 + \frac{2}{3}p^i\gamma^i\right]. \tag{55b}$$

[2] The case of general q^μ is briefly discussed in Appendix B.

Their sum

$$i\Sigma_q(p)|_{\rm div} = i\frac{g^2 C_F}{(4\pi)^2\epsilon}[-\not p] \tag{56}$$

contributes to the vertex correction term (f) as

$$i\Gamma^{(f)\mu}|_{\rm div} = \frac{iC_F g^3}{(4\pi)^2\epsilon}[-\gamma^\mu]. \tag{57}$$

Equation (57) cancels the $O(C_F)$ terms of $i\Gamma^{(e)}$ (54). It is worth nothing that the sum of T and L components of the FD gauge propagator gives sensible correction to the vertex corrections (e) and the quark self-energy correction in (f). The total vertex correction $(d+e+f)$ is

$$i\Gamma^{(d+e+f)[0,i]}|_{\rm div} = i\frac{g^3 C_A}{(4\pi)^2\epsilon}\left[-\frac{4i}{\pi}\gamma^0, \left(-\frac{2}{3}+\frac{4i}{3\pi}\right)\gamma^i\right], \tag{58}$$

which agrees with the result (44).

In calculating corrections to the scattering amplitude, we cannot use Eq. (19) since $|\vec{q}| = 0$. Instead, by using $\bar{v}(p_2)\gamma^0 u(p_1) = \bar{u}(p_3)\gamma^0 v(p_4) = 0$ from the quark EOM, $q^\mu \Pi_{\mu\nu}(q) = 0$, and $q_\mu \Gamma^\mu \propto \gamma^0$, we find

$$i\mathcal{M}^{(b+c)}\Big|_{\rm div} = i\frac{C_A g^4}{(4\pi)^2\epsilon}\bar{v}(p_2)(T^a)_{ji}\gamma^i u(p_1)\frac{1}{Q^2}\bar{u}(p_3)(T^a)_{lm}\gamma^i v(p_4) \times \left(-\frac{7}{3}-\frac{8i}{3\pi}\right), \tag{59}$$

from the gluon self-energy, and

$$i\mathcal{M}^{(d+e+f)}\Big|_{\rm div} = i\frac{C_A g^4}{(4\pi)^2\epsilon}\bar{v}(p_2)(T^a)_{ji}\gamma^j u(p_1)\frac{1}{Q^2}\bar{u}(p_3)(T^a)_{lm}\gamma^j v(p_4) \times \left(-\frac{4}{3}+\frac{8i}{3\pi}\right), \tag{60}$$

from the initial and final vertex corrections, respectively. Both (59) and (60) are independent of $n^\mu(q)$, especially of its undetermined space components $n^i(q)$. The UV-divergent part of the gluon loop corrections to the amplitude is, in total,

$$\begin{aligned} i\mathcal{M}^{\rm FD}({\rm corr})|_{\rm div} &= i\frac{C_A g^4}{(4\pi)^2\epsilon}\bar{v}(p_2)(T^a)_{ji}\gamma^j u(p_1)\frac{1}{Q^2}\bar{u}(p_3)(T^a)_{lm}\gamma^j v(p_4) \times \left(-\frac{11}{3}\right) \\ &= i\mathcal{M}^{(a)} \times \left(\frac{11}{3}\frac{C_A g^2}{(4\pi)^2\epsilon}\right). \end{aligned} \tag{61}$$

This result is again identical to the one in the covariant gauges for $q^\mu = (Q, \vec{0})$.

C. Equivalence of the amplitudes in the FD and Feynman gauges

Up to now, we have only considered the UV-divergent parts of the gluon loop corrections. However, on-shell amplitudes in gauge theories should be independent of the gauge fixing methods. In this subsection, we show how all the n-dependent terms of the loop correction (15) to the $q\bar{q} \to g \to q'\bar{q}'$ process in the FD gauge cancel among each other, including finite parts, to leave the amplitudes the same as in the Feynman gauge. Here, we work on the level of the integrands, without explicit evaluation of loop integration.

We start from the box diagrams (g, h) in Fig. 1. The contribution from (g) is

$$\begin{aligned} i\mathcal{M}^{(g)} &= g^4(T^aT^b)_{ji}(T^bT^a)_{lm}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2(k+q)^2(k+p_2)^2(k+p_4)^2} \\ &\quad \times \bar{v}(p_2)\gamma^\mu(-\not k-\not p_2)\gamma^\lambda u(p_1)\cdot\bar{u}(p_3)\gamma^\sigma(-\not k-\not p_4)\gamma^\nu v(p_4)P^{\rm FD}_{\mu\nu}(k)P^{\rm FD}_{\lambda\sigma}(k+q). \end{aligned} \tag{62}$$

Equation (62) is UV finite and has not been discussed in the previous subsections.

Now we focus on the n-dependent parts of Eq. (62), which give the difference between the Feynman and the FD gauges. It is seen that the gluon momenta in the n-dependent parts of the gluon propagators cancel the attached quark propagators, or "pinch," reducing the kinematic structure to that of the vertex or gluon self-energy contributions [11,12]. For example, $k_\mu n_\nu(k)$ part of $P^{\rm FD}_{\mu\nu}(k)$ in Eq. (62) reduces the integrand as, by using the EOMs for external quarks,

$$\begin{aligned} &\bar{v}(p_2)\gamma^\mu \frac{-\not{k}-\not{p}_2}{(k+p_2)^2}\gamma^\lambda u(p_1)\cdot(k_\mu n_\nu(k))\cdot\bar{u}(p_3)\gamma^\sigma\frac{-\not{k}-\not{p}_4}{(k+p_4)^2}\gamma^\nu v(p_4) \\ &= \bar{v}(p_2)\not{k}\frac{-\not{k}-\not{p}_2}{(k+p_2)^2}\gamma^\lambda u(p_1)\cdot\bar{u}(p_3)\gamma^\sigma\frac{-\not{k}-\not{p}_4}{(k+p_4)^2}\not{n}(k)v(p_4) \\ &= -\bar{v}(p_2)\gamma^\lambda u(p_1)\cdot\bar{u}(p_3)\gamma^\sigma\frac{-\not{k}-\not{p}_4}{(k+p_4)^2}\not{n}(k)v(p_4), \end{aligned} \tag{63}$$

times $P^{\rm FD}_{\lambda\sigma}(k+q)/[k^2(n(k)\cdot k)(k+q)^2]$. The last line of Eq. (63) is independent of p_2, giving a contribution with the kinematic structure of the vertex correction to the final $q'q'g$ coupling.

After successively applying the “pinch” method, the (gn) and (nn) parts of the box contributions $i\mathcal{M}^{(g+h)}$ can be expressed as

$$i\mathcal{M}^{(g+h,gn+nn)} = i\mathcal{M}_1^{\rm box} + i\mathcal{M}_2^{\rm box} + i\mathcal{M}_3^{\rm box}, \tag{64}$$

where

$$\begin{aligned} i\mathcal{M}_1^{\rm box} = &-\frac{1}{2}C_A g^4 (T^a)_{ji}(T^a)_{lm}\bar{u}(p_3)\gamma_\mu v(p_4)\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2(k+q)^2(k+p_2)^2} \\ &\times\bar{v}(p_2)\left\{\frac{\gamma^\mu(\not{k}+\not{p}_2)\not{n}(k+q)}{n(k+q)\cdot(k+q)}+\frac{\not{n}(k)(\not{k}+\not{p}_2)\gamma^\mu}{n(k)\cdot k}-\frac{k^\mu\not{n}(k)(\not{k}+\not{p}_2)\not{n}(k+q)}{(n(k)\cdot k)(n(k+q)\cdot(k+q))}\right\}u(p_1), \end{aligned} \tag{65a}$$

$$\begin{aligned} i\mathcal{M}_2^{\rm box} = &-\frac{1}{2}C_A g^4 (T^a)_{ji}(T^a)_{lm}\bar{v}(p_2)\gamma_\mu u(p_1)\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2(k+q)^2(k+p_3)^2} \\ &\times\bar{u}(p_3)\left\{\frac{\gamma^\mu(\not{k}+\not{p}_3)\not{n}(k+q)}{n(k+q)\cdot(k+q)}+\frac{\not{n}(k)(\not{k}+\not{p}_3)\gamma^\mu}{n(k)\cdot k}-\frac{k^\mu\not{n}(k)(\not{k}+\not{p}_3)\not{n}(k+q)}{(n(k)\cdot k)(n(k+q)\cdot(k+q))}\right\}v(p_4), \end{aligned} \tag{65b}$$

$$i\mathcal{M}_3^{\rm box} = \frac{1}{2}C_A g^4 (T^a)_{ji}(T^a)_{lm}\bar{v}(p_2)\gamma^\mu u(p_1)\cdot\bar{u}(p_3)\gamma^\nu v(p_4)\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2(k+q)^2}\frac{n_\mu(k+q)n_\nu(k)+n_\mu(k)n_\nu(k+q)}{(n(k)\cdot k)(n(k+q)\cdot(k+q))}. \tag{65c}$$

Examining the dependence of $i\mathcal{M}^{\rm box}_{1\text{–}3}$ in (65a)–(65c) on the external momenta q and p_i ($i = 1$ to 4), we find that these three parts kinematically behave as the corrections on the initial qqg vertex, on the final $q'q'g$, and on the gluon self-energy, respectively.

Next, we examine the n-dependent parts of the vertex correction contributions $\mathcal{M}^{(d+e+f)}$, coming from the gluon propagators in the initial qqg and the final $q'q'g$ vertex functions Γ^μ, and also the $n(q)$ dependence coming from the FD gauge propagator $iP^{\rm FD}_{\mu\nu}(q)$ in (20).

The (gn, nn) parts of the vertex function $i\Gamma^{(d)\mu}$ for the initial qqg vertex are written as, after applying the EOMs for external quarks,

$$i\Gamma^{(d,gn+nn)\mu} = i\Gamma_1^{(d)\mu} + i\Gamma_2^{(d)\mu} + i\Gamma_3^{(d)\mu}, \tag{66}$$

where

$$\begin{aligned} i\Gamma_1^{(d)\mu} = &-\frac{1}{2}C_A g^3\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2(k+q)^2(k+p_2)^2}\left[\frac{\not{n}(k)(\not{k}+\not{p}_2)(q^\mu\not{k}+q^2\gamma^\mu)}{n(k)\cdot k}+\frac{(-q^\mu(\not{k}+\not{q})+q^2\gamma^\mu)(\not{k}+\not{p}_2)\not{n}(k+q)}{n(k+q)\cdot(k+q)}\right. \\ &\left.+\frac{-k^\mu q^2+q^\mu(q\cdot k)}{(n(k)\cdot k)(n(k+q)\cdot(k+q))}\not{n}(k)(\not{k}+\not{p}_2)\not{n}(k+q)\right], \end{aligned} \tag{67a}$$

$$i\Gamma_2^{(d)\mu} = -\frac{1}{2}C_A g^3\int\frac{d^Dk}{(2\pi)^D}\left[-\frac{\not{n}(k)(\not{k}+\not{p}_2)\gamma^\mu}{k^2(n(k)\cdot k)(k+p_2)^2}-\frac{\gamma^\mu(\not{k}+\not{p}_2)\not{n}(k+q)}{(k+q)^2(n(k+q)\cdot(k+q))(k+p_2)^2}\right], \tag{67b}$$

$$i\Gamma_3^{(d)\mu} = -\frac{1}{2}C_A g^3 \int \frac{d^D k}{(2\pi)^D}\frac{1}{k^2(k+q)^2}\left[\frac{1}{n(k)\cdot k}(-n(k)\cdot(2q+k)\gamma^\mu - n^\mu(k)\not{k} + (3k+q)^\mu \not{n}(k))\right.$$
$$+\frac{1}{n(k+q)\cdot(k+q)}(-n(k+q)\cdot(k-q)\gamma^\mu - n^\mu(k+q)\not{k} + (3k+2q)^\mu \not{n}(k+q))$$
$$+\frac{1}{(n(k)\cdot k)(n(k+q)\cdot(k+q))}\{((k^2-q^2)n^\mu(k) - k^\mu(n(k)\cdot k) + q^\mu(n(k)\cdot q))\not{n}(k+q)$$
$$+((k^2+2q\cdot k)n^\mu(k+q) - k^\mu(n(k+q)\cdot(k+q)) - q^\mu(n(k+q)\cdot k))\not{n}(k)$$
$$\left.+(n^\mu(k)(k-q)\cdot n(k+q) + n^\mu(k+q)(2q+k)\cdot n(k) - (2k+q)^\mu n(k)\cdot n(k+q))\not{k}\}\right]. \tag{67c}$$

Note that the integral (67a) depends on both q and p_2, the first term in (67b) depends only on p_2, whereas the second term depends only on p_1, after transforming $k \to k+q$. The integrals in (67c) depend only on q.

The $O(n)$ contribution from the vertex function $i\Gamma^{(e)\mu}$ for the initial qqg vertex is

$$i\Gamma^{(e,n)\mu} = \left(C_F - \frac{1}{2}C_A\right)g^3 \int \frac{d^D k}{(2\pi)^D}\frac{1}{k^2(n(k)\cdot k)}\left[\frac{\gamma^\mu(\not{k}+\not{p}_1)\not{n}(k)}{(k+p_1)^2} + \frac{\not{n}(k)(\not{k}+\not{p}_2)\gamma^\mu}{(k+p_2)^2}\right]. \tag{68}$$

From the $O(n)$ part of the quark self-energy

$$i\Sigma_q^{(n)}(p_i) = C_F g^2 \int \frac{d^D k}{(2\pi)^D}\frac{-\not{p}_i(\not{k}+\not{p}_i)\not{n}(k) - \not{n}(k)(\not{k}+\not{p}_i)\not{p}_i}{k^2(n(k)\cdot k)(k+p_i)^2}, \tag{69}$$

we obtain

$$i\Gamma^{(f,n)\mu} = -C_F g^3 \int \frac{d^D k}{(2\pi)^D}\frac{1}{k^2(n(k)\cdot k)}\left[\frac{\gamma^\mu(\not{k}+\not{p}_1)\not{n}(k)}{(k+p_1)^2} + \frac{\not{n}(k)(\not{k}+\not{p}_2)\gamma^\mu}{(k+p_2)^2}\right], \tag{70}$$

for the initial qqg vertex. The $O(C_F)$ part of (68) is exactly canceled by the quark wave function correction (70). The remaining $O(C_A)$ part of (68) cancels $i\Gamma_2^{(d)\mu}$ in Eq. (67b), after momentum transformation $k \to -k-q$ in some terms.

In the remaining parts of Eq. (66), only $i\Gamma_1^{(d)\mu}$ (67a) has p_2 dependence. Its contribution to the amplitude is, from Eq. (20),

$$i\mathcal{M}^{\text{vert}}_{init,1} = \bar{v}(p_2)(T^a)_{ji}i\Gamma_1^{(d)\mu}u(p_1)\frac{i}{q^2}\left(-g_{\mu\nu} + \frac{q_\mu n_\nu(q)}{n(q)\cdot q}\right)(-ig)\bar{u}(p_3)(T^a)_{lm}\gamma^\nu v(p_4)$$
$$= i\mathcal{M}^{\text{vert}}_{init,11} + i\mathcal{M}^{\text{vert}}_{init,12}, \tag{71}$$

where

$$i\mathcal{M}^{\text{vert}}_{init,11} = -\frac{1}{2}C_A g^4 (T^a)_{ji}(T^a)_{lm}\bar{u}(p_3)\gamma_\mu v(p_4)\int \frac{d^D k}{(2\pi)^D}\frac{1}{k^2(k+q)^2(k+p_2)^2}$$
$$\times \bar{v}(p_2)\left[-\frac{\not{n}(k)(\not{k}+\not{p}_2)\gamma^\mu}{n(k)\cdot k} - \frac{\gamma^\mu(\not{k}+\not{p}_2)\not{n}(k+q)}{n(k+q)\cdot(k+q)} + \frac{k^\mu \not{n}(k)(\not{k}+\not{p}_2)\not{n}(k+q)}{(n(k)\cdot k)(n(k+q)\cdot(k+q))}\right]u(p_1), \tag{72a}$$

$$i\mathcal{M}^{\text{vert}}_{init,12} = -\frac{1}{2}C_A g^4 (T^a)_{ji}(T^a)_{lm}\bar{u}(p_3)\not{n}(q)v(p_4)\int \frac{d^D k}{(2\pi)^D}\frac{1}{k^2(k+q)^2}\left[\frac{\bar{v}(p_2)\not{n}(k)u(p_1)}{n(k)\cdot k} + \frac{-\bar{v}(p_2)\not{n}(k+q)u(p_1)}{n(k+q)\cdot(k+q)}\right]. \tag{72b}$$

$i\mathcal{M}^{\text{vert}}_{\text{init},11}$ in (72a) cancels $i\mathcal{M}^{\text{box}}_1$ in (65a), while $i\mathcal{M}^{\text{vert}}_{\text{init},12}$ in Eq. (72b) has no p_2 dependence in the loops and behaves as the gluon self-energy correction.

In the same manner, the n-dependent part of the correction to the final $q'q'g$ vertex cancels $i\mathcal{M}^{\text{box}}_2$ in (65b), leaving only the gluon self-energy-like correction, which we denote as $i\mathcal{M}^{\text{vert}}_{fin,12}$ Also, $i\Gamma_3^{(d)\mu}$ (67c) on the initial and final qqg vertices give self-energy-like contributions to the amplitude, which we denote as $i\mathcal{M}^{\text{vert}}_3$.

Note that the Feynman gauge part of the vertex function, $i\Gamma^{(d+e+f,g)\mu}$, does not give $n(q)$-dependent contribution because of the relation $q_\mu \Gamma^{(d+e+f,g)\mu}(q) = 0$ for the on-shell external quarks.

Therefore, all the remaining n-dependent box/vertex correction parts of the amplitude, $i\mathcal{M}_3^{\text{box}}$ (65c), $i\mathcal{M}_{\text{init},12}^{\text{vert}}$ (72b), $i\mathcal{M}_{\text{fin},12}^{\text{vert}}$, and $i\mathcal{M}_3^{\text{vert}}$ show momentum dependence of that of the gluon self-energy contributions. By lengthy but straightforward calculation, it can be explicitly checked that they exactly cancel the n-dependent part of $i\mathcal{M}^{(b)}$ and the difference of the FP ghost loop contribution $i\mathcal{M}^{(c)}$ between the FD and Feynman gauges.

Summing up, all the n-dependent terms in the scattering amplitudes for the process $q\bar{q} \to q'\bar{q}'$ cancel out exactly, and hence, the FD gauge amplitudes agree exactly with those of the Feynman gauge in the one-loop order.

IV. USE OF BACKGROUND-FIELD GAUGE FIXING

In the preceding section, we have seen that loop integrals in the FD gauge have UV-divergent parts including terms with an unconventional i/π factor. These terms eventually cancel out in the total amplitudes. Moreover, the calculation of the UV-finite parts is even more difficult. These observations suggest that the FD gauge might be, although very useful at the tree level, not suitable for loop calculation.

Here, we introduce an alternative method to include loop correction to the FD gauge amplitudes: the background-field gauge fixing method [13–16], which may avoid the difficulties of the FD gauge loops while keeping its advantages at the tree level, as explained below.

In the background-field gauge, the gluon field A_μ^a is expressed as a sum of the classical field $\tilde{A}_\mu^a$ and the quantum field $\hat{A}_\mu^a$ as $A_\mu^a \to \tilde{A}_\mu^a + \hat{A}_\mu^a$ and perform path integrals over quantum $\hat{A}_\mu^a$ around the background $\tilde{A}_\mu^a$. The effective action $\tilde{\Gamma}[\tilde{A}] \equiv \Gamma[\hat{A} = 0, \tilde{A}]$ is then calculated from 1PI diagrams where all internal propagators are those of quantum fields, while all external fields are classical ones.

In the calculation of $\Gamma[\tilde{A}]$, we need to fix the gauge only for quantum gauge fields. On the other hand, the gauge fixing for $\tilde{A}$ is only necessary to give the propagator for $\tilde{A}$ in constructing scattering amplitudes from the effective action. Therefore, no theoretical problem arises by adopting different gauge fixing methods for classical and quantum gauge fields.

The background-field gauge method adopts the following function to fix the gauge for the quantum field $\hat{A}$:

$$\tilde{F}^a[\hat{A}, \tilde{A}] = (\tilde{D}^\mu \hat{A}_\mu)^a = \partial^\mu \hat{A}_\mu^a - g f^{abc} \tilde{A}^{b\mu} \hat{A}_\mu^c, \quad (73)$$

with the gauge fixing term

$$\mathcal{L}_{\text{GF,BFG}}[\hat{A}, \tilde{A}] = -\frac{1}{2\xi_Q} (\tilde{F}^a[\hat{A}, \tilde{A}])^2, \quad (74)$$

and the corresponding FP ghost Lagrangian

$$\mathcal{L}_{\text{FP,BFG}}[\hat{A}, \tilde{A}] = i\bar{c}^a (\tilde{D}^\mu D_\mu c)^a. \quad (75)$$

This gauge fixing preserves invariance under the "classical" gauge transformation,

$$\tilde{\delta}\hat{A}_\mu^a = -g f^{abc} \omega^b \hat{A}_\mu^c, \quad \tilde{\delta}\tilde{A}_\mu^a = -g f^{abc} \omega^b \tilde{A}_\mu^c - \partial_\mu \omega^a, \quad (76)$$

where $\omega^a(x)$ are infinitesimal phases, but breaks invariance under the "quantum" gauge transformation,

$$\hat{\delta}\hat{A}_\mu^a = -g f^{abc} \omega^b (\tilde{A}_\mu^c + \hat{A}_\mu^c) - \partial_\mu \omega^a, \qquad \hat{\delta}\tilde{A}_\mu^a = 0. \quad (77)$$

As a result, the effective action $\tilde{\Gamma}[\tilde{A}]$ is manifestly invariant under the classical gauge transformation (76). In particular, the gluon self-energy $\tilde{\Pi}_{\mu\nu}(q)$ and $q\bar{q}g$ vertex function $\tilde{\Gamma}^\mu(q, p_1, p_2)$ for on-shell quarks satisfy $q^\mu \tilde{\Pi}_{\mu\nu}(q) = 0$ and $q_\mu \tilde{\Gamma}^\mu(q, p_1, p_2) = 0$ for general q. Furthermore, since calculation of $\tilde{\Gamma}[\tilde{A}]$ is manifestly Lorentz covariant, for an arbitrary ξ_Q, we may express the self-energy as

$$\tilde{\Pi}_{\mu\nu}(q) = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) \tilde{\Pi}_T(q^2). \quad (78)$$

It is then clear that, if $\tilde{\Pi}_{\mu\nu}$ and $\tilde{\Gamma}^\mu$ are used in place of $\Pi_{\mu\nu}$ and Γ^μ, Eqs. (18) and (21) do not depend on whether the Feynman gauge or FD gauge is used for the propagator $iP_{\mu\nu}(q)/q^2$ connecting the 1PI amplitudes.

The one-loop gluon contributions to the gluon self-energy $\tilde{\Pi}_{\mu\nu}$ and the qqg vertex function $\tilde{\Gamma}^\mu$ in the background-field gauge are given in Refs. [13–15]. Their UV divergences are

$$i\tilde{\Pi}_{\mu\nu}^{(b)}(q)\Big|_{\text{div}} = i\frac{C_A g^2}{(4\pi)^2 \epsilon} (-q^2 g_{\mu\nu} + q_\mu q_\nu) \left(-\frac{10}{3}\right), \quad (79)$$

$$i\tilde{\Pi}_{\mu\nu}^{(c)}(q)\Big|_{\text{div}} = i\frac{C_A g^2}{(4\pi)^2 \epsilon} (-q^2 g_{\mu\nu} + q_\mu q_\nu) \left(-\frac{1}{3}\right), \quad (80)$$

and

$$i\tilde{\Gamma}^{(d)\mu}\Big|_{\text{div}} = i\frac{C_A g^3}{(4\pi)^2 \epsilon} \gamma^\mu \left(-\frac{\xi_Q}{2}\right), \quad (81)$$

$$i\tilde{\Gamma}^{(e+f)\mu}\Big|_{\text{div}} = i\frac{C_A g^3}{(4\pi)^2 \epsilon} \gamma^\mu \left(+\frac{\xi_Q}{2}\right), \quad (82)$$

respectively. Note that the UV divergence (79) is independent of the gauge parameter ξ_Q. We also note that the total

qqg vertex function is UV finite and that the renormalization of the gauge coupling is entirely given by the gluon self-energy [15].

We finally comment on the resummation of the gluon self-energy contribution. In the case where the gluon self-energy takes the form (78), we may resum its contributions to the gluon propagator in the FD gauge by the replacement

$$i\frac{P^{\mathrm{FD}}_{\mu\nu}(q)}{q^2} \to i\frac{P^{\mathrm{FD}}_{\mu\nu}(q)}{q^2+\tilde{\Pi}_T(q^2)}. \tag{83}$$

This is proved by using the relation

$$\begin{aligned} &i\frac{P^{\mathrm{FD}}_{\mu\rho}(q)}{q^2}\left(-g^{\rho\sigma}+\frac{q^\rho q^\sigma}{q^2}\right)i\tilde{\Pi}_T(q^2)i\frac{P^{\mathrm{FD}}_{\sigma\nu}(q)}{q^2} \\ &\quad = -i\frac{P^{\mathrm{FD}}_{\mu\nu}(q)}{q^4}\tilde{\Pi}_T(q^2). \end{aligned} \tag{84}$$

Because the self-energy correction $\tilde{\Pi}_T(q^2)$ is the only UV divergent 1PI amplitudes at one-loop, giving the beta function of g, the identity (83) may pave the way to improve the tree-level amplitudes in the FD gauge, given, e.g., in Ref. [1], simply by replacing the gauge couplings by the running couplings.

V. SUMMARY

We have studied radiative corrections in the Feynman-Diagram (FD) gauge [1,2,4], where the gauge boson is quantized along the light cone facing the opposite of its three momentum, Eq. (2). We have calculated the QCD scattering amplitudes for the process $q\bar{q}\to q'\bar{q}'$ at one-loop level, and obtained the following results:

(i) The FP ghosts do not decouple from the scattering amplitudes because the light-cone vector in the FD gauge depends on the three momentum of gluons.
(ii) Loop integrals cannot be done by conventional methods because of the nonanalyticity of the integrand.
(iii) UV singularities with a factor of i/π times the conventional ones appear from the $1/(n(k)\cdot k)$ factor, which does not have a pole in the FD gauge.
(iv) When the FD gauge propagators are expressed as the sum of the transverse (T) and the longitudinal (L) components, all the nonconventional UV singularities appear in the LL combinations of the two virtual gluons in the $q\bar{q}$ rest frame.
(v) All the nonconventional UV singularities cancel in the scattering amplitudes when we sum over terms in the gluon and ghost loop contribution to the propagator corrections, as well as those in the initial qqg and the final $q'q'g$ vertex corrections, reproducing the known QCD beta function.
(vi) We have shown that the finite part of the radiative corrections is identical to that of the Feynman gauge because all the terms which depend on the light-cone vector $n^\mu(q)$ cancel out among two-, three-, and four-point corrections.

Summing up, we have reproduced the known QCD scattering amplitudes for the process $q\bar{q}\to q'\bar{q}'$ at the one-loop level in the FD gauge. This has been proven by showing cancellation of all UV singularities and the finite correction terms, which depend on the light-cone vector $n^\mu(q)$.

Although our findings suggest that the FD gauge is a consistent gauge fixing for quantizing gluons, the lack of covariance and analyticity in the regularized loop integrals does not allow us to take advantage of the standard loop integral tools. Instead, we propose that all the 1PI loop integrals should be done in the Feynman gauge on the FD gauge gluon background. We obtain the same two- and three-point loop functions as those of the conventional background-field gauge, in which both the quantum and background gluons are in the Feynman gauge. Schwinger-Dyson summation of all the one-loop propagator corrections connected by FD gauge gluons gives the one-loop corrected FD gauge propagator. The results may be useful in obtaining improved Born approximation to the tree-level FD gauge amplitudes.

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APPENDIX A: CALCULATION OF LOOP INTEGRALS IN THE FD GAUGE

In this appendix, we explain how we evaluate the UV singular parts of loop integrals with factors $n(k)\cdot k = |k^0|+|\vec{k}|$ in the denominator.

We first note that the UV divergences of loop integrals in our self-energy and vertex corrections should be polynomials of external momenta to appropriate order. For a gluon self-energy loops in $\Pi_{\mu\nu}(q)$, for example, we differentiate the integrands two times by $q^\mu = (q^0, q^i)$ and take $q\to 0$, after regularizing the integrands to avoid infrared divergences generated by these operations. We may then perform loop integrations, which are not Lorentz covariant in general, by known techniques. The UV-divergent parts of the original loops are then easily obtained.

For illustration, we calculate the UV divergence of the 0 component of the integral Eq. (26),

$$I^0(q) = \int \frac{d^Dk}{(2\pi)^D} \frac{\mathrm{sgn}(k^0)}{(|k^0| + |\vec{k}|)((k+q)^2 + i0)}. \quad \text{(A1)}$$

By dimension counting and the $(D-1)$ space dimensional rotational invariance, we can tell that its UV divergent part should take the form $a_0 q^0$ with a q-independent coefficient a_0.

We first differentiate $I^0(q)$ by q^0 to obtain

$$\frac{\partial I^0}{\partial q^0}(q) = \int \frac{d^Dk}{(2\pi)^D} \frac{-2(k^0+q^0)\mathrm{sgn}(k^0)}{(|k^0| + |\vec{k}|)((k+q)^2 + i0)^2}. \quad \text{(A2)}$$

By using the factorization

$$\begin{aligned}(k+q)^2 + i0 &= (k^0+q^0)^2 - |\vec{k}+\vec{q}|^2 + i0 \\ &= (|k^0+q^0| + |\vec{k}+\vec{q}|)(|k^0+q^0| - |\vec{k}+\vec{q}| + i0), \end{aligned} \quad \text{(A3)}$$

the denominators of the integrands become products of $(|l^0| + |\vec{l}|)$ and $(|l^0| - |\vec{l}| + i0)$ (l: a momentum of the propagator). After introducing a fictitious mass parameter $m > 0$ as $(|l^0| \pm |\vec{l}|) \to (|l^0| \pm (|\vec{l}| + m))$ to avoid infrared divergences, we take $q \to 0$ limit to obtain

$$\frac{\partial I^0}{\partial q^0}(0) = \int \frac{d^Dk}{(2\pi)^D} \frac{-2|k^0|}{(|k^0| + |\vec{k}| + m)^3 (|k^0| - |\vec{k}| - m + i0)^2}. \quad \text{(A4)}$$

We then perform $(D-1)$-dimensional space integration by using

$$\frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}} \to \frac{1}{(4\pi)^{\frac{D-1}{2}}} \frac{2}{\Gamma\left(\frac{D-1}{2}\right)} |\vec{k}|^{D-2} d|\vec{k}|, \quad \text{(A5)}$$

after decomposing Eq. (A4) into sum of fractions $1/(|\vec{k}| + A)^n$, where $A = |k^0| + m$ or $-|k^0| + m - i0$. For example, the integration of $1/(|\vec{k}| + A)$ is

$$\begin{aligned}\int \frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}} \frac{1}{|\vec{k}| + A} &= \frac{1}{(4\pi)^{\frac{D-1}{2}}} \frac{2}{\Gamma(\frac{D-1}{2})} \frac{-\pi}{\sin(D\pi)} A^{D-2} \\ &= \frac{1}{(4\pi)^2}\left(\frac{4}{\epsilon} + O(\epsilon^0)\right) A^{D-2}, \end{aligned} \quad \text{(A6)}$$

where $D = 4 - 2\epsilon$. Integration of $1/(|\vec{k}| + A)^n$ for $n \geq 2$ is then obtained by differentiating Eq. (A6) by A.

The space integration in (A4) then takes the form

$$\frac{1}{(4\pi)^2}\left(\frac{4}{\epsilon} + O(\epsilon^0)\right) \int_0^\infty \frac{d|k^0|}{\pi} [f_1(|k^0|, m)(m + |k^0|)^{D-4} + f_2(|k^0|, m)(m - |k^0| - i0)^{D-4}], \quad \text{(A7)}$$

where $f_{1,2}$ are rational functions of $|k^0|$ and m:

$$f_1(|k^0|, m) = \frac{1}{8|k^0|^3}[(-2D^2 + 14D - 23)|k^0|^2 + 2(2D-7)m|k^0| - 3m^2], \quad \text{(A8)}$$

$$f_2(|k^0|, m) = \frac{1}{8|k^0|^3}[(-2D + 7)|k^0|^2 + 2(D-5)m|k^0| + 3m^2], \quad \text{(A9)}$$

for the integral (A4). Note that the term of order $|k^0|^{-1}$ in Eqs. (A8) and (A9) cancel in Eq. (A7) in the $D = 4$ limit, and hence, the $|k^0|$ integral in (A7) is UV finite. The factor $(m - |k^0| - i0)^{D-4}$ for $|k^0| > m$ should be interpreted as $(|k^0| - m)^{D-4} \exp(-i(D-4)\pi)$. The $|k^0|$ integration in Eq. (A7) can then be performed analytically by splitting the integration region into $(0, m)$ and (m, ∞), remembering that D is a general complex number.

The integration of (A7) in $0 \leq |k^0| \leq m$ is, using $|k^0| = mx$,

$$\frac{1}{(4\pi)^2}\left(\frac{4}{\epsilon} + O(\epsilon^0)\right) \frac{1}{\pi} m^{D-4} \int_0^1 dx [f_1(x, 1)(1+x)^{D-4} + f_2(x, 1)(1-x)^{D-4}]. \quad \text{(A10)}$$

It is seen that there is no singularity in (A10), including the boundaries at $x = 0$ and $x = 1$, for $D \simeq 4$. On the other hand, substitution of $D = 4$ into the integrand of (A10) just gives 0. So, the integral should be $O(\epsilon)$, irrelevant in our calculation.

The integration for the other part, $m \leq |k^0| < \infty$, is written as

$$\frac{1}{(4\pi)^2}\left(\frac{4}{\epsilon}+O(\epsilon^0)\right)\frac{1}{\pi}m^{D-4}\int_1^\infty dx[f_1(x,1)(x+1)^{D-4}+f_2(x,1)(x-1)^{D-4}e^{-i(D-4)\pi}]. \tag{A11}$$

Since $f_{1,2}(x,1)\to O(1/x)$ for $x\to\infty$, the integral (A11) is apparently divergent for $D\simeq 4$, but again the integrand vanishes at $D=4$. We therefore expect that integral of (A11) gives a finite result as $(D-4)\times\frac{1}{(D-4)}$.

Let us calculate $(x+1)^{D-4}$ part of the integral in (A11),

$$I_1=\int_1^\infty dx f_1(x,1)(x+1)^{D-4}, \tag{A12}$$

first. By decomposing $f_1(x,1)$ as $C/(x+1)+O(1/(x+1)^2)$, where C is a function of D, the integrand is written as

$$\frac{-2D^2+14D-23}{8}(x+1)^{D-5}+\frac{(-2D^2+18D-37)x^2+(4D-17)x-3}{8x^3}(x+1)^{D-5}. \tag{A13}$$

The first term gives a divergence. By using

$$\int_1^\infty dx(x+1)^{D-5}=-\frac{2^{D-4}}{D-4}, \tag{A14}$$

it is

$$-\frac{1}{8(D-4)}+\frac{1}{4}-\frac{1}{8}\log 2+O(D-4). \tag{A15}$$

The second term behaves as $1/x^2$. One can therefore evaluate its finite term by substituting $D=4$. The result is

$$\frac{1}{16}+\frac{1}{8}\log 2+O(D-4). \tag{A16}$$

By adding (A15) and (A16), we obtain

$$I_1=-\frac{1}{8(D-4)}+\frac{5}{16}+O(D-4). \tag{A17}$$

Next, we calculate $(x-1)^{D-4}$ part of the integral in (A11),

$$I_2=\int_1^\infty dx f_2(x,1)(x-1)^{D-4}, \tag{A18}$$

where the factor $e^{-i(D-4)\pi}$ will be put in later. Although we are going to remove $O(1/(x-1))$ part from $f_2(x,1)$ as was done for f_1, we have to avoid generating a singularity at $x=1$ by this subtraction. For this purpose, we split the integration region into $1\leq x\leq 2$ and $2\leq x<\infty$. The former integral gives

$$\frac{1}{64}-\frac{1}{8}\log 2+O(D-4). \tag{A19}$$

For the latter integral, we split the $C/(x-1)$ part from $f_2(x,1)$. The integrand is then written as

$$\frac{-2D+7}{8}(x-1)^{D-5}+\frac{(4D-17)x^2+(-2D+13)x-3}{8x^3}(x-1)^{D-5}. \tag{A20}$$

The first term gives

$$\frac{1}{8(D-4)}+\frac{1}{4}+O(D-4), \tag{A21}$$

by using

$$\int_2^\infty dx(x-1)^{D-5}=-\frac{1}{D-4}. \tag{A22}$$

The second term can be calculated in $D\to 4$, giving

$$-\frac{5}{64}+\frac{1}{8}\log 2+O(D-4). \tag{A23}$$

Summation of Eqs. (A19), (A21), and (A23) gives

$$I_2=\frac{1}{8(D-4)}+\frac{3}{16}+O(D-4). \tag{A24}$$

By inserting (A17) and (A24) in the integral (A7), we obtain

$$\begin{aligned}\frac{\partial I^0}{\partial q^0}(0)&=\frac{1}{(4\pi)^2}\left(\frac{4}{\epsilon}+O(\epsilon^0)\right)\frac{1}{\pi}m^{-2\epsilon}(I_1+I_2e^{-i(D-4)\pi})\\&=\frac{1}{(4\pi)^2}\left(\frac{4}{\epsilon}+O(\epsilon^0)\right)\frac{1}{\pi}m^{-2\epsilon}\left(\frac{1}{2}-\frac{i\pi}{8}+O(\epsilon)\right)\\&=\frac{1}{(4\pi)^2\epsilon}\left(\frac{2}{\pi}-\frac{i}{2}\right)+O(\epsilon^0).\end{aligned} \tag{A25}$$

The integral $I^0(q)$ of Eq. (A1) is hence

$$I^0(q) = \frac{1}{(4\pi)^2\epsilon}\left(\frac{2}{\pi} - \frac{i}{2}\right)q^0 + O(\epsilon^0). \tag{A26}$$

All of the UV singular parts of the loop integrals involving n^μ, which appear in Sec. III, can be evaluated in the same manner. As another example, we calculate the UV singular part of the space components of the integral (26),

$$I^j(q) = \int\frac{d^Dk}{(2\pi)^D}\frac{1}{(|k^0| + |\vec{k}|)(k+q)^2}\frac{-k^j}{|\vec{k}|}. \tag{A27}$$

Its UV-singular part should take the form $a_1 q^j$ with a q-independent coefficient a_1. In this case, we differentiate $I^j(q)$ by q^i,

$$\frac{\partial I^j}{\partial q^i}(q) = \int\frac{d^Dk}{(2\pi)^D}\frac{-2(k^i+q^i)k^j}{(|k^0| + |\vec{k}|)|\vec{k}|((k+q)^2 + i0)^2}. \tag{A28}$$

Again, by introducing an IR regulator mass m and factorization (A3), and taking the limit $q \to 0$, we obtain

$$\begin{aligned}\frac{\partial I^j}{\partial q^i}(0) &= \int\frac{d^Dk}{(2\pi)^D}\frac{-2k^ik^j}{(|k^0|+|\vec{k}|+m)^3(|k^0|-|\vec{k}|-m+i0)^2|\vec{k}|}\\ &= \frac{1}{D-1}\int\frac{d^Dk}{(2\pi)^D}\frac{-2|\vec{k}|\delta^{ij}}{(|k^0|+|\vec{k}|+m)^3(|k^0|-|\vec{k}|-m+i0)^2}.\end{aligned} \tag{A29}$$

In the second line of Eq. (A29), we replaced k^ik^j by $|\vec{k}|^2\delta^{ij}/(D-1)$ by using the $(D-1)$ dimensional rotational invariance.

The UV singular part of Eq. (A29) can be calculated in the same manner as that of I^0. The final result is

$$I^j(q) = \frac{1}{(4\pi)^2\epsilon}\left(-\frac{2}{3\pi} - \frac{i}{2}\right)q^j + O(\epsilon^0). \tag{A30}$$

APPENDIX B: TRANSVERSE AND LONGITUDINAL CONTRIBUTIONS TO $\Pi_{\mu\nu}(q)$ FOR GENERAL q^μ

The UV divergence of the gluon self-energy $i\Pi_{\mu\nu}(q)$ for general q^μ is separated into TT, TL, and LL parts as

$$i\Pi^{(bTT)}_{[00,0j,ij]}(q)\Big|_{\text{div}} = C_Ag^2\frac{1}{(4\pi)^2\epsilon}\left[\frac{5i}{3}|\vec{q}|^2, -\frac{i}{3}q^0q^j, \frac{i}{3}(q^0)^2\delta^{ij} + \frac{9i}{5}|\vec{q}|^2\delta^{ij} - \frac{31i}{15}q^iq^j\right], \tag{B1}$$

$$i\Pi^{(bTL)}_{[00,0j,ij]}(q)\Big|_{\text{div}} = C_Ag^2\frac{1}{(4\pi)^2\epsilon}\left[-\frac{16i}{3}|\vec{q}|^2, \left(\frac{10i}{3} + \frac{4}{3\pi}\right)q^0q^j, -\frac{8i}{3}(q^0)^2\delta^{ij} + \left(\frac{8i}{15} - \frac{16}{15\pi}\right)|\vec{q}|^2\delta^{ij} + \left(-\frac{4i}{15} + \frac{8}{15\pi}\right)q^iq^j\right], \tag{B2}$$

$$i\Pi^{(bLL)}_{[00,0j,ij]}(q)\Big|_{\text{div}} = C_Ag^2\frac{1}{(4\pi)^2\epsilon}\left[\frac{10}{3\pi}(q^0)^2 - \frac{74}{9\pi}|\vec{q}|^2, \frac{8}{9\pi}q^0q^j, \frac{22}{9\pi}(q^0)^2\delta^{ij} - \frac{86}{45\pi}|\vec{q}|^2\delta^{ij} + \frac{68}{45\pi}q^iq^j\right]. \tag{B3}$$

In contrast to the $q^\mu = (Q, \vec{0})$ case (49b), the TL contribution (B2) has both $O(i)$ and $O(1/\pi)$ terms.

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