Baryon junction and string interactions

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We study junctions between confining strings. We show that the effective theory of such junctions is very predictive, with only one new parameter, the junction's mass, controlling the first couple of terms in the expansion in the system size. By open-closed string duality, these considerations about the baryon junction map to interaction vertices of closed strings. Therefore, we calculate the interaction vertices of closed strings in theories such as Yang-Mills theory. We find some surprising selection rules for string interactions in $3+1$ dimensions. Requiring perturbative stability and that the string coupling is weak, we suggest constraints on the junction's mass.

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I. INTRODUCTION

Many theories admit unbreakable string-like excitations of nonzero tension $T = l_s^{-2}$. The quintessential example is the Abrikosov string in superconductors [1] and the closely related Nielsen-Olesen strings in the Abelian Higgs model [2]. The string stability is a consequence of unbroken 1-form symmetries in the system. In the Abelian Higgs model, it is the magnetic $U(1)$ 1-form symmetry and magnetic flux is confined in the Abrikosov-Nielsen-Olesen string. Another model is the $SU(N)$ Yang-Mills theory with $\mathbb{Z}_N$ electric 1-form symmetry, where string-like excitations confine electric flux.

The examples above are gapped theories in the bulk. In the presence of a long confining string, the only low-energy modes are the string fluctuations (phonons). The effective theories of such fluctuations are well studied for both closed and open strings; see, for instance, [3–18]. Some results were nicely reviewed in [19]. There is also extensive literature on simulations of the confining string and the comparison with theoretical predictions [20–31] (many more references on this subject can be found therein). Many papers also explored the subject of quantizing the string with dynamical end points and made contact with Regge physics (e.g., [32–40]).

In the open string effective field theory (EFT), a Dirichlet boundary condition physically represents external static particles on which the string can end, and a Neumann boundary condition models branes within which the string end point can roam freely. In gauge theories, the point particles on which the strings can end are just quarks. One physical example of the Neumann condition is in Yang-Mills theories when the flux tube ends on the "Janus" interface that is created by changing the theta angle $\theta \to \theta + 2\pi$. The dynamics of these interfaces and some aspects of the strings ending on them were discussed in [41].

In theories with $\mathbb{Z}_N$ 1-form symmetry, an interesting string configuration is the "baryon," where N strings are tied at a vertex point. The string-and-junction configuration was first discussed in [42,43]. This configuration plays an important role in particle scattering, as it tracks baryon number (see, for instance, [44–47]) and is also essential for understanding exotic hadrons (e.g., [48–50]). In this paper, we focus on confining theories with $\mathbb{Z}_3$ 1-form symmetry in a $(d+1)$-dimensional spacetime. Our analysis is independent of the UV physics and bears easy generalization to other string-vertex configurations. In particular, our study applies to the confined phase of $SU(3)$ Yang-Mills theory in $d = 2, 3$. The baryon vertex is clearly observed in simulations [51–55] where our predictions should be testable.

We investigate the "baryon" from two perspectives. In the open string channel, we consider three static quarks positioned at the vertices of a triangle. In this channel, confining strings meet at the Fermat point, as in Fig. 1, and the time direction is perpendicular to the triangle. Equivalently, we can perform a double Wick rotation and define the closed string channel, in which the time direction lies on the plane. The vertex is then interpreted as an interaction vertex where three strings meet. We can call such a vertex a D-instanton as it is localized in time.

Let L be the length of confining strings. We denote an EFT operator to be of order k if it scales as $O(1/L^k)$ in the

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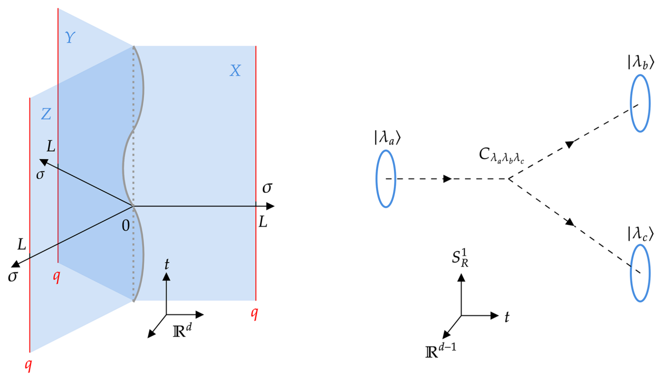

FIG. 1. The "baryon" configuration. In the open string channel (left), three strings are tied at the Fermat point at $\sigma = 0$ and end at the Dirichlet boundaries (quarks) at $\sigma = L$. In the closed string channel (right), three closed string states $\lambda_{a,b,c}$ have an interaction vertex $C_{\lambda_a\lambda_b\lambda_c}$.

action and contributes $O(1/L^{k+1})$ corrections to the spectrum. We argue that the EFT is uniquely determined by M and l_s up to order 2. Up to order -1, the physics is classical. The fact that the "baryon" configuration only breaks Poincaré symmetry spontaneously imposes strong constraints on the EFT, which is why the EFT expansion is robust with no new parameters up to order 2. The two parameters M and l_s can be obtained (for instance) by measuring the ground-state energy of the junction at zero temperature, extending [56] (and related calculations in [57])

$$E_{\text{gs}} = \frac{3L}{l_s^2} + M - \frac{(d-2)\pi}{16L} - \frac{(d+2)\pi M l_s^2}{144L^2} + O(1/L^3). \quad (1)$$

Note that in (1) the static quark masses m_q have been subtracted. The static quark masses m_q are counterterms on the Dirichlet boundaries corresponding to the external quarks, and hence they are scheme dependent. In any specific implementation, they can be measured from the quark-antiquark pair and then subtracted to obtain (1), which is scheme independent.

In this paper, we explicitly demonstrate the implications of open-closed string duality up to order 1. We will see that the string interaction vertices $C_{\lambda_a\lambda_b\lambda_c}$ can be obtained, in some cases completely unambiguously, just from consistency. We find that the coupling constant among generic closed string states of strings of length $2\pi R$ is

$$C_{\lambda_a\lambda_b\lambda_c} \sim e^{-2\pi MR}. \quad (2)$$

We see that if $M > 0$, long strings are weakly coupled, and strongly coupled otherwise. We also show that, on top of the strong coupling catastrophe, there is a classical perturbative instability for sufficiently large negative M. With the same framework, many extensions of the calculations we do are possible, and we comment on some of the open problems towards the end.

II. REVIEW OF EFFECTIVE STRING THEORIES

Here we review the EFT of a long confining string, which physically describes small fluctuations of the string. In the low-energy limit, we assume all massive modes have been integrated out, leaving an effective action of massless modes. A long string configuration spontaneously breaks spacetime Poincaré symmetry as $ISO(1,d) \to ISO(1,1) \times SO(d-1)$, leading to $(d-1)$ Nambu-Goldstone bosons (NGBs) [58] as the only massless degrees of freedom in the string interior.

To be concrete, we study a closed string wrapped on $\mathbb{S}^1_R$, whose world-sheet embedding in spacetime is $X_\mu(\Sigma)$ and $\Sigma_\alpha = (t,\sigma)$. The effective action is constrained by Poincaré symmetry and diffeomorphism invariance on the world sheet. An important action compatible with these requirements is the Nambu-Goto action [59,60],

$$S_{\text{NG}} = -\frac{1}{l_s^2}\int dtd\sigma\sqrt{-\det \partial_\alpha X^\mu \partial_\beta X_\mu}, \quad (3)$$

where we chose a mostly negative signature. Without loss of generality, we choose a static gauge where $X_0 = t$, $X_1 = \sigma \in \mathbb{S}^1_R$, and $X_i = l_s x_i(t,\sigma)$ for $2 \le i \le d$. The Nambu-Goto action admits an expansion in transverse fluctuations x_i, and in this gauge it reads

$$S_{\text{NG}} = \int dtd\sigma\left\{-\frac{1}{l_s^2} + \frac{1}{2}[(\partial_t x_i)^2 - (\partial_\sigma x_i)^2] + \frac{l_s^2}{8}[(\partial_t x_i - \partial_\sigma x_i)^2(\partial_t x_{i'} + \partial_\sigma x_{i'})^2]\right\} + O\left(\frac{1}{R^4}\right). \quad (4)$$

Equation (4) is the unique effective action of NGBs x_i to order 2 [6,9,12]. There are corrections to (3) and (4) with new Wilson coefficients, which start at order 6 for $d = 2$ and order 4 for $d \ge 3$; we will not be concerned with such high-order contributions here.

From the order -2 term in (4), we interpret $T = l_s^{-2}$ as the classical string tension. The order 0 term consists of $(d-1)$ free NGBs, whose modes are left- or right-moving. The left and right mode occupation numbers $n_\lambda^{\text{L,R}} \in \mathbb{N}$ determine the order 0 energy level of a closed string state λ:

$$E_\lambda^{\text{c}} = \frac{2\pi R}{l_s^2} - \frac{(d-1)}{12R} + \frac{n_\lambda^{\text{L}} + n_\lambda^{\text{R}}}{R} + O(1/R^3). \quad (5)$$

The order 2 term is a $T\bar{T}$ deformation of the free action and hence preserves integrability [61–63], and it leads to $O(1/R^3)$ energy corrections in (5). For later convenience, we denote the ground state by $\mathbf{0}$; it has $n_{\mathbf{0}}^{\text{L}} = n_{\mathbf{0}}^{\text{R}} = 0$ and the lowest-lying nonchiral $O(d-1)$ symmetric state is denoted by $\mathbf{1}$—it has $n_{\mathbf{1}}^{\text{L}} = n_{\mathbf{1}}^{\text{R}} = 1$.

The analysis is similar for open strings, except we need to consider boundary conditions and boundary operators. We take an open string of length L. At order 0, two canonical choices for the boundary condition at $\sigma = 0$ are as follows:

(1) Dirichlet: $\partial_t x_i = 0$. Among all boundary operators, we note that $\int_{\sigma=0} dt(\partial_\sigma x_i)^2$ is forbidden by the spontaneously broken Poincaré symmetry [4,8]. The leading operator is $\int_{\sigma=0} dt(\partial_t \partial_\sigma x_i)^2$ and it is of order 3. It represents the moment of inertia of the boundary end point.

(2) Neumann: $\partial_\sigma x_i = 0$. The leading boundary operator is $\int_{\sigma=0} dt(\partial_t x_i)^2$ and it is of order 1. The Poincaré symmetry requires this operator to be accompanied by an order -1 constant term, such that

$$S_{\mathrm{N}} = -M' \int_{\sigma=0} dt\left[1 - \frac{l_s^2}{2}(\partial_t x_i)^2\right] + O(1/L^3). \quad (6)$$

We interpret M' as the classical mass of the end point, which is free to roam on the brane at the boundary. The next-order operators, such as $\int_{\sigma=0} dt[(\partial_t x_i)^2]^2$, are order 3.

Finally, it is important to remark that the EFT expansion means that the frequency cannot exceed l_s^{-1}; otherwise, the expansion in the inverse string size breaks down.

A. Open-closed string duality for a "meson"

As a warm-up, we review the "meson," which is the configuration of an open string connecting a static quark-antiquark pair ($q\bar{q}$ pair). We denote the string spatial length to be L and let boundary conditions be Dirichlet at $\sigma = 0$ and $\sigma = L$. Furthermore, we compactify the time direction on $\mathbb{S}_R^1$ so as to put the open string at finite temperature. From (4) we learn that up to order 1 the unique meson action reads $S_{\mathrm{m}} = S_{\mathrm{m}}^{(-2)} + S_{\mathrm{m}}^{(0)} + O(1/L^2)$, where the order -2 constant $S_{\mathrm{m}}^{(-2)} = \frac{2\pi RL}{l_s^2}$ and the order 0 fluctuations read

$$S_{\mathrm{m}}^{(0)} = \frac{1}{2}\int d\tau d\sigma[(\partial_\tau x_i)^2 + (\partial_\sigma x_i)^2]. \quad (7)$$

In the long-string limit, higher operators are suppressed. For the meson case, the EFT partition function can be obtained as

$$\begin{aligned}\mathcal{Z}_{\mathrm{m}} &= e^{-S_{\mathrm{m}}^{(-2)}} \int \mathcal{D}x_i e^{-S_{\mathrm{m}}^{(0)}}[1 + O(1/L^2)] \\ &= \frac{e^{-\mu L}}{[\eta(q)]^{(d-1)}}[1 + O(1/L^2)],\end{aligned} \quad (8)$$

where the modular parameter $q \equiv e^{-\frac{2\pi^2 R}{L}}$ and $\eta(q)$ is the Dedekind eta function. $e^{-\mu L}$ represents the contribution from the classical energy of the string, where $\mu \equiv \frac{2\pi R}{l_s^2}$.

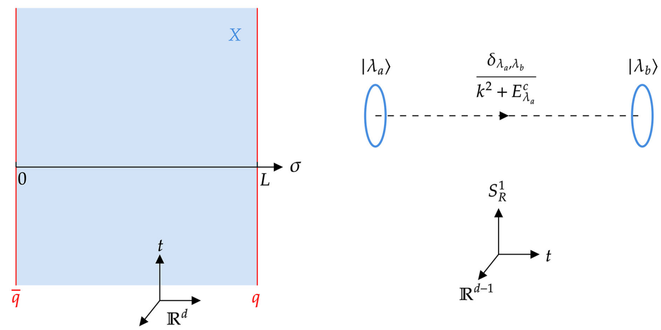

FIG. 2. The "meson" configuration. In the open string channel (left), a single string is connected to Dirichlet boundaries (quark-antiquark pair) at $\sigma = 0$ and $\sigma = L$. In the closed string channel (right), closed string states propagate in spacetime.

In the open string channel, each time slice of the world sheet is an open string, as in Fig. 2, and (8) is interpreted as the thermal partition function

$$\mathcal{Z}_{\mathrm{m}} = \sum_\lambda e^{-2\pi R E_\lambda^{\mathrm{o}}}. \quad (9)$$

Indeed, (9) from the open string spectrum agrees with (8). On the other hand, we may perform a Wick rotation and take the time to be horizontal in Fig. 2. In this case, (8) is interpreted as the two-point function of Polyakov loops Ω and Ω^*, separated by $\vec{X} = (L, 0, \ldots, 0)$. When Ω acts on the vacuum it creates a certain combination of closed string energy eigenstates, each of which behaves as a massive particle in $\mathbb{R}^d$:

$$|\Omega\rangle = \sum_\lambda v_\lambda |\lambda\rangle, \quad \text{such that}$$

$$\langle\lambda'(\vec{X})|\lambda(0)\rangle = \delta_{\lambda,\lambda'} \frac{(E_\lambda^{\mathrm{c}})^{\frac{d}{2}} l_s^{d-1}}{\sqrt{\pi}(2L)^{\frac{d-2}{2}}} K_{\frac{d-2}{2}}(E_\lambda^{\mathrm{c}} L), \quad (10)$$

where we have used the massive propagator in $\mathbb{R}^d$ [4,8]. Therefore, the closed string channel representation of (8) reads [64]

$$\begin{aligned}\mathcal{Z}_{\mathrm{m}} &= \langle\Omega^*(\vec{X})\Omega(0)\rangle \\ &= \sum_\lambda |v_\lambda|^2 \frac{(E_\lambda^{\mathrm{c}})^{\frac{d}{2}} l_s^{d-1}}{\sqrt{\pi}(2L)^{\frac{d-2}{2}}} K_{\frac{d-2}{2}}(E_\lambda^{\mathrm{c}} L).\end{aligned} \quad (11)$$

Consistency requires that there exists a set of the $v_\lambda \in \mathbb{C}$ such that, together with the closed string energies E_λ^{c}, we can match (8) with (11). This is in general a highly nontrivial condition that constrains the effective action and relates the closed string spectrum with the open string spectrum. However, to the order we are working, we can see that an appropriate v_λ exists by performing a modular transformation using the identity in (B3),

$$\mathcal{Z}_{\mathrm{m}} = \frac{(\pi R/L)^{\frac{d-1}{2}} e^{-\mu L}}{[\eta(\tilde{q})]^{(d-1)}} + O(1/R^2), \tag{12}$$

where $\tilde{q} = e^{-\frac{2L}{R}}$ is the dual modular parameter. Note that the closed string states that can be created from Ω acting on vacuum have $n_\lambda^{\mathrm{L}} = n_\lambda^{\mathrm{R}} = n_\lambda$, as they do not carry longitudinal momentum. Additionally, they are singlets under the rotations in the transverse plane. We take $L, R \gg l_s$ and the fixed ratio L/R. The ratio L/R could be large or small, which is convenient for series expansion in the closed or open string channel, respectively. In this limit, the closed string channel representation (11) admits the following expansion:

$$\mathcal{Z}_{\mathrm{m}} = \frac{(\pi R/L)^{\frac{d-1}{2}} e^{-\mu L}}{\tilde{q}^{\frac{d-1}{24}}} \sum_\lambda |v_\lambda|^2 \tilde{q}^{n_\lambda} + (1/R^2). \tag{13}$$

We can choose the convention such that $v_\lambda > 0$, and by comparing (12) with (13) we obtain

$$\begin{aligned} v_{\mathbf{0}} &= 1 + O(1/R^2), \\ v_{\mathbf{1}} &= \sqrt{d-1}[1 + O(1/R^2)], \end{aligned} \tag{14}$$

etc., where the order 0 result is also known from the CFT literature [65] as the Dirichlet boundary state.

III. “BARYON” IN THE OPEN STRING CHANNEL

In this section, we study the three strings tied at a junction, as in Fig. 1. The string end points in $\mathbb{R}^d$ are positioned at $\vec{X} = (L, 0, \ldots, 0)$, $\vec{Y} = (-\frac{L}{2}, -\frac{\sqrt{3}L}{2}, \ldots, 0)$, and $\vec{Z} = (-\frac{L}{2}, \frac{\sqrt{3}L}{2}, \ldots, 0)$, with Dirichlet boundary conditions on the vertices of the equilateral triangle (see [66] for a discussion of the collinear case.). Classically, strings are straight lines meeting at the origin, which is the Fermat-Torricelli point that minimizes the sum of distances to the vertices. Unlike the EFT boundaries we reviewed above, the vertex's location oscillates along the longitudinal directions X_1, Y_1, and Z_1 of the strings. For instance, let $X_1 = l_s x_1(t)$ be the fluctuating position of the vertex in the longitudinal direction of the first string; then, the Nambu-Goto action (3) in static gauge (see Appendix A) reads

$$S_X = \int dt \int_{l_s x_1(t)}^{L} d\sigma \left[-\frac{1}{l_s^2} + \frac{1}{2}(\partial_t x_i)^2 - \frac{1}{2}(\partial_\sigma x_i)^2 + O(1/L^4) \right], \tag{15}$$

and similarly for the Y and Z world sheets. We see that to the leading order, one only has to modify the integration domain in the action by the time-dependent function $l_s x_1(t)$.

The vertex is point-like in the IR, and the geometric condition that three strings are tied at the same point in space is rigid. At the vertex $\sigma = 0$, by solving linear geometric equations we find that $x_1 = (z_2 - y_2)/\sqrt{3}$, $y_1 = (x_2 - z_2)/\sqrt{3}$, $z_1 = (y_2 - x_2)/\sqrt{3}$, and for the transverse fluctuations

$$\begin{cases} x_2 + y_2 + z_2 = 0, \\ x_j = y_j = z_j, \text{ for } 3 \le j \le d. \end{cases} \tag{16}$$

Since the longitudinal fluctuations x_1, y_1, and z_1 are related to the transverse fluctuations, we can easily find the normal modes and perform a perturbative expansion.

The string bulk action includes an order -2 classical part $S_{\mathrm{b}}^{(-2)} = -\frac{3L}{l_s^2}\int dt$, and an order 0 quadratic part

$$\begin{aligned} S_{\mathrm{b}}^{(0)} &= \frac{1}{2}\int_{\mathbb{R}\times[0,L]} dt d\sigma [(\partial_t x_i)^2 - (\partial_\sigma x_i)^2 + \text{cyclic}] \\ &= \frac{1}{2}\sum_{a=1}^{3}\int_{\mathbb{R}\times[0,L]} dt d\sigma [(\partial_t \xi_i^{[a]})^2 - (\partial_\sigma \xi_i^{[a]})^2], \end{aligned} \tag{17}$$

where we have applied a field redefinition $\xi_i^{[1]} = (x_i + y_i + z_i)/\sqrt{3}$, $\xi_i^{[2]} = (x_i - y_i)/\sqrt{2}$, and $\xi_i^{[3]} = (x_i + y_i - 2z_i)/\sqrt{6}$ to diagonalize the system. At order 0, (16) is the Neumann boundary condition for $\xi_2^{[2,3]}$, $\xi_{j\ge3}^{[1]}$ and the Dirichlet boundary condition for $\xi_2^{[1]}$, $\xi_{j\ge3}^{[2,3]}$. In summary, the junction in the equilateral case behaves as a tensor product of Neumann and Dirichlet boundaries to the leading order,

$$(\text{Neumann})^{\otimes d} \otimes (\text{Dirichlet})^{\otimes(2d-3)}. \tag{18}$$

Note that when $d = 3$, Neumann and Dirichlet conditions are assigned to an equal number of polarizations, which will have some consequences below. Physically, $\xi_2^{[2,3]}$ and $\xi_{j\ge3}^{[1]}$ are the spatial displacement, while $\partial_\sigma \xi_2^{[1]}$ and $\partial_\sigma \xi_{j\ge3}^{[2,3]}$ are small rotation angles of the vertex.

At the vertex, we can write down an order -1 constant action $S_{\mathrm{b}}^{(-1)} = -M\int_{\sigma=0} dt$, where M is interpreted as the vertex mass as in (1). The nonlinear realization of the Lorentz group requires $S_{\mathrm{b}}^{(-1)}$ to be accompanied by an order 1 quadratic term,

$$\begin{aligned} S_{\mathrm{b}}^{(1)} &= \frac{M l_s^2}{3}\int_{\sigma=0} dt [(\partial_t \xi_2^{[2]})^2 + (\partial_t \xi_2^{[3]})^2] \\ &\quad + \frac{M l_s^2}{6}\int_{\sigma=0} dt (\partial_t \xi_j^{[1]})^2, \end{aligned} \tag{19}$$

which is unique and agrees with the expansion of a standard world-line action. Note that expanding (15) with respect to the longitudinal fluctuation yields another order 1 term, which is cubic in the fluctuations,

$$\tilde{S}_{\mathrm{b}}^{(1)} = -\frac{l_s}{2}\int_{\sigma=0} dt\{x_1[(\partial_t x_i)^2 - (\partial_\sigma x_i)^2] + \text{cyclic}\}$$
$$= -\frac{l_s}{2\sqrt{6}}\int_{\sigma=0} dt\{\xi_2^{[2]}[(\partial_t\xi_2^{[3]})^2 - (\partial_t\xi_2^{[2]})^2 - (\partial_\sigma\xi_j^{[3]})^2 + (\partial_\sigma\xi_j^{[2]})^2] + 2\xi_2^{[3]}(\partial_t\xi_2^{[2]}\partial_t\xi_2^{[3]} - \partial_\sigma\xi_j^{[2]}\partial_\sigma\xi_j^{[3]})\}. \quad (20)$$

We remark that (20) has important implications at higher orders, but here it will not play any further role.

To obtain how (18) is modified due to the mass of the junction (19), we recompute the dispersion relation of the polarizations $\xi_2^{[2,3]}$, $\xi_{j\geq 3}^{[1]}$ to find

$$\cos(\omega L) = c_{\parallel,\perp}\omega\sin(\omega L), \quad (21)$$

where $c_\parallel = \frac{2Ml_s^2}{3}$ for planar modes $\xi_2^{[2,3]}$, $c_\perp = \frac{Ml_s^2}{3}$ for vertical modes $\xi_j^{[1]}$, and ω is the frequency. Depending on the scale and sign of M, there are essential differences in solutions to (21).

(1) $M < 0$: When $|M| \ll \frac{L}{l_s^2}$, (21) admits an imaginary solution $\omega \approx i|c_{\parallel,\perp}|^{-1}$. Such a tachyon is out of the EFT regime as long as $|M| \lesssim \frac{1}{l_s}$, where the stability needs to be examined in the full nonlinear theory. However, if the junction mass is negative and parametrically large $|M| \gg \frac{1}{l_s}$, the instability is perturbative and within the EFT. We do not discuss the end point of such an instability, but we will get back to the question of whether slightly negative mass $-\frac{1}{l_s} \lesssim M < 0$ is indeed allowed. Interestingly, a negative baryon vertex mass is also found in certain large-N gauge theories [67], with the property $|M| \lesssim \frac{1}{l_s}$ [68]. Anecdotally, a remote analogue is that soap films form plateau borders with a negative tension [70]; of course, one need not worry about zero-temperature instabilities in that setup.

(2) $M \sim \frac{1}{l_s}$: In this case, solutions to (21) are Neumann-like with small corrections:

$$\omega_r = r\frac{\pi}{L}\left[1 - \frac{c_{\parallel,\perp}}{L} + O(1/L^2)\right], \quad (22)$$

where $r \in \mathbb{N} + \frac{1}{2}$. This is the most relevant regime to Yang-Mills theory, where we do not expect a hierarchy between the baryon junction mass and the string tension.

(3) $\frac{1}{l_s} \ll M \ll \frac{L}{l_s^2}$: The spectrum of polarizations $\xi_2^{[2,3]}$ and $\xi_{j\geq 3}^{[1]}$ is divided into two regimes: low-energy modes with $\omega \ll \frac{1}{Ml_s^2}$ follow the dispersion (22), while high-energy modes with $\frac{1}{Ml_s^2} \ll \omega \ll \frac{1}{l_s}$ admit another expansion,

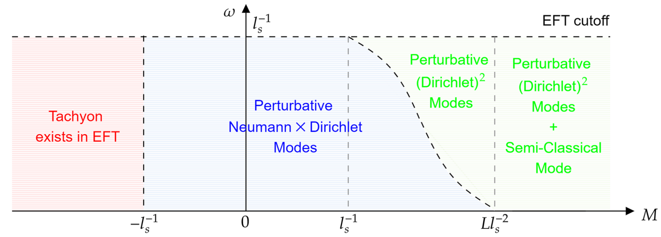

FIG. 3. Spectrum as a function of the vertex mass and system size.

$$\omega_n = n\frac{\pi}{L} + \frac{1}{n\pi c_{\parallel,\perp}} + O(1/n^3), \quad (23)$$

where $n \in \mathbb{N} \gg \frac{L}{Ml_s^2}$. For those $\omega \gg \frac{1}{Ml_s^2}$ modes, the vertex condition becomes approximately Dirichlet for $\xi_2^{[2,3]}$ and $\xi_{j\geq 3}^{[1]}$.

(4) $M \gg \frac{L}{l_s^2}$: In addition to the Dirichlet-Dirichlet modes as in (23), there exist low-frequency semi-classical modes with $\omega \approx (c_{\parallel,\perp}L)^{-1/2}$. These modes correspond to a heavy vertex oscillating in the classical potential without creating waves on the string.

These regimes are summarized in Fig. 3. In the following, we focus on the regime $M \sim l_s^{-1}$ where the string fluctuations dominate the physics. We expect $M \sim l_s^{-1}$ to be the case in Yang-Mills theory. Let us mention that $M \gg l_s^{-1}$ might be interesting as well for other applications; a physical example of a heavy junction could be in a variant of the Abelian Higgs model, where the 1-form symmetry is broken to $\mathbb{Z}_3$ via a heavy charge 3 monopole.

A. Partition function

Applying the method of Sec. II A, we calculate the thermal partition function of the junction by compactifying time on $\mathbb{S}_R^1$. We first consider the classical action $S_{\mathrm{b}}^{(-2)} = \frac{6\pi RL}{l_s^2} = 3\mu L$, $S_{\mathrm{b}}^{(-1)} = 2\pi RM$, and the quadratic fluctuations (17):

$$\mathcal{Z}_{\mathrm{b}}^{(0)} = e^{-S_{\mathrm{b}}^{(-2)} - S_{\mathrm{b}}^{(-1)}}\int \mathcal{D}x_i\mathcal{D}y_i\mathcal{D}z_i e^{-S_{\mathrm{b}}^{(0)}}$$
$$= \frac{e^{-3\mu L - 2\pi RM}}{[\eta(\sqrt{q})]^d[\eta(q)]^{d-3}}, \quad (24)$$

where $q = e^{-\frac{2\pi^2 R}{L}}$. This is the thermal partition function to the order 0, where the junction can be treated by the tensor product of Neumann and Dirichlet boundaries (18).

In the long-string limit, the order 1 action can be treated perturbatively. We have at this order a contribution from (19) and (20):

$$\mathcal{Z}_{\mathrm{b}} = \mathcal{Z}_{\mathrm{b}}^{(0)}[1 - \langle S_{\mathrm{b}}^{(1)}\rangle - \langle\tilde{S}_{\mathrm{b}}^{(1)}\rangle + O(1/L^2)], \quad (25)$$

where $\langle \ldots \rangle$ is the vacuum expectation value in the order 0 theory. Note that the cubic term $\langle \tilde{S}_{\mathrm{b}}^{(1)} \rangle = 0$ as it is odd under parity. Furthermore, we remark that the cubic operator $\tilde{S}_{\mathrm{b}}^{(1)}$ does not perturb the open string channel spectrum at order 1.

To work out $\langle S_{\mathrm{b}}^{(1)} \rangle$, we denote by $\tilde{G}(\Sigma, \Sigma')$ the free-field world-sheet propagator with Dirichlet condition at $\sigma = 0$ and Neumann condition at $\sigma = L$, and by $\tilde{G}_{\alpha\beta} \equiv \lim_{\Sigma \to \Sigma'} \partial_{\Sigma_\alpha} \partial_{\Sigma'_\beta} \tilde{G}(\Sigma, \Sigma')$ its coincident point function. By the Wick theorem, we obtain the one-loop result

$$\begin{aligned} \langle S_{\mathrm{b}}^{(1)} \rangle &= \frac{(d+2)Ml_s^2}{6} \int_{\sigma=0} d\tau \tilde{G}_{\tau\tau} \\ &= \frac{(d+2)Ml_s^2}{144L} \log q [2E_2(q) - E_2(\sqrt{q})], \end{aligned} \tag{26}$$

where $E_2(q)$ is the Eisenstein series; see also Appendix B. As a consistency check, (26) agrees with the ground-state energy (1) and perturbed frequencies (22).

For what comes next, we remark that the open string channel partition function also admits a dual representation,

$$\begin{aligned} \mathcal{Z}_{\mathrm{b}}^{(0)} &= \frac{(\pi R/L)^{d-\frac{3}{2}} e^{-3\mu L - 2\pi R M}}{2^{d/2} [\eta(\tilde{q}^2)]^d [\eta(\tilde{q})]^{d-3}}, \\ \langle S_{\mathrm{b}}^{(1)} \rangle &= -\frac{(d+2)Ml_s^2}{36R} [2E_2(\tilde{q}^2) - E_2(\tilde{q})], \end{aligned} \tag{27}$$

following from (B3) and $\tilde{q} = e^{-\frac{2L}{R}}$.

IV. STRING INTERACTION VERTEX

In this section, we discuss the closed string channel interpretation of (27) as a correlation function of Polyakov loops Ω in $\mathbb{R}^d \times \mathbb{S}_R^1$. From the point of view of the noncompact $\mathbb{R}^d$, the Polyakov loops are point operators and the 1-form $\mathbb{Z}_3$ is reduced to a 0-form $\mathbb{Z}_3$, with the fundamental loop carrying charge 1 mod 3. Symmetry-preserving three-point functions are $\langle \Omega\Omega\Omega \rangle$ and $\langle \Omega^*\Omega^*\Omega^* \rangle$. We now explain how to reinterpret the junction partition function in terms of the three-point function

$$\mathcal{Z}_{\mathrm{b}} = \langle \Omega(\vec{X}) \Omega(\vec{Y}) \Omega(\vec{Z}) \rangle. \tag{28}$$

If we take the time direction to be along $\mathbb{R}^d$, then $\langle \Omega(\vec{X}) \Omega(\vec{Y}) \Omega(\vec{Z}) \rangle$ describes an interaction vertex of three closed strings. As in (10), we start from the idea that the Ω operator creates a superposition of energy eigenstates when acting on the vacuum. The coefficients of the energy eigenstates are (14). When $R, L \gg l_s$, the closed string states are heavy particles, which travel for a long distance. Therefore, we assume that in the limit $L, R \gg l_s$, (28) can be interpreted as strings scattering via a local interaction. We further show that, to the order we compute, such an interaction is purely a contact interaction and the interaction strength $C_{\lambda_x \lambda_y \lambda_z}$ between the energy eigenstates λ_x, λ_y, and λ_z can be determined unambiguously.

We remark that the nonlocality of the scattering, which we expect to be on the scale l_s modulo logarithmic corrections, leads to higher contact couplings between the states' λ's. From Lorentz invariance, it follows that these non-s-wave scatterings contribute at least at order 2 and hence can be neglected for the analysis here. We conclude that we expect the partition function to be reproduced by a simple contact interaction between the string states:

$$\mathcal{Z}_{\mathrm{b}} = \mathcal{Z}_{s\text{-wave}} + O(1/R^2). \tag{29}$$

Let $\vec{W} \in \mathbb{R}^d$ be the location of the contact interaction vertex and $L_{xw} = |\vec{X} - \vec{W}|$; the s-wave amplitude is an integration against the propagators (10) that reads

$$\begin{aligned} \mathcal{Z}_{s\text{-wave}} = \sum_{\lambda_{x,y,z}} \int \frac{d^d \vec{W}}{l_s^d} C_{\lambda_x \lambda_y \lambda_z} \\ \times \prod_{a=x,y,z} \left[\frac{v_{\lambda_a} (E_{\lambda_a}^{\mathrm{c}})^{\frac{d}{2}} l_s^{d-1}}{\sqrt{\pi} (2L_{wa})^{\frac{d-2}{2}}} K_{\frac{d-2}{2}} (E_{\lambda_a}^{\mathrm{c}} L_{wa}) \right]. \end{aligned} \tag{30}$$

This is just the tree-level diagram in a theory in $\mathbb{R}^d$ with cubic interactions

$$\sim \sum_{\lambda_{x,y,z}} \int \frac{d^d \vec{W}}{l_s^d} C_{\lambda_x \lambda_y \lambda_z} \Phi_{\lambda_x} \Phi_{\lambda_y} \Phi_{\lambda_z}. \tag{31}$$

The fields Φ_{λ_a} are the string fields in $\mathbb{R}^d$ that create the energy eigenstates $|\lambda_a\rangle$ wrapped on $\mathbb{S}_R^1$. These fields have mass $E_{\lambda_a}^{\mathrm{c}}$ and hence the propagators as in (10) and (30).

The integral (30) is heavily dominated by a saddle point near the origin, similar to [71,72]. To the order we are concerned, the saddle point solves a generalization of the Fermat-Torricelli problem where each edge is weighted by $E_{\lambda_a}^{\mathrm{c}}$ given in (5). The saddle-point value reads

$$\begin{aligned} F_{\lambda_x \lambda_y \lambda_z} &= \max_{\vec{W}} \left[\prod_a \frac{(E_{\lambda_a}^{\mathrm{c}})^{\frac{d}{2}} l_s^{d-1}}{\sqrt{\pi} (2L_{wa})^{\frac{d-2}{2}}} K_{\frac{d-2}{2}} (E_{\lambda_a}^{\mathrm{c}} L_{wa}) \right] \\ &= \left(\frac{\pi R}{L} \right)^{\frac{3(d-1)}{2}} \frac{e^{-3\mu L}}{\tilde{q}^{\frac{d-1}{8}}} \tilde{q}^{n_x + n_y + n_z} [1 + O(1/R^2)]. \end{aligned} \tag{32}$$

Around the saddle point, we take the Gaussian integral; this is crucial to the order we compute, but corrections to the Gaussian integral are of order 2 and higher.

We find that the s-wave amplitude agrees with the open string channel partition function in its leading L dependence:

$$\mathcal{Z}_{s\text{-wave}} = (\pi R/L)^{d-\frac{3}{2}} \frac{2\pi^d e^{-3\mu L}}{3^{\frac{d}{2}}\tilde{q}^{\frac{d-1}{8}}} \times \sum_{\lambda_{x,y,z}} C_{\lambda_x\lambda_y\lambda_z} v_{\lambda_x} v_{\lambda_y} v_{\lambda_z} \tilde{q}^{n_x+n_y+n_z}[1+O(1/R^2)]. \tag{33}$$

By comparing (33) with (25) and (27), we find the cubic coupling between the three lowest-lying string states **0**:

$$C_{\mathbf{000}} = \frac{e^{-2\pi MR}}{2(2\pi^2/3)^{\frac{d}{2}}}\left[1 + \frac{(d+2)Ml_s^2}{36R} + O(1/R^2)\right]. \tag{34}$$

We can also identify the coupling between the two lowest-lying ones and the second-lowest string **1**:

$$C_{\mathbf{001}} = \frac{e^{-2\pi MR}}{2(2\pi^2/3)^{\frac{d}{2}}}\left[\frac{d-3}{3\sqrt{d-1}} + \frac{(d+2)(d+21)Ml_s^2}{108\sqrt{d-1}R} + O(1/R^2)\right]. \tag{35}$$

At the order we are computing, among the higher string states there are degeneracies and one cannot distill the interaction vertices of each state. Instead, one can obtain average predictions from (27) and (30). At the next order, which we do not elaborate on here, it is possible to go further.

One immediate lesson is that the overall strength of the interaction among closed strings is $C_{\lambda_a\lambda_b\lambda_c} \sim e^{-2\pi MR}$, which is essentially the string coupling constant among strings of length $2\pi R$. It can be interpreted that the cubic interaction among segments of size l_s is $e^{-2\pi M l_s}$, and such a probability is raised to the power of the number of segments, $2\pi R/l_s$. These estimates should be general and applicable to other cases, such as excited glueballs [73]. We remark that in QCD, where strings (flux tubes) are breakable, the EFT predictions are still reliable as far as the string is concerned [75].

For $M > 0$, we see that the interaction is extremely weak for long closed strings, while for $-\frac{1}{l_s} \lesssim M < 0$, the interaction becomes strong when $R \gg \frac{1}{|M|}$. We comment that a negative M that is parametrically larger than l_s^{-1} is ruled out by perturbative stability and $-\frac{1}{l_s} \lesssim M < 0$ leads to strong couplings between long strings and hence requires further analysis of unitarity. This is schematically summarized in Fig. 4.

A central assumption is that when we act with the Polyakov loop Ω on the vacuum, we create single-string eigenstates. This assumption appears to be jeopardized when the cubic coupling is strong. As far as we are aware, lattice simulations, including the extensive simulations we referred to in the Introduction, showed no sign that the Polyakov loop mixes single-string and two-(anti)string states. Therefore, it follows that $-\frac{1}{l_s} \lesssim M < 0$ is strongly disfavored by unitarity arguments and one is compelled to suggest $M \geq 0$. It remains puzzling that in some large-N gauge theories negative values of M were reported in [67].

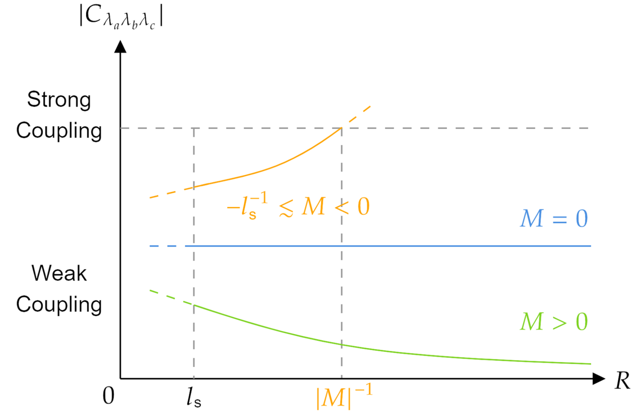

FIG. 4. Contact interaction with various M and R. The EFT breaks down when $R \lesssim l_s$ (left dashed lines) and the perturbative description fails for long strings if $M < 0$ (upper dashed line).

When $d = 3$ and $M = 0$, we notice that (27) contains only even powers of $\tilde{q}$. This is a consequence of a chiral $\mathbb{Z}_2$ symmetry of the NGBs. Explicitly, in the open string channel, this chiral $\mathbb{Z}_2$ is a combination of the transformation $\partial_t x_i \to i\partial_\sigma x_i$, $\partial_\sigma x_i \to -i\partial_t x_i$, which is T-duality-like, and the spatial reflection $x_2 \to x_3$, $x_3 \to x_2$. Such a chiral $\mathbb{Z}_2$ symmetry is preserved by the string bulk action (4) on X, Y, and Z world sheets up to order 2, and at higher orders it is possible to write an EFT term that violates it. Note that the chiral $\mathbb{Z}_2$ exchanges the Neumann (Dirichlet) boundary condition in the X_2 direction with that of Dirichlet (Neumann) in the X_3 direction. Therefore, from (18) we find that this chiral $\mathbb{Z}_2$ is preserved by the vertex only when $d = 3$.

In the closed string channel, the Polyakov operators Ω create $\mathbb{R}^3$ scalars that are charged under the chiral $\mathbb{Z}_2$. Let α_{-n}^i ($\tilde{\alpha}_{-n}^i$) be the closed string left (right) moving modes' creation operators at order 0, where $n \in \mathbb{N}^+$ and $i = 2, 3$. A generic scalar state is created by acting with $(\alpha_{-n}\cdot\alpha_{-n'})$, $(\alpha_{-n}\cdot\tilde{\alpha}_{-n'})$, and $(\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_{-n'})$ on the ground state. Obviously, a scalar state is not charged under the reflection, and the number of α_{-n}^i and $\tilde{\alpha}_{-n}^i$ operators satisfies $m = \tilde{m}$ mod 2. The transformation $\partial_t x_i \to i\partial_\sigma x_i$, $\partial_\sigma x_i \to -i\partial_t x_i$ when acting on mode operators reads

$$\begin{cases} \alpha_{-n}^i \to -\alpha_{-n}^i, \\ \tilde{\alpha}_{-n}^i \to \tilde{\alpha}_{-n}^i. \end{cases} \tag{36}$$

Therefore, each scalar state under the chiral $\mathbb{Z}_2$ acquires $(-1)^m = (-1)^{\tilde{m}}$, which depends on the number of left (right) moving mode operators. We conclude that the s-wave coupling $C_{\lambda_a\lambda_b\lambda_c}$ vanishes up to order 2 if

$$m_{\lambda_a} + m_{\lambda_b} + m_{\lambda_c} = 1 \text{ mod } 2. \tag{37}$$

This explains the selection rule we found when $d = 3$ and $M = 0$, where $m_{\mathbf{0}} = 0$ and $m_{\mathbf{1}} = 1$. We remark that this suggests that M is potentially a symmetry-breaking parameter in $(3+1)$ dimensions.

V. CONCLUSION AND OUTLOOK

In this paper, we demonstrated the open-closed string duality for a "baryon" configuration, as in Fig. 1. We claimed that up to order 1, the effective theory has two parameters: the string tension l_s^{-2} and the vertex mass M. We specified the action and boundary conditions in (16), (17), (19), and (20).

We showed that in the closed string channel the "baryon" is mapped to s-wave scattering of closed strings, and we extracted the universal s-wave couplings (34) and (35). These couplings have important implications for IR physics. We found that $M < 0$ suggests strong coupling and possible unitarity violation, while $M > 0$ implies weak coupling and is stable. Intriguingly, in $(3+1)$ dimensions the interaction is subject to a selection rule, which we argue is a consequence of a world-sheet chiral $\mathbb{Z}_2$ symmetry. We pointed out that $M \neq 0$ in $(3+1)$ dimensions breaks this symmetry and violates the selection rule.

Finally, we list a few important questions.

(1) Nonequilateral configurations: The interaction vertex (31) is independent of the location from which the closed strings are propagating. This is a trivial consequence of locality. To test this simple fact, we must consider the case where the end-point quarks of the baryon junction are positioned on a nonequilateral triangle, i.e., such that $\vec{X} = (L_x, 0, 0\ldots, 0)$, $\vec{Y} = (-\frac{L_y}{2}, \frac{\sqrt{3}L_y}{2}, 0\ldots, 0)$, and $\vec{Z} = (-\frac{L_z}{2}, -\frac{\sqrt{3}L_z}{2}, 0\ldots, 0)$. At order 0 and when $M = 0$, the quantization condition follows from (17) and the rigid condition (16). For a planar mode it reads

$$\cos(\omega L_x)\sin(\omega L_y)\sin(\omega L_z) + \text{cyclic} = 0, \tag{38}$$

while for a vertical mode it reads

$$\sin(\omega L_x)\cos(\omega L_y)\cos(\omega L_z) + \text{cyclic} = 0. \tag{39}$$

In the open string channel interpretation (i.e., the baryon junction channel), the partition function is evaluated as

$$\mathcal{Z}_{\mathrm{b}} = \sum_{\omega} e^{-2\pi R\omega}, \tag{40}$$

which is a function of modular parameters $q_{x,y,z} = e^{-\frac{2\pi^2 R}{L_{x,y,z}}}$. On the other hand, in the closed string channel, all we have to do in (30) is change the points from which we propagate the closed strings. The amplitudes v_{λ_a} and $C_{\lambda_a\lambda_b\lambda_c}$ are insensitive to where the strings are coming from, by locality. Hence, the closed string channel prediction is

$$\begin{aligned}\mathcal{Z}_{s\text{-wave}} = {} & \left(\frac{3\pi R}{L_x + L_y + L_z}\right)^{d-\frac{3}{2}} \frac{2\pi^d e^{-3\mu L}}{3^{\frac{d}{2}}(\tilde{q}_x\tilde{q}_y\tilde{q}_z)^{\frac{d-1}{24}}} \\ & \times \left[\frac{(L_x + L_y + L_z)^2}{3(L_xL_y + L_yL_z + L_zL_x)}\right]^{\frac{d}{2}-1} \\ & \times \sum_{\lambda_{x,y,z}} C_{\lambda_x\lambda_y\lambda_z} v_{\lambda_x} v_{\lambda_y} v_{\lambda_z} \tilde{q}_x^{n_{\lambda_x}} \tilde{q}_y^{n_{\lambda_y}} \tilde{q}_z^{n_{\lambda_z}},\end{aligned} \tag{41}$$

as a function of the dual variables $\tilde{q}_{x,y,z} = e^{-\frac{2L_{x,y,z}}{R}}$. We expect (41) to be identified with (40) following (38) and (39) through a multivariable modular transformation. It would be nice to carry this out. On the other hand, if the $\vec{X}\,\vec{Y}\,\vec{Z}$ triangle has an inner angle greater than 120°, then the Fermat point coincides with the obtuse vertex, as in [66], and a separate discussion of the junction condition and operators is necessary.

(2) Chiral $\mathbb{Z}_2$ symmetry: It would be interesting to understand how exactly this symmetry is broken if $M = 0$. In [67] it was suggested that $M = 0$ in the large-N Maldacena-Nunez solution. It would be nice to understand if the chiral $\mathbb{Z}_2$ symmetry we discussed is always an IR accidental symmetry or if it could be related to some microscopic symmetry.

(3) Instability: We encountered a perturbative instability when $M \lesssim -l_s^{-1}$. From an renormalization-group consideration, the end point of the instability cannot be a point-like junction. It would be interesting to find if there is a fat junction solution to the full nonlinear Nambu-Goto theory.

(4) Higher orders and non-s-wave scattering: It is straightforward to push the precision of this paper to order 2 and higher. Up to (and including) order 2, l_s and M are the only two parameters of the EFT. We would like to know if at this order non-s-wave contact interactions appear.

(5) It would be nice to know the junction mass in Yang-Mills theories and other similar theories and to test the theory we have discussed.

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APPENDIX A: GAUGE FIXING

We briefly explain the gauge we use in (15), as it is not completely standard. First, we notice that diffeomorphism invariance allows us to choose

$$\begin{cases} X_0 = \tilde{t}, \\ X_1 = \tilde{\sigma} + l_s f(\tilde{\sigma}) x_1(\tilde{t}), \\ X_i = l_s x_i(\tilde{t}, \tilde{\sigma}), \text{ for } 2 \le i \le d, \end{cases} \tag{A1}$$

where $\tilde{\sigma} \in [0, L]$, $x_1(\tilde{t})$ is the longitudinal displacement, and $f(\tilde{\sigma})$ is a smooth monotonic function such that $f(0) = 1$ and $f(L) = 0$.

As in the static gauge, we would like to pick a new coordinate $t = X_0 = \tilde{t}$ and $\sigma = X_1 = \tilde{\sigma} + l_s f(\tilde{\sigma}) x_1(\tilde{t})$. Following the chain rule

$$\begin{aligned} \partial_t x_i &= \partial_{\tilde{t}} x_i - l_s f \partial_t x_1 \partial_{\tilde{\sigma}} x_i, \\ \partial_\sigma x_i &= (1 + l_s \partial_{\tilde{\sigma}} f x_1)^{-1} \partial_{\tilde{\sigma}} x_i, \end{aligned} \tag{A2}$$

one can verify that the bulk action is as in (4), yet the domain of integration becomes dynamical, $\sigma \in [l_s x(t), L]$.

APPENDIX B: GREEN'S FUNCTION AND REGULARIZATION

The Dedekind η function is defined as

$$\eta(q) \equiv q^{\frac{1}{24}} \prod_{n \in \mathbb{N}^+} (1 - q^n). \tag{B1}$$

The Eisenstein series is defined as

$$E_{2k}(q) \equiv 1 + \frac{2}{\zeta(1-2k)} \sum_{n \in \mathbb{N}^+} \frac{n^{2k-1} q^n}{1 - q^n}, \quad k \in \mathbb{N}^+. \tag{B2}$$

Let $q = e^{2\pi i \tau}$ and $\tilde{\tau} = -\frac{1}{\tau}$; we use the following modular transformation of these functions:

$$\begin{aligned} \eta(q) &= \sqrt{-i\tilde{\tau}}\, \eta(\tilde{q}), \\ E_2(q) &= -\frac{6i}{\pi} \tilde{\tau} + \tilde{\tau}^2 E_2(\tilde{q}), \\ E_{2k}(q) &= \tilde{\tau}^{2k} E_{2k}(\tilde{q}). \end{aligned} \tag{B3}$$

More practically, we need

$$\begin{aligned} \log q E_2(q) &= -12 - \log \tilde{q} E_2(\tilde{q}), \\ \log q E_2(\sqrt{q}) &= -24 - 4 \log \tilde{q} E_2(\tilde{q}^2). \end{aligned} \tag{B4}$$

We denote the modular parameter as $q = e^{-\frac{2\pi^2 R}{L}}$, and a useful infinite sum reads

$$\begin{aligned} \sum_{r \in \mathbb{N}+\frac{1}{2}} \sum_{m \in \mathbb{Z}} \frac{\frac{\pi^2 r^2}{L^2}}{\frac{\pi^2 r^2}{L^2} + \frac{m^2}{R^2}} &= -\frac{\log q}{2} \sum_{r \in \mathbb{N}+\frac{1}{2}} r \frac{1 + q^r}{1 - q^r} \\ &= -\frac{\log q}{48} [2E_2(q) - E_2(\sqrt{q})], \\ \sum_{r \in \mathbb{N}+\frac{1}{2}} \sum_{m \in \mathbb{Z}} \frac{\frac{m^2}{R^2}}{\frac{\pi^2 r^2}{L^2} + \frac{m^2}{R^2}} &= -\sum_{r \in \mathbb{N}+\frac{1}{2}} \sum_{m \in \mathbb{Z}} \frac{\frac{\pi^2 r^2}{L^2}}{\frac{\pi^2 r^2}{L^2} + \frac{m^2}{R^2}} \\ &= \frac{\log q}{48} [2E_2(q) - E_2(\sqrt{q})], \end{aligned} \tag{B5}$$

where we used $\sum_{m \in \mathbb{Z}} 1 = 1 + 2\zeta(0) = 0$.

For the Neumann-Dirichlet boundary condition on $\mathbb{S}^1_R \times [0, L]$, Green's function reads

$$\begin{aligned} \tilde{G}(\sigma, \sigma', \tau - \tau') = \frac{1}{\pi R L} \sum_{r \in \mathbb{N}+\frac{1}{2}} \sum_{m \in \mathbb{Z}} \frac{e^{i\frac{m}{R}(\tau - \tau')}}{\frac{\pi^2 r^2}{L^2} + \frac{m^2}{R^2}} \\ \times \cos\left(\frac{r\pi\sigma}{L}\right) \cos\left(\frac{r\pi\sigma'}{L}\right). \end{aligned} \tag{B6}$$

The coincident point derivatives of (B6) are

$$\tilde{G}_{\tau\tau}(\sigma) = \frac{1}{\pi R L} \sum_{r \in \mathbb{N}+\frac{1}{2}} \sum_{m \in \mathbb{Z}} \frac{\frac{m^2}{R^2}}{\frac{\pi^2 r^2}{L^2} + \frac{m^2}{R^2}} \cos^2\left(\frac{r\pi\sigma}{L}\right). \tag{B7}$$

Using (B5), we obtain

$$\begin{aligned} \int_{\sigma=0} d\tau \tilde{G}_{\tau\tau} &= \frac{2}{L} \sum_{r \in \mathbb{N}+\frac{1}{2}} \sum_{m \in \mathbb{Z}} \frac{\frac{m^2}{R^2}}{\frac{\pi^2 r^2}{L^2} + \frac{m^2}{R^2}} \\ &= \frac{\log q}{24L} [2E_2(q) - E_2(\sqrt{q})]. \end{aligned} \tag{B8}$$

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