Temperature fluctuations in a relativistic gas: Pressure corrections and possible consequences in the deconfinement transition

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In this work, we study the effects of random temperature fluctuations on the partition function of a quantum system by means of the *replica method*. This picture provides a conceptual model for a quantum nonequilibrium system, depicted as an ensemble of subsystems at different temperatures, randomly distributed with respect to a given mean value. We then assume the temperature displays stochastic fluctuations $T = T_0 + \delta T$ with respect to its ensemble average value T_0, with zero mean $\overline{\delta T} = 0$ and standard deviation $\overline{\delta T^2} = \Delta$. By means of the replica method, we obtain the average grand canonical potential, leading to the equation of state and the corresponding excess pressure caused by these fluctuations with respect to the equilibrium system at a uniform temperature. Our findings reveal an increase in pressure as the system's ensemble average temperature T_0 rises, consistently exceeding the pressure observed in an equilibrium state. We applied our general formalism to three paradigmatic physical systems; the relativistic Fermi gas, the ideal gas of photons, and a gas of non-Abelian gauge fields (gluons) in the noninteracting limit. Finally, we explore the implications for the deconfinement transition in the context of the simple bag model, where we show that the critical temperature decreases.

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I. INTRODUCTION

Finite temperature quantum field theory provides a general framework to study the statistical and thermodynamic properties of quantum matter, from condensed matter systems to applications in high-energy physics [1–3]. However, many experimental systems of interest are not in strict thermodynamic equilibrium. For instance, in high-energy experiments such as ultrarelativistic heavy-ion collisions, the emergence of the quark-gluon plasma is a broadly accepted phenomenon [4–6]. This state involves the coexistence of the fundamental degrees of freedom in quantum chromodynamics, in principle assumed to be in thermal equilibrium, so that finite temperature field theory has been instrumental in successfully explaining and predicting a wide array of observables arising from such systems. Nevertheless, the initial and hadronization phases of a heavy-ion collision are not in thermal equilibrium [7–10]. This leads to essential questions: How do thermal fluctuations impact observables during these stages? And what theoretical approaches can correctly capture those effects?

A number of theoretical approximations have been developed to represent nonequilibrium conditions in quantum systems. For instance, the Keldysh contour path emerged as a formalism to describe the quantum mechanical evolution of systems under time-varying external fields [11]. This formalism has found notable applications, particularly within strongly correlated electron systems [12–16], offering insights into their many-body properties and nonequilibrium dynamics [17,18]. Another approach used to model the fluctuations of intensive thermodynamic parameters is the so-called *superstatistics*. This method assumes an ensemble of subsystems, each of them individually in local thermal equilibrium [19]. Although this approximation represents a semiclassical description of the thermal fluctuations,

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it has been applied extensively in condensed matter systems [20,21], information theory [22–24], and more recently to the study of the QCD phase diagram [25,26].

There is an alternative perspective to address fluctuations or disorder within a system; the well-known *replica trick*, introduced by Parisi as a method to average the free energy, defined via the logarithm of the partition function $\ln Z$, of a system over quenched (or frozen) disorder [27]. This method builds on the mathematical identity,

$$\overline{\ln Z} = \lim_{n\to 0} \frac{\overline{Z^n} - 1}{n}, \tag{1}$$

which in practical implementation involves replicating the system n-times, with the corresponding partition function as an effective coupling of the n-replicas of the same Lagrangian. This procedure can be performed either at the level of the canonical ensemble, leading to an statistical average of the Helmholtz free-energy $\overline{\mathcal{F}}(\mathcal{N}, \mathcal{V}, T) = -T\overline{\ln Z}(\mathcal{N}, \mathcal{V}, T)$ or the grand canonical ensemble, where in such case one obtains the corresponding average of the grand potential $\bar{\Omega}(\mu, \mathcal{V}, T) = -T\overline{\ln Z}(\mu, \mathcal{V}, T)$.

In a series of two recent articles [28,29], we applied this formalism to investigate the effects of stochastic fluctuations in a classical background magnetic field on the properties of a quantum electrodynamics (QED) medium. Specifically, we showed that magnetic fluctuations lead to an effective interaction between the fermions. This approach, at the perturbative level, results in quasi-particles propagating through a dispersive medium [28]. Moreover, our mean-field analysis predicted the emergence of order parameters representing the components of a vector current [29]. Both perspectives revealed deviations from $U(1)$ symmetry, highlighting the influence of stochastic fluctuations within the background magnetic field on the properties of the QED medium.

In this study, we shall assume a nonequilibrium scenario, where temperature is then not defined uniformly through the whole system, but smaller regions may still be pictured as nearly thermalized subsystems. Therefore, we model this situation by an ensemble of subsystems whose individual temperatures $T = T_0 + \delta T$ are subjected to stochastic fluctuations with zero mean $\overline{\delta T} = 0$, but finite variance $\overline{\delta T^2} = \Delta$.

For technical reasons, it is more convenient to represent these fluctuations in terms of the inverse temperature,

$$\beta = (T_0 + \delta T)^{-1} = T_0^{-1} - \frac{\delta T}{T_0^2} = \beta_0 + \delta\beta, \tag{2}$$

where clearly we have the corresponding relations,

$$\begin{aligned} \delta\beta &= -\frac{\delta T}{T_0^2}, \\ \overline{\delta\beta} &= -T_0^{-2}\overline{\delta T} = 0, \\ \overline{\delta\beta^2} &= T_0^{-4}\overline{\delta T^2} = \beta_0^4 \Delta = \Delta_\beta. \end{aligned} \tag{3}$$

We capture these statistical features by assuming a Gaussian distribution with zero mean, i.e.

$$dP[\delta\beta] = \frac{d(\delta\beta)}{\sqrt{2\pi\Delta_\beta}} e^{-\frac{\delta\beta^2}{2\Delta_\beta}}. \tag{4}$$

Therefore, the corresponding moments of the thermal fluctuations are given by the exact expressions ($\forall \ j \in \mathbb{N}$)

$$\begin{aligned} \overline{\delta\beta} &= \overline{\delta\beta^{2j-1}} = 0, \\ \overline{\delta\beta^2} &= \Delta_\beta, \\ \overline{\delta\beta^{2j}} &= \Delta_\beta^j (2j-1)!!. \end{aligned} \tag{5}$$

We will further apply the replica formalism to average over these temperature fluctuations. Subsequently, in the grand canonical ensemble at finite chemical potential, by using the Matsubara imaginary time formalism, we calculate the system's grand potential, and its equation of state.

II. THE GRAND CANONICAL PARTITION FUNCTION

Let us start by considering the general definition of the partition function in the grand canonical ensemble,

$$Z(\mu, \mathcal{V}, T) = \mathrm{Tr}[e^{-\beta(\hat{H} - \mu\hat{N})}]. \tag{6}$$

As we stated in the Introduction, we shall assume that our system is not fully thermalized, but for a quenched distribution of local temperatures, with the statistical properties described by Eq. (5). Moreover, the statistical average over such distribution of temperatures is calculated via the replica trick Eq. (1),

$$\overline{\ln Z} = \lim_{n\to 0} \frac{1}{n} \left(\overline{\mathrm{Tr}\left[\exp\left\{ -(\beta_0 + \delta\beta) \sum_{r=1}^{n} \hat{K}^{(r)} \right\} \right]} - 1 \right), \tag{7}$$

where here each replica $1 \le r \le n$ has an associated operator $\hat{K}^{(r)}$.

Let us now expand the exponential inside the trace in Eq. (7) in powers of the fluctuation $\delta\beta$, and then take the statistical average of each term using the properties of the distribution in Eq. (5) as follows:

$$\begin{aligned}\overline{Z^n} = \overline{\mathrm{Tr}\left[\exp\left\{-(\beta_0+\delta\beta)\sum_{r=1}^{n}\hat{K}^{(r)}\right\}\right]} &= \int dP[\delta\beta]\mathrm{Tr}\left[e^{-\beta_0\sum_{r=1}^{n}\hat{K}^{(r)}}\left(1+\sum_{j=1}^{\infty}\frac{(-1)^j(\delta\beta)^j}{j!}\left(\sum_{r=1}^{n}\hat{K}^{(r)}\right)^j\right)\right]\\ &= \mathrm{Tr}\left[e^{-\beta_0\sum_{r=1}^{n}\hat{K}^{(r)}}\left(1+\sum_{j=1}^{\infty}\frac{\Delta_\beta^j}{(2j)!}(2j-1)!!\left(\sum_{r=1}^{n}\hat{K}^{(r)}\right)^{2j}\right)\right]\\ &= \left(1+\sum_{j=1}^{\infty}\frac{\Delta_\beta^j}{(2j)!}(2j-1)!!\frac{\partial^{2j}}{\partial\beta_0^{2j}}\right)\mathrm{Tr}\left[e^{-\beta_0\sum_{r=1}^{n}\hat{K}^{(r)}}\right]. \qquad (8)\end{aligned}$$

Remarkably, after the statistical average over fluctuations was taken, the power expansion involving the trace can be expressed as temperature derivatives of the partition function of the reference system, defined by

$$Z_0^n = \mathrm{Tr}\left[e^{-\beta_0\sum_{r=1}^{n}\hat{K}^{(r)}}\right], \qquad (9)$$

as follows:

$$\begin{aligned}\overline{Z^n} &= \left(1+\sum_{j=1}^{\infty}\frac{(\Delta_\beta/2)^j}{j!}\frac{\partial^{2j}}{\partial\beta_0^{2j}}\right)Z_0^n\\ &= \exp\left[\frac{\Delta_\beta}{2}\frac{\partial^2}{\partial\beta_0^2}\right]Z_0^n, \qquad (10)\end{aligned}$$

where in the first line we used the identity $(2j-1)!!/(2j)! = 2^{-j}/j!$, and we finally reassembled the series in the form of the exponential differential operator.

For the thermodynamics analysis of different systems, we shall be interested in the statistical average over noise of the grand potential $-\beta_0\overline{\Omega} = \overline{\ln Z}$, such that applying the appropriate limit over the number of replicas $n\to 0$, we obtain

$$\begin{aligned}-\beta_0\bar{\Omega} = \overline{\ln Z} = \lim_{n\to 0}\frac{\overline{Z^n}-1}{n} &= \exp\left[\frac{\Delta_\beta}{2}\frac{\partial^2}{\partial\beta_0^2}\right]\lim_{n\to 0}\frac{Z_0^n-1}{n}\\ &= \exp\left[\frac{\Delta_\beta}{2}\frac{\partial^2}{\partial\beta_0^2}\right]\lim_{n\to 0}\frac{e^{n\ln Z_0}-1}{n}\\ &= \exp\left[\frac{\Delta_\beta}{2}\frac{\partial^2}{\partial\beta_0^2}\right]\ln Z_0. \qquad (11)\end{aligned}$$

Even though Eq. (11) is an exact result, we shall assume that the temperature fluctuations are weak. Therefore, up to first order in the fluctuation Δ, the Taylor expansion of the exponential differential operator leads to

$$\begin{aligned}\overline{\ln Z/Z_0} &= \frac{\Delta_\beta}{2}\frac{\partial^2}{\partial\beta_0^2}\ln Z_0 + O(\Delta_\beta^2)\\ &= \beta_0(\mathcal{P}\mathcal{V}-(\mathcal{P}\mathcal{V})_{ig}), \qquad (12)\end{aligned}$$

where $\beta_0(\mathcal{P}\mathcal{V})_{ig} = \ln Z_0$ is the equation of state for the corresponding ideal gas. Therefore, up to $O(\Delta^2)$, the *excess pressure* $\delta\mathcal{P}\equiv\mathcal{P}-\mathcal{P}_{\mathrm{ig}}$ of the gas due to the average effect of the temperature fluctuation is

$$\delta\mathcal{P}\equiv\mathcal{P}-\mathcal{P}_{\mathrm{ig}} = \frac{\Delta_\beta}{2\mathcal{V}\beta_0}\frac{\partial^2}{\partial\beta_0^2}\ln Z_0 + O(\Delta^2). \qquad (13)$$

It is interesting to analyze the physical interpretation of this lowest order contribution using general thermodynamics relations. From the differential form of the grand potential for the ideal reference system, we have

$$d\Omega_0 = -\mathcal{P}d\mathcal{V} - SdT_0 - Nd\mu, \qquad (14)$$

from which we conclude that

$$S = -\left.\frac{\partial\Omega_0}{\partial T_0}\right|_{\mu,\mathcal{V}}. \qquad (15)$$

On the other hand, using $T_0=\beta_0^{-1}$ and the definition $\Omega_0 = -T_0\ln Z_0$, it is possible to show the identity (see Appendix A for details)

$$\begin{aligned}\frac{\Delta_\beta}{2\beta_0\mathcal{V}}\frac{\partial^2}{\partial\beta_0^2}\ln Z_0 &= \frac{\Delta_\beta T_0^2}{2\mathcal{V}}T_0\left.\frac{\partial S}{\partial T_0}\right|_{\mu,\mathcal{V}}\\ &= \frac{\Delta_\beta T_0^2}{2\mathcal{V}}\left[C_v + \frac{(T_0\frac{\partial N}{\partial T_0}|_{\mu,\mathcal{V}})^2}{\langle(\delta\hat{N})^2\rangle}\right]\\ &\geq 0. \qquad (16)\end{aligned}$$

Therefore, based on general thermodynamics considerations for the ideal reference system, we expect for the excess pressure due to random temperature fluctuations in the ensemble to be positive $\delta\mathcal{P}\geq 0$ at first order in Δ.

In the next sections, we shall apply this general identity to three basic but rather fundamental examples: The relativistic Fermi gas, a gas of Abelian gauge fields (photons), and a gas of non-Abelian gauge fields (gluons).

III. RELATIVISTIC FERMI GAS WITH THERMAL NOISE

Let us first focus on a system of QED fermions, described by the Hamiltonian operator (including the chemical potential)

$$\hat{H} - \mu\hat{N} = \int d^3x \hat{\psi}^\dagger(\mathbf{x})\gamma^0[\boldsymbol{\gamma}\cdot(-\mathrm{i}\nabla) + m - \gamma^0\mu]\hat{\psi}(\mathbf{x}) \equiv \hat{K}. \tag{17}$$

The corresponding partition function $Z_0^n \equiv Z_{F0}^n$ in Eq. (10) corresponds to n-replicas of this ideal gas of relativistic fermions, represented by the Grassmann fields $\psi_r(x)$, with $1 \le r \le n$ the replica index. The corresponding standard functional integral representation is thus given by

$$\begin{aligned} Z_{F0}^n &= \prod_{r=1}^n \int \mathcal{D}[\psi_r^\dagger, \psi_r] \exp\left[-\int_0^{\beta_0} d\tau \sum_{r=1}^n \psi_r^\dagger(\mathbf{x},\tau)\gamma^0(\gamma^0(\partial_\tau - \mu) + \boldsymbol{\gamma}\cdot\mathbf{p} + m)\psi_r(\mathbf{x},\tau)\right] \\ &= \det\left[\partial_\tau - \mu + \gamma^0\boldsymbol{\gamma}\cdot\mathbf{p} + m\gamma^0\right]^n \\ &= \exp\left\{n\mathrm{Tr}\ln\left[\partial_\tau - \mu + \gamma^0\boldsymbol{\gamma}\cdot\mathbf{p} + m\gamma^0\right]\right\} \\ &= \exp\left(n\ln Z_{F0}\right), \end{aligned} \tag{18}$$

where the symbol Tr stands for the functional trace (integral over phase-space) and the trace in the space of Dirac matrices. Here, we also defined the partition function for the fermion gas,

$$\begin{aligned} \ln Z_{F0} &= \mathrm{Tr}\ln\left[\partial_\tau - \mu + \gamma^0\boldsymbol{\gamma}\cdot\mathbf{p} + m\gamma^0\right] \\ &= \mathcal{V}\int\frac{d^3p}{(2\pi)^3}\sum_{k\in\mathbb{Z}}\mathrm{tr}\ln\left[\mathrm{i}\omega_k - \mu + \gamma^0\boldsymbol{\gamma}\cdot\mathbf{p} + m\gamma^0\right], \end{aligned} \tag{19}$$

where we diagonalized the operator in Matsubara-momentum space, and $\omega_k = (2k+1)\pi/\beta_0$ $(k \in \mathbb{Z})$ are the Fermi Matsubara frequencies. Using the elementary identity, valid for any diagonalizable matrix $\hat{A}$,

$$\begin{aligned} \mathrm{tr}\ln\hat{A} &= \mathrm{tr}[\hat{P}^{-1}(\ln\hat{A})\hat{P}] = \mathrm{tr}[\ln(\hat{P}^{-1}\hat{A}\,\hat{P})] \\ &= \sum_i \ln\lambda_i, \end{aligned} \tag{20}$$

where $\hat{P}$ is the unitary transformation that diagonalizes $\hat{A}$ and λ_i its eigenvalues.

We evaluate the trace in Eq. (20) from the (double-degenerate) eigenvalues of the matrix in the argument of the logarithm: $\mathrm{i}\omega_k - \mu \pm E_\mathbf{p}$, with $E_\mathbf{p} = \sqrt{p^2 + m^2}$, to obtain

$$\begin{aligned} \ln Z_{F0} &= 2\mathcal{V}\int\frac{d^3p}{(2\pi)^3}\sum_{k\in\mathbb{Z}}\{\ln\left[\mathrm{i}\omega_k - \mu + E_\mathbf{p}\right] \\ &\quad + \ln\left[\mathrm{i}\omega_k - \mu - E_\mathbf{p}\right]\} \\ &= 2\mathcal{V}\int\frac{d^3p}{(2\pi)^3}\left\{\ln\left(1 + e^{\beta_0(\mu - E_\mathbf{p})}\right)\right. \\ &\quad \left. + \ln\left(1 + e^{\beta_0(\mu + E_\mathbf{p})}\right)\right\}. \end{aligned} \tag{21}$$

The evaluation of the Matsubara sum in the final step is presented in detail in Appendix B. Finally, inserting this result into Eq. (11), we obtain

$$\begin{aligned} \overline{\ln Z_F} &= \lim_{n\to 0}\frac{\overline{Z_F^n} - 1}{n} = \exp\left[\frac{\Delta_\beta}{2}\frac{\partial^2}{\partial\beta_0^2}\right]\lim_{n\to 0}\frac{Z_{F0}^n - 1}{n} \\ &= \exp\left[\frac{\Delta_\beta}{2}\frac{\partial^2}{\partial\beta_0^2}\right]\lim_{n\to 0}\frac{e^{n\ln Z_{F0}} - 1}{n} \\ &= \exp\left[\frac{\Delta_\beta}{2}\frac{\partial^2}{\partial\beta_0^2}\right]\ln Z_{F0}. \end{aligned} \tag{22}$$

A. The equation of state

Applying Eq. (13), we obtain the explicit formula for the excess pressure $\delta\mathcal{P} = \mathcal{P} - \mathcal{P}_{ig}$,

$$\begin{aligned} \delta\mathcal{P} &= \frac{\Delta_\beta}{\beta_0}\sum_{s=\pm 1}\int\frac{d^3p}{(2\pi)^3}(E_\mathbf{p} + s\mu)^2 n_\mathrm{F}\left(\frac{E_\mathbf{p} + s\mu}{T_0}\right) \\ &\quad \times\left[1 - n_\mathrm{F}\left(\frac{E_\mathbf{p} + s\mu}{T_0}\right)\right], \end{aligned} \tag{23}$$

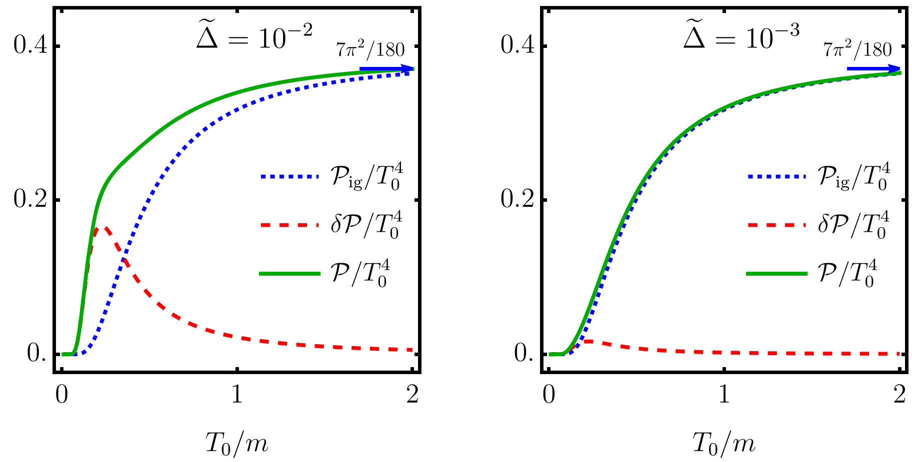

FIG. 1. Pressure normalized to T_0^4 when $\mu = 0$ for the ideal fermion gas (dotted line), the excess pressure of Eq. (25) (dashed line), and the total pressure (continuous line). The arrow indicates the asymptotic ideal gas limit at high temperatures.

where $n_F(x) = (e^x + 1)^{-1}$ is the Fermi distribution, allows us to verify that it is clearly positive definite, in agreement with our general analysis in Eq. (16).

In order to evaluate the integral, it is convenient to perform the change of variable (here $E = E_{\mathbf{p}}$ for short notation)

$$\mathbf{p}^2 = E^2 - m^2 \rightarrow d^3p = 4\pi p^2 dp = 4\pi\sqrt{E^2 - m^2}EdE, \quad (24)$$

and later define the dimensionless variables $x \equiv E/m$, $y \equiv T_0/m$, $z = \mu/m$, and $\tilde{\Delta} \equiv \Delta/m^2$, with m the bare mass (it is an ideal Fermi gas without interactions), such that we have

$$\delta P = \frac{m^4\tilde{\Delta}}{2\pi^2 y^3}\sum_{s=\pm1}\int_1^\infty dx\, x(x+sz)^2\sqrt{x^2-1} \times n_{\rm F}\left(\frac{x+sz}{y}\right)n_{\rm F}\left(-\frac{x+sz}{y}\right), \quad (25)$$

where we used the property $n_{\rm F}(-x) = 1 - n_{\rm F}(x)$.

We represent the ratio between the total ensemble-average pressure $\mathcal{P} = \mathcal{P}_{\rm ig} + \delta\mathcal{P}$ and the fourth power of the average temperature, i.e. $\mathcal{P}/T_0^4$ in Fig. 1, for the specific case of vanishing chemical potential. For the sake of comparison, we included the well known temperature dependence of the ideal relativistic Fermi gas, that asymptotically attains the limit $\mathcal{P}/T_0^4 \sim 7\pi^2/180$ as $T_0/m > 1$ [2], where m is the bare mass, consistent with our noninteracting model of an ideal Fermi gas. We also represent the excess pressure contribution $\delta\mathcal{P}$ arising from stochastic fluctuations in the temperature, calculated after Eq. (25). As can be noticed, the excess pressure represents a significant contribution at low equilibrium temperatures $T_0 \lesssim m$, but it rapidly decreases at higher temperatures $T_0 > m$, with m the bare mass consistent with our noninteracting fermion model. Moreover, the magnitude of this deviation from the ideal gas pressure in actual thermal equilibrium is proportional to the parameter $\tilde{\Delta}$ representing the standard deviation in the temperature distribution across the ensemble of subsystems. The effects of a finite chemical potential $\mu > 0$ over the excess pressure are displayed in Fig. 2. Clearly, the chemical potential represents a small effect over the overall temperature dependence already discussed in Fig. 1

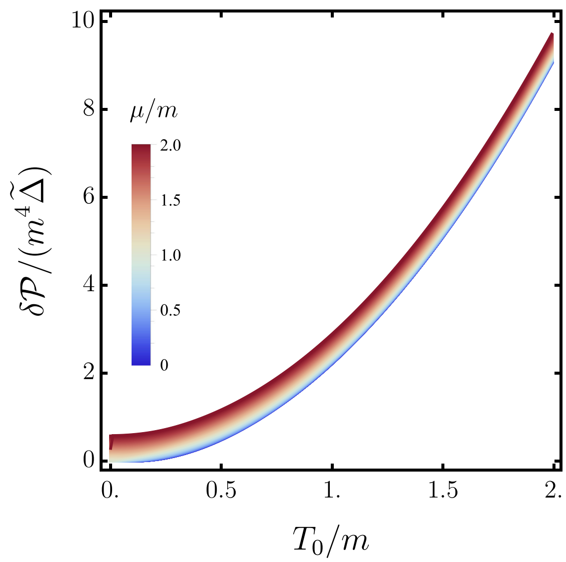

FIG. 2. Excess pressure of the Fermi gas, computed up to order $O(\Delta)$ from Eq. (25), as a function of the average temperature T_0, and the chemical potential μ.

IV. THE PHOTON GAS WITH THERMAL NOISE

In this section, we shall apply our general result as expressed by Eq. (10) to the particular case of a system of free Abelian gauge fields $A_\mu(x)$, whose gauge invariant Lagrangian is defined by

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{26}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ the strength tensor. As is well-established [3], the partition function is obtained by the Fadeev-Popov technique

$$\begin{aligned} Z_{B0} &= \mathrm{Tr}e^{-\beta_0 \hat{H}} \\ &= \int \mathcal{D}[A_\mu]\delta[\mathcal{F}]\det\left(\frac{\partial\mathcal{F}}{\partial\alpha}\right)e^{\int_0^{\beta_0}d\tau\int d^3x\mathcal{L}}, \end{aligned} \tag{27}$$

where the general gauge-fixing condition is determined by the functional equation $\mathcal{F}[A_\mu] = 0$. A convenient way to perform the integral is to consider the family of covariant gauges $\mathcal{F}[A_\mu] = \partial^\mu A_\mu - f(x,\tau) = 0$. Therefore, under a gauge transformation $A^\mu \to A^\mu - \partial^\mu\alpha$, the former equation becomes

$$\mathcal{F}[A_\mu - \partial^\mu\alpha] = \partial^\mu A_\mu - f(x,\tau) - \partial^2\alpha, \tag{28}$$

thus leading to the result

$$\det\left(\frac{\partial\mathcal{F}}{\partial\alpha}\right) = \det(-\partial^2), \tag{29}$$

such that

$$Z_{B0} = \int \mathcal{D}[A_\mu]\delta[\partial^\mu A_\mu - f]\det(-\partial^2)e^{\int_0^{\beta_0}d\tau\int d^3x\mathcal{L}}. \tag{30}$$

Integrating over a functional distribution of gauge-fixing conditions with weight $\exp[-\frac{1}{2\xi}\int d\tau\int d^3x f^2(x,\tau)]$, we arrive at

$$Z_{B0} = \int \mathcal{D}[A_\mu]\det(-\partial^2)e^{\int_0^{\beta_0}d\tau\int d^3x(\mathcal{L}-\frac{1}{2\xi}(\partial^\mu A_\mu)^2)}. \tag{31}$$

Finally, we introduce a set of Grassmann "ghost" fields $\bar{\eta}, \eta$, to represent the determinant of the Laplacian operator, such that

$$\begin{aligned} Z_{B0} &= \int \mathcal{D}[A_\mu]\int \mathcal{D}[\bar{\eta},\eta]e^{\int_0^{\beta_0}d\tau\int d^3x(\mathcal{L}-\frac{1}{2\xi}(\partial^\mu A_\mu)^2-\bar{\eta}\partial^2\eta)} \\ &= \int \mathcal{D}[A_\mu]\int \mathcal{D}[\bar{\eta},\eta]e^{\int_0^{\beta_0}d\tau\int d^3x[-\frac{1}{2}A_\mu(-g^{\mu\nu}\partial^2+(1-\frac{1}{\xi})\partial^\mu\partial^\nu)A_\nu-\bar{\eta}\partial^2\eta]}, \end{aligned} \tag{32}$$

where in the second step we integrated by parts. The corresponding partition function for n-replicas of this system (assuming the Feynman gauge for definiteness $\xi = 1$) requires the introduction of the replica gauge fields $A_\mu^a(x)$, as well as the replica ghosts $\bar{\eta}^r$, η^r, for $1 \le r \le n$, such that

$$\begin{aligned} Z_{B0}^n &= \int \prod_{r=1}^n \mathcal{D}[A_\mu^r]\int \mathcal{D}[\bar{\eta}^r,\eta^r]e^{\int_0^{\beta_0}d\tau\int d^3x[-\frac{1}{2}\sum_{r=1}^n A_\mu^r(-\partial^2)A_\mu^r-\sum_{r=1}^n\bar{\eta}^r\partial^2\eta^r]} \\ &= (\det[-\partial^2])^{-4n/2}\times(\det[-\partial^2])^{2n/2} = \exp[-n\mathrm{Tr}\ln[-\partial^2]] \\ &= \exp[n\ln Z_{B0}], \end{aligned} \tag{33}$$

where we identify the grand partition function of an ideal gas of massless bosons (with zero chemical potential $\mu_B = 0$)

$$\begin{aligned} \ln Z_{B0} &= -\mathrm{Tr}\ln[-\partial^2] = -\mathrm{Tr}\ln[-\partial_\tau^2 - \nabla^2] \\ &= -\mathcal{V}\int\frac{d^3p}{(2\pi)^3}\sum_{k\in\mathbb{Z}}\ln[\omega_k^2 + \mathbf{p}^2] \\ &= -\mathcal{V}\int\frac{d^3p}{(2\pi)^3}\sum_{k\in\mathbb{Z}}\{\ln[|\mathbf{p}| + \mathrm{i}\omega_k] + \ln[|\mathbf{p}| - \mathrm{i}\omega_k]\} \\ &= -\mathcal{V}\int\frac{d^3p}{(2\pi)^3}[\ln(1 - e^{-\beta_0|\mathbf{p}|}) + \ln(e^{\beta_0|\mathbf{p}|} - 1)] \\ &= -2\mathcal{V}\int\frac{d^3p}{(2\pi)^3}\left[\frac{\beta_0}{2}|\mathbf{p}| + \ln(1 - e^{-\beta_0|\mathbf{p}|})\right]. \end{aligned} \tag{34}$$

Here, the Matsubara sum is over bosonic even frequencies $\omega_k = 2k\pi/\beta_0$, for $k\in\mathbb{Z}$ (details in Appendix B). In addition, the overall prefactor represents the $\nu_B = 2$ degrees of freedom, that for photons represents the two transverse physical polarization modes [the other two were naturally removed upon integrating the ghosts, as seen in Eq. (33)]. The first, linear term in the integral Eq. (34) represents the (divergent) vacuum energy, and can therefore be subtracted. For the remaining logarithmic expression, it is convenient to use polar coordinates $d^3p = 4\pi|\mathbf{p}|^2d|\mathbf{p}|$, and then change the integration variable by introducing $x = \beta_0|\mathbf{p}|$ to obtain the grand potential,

$$\begin{aligned} \Omega_0^B &= -T_0\ln Z_{B0} = \nu_B\frac{\mathcal{V}T_0^4}{2\pi^2}\int_0^\infty dx\,x^2\ln(1 - e^{-x}) \\ &= -\nu_B\frac{\mathcal{V}T_0^4}{6\pi^2}\int_0^\infty dx\frac{x^3}{e^x - 1} \\ &= -\nu_B\mathcal{V}\frac{\pi^2T_0^4}{90}, \end{aligned} \tag{35}$$

where in the second line we integrated by parts, and we finally generalized the result by introducing ν_B as the total number of discrete degrees of freedom. The ideal gas pressure for bosons in equilibrium at temperature T_0 is thus

$$\mathcal{P}_{\rm ig}^{B} = \nu_B \frac{\pi^2 T_0^4}{90}. \tag{36}$$

To obtain the statistical average over noise, we again apply the identity Eq. (10), such that

$$\begin{aligned}\overline{\ln Z_B} &= \lim_{n\to 0}\frac{\overline{Z_B^n}-1}{n}\\ &= \exp\left[\frac{\Delta_\beta}{2}\frac{\partial^2}{\partial\beta_0^2}\right]\lim_{n\to 0}\frac{Z_{B0}^n-1}{n}\\ &= \exp\left[\frac{\Delta_\beta}{2}\frac{\partial^2}{\partial\beta_0^2}\right]\ln Z_{B0}.\end{aligned} \tag{37}$$

Therefore, we have

$$\begin{aligned}\overline{\ln Z_B/Z_{B0}} &= \frac{\Delta_\beta}{2}\frac{\partial^2}{\partial\beta_0^2}\ln Z_{B0} + O(\Delta_\beta^2)\\ &= \beta_0(\mathcal{P}\mathcal{V} - (\mathcal{P}\mathcal{V})_{ig}).\end{aligned} \tag{38}$$

The corresponding excess pressure due to nonequilibrium thermal fluctuations will be, after Eq. (13)

$$\begin{aligned}\delta\mathcal{P}^B = \mathcal{P} - \mathcal{P}_{\rm ig}^B &= \frac{\Delta_\beta}{2\beta_0\mathcal{V}}\frac{\partial^2}{\partial\beta_0^2}\ln Z_{B0}\\ &= \nu_B\frac{\pi^2}{15}\Delta_\beta\beta_0^{-6} = \nu_B\frac{\pi^2}{15}\Delta T_0^2 > 0,\end{aligned} \tag{39}$$

a positive quantity as well, in agreement with the general proof presented in Eq. (16).

V. THE GLUON GAS WITH THERMAL NOISE

The Lagrangian for a system of non-Abelian, $SU(N)$ gauge fields A_a^μ is given by

$$\mathcal{L} = -\frac{1}{4}F_a^{\mu\nu}F_{a,\mu\nu}, \tag{40}$$

where $1 \le a \le N^2-1$ represents the color index, with $N=3$ for gluons in QCD. The corresponding N^2-1 Lie group generators t^a satisfy the algebra $[t^a, t^b] = \mathrm{i}f^{abc}t^c$, with f^{abc} the structure constants. The field strength tensor in the Lagrangian equation (40), for a color charge g, is thus defined by

$$F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g f^{abc}A_b^\mu A_c^\nu. \tag{41}$$

The corresponding partition function is expressed by means of the Fadeev-Popov technique [2,3],

$$Z_G = \int \mathcal{D}[A_a^\mu]\delta[\mathcal{F}^a]\det\left(\frac{\delta\mathcal{F}^a}{\delta\alpha^c}\right)e^{\int_0^{\beta_0}d\tau\int d^3x\mathcal{L}}. \tag{42}$$

In this case, we have a set of N^2-1 gauge-fixing conditions $\mathcal{F}^a[A_a^\mu]=0$. By choosing a family of covariant gauges $\mathcal{F}^a = \partial_\mu A_a^\mu - f^a(x,\tau) = 0$, such that under a gauge transformation $A_a^\mu \to A_a^\mu + g f^{abc}A_b^\mu\alpha^c - \partial^\mu\alpha^a$, one obtains

$$\det\left(\frac{\delta\mathcal{F}^a}{\delta\alpha^c}\right) = \det\left(-\partial^2\delta_c^a + g f^{abc}\partial_\mu A_b^\mu\right). \tag{43}$$

As in the case already discussed for photons, we integrate over a functional distribution of gauge fixing conditions $\exp\left[-\frac{1}{2\xi}\int d\tau\int d^3x f_a^2(x,\tau)\right]$, to obtain

$$\begin{aligned}Z_G = \int &\mathcal{D}[A_a^\mu]\det\left(-\partial^2\delta_c^a + g f^{abc}\partial_\mu A_b^\mu\right)\\ &\times e^{\int_0^{\beta_0}d\tau\int d^3x(\mathcal{L}-\frac{1}{2\xi}(\partial_\mu A_a^\mu)^2)}.\end{aligned} \tag{44}$$

The functional determinant is expressed in terms of an integral over Grassmann "ghosts" fields $\bar{\eta}_a$, η_a, for each color index $1 \le a \le N^2-1$, to obtain (choosing the Feynman gauge $\xi = 1$ for definiteness, and integrating by parts in the gauge field sector)

$$Z_G = \int\mathcal{D}[A_a^\mu]\int\mathcal{D}[\bar{\eta}_a,\eta_a]e^{\int_0^{\beta_0}d\tau\int d^3x\mathcal{L}_{\rm eff}[A_a^\mu,\bar{\eta}_a,\eta_a]} \tag{45}$$

with the effective Lagrangian

$$\begin{aligned}\mathcal{L}_{\rm eff} = &-\frac{1}{2}A_a^\mu(-\delta_b^a g_{\mu\nu}\partial^2)A_b^\nu + \bar{\eta}_a(-\delta_b^a\partial^2)\eta_b\\ &-\frac{g^2}{4}(f^{eab}A_{a\mu}A_{b\nu})(f^{ecd}A_c^\mu A_d^\nu)\\ &-g f^{abc}(\partial_\mu A_{a\nu})A_b^\mu A_c^\nu - g\bar{\eta}_a f^{abc}\partial^\mu A_{b\mu}\eta_c.\end{aligned} \tag{46}$$

The first two terms in the effective Lagrangian represent a noninteracting system of N^2-1 free gluons and ghosts, respectively. The remaining three contributions represent the self-interaction between gluons, of first and second order in the color charge, respectively, as well as the interaction between gluons and ghosts, which is first order in the charge. In what follows, to illustrate our method via explicit analytical results, we shall consider the effects of thermal noise over the noninteracting, ideal gas of gluons, such that we shall restrict ourselves to zero order in the color charge, corresponding to the first two terms in the effective Lagrangian. Applying then the replica trick for this noninteracting system, we have (adding an additional replica index $1 \le r \le n$ to all the fields)

$$
\begin{aligned}
Z_{G0}^n &= \int \prod_{r=1}^{n} \mathcal{D}[A_a^{r\mu}] \int \mathcal{D}[\bar{\eta}_a^r, \eta_a^r] e^{\int_0^{\beta_0} d\tau \int d^3x [-\frac{1}{2}\sum_{r=1}^{n} A_a^{r\mu}(-\delta_b^a g_{\mu\nu}\partial^2)A_b^{r\nu} + \bar{\eta}_a(-\delta_a^b \partial^2)\eta_b]} \\
&= (\det[-\partial^2])^{-4n(N^2-1)/2} \times (\det[-\partial^2])^{2(N^2-1)n/2} = \exp[-n(N^2-1)\mathrm{Tr}\ln[-\partial^2]] \\
&= \exp[n(N^2-1)\ln Z_{B0}],
\end{aligned} \tag{47}
$$

where $\ln Z_{B0} = -\mathrm{Tr}\ln[-\partial^2]$ is just the partition function for the ideal gas of bosons, already defined and calculated explicitly in Eqs. (34) and (35). Therefore, as in the previous examples we apply Eq. (11) to obtain

$$
\begin{aligned}
\overline{\ln Z_G} &= \lim_{n\to 0} \frac{\overline{Z_{G0}^n} - 1}{n} \\
&= \exp\left[\frac{\Delta_\beta}{2}\frac{\partial^2}{\partial\beta_0^2}\right] \lim_{n\to 0} \frac{e^{n(N^2-1)\ln Z_{B0}} - 1}{n} \\
&= (N^2-1)\exp\left[\frac{\Delta_\beta}{2}\frac{\partial^2}{\partial\beta_0^2}\right] \ln Z_{B0}.
\end{aligned} \tag{48}
$$

Therefore, repeating the same steps already discussed in the gas of photons, the corresponding equation of state in thermal equilibrium for the ideal gluon gas is again given by Eq. (36), but with a total number of degrees of freedom $\nu_B = 2(N^2 - 1)$. Similarly, the excess pressure due to stochastic fluctuations in temperature will be given by Eq. (39), with $\nu_B = 2(N^2 - 1)$ (with $N = 3$ for gluons in QCD). Despite this explicit result is exact at zero order in the color charge, one can extend the applicability of the formalism to incorporate interaction effects at the level of the effective action, for instance, with a running coupling constant $g \to g(k)$ determined by the standard $\beta(g) = \partial g/\partial \log M$ function in perturbation theory [30]. The technical aspects of such procedure are beyond the scope of the present article, and will be communicated in a separate contribution.

VI. IMPLICATIONS FOR THE DECONFINEMENT TRANSITION

Our previous results, although exact, were explicitly applied to noninteracting systems represented by ideal Fermi and Bose gases, respectively. Nevertheless, it is interesting to explore their consequences in the context of the deconfinement transition, within the simple bag model considerations. The Bag model represents an elementary approach, where the corresponding hadronic and plasma phases are modeled as ideal quantum gases [2], just as developed in our former results, and hence it provides a toy model to explore the effects of stochastic thermal fluctuations in the deconfinement transition. Assuming that the hadronic phase is mainly constituted by pions [2] (bosons with $\mu = 0$ and $\nu_B = 3$ for charged states $0, \pm$), applying Eq. (36) we have that its pressure, including the excess pressure effect due to temperature fluctuations, would be

$$
\mathcal{P}_{\mathrm{Had}} = 3\frac{\pi^2 T_0^4}{90} + \delta\mathcal{P}^{\mathrm{Had}}. \tag{49}
$$

On the other hand, for the plasma phase we have $\nu_F = 2 \times 3 \times 2 = 12$ for quarks (fermions), and $\nu_B = 2 \times (3^2 - 1) = 16$ for gluons (non-Abelian gauge fields), such that

$$
\begin{aligned}
\mathcal{P}_{\mathrm{Plasma}} &= \left(\nu_B + \frac{7}{4}\nu_F\right)\frac{\pi^2 T_0^4}{90} + \delta\mathcal{P}^{\mathrm{Plasma}} - B \\
&= \frac{37\pi^2}{90} T_0^4 + \delta\mathcal{P}^{\mathrm{Plasma}} - B.
\end{aligned} \tag{50}
$$

Here, we included the bag constant $B \sim 200$ MeV [2], and the excess pressure due to temperature fluctuations associated to both quarks and gluons $\delta\mathcal{P}^{\mathrm{Plasma}} = \delta\mathcal{P}^G + \delta\mathcal{P}^Q > 0$.

The critical temperature T_c is obtained by imposing the condition of equal pressures at both phases at the phase transition, i.e.

$$
3\frac{\pi^2 T_c^4}{90} = \frac{37\pi^2}{90} T_c^4 + \delta\mathcal{P}^{\mathrm{Net}} - B, \tag{51}
$$

where we defined the net excess pressure as

$$
\begin{aligned}
\delta\mathcal{P}^{\mathrm{Net}} &= \delta\mathcal{P}^{\mathrm{Plasma}} - \delta\mathcal{P}^{\mathrm{Had}} = \delta\mathcal{P}^G - \delta\mathcal{P}^{\mathrm{Had}} + \delta\mathcal{P}^Q \\
&= 13\frac{\pi^2}{15}\Delta T_0^2 + \delta\mathcal{P}^Q > 0,
\end{aligned} \tag{52}
$$

which is clearly a positive definite quantity.

Finally, solving for T_c in Eq. (51), we obtain

$$
T_c = T_c^0\left(1 - \frac{\delta\mathcal{P}^{\mathrm{Net}}}{(T_c^0)^4}\right)^{1/4} \leq T_c^0, \tag{53}
$$

with $T_c^0 = (45B/17\pi^2)^{1/4} \sim 144$ MeV [2] the critical temperature for a homogenous thermalized system. Therefore, we conclude that nonequilibrium temperature fluctuations will in principle decrease the critical temperature for the deconfinement transition.

VII. SUMMARY AND CONCLUSIONS

As an approximation to the nonequilibrium conditions arising in several relativistic quantum systems, such as heavy-ion collisions, we have considered an ensemble of subsystems at different temperatures $T = T_0 + \delta T$, with average T_0 and standard deviation $\overline{\delta T^2} = \Delta$. These statistical properties imply that the inverse temperature $\beta = \beta_0 + \delta\beta$ can be modeled by a Gaussian distributed stochastic fluctuation $\delta\beta$, with zero mean and standard deviation $\overline{\delta\beta^2} = \Delta_\beta = \beta_0^4\Delta$. We applied the replica trick to obtain the statistical average of the grand potential as a series expansion at all orders in the parameter Δ, that can be expressed in compact form as an exponential differential operator acting upon the partition function of the system in equilibrium at the average temperature T_0, thus allowing to obtain the excess pressure due to temperature fluctuations. This represents an exact result, that can be applied to quantum systems even in the presence of interactions. In order to present our novel formalism, we applied it explicitly to solve for the equation of state and excess pressure in three paradigmatic physical systems: The relativistic Fermi gas, the photon gas, and a gas of non-Abelian gauge fields (gluons) in the noninteracting limit. In agreement with our previous works [28,29], the statistical average over fluctuating parameters (in this case the temperature) within the replica formalism, can be interpreted as an effective particle-particle interaction introduced to the free Lagrangian [28,29]. The strength of these interactions is here proportional to the parameter Δ, that represents the autocorrelation in the temperature fluctuations. This result may be of significant impact in the interpretation of nonequilibrium effects and thermal fluctuations in several high-energy systems, particularly on the deconfinement transition between hadronic matter and the quark gluon plasma, for which we provided a simple but straightforward analysis based on the bag model, showing that the critical temperature decreases due to such nonequilibrium fluctuations. A more precise assessment of such effects in this context can be performed, by applying our formalism to the corresponding interacting field theories, which is work currently in progress.

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APPENDIX A: THERMODYNAMIC RELATIONS FOR THE FLUCTUATIONS

As clearly stated in the main text, the first order in Δ contribution to the excess pressure is proportional to the second derivative of the grand partition function of the reference system, constituted by the relativistic Fermi gas. Using the relation $\beta_0 = T_0^{-1}$, and the definition $\ln Z_0 = -\Omega_0/T_0$, we have

$$\begin{aligned}\frac{\partial^2}{\partial\beta_0^2}\ln Z_0 &= -T_0\frac{\partial}{\partial T_0}\left(T_0\frac{\partial\Omega_0}{\partial T_0}\bigg|_{\mu,\mathcal{V}} - \Omega_0\right)_{\mu,\mathcal{V}}\\ &= -T_0^3\frac{\partial^2\Omega_0}{\partial T_0^2}\bigg|_{\mu,\mathcal{V}}.\end{aligned} \tag{A1}$$

On the other hand, from the differential form of the grand potential,

$$d\Omega_0 = -\mathcal{P}d\mathcal{V} - SdT_0 - Nd\mu, \tag{A2}$$

we have the relations

$$\begin{aligned}S &= -\frac{\partial\Omega_0}{\partial T_0}\bigg|_{\mu,\mathcal{V}},\\ \frac{\partial S}{\partial\mu}\bigg|_{T_0,\mathcal{V}} &= \frac{\partial N}{\partial T_0}\bigg|_{\mu,\mathcal{V}}.\end{aligned} \tag{A3}$$

Inserting the first expression into Eq. (A1) we obtain

$$\frac{\partial^2}{\partial\beta_0^2}\ln Z_0 = T_0^3\frac{\partial S}{\partial T_0}\bigg|_{\mu,\mathcal{V}}. \tag{A4}$$

We remark that the entropy derivative in Eq. (A4) is related to the specific heat at constant volume C_v, and hence a positive definite quantity, as we show as follows. By definition, we have (using the Jacobian notation)

$$\begin{aligned}C_v &= T_0\frac{\partial S}{\partial T_0}\bigg|_{\mathcal{V},N} = T_0\frac{\partial(S,N)}{\partial(T_0,N)} = T_0\frac{\frac{\partial(S,N)}{\partial(T_0,\mu)}}{\frac{\partial(T_0,N)}{\partial(T_0,\mu)}}\\ &= \frac{T_0}{\frac{\partial N}{\partial\mu}\Big|_{T_0,\mathcal{V}}}\begin{vmatrix}\frac{\partial S}{\partial T_0}\Big|_{\mu,\mathcal{V}} & \frac{\partial S}{\partial\mu}\Big|_{T_0,\mathcal{V}}\\ \frac{\partial N}{\partial T_0}\Big|_{\mu,\mathcal{V}} & \frac{\partial N}{\partial\mu}\Big|_{T_0,\mathcal{V}}\end{vmatrix}.\end{aligned} \tag{A5}$$

Evaluating the determinant, we obtain after some elementary algebra

$$\frac{C_v}{T_0} = \frac{\partial S}{\partial T_0}\bigg|_{\mu,\mathcal{V}} - \frac{\frac{\partial S}{\partial \mu}\Big|_{T_0,\mathcal{V}} \cdot \frac{\partial N}{\partial T_0}\Big|_{\mu,\mathcal{V}}}{\frac{\partial N}{\partial \mu}\Big|_{T_0,\mathcal{V}}} = \frac{\partial S}{\partial T_0}\bigg|_{\mu,\mathcal{V}} - \frac{\left(\frac{\partial N}{\partial T_0}\Big|_{\mu,\mathcal{V}}\right)^2}{\frac{\partial N}{\partial \mu}\Big|_{T_0,\mathcal{V}}}, \tag{A6}$$

where in the second line we substituted the second relation in Eq. (A3). From Eq. (A6) we obtain

$$\frac{\partial S}{\partial T_0}\bigg|_{\mu,\mathcal{V}} = \frac{C_v}{T_0} + \frac{\left(\frac{\partial N}{\partial T_0}\Big|_{\mu,\mathcal{V}}\right)^2}{\frac{\partial N}{\partial \mu}\Big|_{T_0,\mathcal{V}}}. \tag{A7}$$

Therefore, substituting this result into Eq. (A4), we obtain

$$\frac{\partial^2}{\partial \beta_0^2}\ln Z_0 = T_0^2\left(C_v + T_0\frac{\left(\frac{\partial N}{\partial T_0}\Big|_{\mu,\mathcal{V}}\right)^2}{\frac{\partial N}{\partial \mu}\Big|_{T_0,\mathcal{V}}}\right). \tag{A8}$$

Finally, applying the statistical-mechanical definition fo the average particle number $N = \langle \hat{N} \rangle$ in the grand canonical ensemble, we have

$$\frac{\partial N}{\partial \mu}\bigg|_{T_0,\mathcal{V}} = \frac{1}{T_0}(\langle \hat{N}^2\rangle - \langle \hat{N}\rangle^2) = \frac{\langle (\delta\hat{N})^2\rangle}{T_0} \geq 0, \tag{A9}$$

which combined with Eq. (A8) leads us to prove the inequality,

$$\frac{\partial^2}{\partial \beta_0^2}\ln Z_0 = T_0^3\frac{\partial S}{\partial T_0}\bigg|_{\mu,\mathcal{V}} = T_0^2\left(C_v + T_0^2\frac{\left(\frac{\partial N}{\partial T_0}\Big|_{\mu,\mathcal{V}}\right)^2}{\langle (\delta\hat{N})^2\rangle}\right) \geq 0, \tag{A10}$$

as stated in the main text.

APPENDIX B: MATSUBARA SUMS

Here we present the detail of the evaluation of the Matsubara sum of the logarithmic functions in the main text. For simplicity, let us define $\xi_p = \mu \pm E_{\mathbf{p}}$, and hence consider the generic sum for $\omega_k = (2k+1)\pi/\beta_0$,

$$S = \sum_{k\in\mathbb{Z}} \ln(\mathrm{i}\omega_k - \xi_p). \tag{B1}$$

To evaluate the sum, we shall construct an integration path on the complex contour, by using the meromorphic function

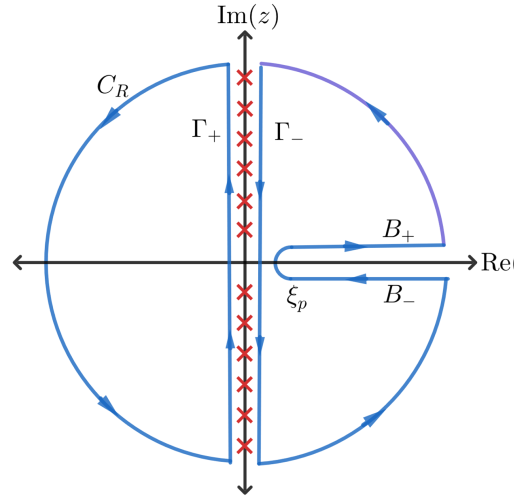

FIG. 3. Integration contour in Eq. (B4).

$$g_F(z) = \frac{\beta_0}{e^{\beta_0 z} + 1}, \tag{B2}$$

that possesses infinitely many single poles along the imaginary axis at the Matsubara frequencies $z_k = \mathrm{i}\omega_k$, with residue 1:

$$\begin{aligned}\mathrm{Res} g_F(z)|_{z=\mathrm{i}\omega_k} &= \lim_{z\to \mathrm{i}\omega_k}(z-\mathrm{i}\omega_k)g_F(z)\\ &= \beta_0 \lim_{z\to\mathrm{i}\omega_k}\frac{z-\mathrm{i}\omega_k}{1+e^{\mathrm{i}\beta_0\omega_k}e^{\beta_0(z-\mathrm{i}\omega_k)}}\\ &= \beta_0 \lim_{z\to\mathrm{i}\omega_k}\frac{z-\mathrm{i}\omega_k}{1-e^{\beta_0(z-\mathrm{i}\omega_k)}}\\ &= 1.\end{aligned} \tag{B3}$$

Therefore, we consider the complex integral over the contour illustrated in Fig. 3,

$$\begin{aligned}\frac{1}{2\pi\mathrm{i}}\oint dz\, g_F(z)\ln(z-\xi_p) &= \int_{C_R}\frac{dz}{2\pi\mathrm{i}}g_F(z)\ln(z-\xi_p)\\ &+ \int_{B_+\cup B_-}\frac{dz}{2\pi\mathrm{i}}g_F(z)\ln(z-\xi_p)\\ &+ \int_{\Gamma_+\cup\Gamma_-}\frac{dz}{2\pi\mathrm{i}}g_F(z)\ln(z-\xi_p)\\ &= 0,\end{aligned} \tag{B4}$$

where we have surrounded the branch cut of the logarithm starting at $z = \xi_p$, and no poles are enclosed inside the contour. Now, we calculate separately each component. Clearly, after the exponential contribution in the denominator of $g_F(z)$, we have

$$\lim_{R\to\infty}\int_{C_R}\frac{dz}{2\pi\mathrm{i}}g_F(z)\ln(z-\xi_p) = 0. \tag{B5}$$

The integral that surrounds the Matsubara poles in the imaginary axis is calculated using the residue theorem,

$$\lim_{R\to\infty}\int_{\Gamma_+\cup\Gamma_-}\frac{dz}{2\pi\mathrm{i}}g_F(z)\ln(z-\xi_p)$$
$$=-\sum_{k\in\mathbb{Z}}\lim_{z\to\mathrm{i}\omega_k}(z-\mathrm{i}\omega_k)g_F(z)\ln(z-\xi_p)$$
$$=-\sum_{k\in\mathbb{Z}}\ln(\mathrm{i}\omega_k-\xi_p)=-S, \tag{B6}$$

where we applied Eq. (B3) for the residues of $g(z)$. Substituting Eq. (B6) into Eq. (B4), we have that the Matsubara sum is given by the integral around the branch cut,

$$\begin{aligned}
S=\sum_{k\in\mathbb{Z}}\ln(\mathrm{i}\omega_k-\xi_p)&=\int_{B_+\cup B_-}\frac{dz}{2\pi\mathrm{i}}g_F(z)\ln(z-\xi_p)\\
&=-\frac{1}{2\pi\mathrm{i}}\int_{\xi_p}^{\infty}dx\,g_F(x)[\ln(x-\xi_p+\mathrm{i}\epsilon^+)-\ln(x-\xi_p-\mathrm{i}\epsilon^+)]\\
&=-\frac{1}{2\pi\mathrm{i}}\int_{\xi_p}^{\infty}dx\frac{\beta_0e^{-\beta_0x}}{e^{-\beta_0x}+1}[\ln(x-\xi_p+\mathrm{i}\epsilon^+)-\ln(x-\xi_p-\mathrm{i}\epsilon^+)]\\
&=\frac{1}{2\pi\mathrm{i}}\int_{\xi_p}^{\infty}dx\frac{\partial}{\partial x}\ln(e^{-\beta_0x}+1)[\ln(x-\xi_p+\mathrm{i}\epsilon^+)-\ln(x-\xi_p-\mathrm{i}\epsilon^+)]\\
&=-\frac{1}{2\pi\mathrm{i}}\int_{\xi_p}^{\infty}dx\ln(e^{-\beta_0x}+1)\left[\frac{1}{x-\xi_p+\mathrm{i}\epsilon^+}-\frac{1}{x-\xi_p-\mathrm{i}\epsilon^+}\right]\\
&=\int_{\xi_p}^{\infty}dx\;\ln(e^{-\beta_0x}+1)\delta(x-\xi_p)\\
&=\ln(e^{-\beta_0\xi_p}+1),
\end{aligned}\tag{B7}$$

where in the fourth line we integrated by parts, and in the third step we used the identity

$$\lim_{\epsilon\to0^+}\frac{1}{A\pm\mathrm{i}\epsilon}=PV(1/A)\mp\mathrm{i}\pi\delta(A). \tag{B8}$$

For the case of bosons, an identical contour integration as depicted in Fig. 3 can be performed, but using instead the meromorphic function

$$g_B(z)=\frac{\beta_0}{e^{\beta_0z}-1}, \tag{B9}$$

that possesses infinitely many simple poles at the $z_k=\mathrm{i}\omega_k$, with $\omega_k=2\pi k/\beta_0$ ($k\in\mathbb{Z}$) the even Matsubara frequencies. The residues of this function at each pole are

$$\begin{aligned}
\mathrm{Res}g_B(z)|_{z=\mathrm{i}\omega_k}&=\lim_{z\to\omega_k}(z-\mathrm{i}\omega_k)g_B(z)\\
&=\beta_0\lim_{z\to\omega_k}\frac{z-\mathrm{i}\omega_k}{e^{\mathrm{i}\beta_0\omega_k}e^{\beta_0(z-\mathrm{i}\omega_k)}-1}\\
&=\beta_0\lim_{z\to\omega_k}\frac{z-\mathrm{i}\omega_k}{e^{\beta_0(z-\mathrm{i}\omega_k)}-1}\\
&=1.
\end{aligned}\tag{B10}$$

Integrating along the contour Fig. 3, and following the same steps in the calculation, one arrives in this second case for bosonic frequencies $\omega_k=2\pi k\beta_0$ to the result

$$S=\sum_{k\in\mathbb{Z}}\ln(\mathrm{i}\omega_k-\xi_p)=\ln(1-e^{-\beta_0\xi_p}). \tag{B11}$$

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