Fermion self-energy and effective mass in a noisy magnetic background

Jorge David Castaño-Yepes [1,*] and Enrique Muñoz [1,2,†]

[1] *Facultad de Física, Pontificia Universidad Católica de Chile, Vicuña Mackenna 4860, Santiago, Chile*
[2] *Center for Nanotechnology and Advanced Materials CIEN-UC, Avenida Vicuña Mackenna 4860, Santiago, Chile*

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In this article, we consider the propagation of QED fermions in the presence of a classical background magnetic field with white-noise stochastic fluctuations. The effects of the magnetic field fluctuations are incorporated into the fermion and photon propagators in a quasiparticle picture, which we developed in previous works using the *replica trick*. By working in the very strong-field limit, here we explicitly calculate the fermion self-energy involving radiative contributions at first order in α_{em}, in order to obtain the noise-averaged mass of the fermion propagating in the fluctuating magnetized medium. Our analytical results reveal a leading double-logarithmic contribution $\sim[\ln(|eB|/m^2)]^2$ to the mass, with an imaginary part representing a spectral broadening proportional to the magnetic noise autocorrelation Δ. While a uniform magnetic field already breaks Lorentz invariance, inducing the usual separation into two orthogonal subspaces (perpendicular and parallel with respect to the field), the presence of magnetic noise further breaks the remaining symmetry, thus leading to distinct spectral widths associated with fermion and antifermion, and their spin projection in the quasiparticle picture.

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I. INTRODUCTION

The presence of strong, transient and inhomogeneous magnetic fields and their effects over elementary particles is a subject of great interest in several physical scenarios, such as ultrarelativistic heavy-ion collisions [1–5] and the corresponding genesis of the quark-gluon plasma [6–9]. In such a magnetized medium, fermions (charged leptons and quarks) as well as neutral gauge fields (photons and gluons) develop nontrivial responses due to vacuum polarization effects. The case of a classical, static and homogeneous background magnetic field, since the seminal work of Schwinger [10], has been discussed extensively in the literature [11,12]. This idealized scenario has been studied in the context of the gluon polarization tensor [7,8,13,14], where the presence of the magnetic field breaks the Lorentz invariance, thus generating three tensor components with their corresponding refraction indexes, the so-called vacuum birefringence phenomena [13,14]. Similarly, the propagation of fermions in such a uniform magnetized background [15,16], as expressed by the self-energy, leads to the definition of a magnetic mass with a leading double-logarithmic dependence $\sim[\ln(|eB|/m^2)]^2$, where $|eB|$ and m are the background field and bare fermion mass, respectively [11,16–18]. In addition, an spectral broadening arising from imaginary parts in the fermion self-energy was recently predicted due to Landau level mixing [16]. Since spatiotemporal fluctuations in the background magnetic field should indeed exist in the physical scenarios of interest [3], we recently developed a theoretical formalism to include random fluctuations with respect to a uniform and constant magnetic field background, as a next step towards a more realistic approximation [19]. For the gauge fields $A^\mu(x)$, we distinguish three physically different contributions

$$A^\mu(x) \to A^\mu(x) + A^\mu_{\mathrm{BG}}(x) + \delta A^\mu_{\mathrm{BG}}(\mathbf{x}), \tag{1}$$

where $A^\mu(x)$ is the dynamical photonic quantum field, while BG represents the classical "background." This background is assumed to be generated by classical currents $J^\mu_{\mathrm{class}}(x)$

$$\Box(A^\mu_{\mathrm{BG}} + \delta A^\mu_{\mathrm{BG}}) = J^\mu_{\mathrm{class}}(x) + \delta J^\mu_{\mathrm{class}}(x) \tag{2}$$

that exhibit stochastic fluctuations $\delta J^\mu_{\mathrm{class}}(x)$, as a model for nonequilibrium scenarios such as the very early stages of ultrarelativistic heavy-ion collisions. As a consequence, the BG gauge field will develop spatial fluctuations $\delta A^\mu_{\mathrm{BG}}(\mathbf{x})$

[*] Contact author: jcastano@uc.cl
[†] Contact author: ejmunozt@uc.cl

with respect to the mean value $A^{\mu}_{\rm BG}(x)$, which we describe as white noise satisfying the statistical properties

$$\langle \delta A^{j}_{\rm BG}(\mathbf{x})\delta A^{k}_{\rm BG}(\mathbf{x}')\rangle = \Delta\delta_{j,k}\delta^3(\mathbf{x}-\mathbf{x}'),$$
$$\langle \delta A^{\mu}_{\rm BG}(\mathbf{x})\rangle = 0, \quad (3)$$

such that the corresponding classical magnetic field background is $\mathbf{B}+\delta\mathbf{B}(x) = \nabla\times(\mathbf{A}_{\rm BG}(x)+\delta\mathbf{A}_{\rm BG}(x))$, with a uniform mean value $\mathbf{B}$. In our formalism, these statistical properties are reproduced by a Gaussian functional distribution

$$dP[\delta\mathbf{A}_{\rm BG}] = \mathcal{N}e^{-\int d^3x\frac{[\delta\mathbf{A}_{\rm BG}(\mathbf{x})]^2}{2\Delta}}\mathcal{D}[\delta\mathbf{A}_{\rm BG}(\mathbf{x})]. \quad (4)$$

The Lagrangian for the model is expressed as a superposition of two terms

$$\mathcal{L} = \mathcal{L}_{\rm FBG} + \mathcal{L}_{\rm NBG}, \quad (5)$$

where the first represents the system of Fermions and photons immersed in the deterministic background field (FBG)

$$\mathcal{L}_{\rm FBG} = \bar{\psi}(\mathrm{i}\partial\!\!\!/ - eA\!\!\!/_{\rm BG} - eA\!\!\!/ - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad (6)$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\nu$ the electromagnetic tensor for the quantum photon gauge fields. The second term in the Lagrangian equation (6) represents the interaction between the Fermions and the static classical noise (NBG), represented by the spatial fluctuations $\delta A^{\mu}_{\rm BG}(x)$

$$\mathcal{L}_{\rm NBG} = \bar{\psi}(-e\delta A\!\!\!/_{\rm BG})\psi. \quad (7)$$

The generating functional (in the absence of sources) for a given realization of the noisy fields is given by

$$Z[A_{\rm BG}+\delta A_{\rm BG}] = \int \mathcal{D}A^{\mu}\mathcal{D}[\bar{\psi},\psi]e^{\mathrm{i}\int d^4x[\mathcal{L}_{\rm FBG}+\mathcal{L}_{\rm NBG}]}. \quad (8)$$

To study the connected correlation functions in this system, among them the fermion propagator, we need to calculate the statistical average over the magnetic background noise $\delta\mathbf{A}_{\rm BG}$ of the $\ln Z[A_{\rm BG}+\delta A_{\rm BG}]$ over the functional measure defined in Eq. (4),

$$\overline{\ln Z[A_{\rm BG}]} = \int dP[\delta A_{\rm BG}]\ln Z[A_{\rm BG}+\delta A_{\rm BG}]. \quad (9)$$

This procedure is mathematically implemented by means of the replica trick [20,21]

$$\overline{\ln Z[A_{\rm BG}]} = \lim_{n\to 0}\frac{\overline{Z^n[A_{\rm BG}]}-1}{n}, \quad (10)$$

where Z^n is obtained by incorporating an additional "replica" component for each of the Fermion fields, i.e. $\psi(x)\to\psi^a(x)$, for $1\le a\le n$. The "replicated" Lagrangian then has the same form as Eqs. (6) and (7), but with an additional sum over the replica components of the Fermion fields. Therefore, the averaging procedure leads to

$$\overline{Z^n[A_{\rm BG}]} = \int\mathcal{D}A^{\mu}\prod_{a=1}^{n}\mathcal{D}[\bar{\psi}^a,\psi^a]e^{\mathrm{i}\int d^4x\sum_{a=1}^{n}\mathcal{L}_{\rm FBG}[\bar{\psi}^a,\psi^a]}$$
$$\times\int\mathcal{D}[\delta\mathbf{A}_{\rm BG}]e^{-\int d^3x\frac{[\delta\mathbf{A}_{\rm BG}(\mathbf{x})]^2}{2\Delta}}e^{\mathrm{i}\int d^4x\sum_{a=1}^{n}\mathcal{L}_{\rm NBG}[\bar{\psi}^a,\psi^a]},$$
$$= \int\mathcal{D}A^{\mu}\prod_{a=1}^{n}\mathcal{D}[\bar{\psi}^a,\psi^a]e^{\mathrm{i}\bar{S}[\bar{\psi}^a,\psi^a,A;A_{\rm BG}]}, \quad (11)$$

where in the last step we explicitly performed the Gaussian integral over the background noise, leading to the definition of the effective averaged action for the system

$$\bar{S}[\bar{\psi}^a,\psi^a,A;A_{\rm BG}] = \int d^4x\left(\sum_{a=1}^{n}\bar{\psi}^a(x)(\mathrm{i}\partial\!\!\!/ - eA\!\!\!/_{\rm BG} - eA\!\!\!/ - m)\psi^a(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right)$$
$$+\mathrm{i}\frac{e^2\Delta}{2}\int d^4x\int d^4y\sum_{a,b}^{n}\sum_{j=1}^{3}\bar{\psi}^a(x)\gamma^j\psi^a(x)\bar{\psi}^b(y)\gamma_j\psi^b(y)\delta^3(\mathbf{x}-\mathbf{y}). \quad (12)$$

As clearly seen in Eq. (12), the average over stochastic magnetic fluctuations, in the replica formalism, leads to an effective interacting theory for the fermion sector. The corresponding interaction couples the fermion vector currents, it is proportional to the noise autocorrelation Δ and, moreover, it is strictly local in spatial coordinates as is evident from the presence of the $\delta^3(\mathbf{x}-\mathbf{y})$. In our previous work [19,22], we analyzed the effects of this interaction as a perturbative expansion in terms of the magnetic noise autocorrelation $\Delta\ge 0$ for the fermion [19] and photon [22] propagators, respectively. Since the reference system is characterized by the presence of an average and uniform background magnetic field $\mathbf{B}$, fermions are described by the Schwinger propagator [10] at zero order in the noise Δ, with corrections incorporated via Dyson equation depending on the self-energy due to the effective interaction at higher orders in Δ [19]. Moreover, the $\delta^3(\mathbf{x}-\mathbf{y})$ character of this effective interaction collapses the products of

Schwinger phases in internal diagrammatic lines to a single point, such that only an overall phase connecting the external points of the fermion propagator survives. This feature considerably simplifies the perturbation theory, such that only the translational invariant parts of the self-energy need to be calculated in momentum space [19]. In particular, we have shown that the effects of magnetic noise over the fermion propagator, at first order in Δ, is equivalent to a dispersive media, with an effective refraction index that modifies the group velocity of the otherwise free particles but not their mass [19]. In contrast, we have recently shown [22] that, as a consequence of the magnetic fluctuations combined with charged vacuum polarization in QED, photons develop anisotropic magnetic masses $M_\perp$ and $M_\parallel$, which are proportional to Δ [22]. Therefore, a related question remains opened: What are the effects of the background magnetic noise over the fermion self-energy and its corresponding effective mass, when radiative effects are taken into account? In this article, we shall address this question from a perturbation theory perspective within the framework of QED as captured by Eq. (12) by applying our previous results for the noise-averaged fermion [19] and photon [22] propagators, respectively, in the very strong magnetic field limit $|eB| \gg m^2$ which is relevant for heavy-ion collisions.

II. THE QED FERMION SELF-ENERGY AT 1-LOOP

In the configuration space, for a QED fermion with charge $-e$ and mass m propagating through a magnetized medium, its self-energy due to radiative effects at 1-loop (as depicted in Fig. 1) can be expressed as follows:

$$-\mathrm{i}\Sigma(x,x') = (-\mathrm{i}e)^2\gamma^\mu \mathrm{i}S(x,x')\gamma^\mu D_{\mu\nu}(x-x'). \quad (13)$$

Here, the fermion propagator is given by

$$\mathrm{i}S(x,x') = \Phi(x,x')\int\frac{d^4k}{(2\pi)^4}e^{-\mathrm{i}k\cdot(x-x')}\mathrm{i}S(k), \quad (14)$$

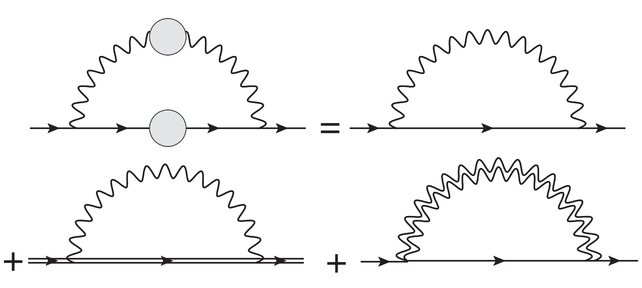

FIG. 1. Feynman diagrams depicting the contributions to fermion self-energy up to order $O(\Delta^2)$. The single lines represent fermions and photons in a constant and intense magnetic field, while double lines depict fermions and photons in a fluctuating background magnetic field.

while

$$D_{\mu\nu}(x-x') = \int\frac{d^4q}{(2\pi)^4}e^{-\mathrm{i}q\cdot(x-x')}D_{\mu\nu}(q) \quad (15)$$

represents the photon propagator.

As discussed in the literature [10,11], the presence of the magnetic field breaks the translational invariance of the propagators, but gauge-covariance is granted by the Schwinger phase factor $\Phi(x,x')$, which takes the following form:

$$\Phi(x,x') = \exp\left\{\mathrm{i}e\int_x^{x'} d\xi^\mu\left[A_\mu + \frac{1}{2}F_{\mu\nu}(\xi - x')^\nu\right]\right\}. \quad (16)$$

For a magnetic field $\mathbf{B}$ oriented along the $\hat{z}$ direction, in the symmetric gauge

$$A_\mu = \frac{B}{2}(0,-x_2,x_1,0), \quad (17)$$

we obtain the phase explicitly as

$$\Phi(x,x') = \exp\left(\frac{\mathrm{i}eB}{2}\epsilon_{ij}x_i x'_j\right), \quad (18)$$

where ϵ_{ij} is the two-dimensional Levy-Civita tensor.

Similarly, and in consistency with the Dyson equation for the fermion propagator in configuration space, the fermion self-energy also involves the Schwinger phase

$$-\mathrm{i}\Sigma(x,x') = \Phi(x,x')\int\frac{d^4p}{(2\pi)^4}e^{-\mathrm{i}p\cdot(x-x')}[-\mathrm{i}\Sigma(p)], \quad (19)$$

where the translational-invariant factor is given by the expression

$$-\mathrm{i}\Sigma(p) \equiv (-\mathrm{i}e)^2\int\frac{d^4k}{(2\pi)^4}\gamma^\mu \mathrm{i}S(k)\gamma^\nu D_{\mu\nu}(p-k). \quad (20)$$

For the effective interaction considered in this theory, as we explained in the Introduction, the Schwinger phase becomes a global common factor in the Dyson equation for the propagator. Therefore, we only need to work with the translational-invariant, i.e. momentum-space, components of the Dyson equation.

The presence of a constant magnetic field background breaks the Lorentz invariance. Therefore, the phase space is splitted into two subspaces according to the parallel and perpendicular directions with respect to the background field. The metric tensor is thus splitted accordingly $g^{\mu\nu} = g_\parallel^{\mu\nu} + g_\perp^{\mu\nu}$, where

$$g_\parallel^{\mu\nu} = \mathrm{diag}(1,0,0,-1),$$
$$g_\perp^{\mu\nu} = \mathrm{diag}(0,-1,-1,0). \quad (21)$$

The latter implies that for any four-vector

$$p^\mu = p_\parallel^\mu + p_\perp^\mu, \tag{22}$$

the inner product also splits accordingly $p^2 = p_\parallel^2 - \mathbf{p}_\perp^2$, with

$$p_\parallel^2 = g_\parallel^{\mu\nu} p_\mu p_\nu = p_0^2 - p_3^2,$$
$$p_\perp^2 = g_\perp^{\mu\nu} p_\mu p_\nu = -\mathbf{p}_\perp^2 = -(p_1^2 + p_2^2). \tag{23}$$

A. The noise-averaged propagators

In two of our recent articles, we showed that the fermion propagator in a media with fluctuations in a external and very intense magnetic field can be expressed as [19,22]

$$\mathrm{i}S(p) = \mathrm{i}S_0(p) + \Delta \cdot \mathrm{i}S_1(p) + O(\Delta^2), \tag{24}$$

where the noiseless part of the fermion propagator is provided by its expression in the so-called lowest Landau level [23,24]:

$$\mathrm{i}S_0(p) = 2\mathrm{i}\frac{e^{-\mathbf{p}_\perp^2/|eB|}}{p_\parallel^2 - m^2 + \mathrm{i}\epsilon}(\not{p}_\parallel + m)\mathcal{O}^{(\uparrow)}, \tag{25}$$

and the noise-averaged part is given by [19,22]

$$\mathrm{i}S_1(p) \equiv \mathrm{i}\left(\frac{|eB|}{2\pi}\right)[\Theta_1(p)(\not{p}_\parallel + m)\mathcal{O}^{(\uparrow)} - \Theta_2(p)\gamma^3\mathcal{O}^{(\uparrow)} + \Theta_3(p)\mathrm{sign}(eB)\mathrm{i}\gamma^1\gamma^2(\not{p}_\parallel + m)], \tag{26}$$

where

$$\mathcal{O}^{(\uparrow,\downarrow)} = \frac{1}{2}[1 \mp \mathrm{sign}(eB)\mathrm{i}\gamma^1\gamma^2], \tag{27a}$$

$$\Theta_1(p) \equiv \frac{3(p_\parallel^2 + m^2)e^{-2\mathbf{p}_\perp^2/|eB|}}{(p_\parallel^2 - m^2)^2\sqrt{p_0^2 - m^2}}, \tag{27b}$$

$$\Theta_2(p) \equiv \frac{3p_3 e^{-2\mathbf{p}_\perp^2/|eB|}}{(p_\parallel^2 - m^2)\sqrt{p_0^2 - m^2}}, \tag{27c}$$

$$\Theta_3(p) \equiv \frac{e^{-2\mathbf{p}_\perp^2/|eB|}}{(p_\parallel^2 - m^2)\sqrt{p_0^2 - m^2}}. \tag{27d}$$

On the other hand, in our recent work [22] we calculated the photon propagator in the same strong magnetic field regime, such that its average over magnetic noise takes the following form:

$$D^{\mu\nu}(q) = \frac{-\mathrm{i}g_\parallel^{\mu\nu}}{q^2 + \mathrm{i}M_\parallel^2 + \mathrm{i}\epsilon} + \frac{-\mathrm{i}g_\perp^{\mu\nu}}{q^2 - \mathrm{i}M_\perp^2 + \mathrm{i}\epsilon} - \frac{2M_\perp^2\delta_3^\mu\delta_3^\nu}{(q^2 + \mathrm{i}M_\parallel^2 + \mathrm{i}\epsilon)(q^2 + \mathrm{i}(M_\parallel^2 - 3M_\perp^2) + \mathrm{i}\epsilon)}, \tag{28}$$

thus revealing that photons in such dispersive media may acquire magnetic masses proportional to the noise Δ, i.e. $M_\parallel$, and $M_\perp$ given by [22]

$$M_\parallel^2 \equiv \frac{59\alpha_{\mathrm{em}}|eB|^2\Delta}{96\pi m}, \qquad M_\perp^2 \equiv \frac{\alpha_{\mathrm{em}}|eB|^2\Delta}{3\pi m}, \tag{29}$$

where $\alpha_{\mathrm{em}} \equiv \frac{e^2}{4\pi}$ is the QED fine structure constant. For practical purposes, we shall expand the propagator in Eq. (28) up to first order in Δ

$$D^{\mu\nu}(q) = D_0^{\mu\nu}(q) + \Delta \cdot D_1^{\mu\nu}(q) + O(\Delta^2), \tag{30}$$

with

$$D_0^{\mu\nu}(q) = \frac{-\mathrm{i}g^{\mu\nu}}{q^2 + \mathrm{i}\epsilon}, \tag{31}$$

the free photon propagator in the Feynman gauge, and

$$D_1^{\mu\nu}(q) = -\frac{\alpha_{\mathrm{em}}|eB|^2}{96\pi m(q^2 + \mathrm{i}\epsilon)^2}(59g_\parallel^{\mu\nu} - 32g_\perp^{\mu\nu} + 64\delta_3^\mu\delta_3^\nu), \tag{32}$$

the first order contribution due to the magnetic background noise.

B. The noise-averaged fermion self-energy

From a perturbation theory analysis, and considering the system subjected to a uniform magnetic filed background as a reference, the self-energy averaged over magnetic noise, up to first-order in the noise autocorrelation Δ, is expressed by the following terms at first order in α_{em} and Δ:

$$\Sigma(p, B, \Delta) = \Sigma_0^r(p, B) + \Sigma_\Delta^r(p, B) + O(\Delta^2). \tag{33}$$

These contributions are represented by the Feynman diagrams depicted in Fig. 1, and defined by the following algebraic expressions, plus the corresponding counterterms

$$-\mathrm{i}\Sigma_0(p, B) \equiv (-\mathrm{i}e)^2 \int \frac{d^4k}{(2\pi)^4}\gamma^\mu \mathrm{i}S_0(k)\gamma^\nu D_{\mu\nu}^0(p - k), \tag{34a}$$

$$-\mathrm{i}\Sigma_\Delta(p, B) \equiv (-\mathrm{i}e)^2\Delta \int \frac{d^4k}{(2\pi)^4}\gamma^\mu \mathrm{i}S_1(k)\gamma^\nu D_{\mu\nu}^0(p - k). \tag{34b}$$

Even though there are three Feynman diagrams depicted in Fig. 1, the third one is $O(\alpha_{\rm em}^2)$ and hence is not included in our approximation for overall consistency in perturbation theory.

The counterterms must be determined in order to impose the physically appropriate renormalization conditions. We remark that in the noiseless limit $\Delta = 0$, the full Lorentz invariance of the propagator is already broken by the presence of the uniform background magnetic field, that here we assume is very intense ($|eB| \gg m^2$). In such case, the *magnetic mass operator*, as already discussed in our previous work [16], is given by the expression

$$\hat{M}_B(\Delta = 0) = m + \Sigma_0(p, B)|_{\substack{\not{p}_\parallel = m, \\ \not{p}_\perp = 0}}. \tag{35}$$

On the other hand, we remark that [as seen already in the structure of the propagators Eqs. (26) and (32)], the presence of magnetic noise fully breaks the Lorentz symmetry, such that when $\Delta > 0$ we are restricted to define the physical mass from the condition $\not{p}_\parallel = \gamma^0 m$, $\mathbf{p} = 0$. Therefore, in order for the limit $\Delta \to 0^+$ to correctly match the noiseless result in Eq. (35), we consider the following renormalization conditions

$$\begin{aligned}\lim_{\Delta\to 0^+} \Sigma(p, B, \Delta)|_{\not{p}_\parallel = m\gamma^0, \mathbf{p}=0} &= \Sigma_0^r(p, B)|_{\not{p}_\parallel = m\gamma^0, \mathbf{p}=0}, \\ &= \hat{M}_B(\Delta = 0) - m, \\ &= \Sigma_0(p, B)|_{\not{p}_\parallel = m, \not{p}_\perp = 0},\end{aligned} \tag{36a}$$

and

$$\begin{aligned}&\lim_{\Delta\to 0^+} \frac{\partial}{\partial \not{p}_\parallel} \Sigma(p, B, \Delta)|_{\substack{\not{p}_\parallel = m\gamma^0 \\ \mathbf{p}=0}}, \\ &\quad = \frac{\partial}{\partial \not{p}_\parallel} \Sigma_0^r(p, B)|_{\substack{\not{p}_\parallel = m\gamma^0 \\ \mathbf{p}=0}}, = \frac{\partial}{\partial \not{p}_\parallel} \Sigma_0(p, B)|_{\substack{\not{p}_\parallel = m, \\ \mathbf{p}_\perp = 0}}\end{aligned} \tag{36b}$$

which ensure that the pole and the residue of the propagator correspond to the physical fermion mass in the limit of zero magnetic noise but finite background field.

These conditions allow us to determine the corresponding counterterms δ_Z and δ_m in the renormalized expression

$$\Sigma_0^r(p, B) = \Sigma_0(p, B) + \delta_Z(\not{p}_\parallel - m\gamma^0) - \delta_m, \tag{37}$$

such that we have

$$\begin{aligned}\delta_m &= \Sigma_0(p, B)\Big|_{\substack{\not{p}_\parallel = m\gamma^0 \\ \mathbf{p}=0}}, - \Sigma_0(p, B)\Big|_{\substack{\not{p}_\parallel = m, \\ \mathbf{p}_\perp = 0}} \\ \delta_Z &= \frac{\partial}{\partial \not{p}_\parallel} \Sigma_0(p, B)\Big|_{\substack{\not{p}_\parallel = m, \\ \mathbf{p}_\perp = 0}} - \frac{\partial}{\partial \not{p}_\parallel} \Sigma_0(p, B)\Big|_{\substack{\not{p}_\parallel = m\gamma^0 \\ \mathbf{p}=0}}.\end{aligned} \tag{38}$$

The expressions for $-\mathrm{i}\Sigma_i(p, B)$ for $i = 0, \Delta$ are computed in Appendices A and B, in order to determine the magnetic mass operator $\hat{M}_B(\Delta)$ according to the definition and renormalization prescriptions described above, as will be shown in the next section.

III. THE NOISE-AVERAGED MAGNETIC MASS OPERATOR AT FIRST ORDER IN $\alpha_{\rm em}$ AND Δ

The noise-averaged magnetic mass operator of the fermion is obtained from the corresponding expression for the self-energy at $\not{p}_\parallel = \gamma^0 m$, $\mathbf{p} = 0$, as discussed in the previous section

$$\begin{aligned}\hat{M}_B(\Delta) - m &= \Sigma(p, B, \Delta)|_{\not{p}_\parallel = m\gamma^0, \mathbf{p}=0}, \\ &= \Sigma_0^r(p, B)|_{\substack{\not{p}_\parallel = m\gamma^0 \\ \mathbf{p}=0}}, + \Sigma_\Delta^r(p, B)|_{\substack{\not{p}_\parallel = m\gamma^0 \\ \mathbf{p}=0}}, \\ &= \hat{M}_B(\Delta = 0) - m + \Sigma_\Delta^r(p, B)|_{\substack{\not{p}_\parallel = m\gamma^0 \\ \mathbf{p}=0}},\end{aligned} \tag{39}$$

where the magnetic mass operator in the noiseless limit $\Delta = 0$, as shown in detail in Appendix A, is given by

$$\begin{aligned}\hat{M}_B(\Delta = 0) - m &= \Sigma_0(p, B)|_{\not{p}_\parallel = m, \mathbf{p}_\perp = 0}, \\ &= \mathcal{O}^{(\uparrow)} M_{B0}^{(\uparrow)} + \mathcal{O}^{(\downarrow)} M_{B0}^{(\downarrow)},\end{aligned} \tag{40}$$

where we defined the fermion magnetic mass eigenvalues for the $\uparrow$ and $\downarrow$ spin projections by

$$\begin{aligned}M_{B0}^{(\uparrow)} &= \frac{\alpha_{\rm em} m}{\pi}\left[\ln^2\mathcal{B} - \left(\gamma_e + \mathrm{i}\frac{\pi}{2}\right)\ln\mathcal{B} + \frac{\pi^2}{3}\right] + O(\mathcal{B}^{-1}), \\ M_{B0}^{(\downarrow)} &= \frac{\alpha_{\rm em} m}{\pi}\left[\ln^2\mathcal{B} - \left(1 + \gamma_e + \mathrm{i}\frac{\pi}{2}\right)\ln\mathcal{B}\right. \\ &\quad \left. - \left(2 - \gamma_e - \frac{\pi^2}{3} - \mathrm{i}\frac{\pi}{2}\right)\right] + O(\mathcal{B}^{-1}).\end{aligned} \tag{41}$$

Here, we defined the average magnetic field in dimensionless units by $\mathcal{B} = |eB|/m^2$. The contribution arising from the self-energy terms proportional to the magnetic noise autocorrelation Δ, as shown in detail in Appendix B, are defined in terms of the two projectors,

$$\mathcal{P}^{(\pm)} \equiv \frac{1}{2}(\mathbb{1} \pm \gamma^0) \tag{42}$$

onto the fermion (+) and antifermion (−) subspaces, respectively, in the rest frame $\mathbf{p} = 0$.

Therefore, we can split the noise contribution to the self-energy into four subspaces, namely

$$\lim_{\substack{p_0\to m \\ \mathbf{p}\to 0}} [-\mathrm{i}\Sigma_\Delta(p)]_r = \sum_{\sigma=\uparrow,\downarrow} \sum_{\lambda=\pm 1} [-\mathrm{i}\tilde{\Sigma}_\Delta^{(\sigma,\lambda)} \mathcal{O}^{(\sigma)} \mathcal{P}^{(\lambda)}], \tag{43}$$

where we defined the coefficients:

$$\tilde{\Sigma}_{\Delta}^{(\downarrow,\pm)} \equiv \mathrm{i}\frac{\sqrt{2}}{\pi^{3/2}}(3\ln(2)\pm 8)\alpha_{\mathrm{em}}\Delta\sqrt{|eB|}m + O(m^2), \quad (44a)$$

$$\tilde{\Sigma}_{\Delta}^{(\uparrow,\pm)} \equiv \mathrm{i}\frac{\sqrt{2}}{\pi^{3/2}}(3\ln(2)\mp 2)\alpha_{\mathrm{em}}\Delta\sqrt{|eB|}m + O(m^2), \quad (44b)$$

which are clearly purely imaginary.

In summary, our results indicate that the fermion magnetic mass operator, in the presence of noise $\Delta > 0$, possesses four different eigenvalues depending on the spin $\sigma = \uparrow, \downarrow$ and $\lambda = \pm$ projections, as follows:

$$M_B^{(\sigma,\lambda)}(\Delta) = m + M_{B0}^{(\sigma)} + \tilde{\Sigma}_{\Delta}^{(\sigma,\lambda)}. \quad (45)$$

We notice that these eigenvalues are complex, such that the real parts strictly correspond to the fermion magnetic mass, i.e. $m_B^{(\sigma)} = \mathrm{Re}M_B^{(\sigma,\lambda)}(\Delta)$, which turns out to be noise independent

$$m_B^{(\uparrow)} = m + \frac{\alpha_{\mathrm{em}}m}{\pi}\left[\ln^2\mathcal{B} - \gamma_e\ln\mathcal{B} + \frac{\pi^2}{3}\right] + O(\mathcal{B}^{-1}),$$
$$m_B^{(\downarrow)} = m + \frac{\alpha_{\mathrm{em}}m}{\pi}\left[\ln^2\mathcal{B} - (1+\gamma_e)\ln\mathcal{B} + \frac{\pi^2}{3} + \gamma_e - 2\right] + O(\mathcal{B}^{-1}), \quad (46)$$

which are depicted in Fig. 2. We notice that the magnetic mass is different for each spin projection, as expected from the Zeeman interaction splitting. This effect becomes stronger in very intense magnetic fields and may be of interest in different physical scenarios.

On the other hand, the imaginary parts represent a Breit-Wigner resonance $\Gamma^{(\sigma,\lambda)}(\Delta) = -2\mathrm{Im}M_B^{(\sigma,\lambda)}(\Delta)$ due to the combination of the field and the magnetic noise given by

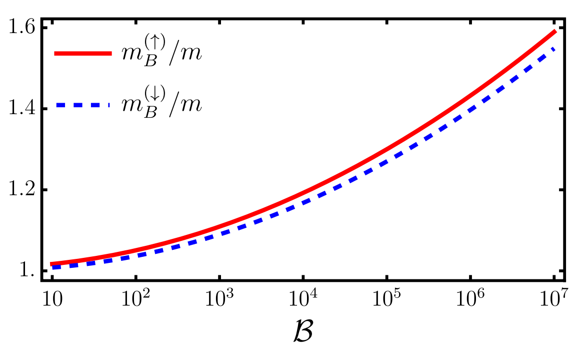

FIG. 2. The magnetic mass of the fermion, calculated from Eq. (46), is shown for the two spin projections as a function of the average background magnetic field (in dimensionless units) $\mathcal{B} = |eB|/m^2$.

$$\Gamma^{(\uparrow,\pm)}(\Delta) = \alpha_{\mathrm{em}}m\left(\ln\mathcal{B} - \frac{2\sqrt{2\mathcal{B}}m\Delta}{\pi^{3/2}}(3\ln(2)\mp 2)\right), \quad (47a)$$

$$\Gamma^{(\downarrow,\pm)}(\Delta) = \alpha_{\mathrm{em}}m\left(\ln\mathcal{B} - 1 - \frac{2\sqrt{2\mathcal{B}}m\Delta}{\pi^{3/2}}(3\ln(2)\pm 8)\right). \quad (47b)$$

Figure 3 shows the behavior of $\Gamma^{(\sigma,\lambda)}(\Delta)$, computed from Eq. (47), as a function of the average magnetic field (in dimensionless units) $\mathcal{B} = |eB|/m^2$, for the four eigenvalues corresponding to each projection $(\uparrow\downarrow, \pm)$, respectively. As can be seen in the figures, deviations from the noiseless limit $\Delta = 0$ (solid line) become appreciable after some critical value $\mathcal{B} > \mathcal{B}_c$ that depends on the magnitude of $m\Delta$ via the product $m\Delta\sqrt{\mathcal{B}}$. In cases represented in Figs. 3(a) and 3(c), for $m\Delta = 10^{-2}$ and $m\Delta = 10^{-3}$ respectively, where the spin projection is parallel $(\uparrow)$ to the direction of the background magnetic field, the imaginary part of the magnetic mass in the subspace given by the projection $\mathcal{P}^{(+)}$ decreases as compared to the corresponding one for the projection $\mathcal{P}^{(-)}$ and also with respect to the noiseless case $\Delta = 0$. The opposite occurs when the spin projection is antiparallel to the direction of the background magnetic field, as depicted in Figs. 3(b) and 3(d), for $m\Delta = 10^{-2}$ and $m\Delta = 10^{-3}$ respectively. This implies that in the quasiparticle picture, the charge conjugation combined with the breaking of Lorentz symmetry provided by the magnetic noise results in different spectral widths for the various modes.

Note that in the heavy-ion collisions scenario, the magnetic background is about the pion-mass squared, so that for electrons we would have $\mathcal{B} \sim 8\times 10^4$, while for light quarks $\mathcal{B} \sim 8\times 10^3$. Therefore, our approximation based on the lowest Landau level expression for the fermion propagator valid for $\mathcal{B} \gg 1$ is well justified. Hence, for some ranges of the noise $m\Delta$, the effects displayed in Fig. 3 might be detected in actual experiments.

In Fig. 4, the imaginary part of the mass eigenvalues, corresponding to the Breit-Wigner resonances $\Gamma^{(\sigma,\lambda)}(\Delta)$ defined in Eq. (47) for each of the four projections, are shown as a function of the magnetic noise autocorrelation $m\Delta$, for a constant average field value of $\mathcal{B} = 10^4$. In terms of the physical interpretation, these Breit-Wigner resonances proportional to $\mathrm{Im}\Sigma(p, B, \Delta)$ lead to a small broadening in the peak of the Lorentzian spectral density distribution, as we discussed in Ref. [16]. As seen in Fig. 4, both parallel $(\uparrow, \pm)$ spin projections exhibit a linear dependence on $m\Delta$ with a negative slope. This effect is milder in the $(\uparrow, +)$ than in the $(\uparrow, -)$ polarization. In contrast, the antiparallel spin polarizations $(\downarrow, \pm)$ display opposite behavior, with $(\downarrow, -)$ showing a positive slope, while $(\downarrow, +)$ exhibits a negative one. Nevertheless, since the spectral broadening depends on the absolute value of these parameters, in all four polarizations the spectral width grows with the

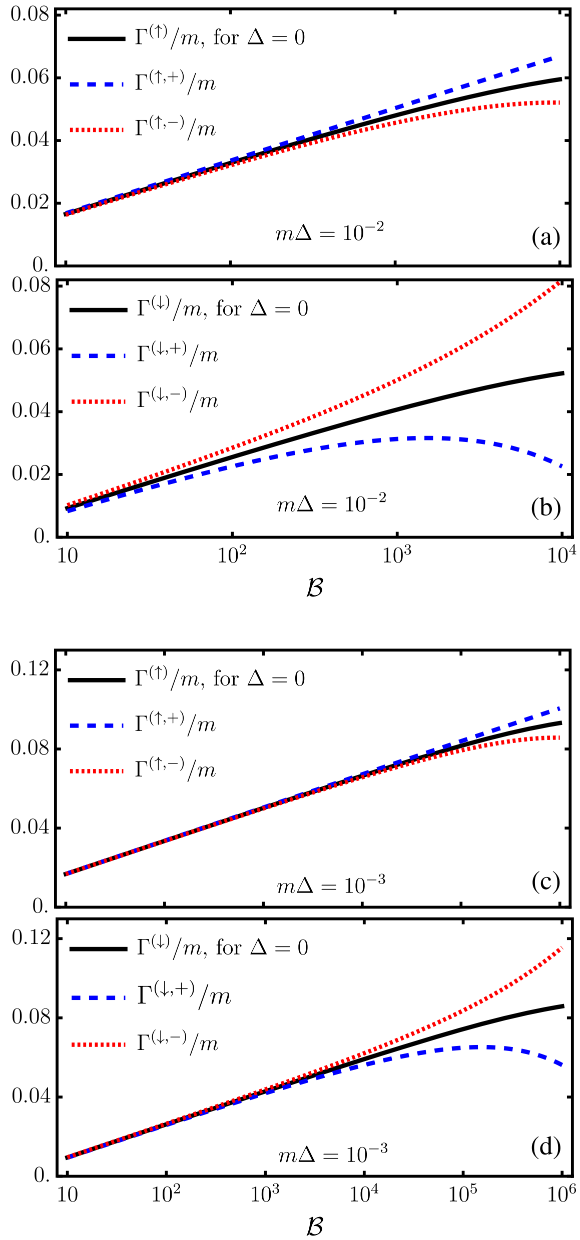

FIG. 3. Breit-Wigner resonance $\Gamma^{(\sigma,\lambda)}$ computed from Eq. (47), as a function of the average background field $\mathcal{B} = |eB|/m^2$. The eigenvalues corresponding to projections onto the four independent subspaces are shown for two different values of the magnetic noise autocorrelation $m\Delta$.

magnetic noise autocorrelation Δ. As discussed in our previous work [16], the spectral density corresponding to each projection is defined by the Lorentzian distributions

$$\tilde{\rho}^{(\sigma,\lambda)}(p^2) = \frac{m_B^{(\sigma)}\Gamma^{(\sigma,\lambda)}/\pi}{[p^2 - (m_B^{(\sigma)})^2]^2 + [m_B^{\sigma}\Gamma^{(\sigma,\lambda)}]^2}, \qquad (48)$$

where the relative spectral width decays for very intense magnetic fields $\mathcal{B} \gg 1$ as $\frac{\Gamma^{(\sigma,\lambda)}}{m_B^{\sigma}} \sim [\ln \mathcal{B}]^{-1}$. The corresponding

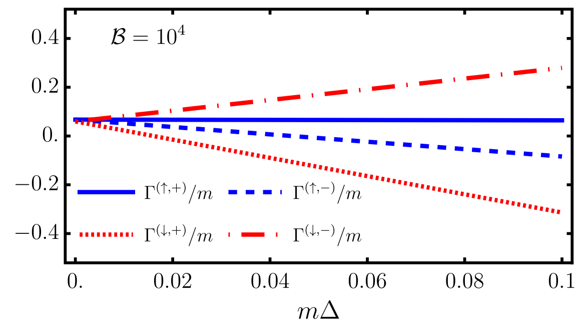

FIG. 4. The Breit-Wigner resonance $\Gamma^{(\sigma,\lambda)}(\Delta)$, computed from Eq. (47), as a function of the noise autocorrelation parameter $m\Delta$, for a fixed intensity of the average background field $\mathcal{B} = |eB|/m^2 = 10^4$. The eigenvalues corresponding to projections onto the four independent subspaces are displayed for comparison.

spectral density distributions, as computed from Eq. (48) for the four different projections, are displayed in Fig. 5, where the spectral width due to the finite value of $\Gamma^{(\sigma,\lambda)}$ is clearly appreciated. Interestingly, this spectral broadening effect induced by the presence of the noise autocorrelation $\Delta > 0$ is different depending on the spin projection $\uparrow, \downarrow$, as well as the projection onto the subspaces $\mathcal{P}^{(\pm)}$. However, the physical magnetic mass representing the center of the spectral distribution only depends on the spin projection, as expected from the usual Zeeman splitting effect due to the spin-magnetic field interaction.

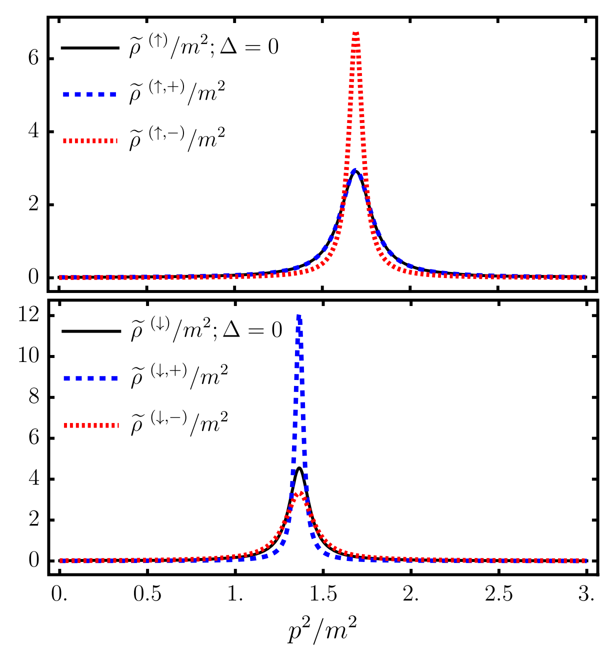

FIG. 5. The spectral density distributions for each of the four projections $\uparrow, \downarrow, \pm$, computed from Eq. (48), as a function of the dimensionless momentum p^2/m^2. The dashed and dotted lines correspond to a noise autocorrelation $m\Delta = 10^{-2}$, while the solid line represents the noiseless limit $\Delta = 0$. The average background field is $\mathcal{B} = 10^5$ for all cases.

IV. DISCUSSION AND CONCLUSIONS

We studied the QED fermion propagator in a strongly magnetized medium with white-noise fluctuations. In a quasiparticle picture based on our previous results for the fermion and photon propagators in such media [19,22], we computed the self-energy contribution due to radiative corrections at first order in the electromagnetic fine structure constant $\alpha_{\rm em}$. The presence of the background magnetic field breaks the Lorentz invariance, thus splitting the metric into two subspaces according to the directions parallel and perpendicular to the field, respectively. Accordingly, the eigenvalues of the magnetic mass operator obtained from the real part of the self-energy are different for the corresponding two spin projections $\uparrow,\downarrow$. Moreover, the presence of the magnetic fluctuations, whose strength is proportional to the noise autocorrelation Δ, fully breaks the Lorentz invariance, thus leading to imaginary components in the mass operator eigenvalues that depend on the spin polarization as well as in the projection onto the $\mathcal{P}^{\pm}=(\mathbb{1}\pm\gamma^0)/2$ subspaces corresponding to fermion $(+)$ and antifermion $(-)$ in their rest frame. The later imaginary contributions correspond to a further spectral broadening effect as we discussed in Ref. [16], here caused by the magnetic noise that, nevertheless, does not renormalize the magnetic mass in agreement with our previous studies [19]. The existence of this further splitting in the spectral width is a clear indicator, at a perturbative level, of the charge conjugation symmetry breaking induced by the statistical model. The physical reason behind it is that, in the presence of a uniform and deterministic background magnetic field, charge conjugation symmetry $\mathcal{C}$ is granted given that *both* the fermion charge $q_f\to\mathcal{C}q_f\mathcal{C}=-q_f$ *and* the direction of the field $\mathbf{B}\to\mathcal{C}\mathbf{B}\mathcal{C}\to-\mathbf{B}$ revert their signs *simultaneously*, the latter due to the reversal of the direction of the classical currents that generate the field $\mathcal{C}\mathbf{J}_{\rm class}\mathcal{C}\to-\mathbf{J}_{\rm class}$. In contrast, if the background classical field possesses a stochastic fluctuating component $\mathbf{B}(x)=\mathbf{B}+\delta\mathbf{B}(x)$ generated by statistically incoherent classical sources $\mathbf{J}_{\rm class}+\delta\mathbf{J}_{\rm class}$, once we perform the average over magnetic fluctuations the system looses track of the direction of the fluctuating part $\delta\mathbf{B}(x)$, whose statistical effect remains present on the scalar coefficient Δ [see the effective interaction in Eq. (12)]. Therefore, in this theory charge conjugation symmetry can only be approximately satisfied by the average background field $\mathcal{C}\mathbf{B}\mathcal{C}\to-\mathbf{B}$, but the presence of any finite magnetic noise $\Delta>0$ will break it. Even though we verified this property explicitly at a perturbative level, the previous argument supports the conclusion that it is a nonperturbative feature of our model.

ACKNOWLEDGMENTS

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APPENDIX A: COMPUTATION OF $-\mathrm{i}\Sigma_0(p,B)$

From Eqs. (25), (31), and (34a), we get

$$-\mathrm{i}\Sigma_0(p,B)=2e^2\int\frac{d^4k}{(2\pi)^4}\frac{e^{-\mathbf{k}_\perp{}^2/|eB|}\gamma^\mu(\not{k}_\parallel+m)\mathcal{O}^{(\uparrow)}\gamma_\mu}{(k_\parallel^2-m^2+\mathrm{i}\epsilon)((k-p)^2+\mathrm{i}\epsilon)},\tag{A1}$$

so that the tensor structure can be spitted as

$$\begin{aligned}\gamma^\mu(\not{k}_\parallel+m)\mathcal{O}^{(\uparrow)}\gamma_\mu&\\ &=\frac{1}{2}[\gamma^\mu\not{k}_\parallel\gamma_\mu-\mathrm{i}\,\mathrm{sign}(eB)\gamma^\mu\not{k}_\parallel\gamma^1\gamma^2\gamma_\mu+m\gamma^\mu\gamma_\mu\\ &\quad-\mathrm{i}\,\mathrm{sign}(eB)m\gamma^\mu\gamma^1\gamma^2\gamma_\mu],\\ &=\frac{1}{2}[-2\not{k}_\parallel+2\mathrm{i}\,\mathrm{sign}(eB)\gamma^2\gamma^1\not{k}_\parallel+4m],\\ &=2[m-\not{k}_\parallel\mathcal{O}^{(\downarrow)}].\end{aligned}\tag{A2}$$

On the other hand, by a Schwinger parametrization (in the Feynman time-ordered prescription $\epsilon\to0^+$),

$$\frac{1}{A+\mathrm{i}\epsilon}=-\mathrm{i}\int_0^\infty d\tau e^{\mathrm{i}(A+\mathrm{i}\epsilon)\tau},\tag{A3}$$

the expression takes the form

$$\begin{aligned}&-\mathrm{i}\Sigma_0(p,B)\\ &=4e^2(-\mathrm{i})^2\int_0^\infty d\tau_1\int_0^\infty d\tau_2\int\frac{d^4k}{(2\pi)^4}[m-\not{k}_\parallel\mathcal{O}^{(\downarrow)}]\\ &\quad\times\exp\left\{-\frac{\mathbf{k}_\perp^2}{|eB|}+\mathrm{i}\tau_1[(k-p)^2+\mathrm{i}\epsilon]+\mathrm{i}\tau_2(k_\parallel^2-m^2+\mathrm{i}\epsilon)\right\}.\end{aligned}\tag{A4}$$

The factor in the exponential can be rearranged as

$$\begin{aligned}&-\frac{\mathbf{k}_\perp^2}{|eB|}+\mathrm{i}\tau_1[(k-p)^2+\mathrm{i}\epsilon]+\mathrm{i}\tau_2(k_\parallel^2-m^2+\mathrm{i}\epsilon)\\ &\quad=-\left(\frac{1}{|eB|}+\mathrm{i}\tau_1\right)\left(\mathbf{k}_\perp^2-\frac{2\mathrm{i}\tau_1k_\perp\cdot p_\perp}{1/|eB|+\mathrm{i}\tau_1}\right)-\mathrm{i}\tau_1\mathbf{p}_\perp^2\\ &\qquad+\mathrm{i}(\tau_1+\tau_2)\left(k_\parallel^2-\frac{2\tau_1k_\parallel\cdot p_\parallel}{\tau_1+\tau_2}\right)\\ &\qquad+\mathrm{i}\tau_1p_\parallel^2-\mathrm{i}\tau_2m^2-\epsilon(\tau_1+\tau_2),\end{aligned}\tag{A5}$$

so that, by performing the following change of variables,

$$\begin{aligned}\ell_\parallel^\mu&\equiv k_\parallel^\mu-\frac{\tau_1}{\tau_1+\tau_2}p_\parallel^\mu,\\ \ell_\perp&\equiv k_\perp^\mu-\frac{\mathrm{i}|eB|\tau_1}{1+\mathrm{i}|eB|\tau_1}p_\perp^\mu,\end{aligned}\tag{A6}$$

we get

$$-\mathrm{i}\Sigma_0(p,B) = -4e^2\int d^2\tau\int\frac{d^4\ell}{(2\pi)^4}\left[m-\left(\ell_\parallel+\frac{\tau_1}{\tau_1+\tau_2}\not{p}_\parallel\right)\mathcal{O}^{(\downarrow)}\right] \times\exp\left\{\mathrm{i}(\tau_1+\tau_2)\ell_\parallel^2-\frac{1+\mathrm{i}|eB|\tau_1}{|eB|}\ell_\perp^2+\mathrm{i}\tau_2\left(\frac{\tau_1 p_\parallel^2}{\tau_1+\tau_2}-m^2\right)-\frac{\mathrm{i}\tau_1\mathbf{p}_\perp^2}{1+\mathrm{i}|eB|\tau_1}-\epsilon(\tau_1+\tau_2)\right\}. \quad (A7)$$

The Gaussian integration over the momenta are calculated as follows:

$$\int\frac{d^2\ell_\parallel}{(2\pi)^2}\exp\left[\mathrm{i}(\tau_1+\tau_2)\ell_\parallel^2\right] = \mathrm{i}\int\frac{d^2\ell_E}{(2\pi)^2}e^{-\mathrm{i}(\tau_1+\tau_2)\ell_E^2}, = \frac{\mathrm{i}}{(2\pi)^2}\frac{\pi}{\mathrm{i}(\tau_1+\tau_2)}, = \frac{1}{(2\pi)^2}\frac{\pi}{(\tau_1+\tau_2)}, \quad (A8a)$$

$$\int\frac{d^2\ell_\perp}{(2\pi)^2}\exp\left[-\frac{1+\mathrm{i}|eB|\tau_1}{|eB|}\ell_\perp^2\right]=\frac{1}{(2\pi)^2}\frac{\pi|eB|}{1+\mathrm{i}|eB|\tau_1}, \quad (A8b)$$

where in the first equation we first performed a Wick rotation $\ell^0\to\mathrm{i}\ell_E^0$ to render the integration coordinates to the Euclidean metric. After this procedure, we are left with the expression

$$-\mathrm{i}\Sigma_0(p,B) = \frac{\alpha_{\mathrm{em}}|eB|}{\pi}\int d^2\tau\frac{m-\frac{\tau_1}{\tau_1+\tau_2}\not{p}_\parallel\mathcal{O}^{(\downarrow)}}{(1+\mathrm{i}|eB|\tau_1)(\tau_1+\tau_2)}\exp\left[\mathrm{i}\tau_2\left(\frac{\tau_1 p_\parallel^2}{\tau_1+\tau_2}-m^2\right)-\frac{\mathrm{i}\tau_1\mathbf{p}_\perp^2}{1+\mathrm{i}|eB|\tau_1}-\epsilon(\tau_1+\tau_2)\right]. \quad (A9)$$

It is convenient to introduce the change of variables

$$\tau_1=\frac{sy}{m^2}\quad\text{and}\quad\tau_2=\frac{s(1-y)}{m^2}\to\left|\frac{\partial(\tau_1,\tau_2)}{\partial(s,y)}\right|=\frac{s}{m^4}, \quad (A10)$$

so that in the dimensionless variables $\mathcal{B}=|eB|/m^2$, $\rho_{\parallel,\perp}=p_{\parallel,\perp}/m$

$$-\mathrm{i}\Sigma_0(p,B)=\frac{\alpha_{\mathrm{em}}\mathcal{B}m}{\pi}\int_0^\infty ds\int_0^1 dy\frac{1-y\not{\rho}_\parallel\mathcal{O}^{(\downarrow)}}{1+\mathrm{i}\mathcal{B}sy}\exp\left[\mathrm{i}s(1-y)(y\rho_\parallel^2-1)-\frac{\mathrm{i}sy\rho_\perp^2}{1+\mathrm{i}\mathcal{B}sy}-\epsilon s\right]. \quad (A11)$$

In order to obtain the expression for the magnetic mass operator in the noiseless limit $\Delta=0$ as stated in the main text, we evaluate this integral at the condition $\not{p}_\parallel=m$, $\mathbf{p}_\perp=0$, as follows:

$$\Sigma_0(p,B)|_{\not{p}_\parallel=m,\mathbf{p}_\perp=0}=\frac{\mathrm{i}\alpha_{\mathrm{em}}\mathcal{B}m}{\pi}\int_0^\infty ds\int_0^1 dy\frac{1-y\mathcal{O}^{(\downarrow)}}{1+\mathrm{i}\mathcal{B}sy}\exp\left(-\mathrm{i}s(1-y)^2-\epsilon s\right), =\frac{\mathrm{i}\alpha_{\mathrm{em}}\mathcal{B}m}{\pi}\left[\left(\int_0^\infty ds\int_0^1 dy\frac{e^{-\mathrm{i}s(1-y)^2-\epsilon s}}{1+\mathrm{i}\mathcal{B}sy}\right)\mathcal{O}^{(\uparrow)}+\left(\int_0^\infty ds\int_0^1 dy\frac{(1-y)e^{-\mathrm{i}s(1-y)^2-\epsilon s}}{1+\mathrm{i}\mathcal{B}sy}\right)\mathcal{O}^{(\downarrow)}\right], =M_{B0}^{(\uparrow)}\mathcal{O}^{(\uparrow)}+M_{B0}^{(\downarrow)}\mathcal{O}^{(\downarrow)}, \quad (A12)$$

where we used the completeness relation for the spin projectors

$$\mathcal{O}^{(\uparrow)}+\mathcal{O}^{(\downarrow)}=\mathbb{1}, \quad (A13)$$

and defined the corresponding fermion magnetic mass eigenvalues $M_{B0}^{\uparrow,\downarrow}$ for the two spin projections $\uparrow,\downarrow$, respectively. In order to explicitly compute such noiseless mass eigenvalues for the intense magnetic field approximation $\mathcal{B}\gg 1$, the integration region is restricted by a lower cutoff [16] $\sim\mathcal{B}^{-1}$, as follows (for $\epsilon\to 0^+$):

$$\int_0^\infty ds\int_0^1 dy\frac{e^{-\mathrm{i}s(1-y)^2-\epsilon s}}{1+\mathrm{i}\mathcal{B}sy}\to\int_{\mathcal{B}^{-1}}^\infty ds\int_{\mathcal{B}^{-1}}^1 dy\frac{e^{-\mathrm{i}s(1-y)^2-\epsilon s}}{1+\mathrm{i}\mathcal{B}sy}=\int_{\mathcal{B}^{-1}}^\infty ds\int_{\mathcal{B}^{-1}}^1 dy e^{-\mathrm{i}s(1-y)^2-\epsilon s}\left(-\frac{\mathrm{i}}{\mathcal{B}sy}-\sum_{n=1}^\infty\left(\frac{\mathrm{i}}{\mathcal{B}sy}\right)^{n+1}\right)$$
$$=\int_{\mathcal{B}^{-1}}^1\frac{dy}{\mathrm{i}\mathcal{B}y}\left\{\Gamma\left[0;\frac{\mathrm{i}(1-y)^2}{\mathcal{B}}\right]+\sum_{n=1}^\infty\left(\frac{\mathrm{i}}{\mathcal{B}}\right)^n\frac{[\mathrm{i}(1-y)^2]^n}{y^n}\Gamma\left[-n;\frac{\mathrm{i}(1-y)^2}{\mathcal{B}}\right]\right\}, \tag{A14}$$

Let us consider the series representation of the incomplete gamma function, defined by

$$\Gamma(0,z)=-\gamma_e-\ln(z)-\sum_{k=1}^\infty\frac{(-z)^k}{k\cdot k!}, \tag{A15}$$

where γ_e is the Euler-Mascheroni constant, and for $n\in\mathbb{Z}^+$ by

$$\Gamma(-n,z)=\frac{(-1)^n}{n!}\Gamma(0,z)+\frac{z^{-n}f_n(z)e^{-z}}{n!}, \tag{A16}$$

with the $n-1$ degree polynomials

$$f_n(z)=\sum_{k=0}^{n-1}(-1)^k(n-k-1)!z^k. \tag{A17}$$

Then, it is convenient to reorganize the infinite series in the integrand of Eq. (A14) in order to group the leading logarithmic contributions in the $\Gamma[0;\frac{\mathrm{i}(1-y)^2}{\mathcal{B}}]$ function as follows:

$$\frac{1}{\mathrm{i}\mathcal{B}y}\left\{\Gamma\left[0;\frac{\mathrm{i}(1-y)^2}{\mathcal{B}}\right]+\sum_{n=1}^\infty\left(\frac{\mathrm{i}}{\mathcal{B}}\right)^n\frac{[\mathrm{i}(1-y)^2]^n}{y^n}\Gamma\left[-n;\frac{\mathrm{i}(1-y)^2}{\mathcal{B}}\right]\right\}$$
$$=\frac{1}{\mathrm{i}\mathcal{B}y}\left\{1+\sum_{n=1}^\infty\frac{1}{n!}\left(\frac{(1-y)^2}{y\mathcal{B}}\right)^n\right\}\Gamma\left[0;\frac{\mathrm{i}(1-y)^2}{\mathcal{B}}\right]+\frac{e^{-\mathrm{i}\frac{(1-y)^2}{\mathcal{B}y}}}{\mathrm{i}\mathcal{B}y}\sum_{n=1}^\infty\frac{(-\mathrm{i}\mathcal{B}y)^{-n}}{n!}f_n(\mathrm{i}(1-y)^2/\mathcal{B})$$
$$=\frac{e^{\frac{(1-y)^2}{\mathcal{B}y}}}{\mathrm{i}\mathcal{B}y}\Gamma\left[0;\frac{\mathrm{i}(1-y)^2}{\mathcal{B}}\right]+\frac{e^{-\mathrm{i}\frac{(1-y)^2}{\mathcal{B}y}}}{\mathrm{i}\mathcal{B}y}\sum_{n=1}^\infty\frac{(-\mathrm{i}\mathcal{B}y)^{-n}}{n!}f_n(\mathrm{i}(1-y)^2/\mathcal{B})\approx\frac{\mathrm{i}(\gamma_e-\ln\mathcal{B}+\ln[\mathrm{i}(1-y)^2])}{y\mathcal{B}}+O(\mathcal{B}^{-2}). \tag{A18}$$

Then (with $\epsilon\to0^+$), we have that at leading order in $\mathcal{B}\gg1$

$$\int_{\mathcal{B}^{-1}}^\infty ds\int_{\mathcal{B}^{-1}}^1 dy\frac{e^{-\mathrm{i}s(1-y)^2-\epsilon s}}{1+\mathrm{i}\mathcal{B}sy}\approx\int_{\mathcal{B}^{-1}}^1 dy\frac{\mathrm{i}(\gamma_e-\ln\mathcal{B}+\ln[\mathrm{i}(1-y)^2])}{y\mathcal{B}}+O(\mathcal{B}^{-2}),$$
$$\approx\frac{\mathrm{i}}{\mathcal{B}}\left\{\left[\gamma_e+\ln\left(\frac{\mathrm{i}(1-\mathcal{B})^2}{\mathcal{B}^3}\right)\right]\ln(\mathcal{B})-2\mathrm{Li}_2\left(\frac{\mathcal{B}-1}{\mathcal{B}}\right)\right\}+O(\mathcal{B}^{-2}),$$
$$\approx-\mathrm{i}\mathcal{B}^{-1}\ln^2\mathcal{B}+\mathrm{i}\left(\gamma_e+\mathrm{i}\frac{\pi}{2}\right)\mathcal{B}^{-1}\ln\mathcal{B}-\mathrm{i}\frac{\pi^2}{3}\mathcal{B}^{-1}+O(\mathcal{B}^{-2}). \tag{A19}$$

Applying the same series expansion to the second term, we have

$$\begin{aligned}\int_{\mathcal{B}^{-1}}^{\infty} ds \int_{\mathcal{B}^{-1}}^{1} dy \frac{(1-y)e^{-\mathrm{i}s(1-y)^2-\epsilon s}}{1+\mathrm{i}\mathcal{B}sy} &\approx \int_{\mathcal{B}^{-1}}^{1} dy(1-y)\frac{\mathrm{i}(\gamma_e - \ln\mathcal{B} + \ln[\mathrm{i}(1-y)^2])}{y\mathcal{B}} + O(\mathcal{B}^{-2}),\\ &\approx -\mathrm{i}\left(\frac{(\gamma_e-2)}{\mathcal{B}} - \gamma_e\mathcal{B}^{-1}\ln\mathcal{B} + \frac{(1-\ln\mathcal{B}-\mathcal{B}^{-1})}{\mathcal{B}}\ln\left[\frac{\mathrm{i}(\mathcal{B}-1)^2}{\mathcal{B}^3}\right]\right)\\ &\quad - 2\mathrm{i}\mathcal{B}^{-1}\mathrm{Li}_2\left(\frac{\mathcal{B}-1}{\mathcal{B}}\right) + O(\mathcal{B}^{-2}),\\ &\approx -\mathrm{i}\mathcal{B}^{-1}\ln^2\mathcal{B} + \mathrm{i}\left(1+\gamma_e+\mathrm{i}\frac{\pi}{2}\right)\mathcal{B}^{-1}\ln\mathcal{B} + \mathrm{i}\left(2-\gamma_e-\frac{\pi^2}{3}-\mathrm{i}\frac{\pi}{2}\right)\mathcal{B}^{-1} + O(\mathcal{B}^{-2}).\end{aligned} \tag{A20}$$

Therefore, we conclude that in the noiseless limit $\Delta = 0$, the magnetic mass eigenvalues for each spin projection are given by the expressions

$$\begin{aligned}M_{B0}^{(\uparrow)} &= \frac{\alpha_{\mathrm{em}} m}{\pi}\left[\ln^2\mathcal{B} - \left(\gamma_e + \mathrm{i}\frac{\pi}{2}\right)\ln\mathcal{B} + \frac{\pi^2}{3}\right] + O(\mathcal{B}^{-1}),\\ M_{B0}^{(\downarrow)} &= \frac{\alpha_{\mathrm{em}} m}{\pi}\left[\ln^2\mathcal{B} - \left(1+\gamma_e+\mathrm{i}\frac{\pi}{2}\right)\ln\mathcal{B} - \left(2-\gamma_e-\frac{\pi^2}{3}-\mathrm{i}\frac{\pi}{2}\right)\right] + O(\mathcal{B}^{-1}).\end{aligned} \tag{A21}$$

APPENDIX B: CALCULATION OF $-\mathrm{i}\Sigma_\Delta(p)$

From Eqs. (26), (31), and (34b), and given that

$$\gamma^\mu(\not{k}_\parallel + m)\mathcal{O}^{(\uparrow)}\gamma_\mu = 2[m - \not{k}_\parallel\mathcal{O}^{(\downarrow)}], \tag{B1a}$$

$$\gamma^\mu\gamma^3\mathcal{O}^{(\uparrow)}\gamma_\mu = -2\gamma^3\mathcal{O}^{(\downarrow)}, \tag{B1b}$$

$$\gamma^\mu\mathrm{i}\gamma^1\gamma^2(\not{k}_\parallel + m)\gamma_\mu = 2\not{k}_\parallel\mathrm{i}\gamma^1\gamma^2, \tag{B1c}$$

we get

$$-\mathrm{i}\Sigma_\Delta(p) = \sum_{i=1}^{3}[-\mathrm{i}\Sigma_\Delta^{(i)}(p)], \tag{B2}$$

where

$$-\mathrm{i}\Sigma_\Delta^{(1)}(p) = -4\alpha_{\mathrm{em}}\Delta|eB|\int\frac{d^4k}{(2\pi)^4}\frac{\Theta_1(k)(m - \not{k}_\parallel\mathcal{O}^{(\downarrow)})}{(k-p)^2 + \mathrm{i}\epsilon}, \tag{B3a}$$

$$-\mathrm{i}\Sigma_\Delta^{(2)}(p) = -4\alpha_{\mathrm{em}}\Delta|eB|\int\frac{d^4k}{(2\pi)^4}\frac{\Theta_2(k)\gamma^3\mathcal{O}^{(\downarrow)}}{(k-p)^2 + \mathrm{i}\epsilon}, \tag{B3b}$$

$$-\mathrm{i}\Sigma_\Delta^{(3)}(p) = -4\alpha_{\mathrm{em}}\Delta|eB|\mathrm{sign}(eB)\int\frac{d^4k}{(2\pi)^4}\frac{\Theta_3(k)\not{k}_\parallel\mathrm{i}\gamma^1\gamma^2}{(k-p)^2 + \mathrm{i}\epsilon}. \tag{B3c}$$

Let us start with $-\mathrm{i}\Sigma_\Delta^{(1)}(p)$, which from Eq. (27b) is

$$-\mathrm{i}\Sigma_\Delta^{(1)}(p) = -12\alpha_{\mathrm{em}}\Delta|eB|\int\frac{d^4k}{(2\pi)^4}\frac{(k_\parallel^2 + m^2)e^{-2\mathbf{k}_\perp^2/|eB|}}{[(k_\parallel^2 - m^2)^2\sqrt{k_0^2 - m^2}][(k-p)^2 + \mathrm{i}\epsilon]}(m - \not{k}_\parallel\mathcal{O}^{(\downarrow)}). \tag{B4}$$

After a Schwinger parametrization in the photon's propagator and by defining the momenta shift

$$\ell_{\perp} = k^{\mu}_{\perp} - \frac{\mathrm{i}|eB|}{2+\mathrm{i}|eB|\tau} p^{\mu}_{\perp}, \tag{B5}$$

the integral takes the form

$$\begin{aligned} -\mathrm{i}\Sigma^{(1)}_{\Delta}(p) = 12\mathrm{i}\alpha_{\mathrm{em}}\Delta|eB| \int_0^{\infty} d\tau \exp\left[-\frac{2\mathrm{i}\tau \mathbf{p}^2_{\perp}}{2+\mathrm{i}\tau|eB|} - \epsilon\tau\right] \int \frac{d^2\ell_{\perp}}{(2\pi)^2} \exp\left[-\left(\frac{2+\mathrm{i}|eB|\tau}{|eB|}\right)\ell^2_{\perp}\right] \\ \times \int \frac{d^2 k_{\parallel}}{(2\pi)^2} \frac{(k_{\parallel}+m^2)e^{\mathrm{i}\tau(k_{\parallel}-p_{\parallel})^2}}{(k^2_{\parallel}-m^2)^2\sqrt{k_0^2-m^2}} (m - \not{k}_{\parallel}\mathcal{O}^{(\downarrow)}). \end{aligned} \tag{B6}$$

By using Eq. (A8a), and defining the parallel momenta shift:

$$\ell^{\mu}_{\parallel} = k^{\mu}_{\parallel} - p^{\mu}_{\parallel}, \tag{B7}$$

$$-\mathrm{i}\Sigma^{(1)}_{\Delta}(p) = \frac{3\mathrm{i}\alpha_{\mathrm{em}}\Delta|eB|^2}{\pi} \int_0^{\infty} d\tau \frac{\exp\left[-\frac{2\mathrm{i}\tau\mathbf{p}^2_{\perp}}{2+\mathrm{i}\tau|eB|} - \epsilon\tau\right]}{2+\mathrm{i}|eB|\tau} \mathcal{I}_1(\tau, p_0, p_3), \tag{B8}$$

where

$$\mathcal{I}_1(\tau, p_0, p_3) \equiv \int \frac{d^2\ell_{\parallel}}{(2\pi)^2} \frac{[(\ell_{\parallel}+p_{\parallel})^2+m^2][m-(\not{\ell}_{\parallel}+\not{p}_{\parallel})\mathcal{O}^{(\downarrow)}]e^{\mathrm{i}\tau\ell^2_{\parallel}}}{[(\ell_{\parallel}+p_{\parallel})^2-m^2]^2\sqrt{(\ell_0+p_0)^2-m^2}}. \tag{B9}$$

We are interested on the limits $p_0 \to m$ and $\mathbf{p} \to 0$, so that

$$\lim_{\substack{p_0\to m \\ \mathbf{p}\to 0}} [-\mathrm{i}\Sigma^{(1)}_{\Delta}(p)] = \frac{3\mathrm{i}\alpha_{\mathrm{em}}\Delta|eB|^2}{\pi} \int_0^{\infty} d\tau \frac{\mathcal{I}_1(\tau, m, 0)e^{-\epsilon\tau}}{2+\mathrm{i}|eB|\tau}, \tag{B10}$$

with

$$\begin{aligned} \mathcal{I}_1(\tau, m, 0) &\equiv \int \frac{d^2\ell_{\parallel}}{(2\pi)^2} \frac{[\ell_0(\ell_0+2m)-\ell_3^2+2m^2][m-((\ell_0+m)\gamma^0+\ell_3\gamma^3)\mathcal{O}^{(\downarrow)}]e^{\mathrm{i}\tau\ell^2_{\parallel}}}{[\ell_0(\ell_0+2m)-\ell_3^2]^2\sqrt{\ell_0(\ell_0+2m)}}, \\ &= -\mathrm{i}\int \frac{d^2\ell_{\mathrm{E}}}{(2\pi)^2} \frac{[\ell_4(\ell_4-2\mathrm{i}m)+\ell_3^2-2m^2][m-((\mathrm{i}\ell_4+m)\gamma^0+\ell_3\gamma^3)\mathcal{O}^{(\downarrow)}]e^{-\mathrm{i}\tau\ell^2_{\mathrm{E}}}}{[\ell_4(\ell_4-2\mathrm{i}m)+\ell_3^2]^2\sqrt{\ell_4(2\mathrm{i}m-\ell_4)}}, \end{aligned} \tag{B11}$$

where in the second line we performed a Wick rotation in order to get the Euclidean space, i.e., $\ell_0 \to \mathrm{i}\ell_4$ and $\ell_{\mathrm{E}} \equiv \ell_4^2 + \ell_3^2$, with the following change of variables

$$\ell_4 = r\sin\theta, \qquad \ell_3 = r\cos\theta, \tag{B12}$$

$$\mathcal{I}_1(\tau, m, 0) = -\mathrm{i}\int_0^{\infty} \frac{rdr}{2\pi} f_1(m, r)e^{-\mathrm{i}\tau r^2}, \tag{B13}$$

where

$$f_1(m, r) \equiv \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{2m^2 - r^2 + 2\mathrm{i}mr\sin\theta}{r^{5/2}(r-2\mathrm{i}m\sin\theta)^2\sqrt{\sin\theta(2\mathrm{i}m - r\sin\theta)}} [(m+\mathrm{i}r\sin\theta)\gamma^0\mathcal{O}^{(\downarrow)} + r\cos\theta\gamma^3\mathcal{O}^{(\downarrow)} - m]. \tag{B14}$$

In order to obtain analytical results, we consider a situation where the fermion mass is a small energy scale, such that $m^2 \ll |eB|$. This condition is physically accessible, given that in heavy-ion collisions, the produced magnetic fields are estimated to be about the pion mass squared. Therefore, for light quarks, this is a good approximation. So, by expanding in a power series around $m = 0$, we get

$$f_1(m,r) \approx \int_0^{2\pi} \frac{d\theta}{2\pi} \left\{ \frac{\mathrm{i}(\cos\theta\gamma^3\mathcal{O}^{(\downarrow)} + \mathrm{i}\sin\theta\gamma^0\mathcal{O}^{(\downarrow)})}{r^2|\sin\theta|} - [(3+\cot^2\theta)\cot\theta\gamma^3\mathcal{O}^{(\downarrow)} + \mathrm{i}(\csc^2\theta + 2\gamma^0\mathcal{O}^{(\downarrow)})]|\sin\theta|\frac{m}{r^3} \right\} + O(m^2), = \frac{2\mathrm{i}(\ln(2) + 2\gamma^0\mathcal{O}^{(\downarrow)})m}{\pi r^3} + O(m^2). \tag{B15}$$

Hence,

$$\mathcal{I}_1(\tau,m,0) = \frac{(\ln(2) + 2\gamma^0\mathcal{O}^{(\downarrow)})m}{\pi^2} \int_0^\infty dr \frac{e^{-\mathrm{i}\tau r^2}}{r^2}. \tag{B16}$$

It is clear that the radial integral has a singularity when $r \to 0$. To avoid it, we regularize this by the following prescription:

$$\int_0^\infty dr \frac{e^{-\mathrm{i}\tau r^2}}{r^2} \to \int_0^\infty dr \frac{e^{-\mathrm{i}\tau r^2}}{r^2 + \mu^2} \tag{B17}$$

so that, as is shown in Appendix C,

$$\mathcal{J}(\tau,\mu) \equiv \int_0^\infty dr \frac{e^{-\mathrm{i}\tau r^2}}{r^2+\mu^2}, = \frac{\pi e^{\mathrm{i}\mu^2\tau}}{2\mu}\left[1 - (1+\mathrm{i})C\left(\sqrt{\frac{2\tau}{\pi}}\mu\right) - (1-\mathrm{i})S\left(\sqrt{\frac{2\tau}{\pi}}\mu\right)\right], \tag{B18}$$

where $S(z)$ and $C(z)$ are the Fresnel integrals. With this in mind, the singularity is isolated by power expanding around $\mu = 0$, i.e.

$$\mathcal{J}(\tau,\mu) = \frac{\pi}{2\mu} - (1+\mathrm{i})\sqrt{\frac{\pi\tau}{2}} + O(\mu), \tag{B19}$$

then we defined the regularized integral as

$$\mathcal{J}^r(\tau,\mu) \equiv \mathcal{J} - \frac{\pi}{2\mu}, \tag{B20}$$

so that

$$\int_0^\infty dr \frac{e^{-\mathrm{i}\tau r^2}}{r^2} \to \lim_{\mu\to 0} \mathcal{J}_R(\tau,\mu) = -(1+\mathrm{i})\sqrt{\frac{\pi\tau}{2}}. \tag{B21}$$

Putting it all together:

$$\lim_{\substack{p_0\to m\\ \mathbf{p}\to 0}} [-\mathrm{i}\Sigma_\Delta^{(1)}(p)]^r = -\frac{3\mathrm{i}(1+\mathrm{i})(\ln(2) + 2\gamma^0\mathcal{O}^{(\downarrow)})\alpha_{\mathrm{em}}\Delta|eB|^2 m}{\pi^3} \times \sqrt{\frac{\pi}{2}} \int_0^\infty d\tau \frac{\sqrt{\tau}e^{-\epsilon\tau}}{2 + \mathrm{i}|eB|\tau}, \tag{B22}$$

where the superscript r stands for *regularized*.

Now, we note that

$$\sqrt{\frac{\pi}{2}} \int_0^\infty d\tau \frac{\sqrt{\tau}e^{-\epsilon\tau}}{2+\mathrm{i}|eB|\tau} \approx -\frac{\mathrm{i}\pi}{|eB|\sqrt{2\epsilon}} + \frac{(1+\mathrm{i})\pi^{3/2}}{\sqrt{2}|eB|^{3/2}} + O(\epsilon^{1/2}), \tag{B23}$$

in such a way that we also absorb the singularity of $\epsilon \to 0^+$ in the counterterms associated to the noise-averaged self-energy. Therefore:

$$\lim_{\substack{p_0\to m\\ \mathbf{p}\to 0}} [-\mathrm{i}\Sigma_\Delta^{(1)}(p)]^r = \frac{3\sqrt{2}}{\pi^{3/2}}(\ln(2) + 2\gamma^0\mathcal{O}^{(\downarrow)})\alpha_{\mathrm{em}}\Delta\sqrt{|eB|}m + O(m^2). \tag{B24}$$

Finally, from the completeness relation

$$\mathcal{O}^{(\uparrow)} + \mathcal{O}^{(\downarrow)} = \mathbb{1}, \tag{B25}$$

we can split the expression in the two spin components as follows:

$$\lim_{\substack{p_0\to m\\ \mathbf{p}\to 0}} [-\mathrm{i}\Sigma_\Delta^{(1)}(p)]^r = -\mathrm{i}\tilde{\Sigma}_\Delta^{(1,\downarrow)}\mathcal{O}^{(\downarrow)} - \mathrm{i}\tilde{\Sigma}_\Delta^{(1,\uparrow)}\mathcal{O}^{(\uparrow)}, \tag{B26}$$

where

$$-\mathrm{i}\tilde{\Sigma}_\Delta^{(1,\downarrow)} \equiv \frac{3\sqrt{2}}{\pi^{3/2}}(\ln(2) + 2\gamma^0)\alpha_{\mathrm{em}}\Delta\sqrt{|eB|}m, \tag{B27a}$$

and

$$-\mathrm{i}\tilde{\Sigma}_\Delta^{(1,\uparrow)} \equiv \frac{3\sqrt{2}}{\pi^{3/2}}\ln(2)\alpha_{\mathrm{em}}\Delta\sqrt{|eB|}m. \tag{B27b}$$

The next structure is

$$-\mathrm{i}\Sigma_\Delta^{(2)}(p) = -4\alpha_{\mathrm{em}}\Delta|eB| \int \frac{d^4k}{(2\pi)^4} \frac{\Theta_2(k)\gamma^3\mathcal{O}^{(+)}}{(k-p)^2 + \mathrm{i}\epsilon}, \tag{B28}$$

which after following the same procedure has the following form:

$$-\mathrm{i}\Sigma_{\Delta}^{(2)}(p) = 12\mathrm{i}\alpha_{\mathrm{em}}\Delta|eB|\int_0^\infty d\tau \exp\left[-\frac{2\mathrm{i}\tau\mathbf{p}_\perp^2}{2+\mathrm{i}\tau|eB|}-\epsilon\tau\right] \times\int\frac{d^2\ell_\perp}{(2\pi)^2}\exp\left[-\left(\frac{2+\mathrm{i}|eB|\tau}{|eB|}\right)\ell_\perp^2\right] \times\int\frac{d^2\ell_\parallel}{(2\pi)^2}\frac{(\ell_3+p_3)e^{\mathrm{i}\tau\ell_\parallel^2}\gamma^3\mathcal{O}^{(+)}}{[(\ell_\parallel+p_\parallel)^2-m^2]\sqrt{(\ell_0+p_0)^2-m^2}}, \tag{B29}$$

so that in the limits $p_0\to m, \mathbf{p}\to 0$ it vanishes by the parity of the parallel integration, i.e.,

$$\lim_{\substack{p_0\to m\\ \mathbf{p}\to 0}}[-\mathrm{i}\Sigma_\Delta^{(2)}(p)] = 0. \tag{B30}$$

The third structure is given by

$$-\mathrm{i}\Sigma_\Delta^{(3)}(p) = -4\alpha_{\mathrm{em}}\Delta|eB|\mathrm{sign}(eB)\int\frac{d^4k}{(2\pi)^4}\frac{\Theta_3(k)\not{k}_\parallel \mathrm{i}\gamma^1\gamma^2}{(k-p)^2+\mathrm{i}\epsilon}, \tag{B31}$$

so that with the described procedure it turns into the expression

$$-\mathrm{i}\Sigma_\Delta^{(3)} = 4\mathrm{i}\alpha_{\mathrm{em}}\Delta|eB|\mathrm{sign}(eB) \times\int_0^\infty d\tau\exp\left[-\frac{2\mathrm{i}\tau\mathbf{p}_\perp^2}{2+\mathrm{i}\tau|eB|}-\epsilon\tau\right] \times\int\frac{d^2\ell_\perp}{(2\pi)^2}\exp\left[-\left(\frac{2+\mathrm{i}|eB|\tau}{|eB|}\right)\ell_\perp^2\right] \times\int\frac{d^2\ell_\parallel}{(2\pi)^2}\frac{(\not{\ell}_\parallel+\not{p}_\parallel)e^{\mathrm{i}\tau\ell_\parallel^2}\mathrm{i}\gamma^1\gamma^2}{[(\ell_\parallel+p_\parallel)^2-m^2]\sqrt{(\ell_0+p_0)^2-m^2}}. \tag{B32}$$

Given the parity of the integrand, the odd terms in ℓ do not contribute to the integral. Hence, we have

$$\lim_{\substack{p_0\to m\\ \mathbf{p}\to 0}}[-\mathrm{i}\Sigma_\Delta^{(3)}(p)] = \frac{\mathrm{i}\alpha_{\mathrm{em}}\Delta|eB|^2\mathrm{sign}(eB)}{\pi}\int_0^\infty d\tau\frac{\mathcal{I}_3(\tau,m,0)e^{-\epsilon\tau}}{2+\mathrm{i}|eB|\tau}, \tag{B33}$$

where

$$\mathcal{I}_3(\tau,p_0,p_3) \equiv\int\frac{d^2\ell_\parallel}{(2\pi)^2}\frac{(\ell_0+m)e^{\mathrm{i}\tau\ell_\parallel^2}\gamma^0(\mathrm{i}\gamma^1\gamma^2)}{[(\ell_\parallel+p_\parallel)^2-m^2]\sqrt{(\ell_0+p_0)^2-m^2}}. \tag{B34}$$

After a Wick rotation and performing the change of variables of Eq. (B12):

$$\mathcal{I}_3(\tau,m,0) = -\mathrm{i}\int\frac{d^2\ell_{\mathrm{E}}}{(2\pi)^2}\frac{(\mathrm{i}\ell_4+m)e^{-\mathrm{i}\tau\ell_{\mathrm{E}}^2}\gamma^0(\mathrm{i}\gamma^1\gamma^2)}{[\ell_4(\ell_4-2\mathrm{i}m)+\ell_3^2]\sqrt{\ell_4(2\mathrm{i}m-\ell_4)}}, = -\mathrm{i}\int_0^\infty\frac{rdr}{2\pi}f_3(m,r)e^{-\mathrm{i}\tau r^2}, \tag{B35}$$

with

$$f_3(m,r)\equiv\int_0^{2\pi}\frac{d\theta}{2\pi}\frac{(\mathrm{i}r\sin\theta+m)\gamma^0(\mathrm{i}\gamma^1\gamma^2)}{r^{3/2}(r-2\mathrm{i}m\sin\theta)\sqrt{\sin\theta(2\mathrm{i}m-r\sin\theta)}}, \tag{B36}$$

so that around $m=0$:

$$f_3(m,r)\approx\int_0^{2\pi}\frac{d\theta}{2\pi}\left[\frac{\sin\theta}{|\sin\theta|r^2}+\frac{2\mathrm{i}|\sin\theta|m}{r^3}\right]\gamma^0(\mathrm{i}\gamma^1\gamma^2) + O(m^2), = \frac{8\mathrm{i}m}{2\pi r^3}\gamma^0(\mathrm{i}\gamma^1\gamma^2)+O(m^2). \tag{B37}$$

Then

$$\mathcal{I}_3(\tau,m,0) = \frac{2m}{\pi^2}\int_0^\infty dr\frac{e^{-\mathrm{i}\tau r^2}}{r^2}\gamma^0(\mathrm{i}\gamma^1\gamma^2) \to -\frac{2(1+\mathrm{i})m}{\pi^2}\sqrt{\frac{\pi\tau}{2}}\gamma^0(\mathrm{i}\gamma^1\gamma^2), \tag{B38}$$

where the prescription of Eq. (B21) was used, and, after using Eq. (B23), we get

$$\lim_{\substack{p_0\to m\\ \mathbf{p}\to 0}}[-\mathrm{i}\Sigma_\Delta^{(3)}(p)]^r = \frac{2\sqrt{2}}{\pi^{3/2}}\alpha_{\mathrm{em}}\Delta\mathrm{sign}(eB)\sqrt{|eB|}m\gamma^0(\mathrm{i}\gamma^1\gamma^2) + O(m^2). \tag{B39}$$

Finally, from the identity

$$\mathrm{i}\,\mathrm{sign}(eB)\gamma^1\gamma^2 = \mathcal{O}^{(\downarrow)}-\mathcal{O}^{(\uparrow)}, \tag{B40}$$

the expression can be written as

$$\lim_{\substack{p_0\to m\\ \mathbf{p}\to 0}}[-\mathrm{i}\Sigma_\Delta^{(3)}(p)]^r = -\mathrm{i}\tilde{\Sigma}_\Delta^{(3)}(\mathcal{O}^{(\downarrow)}-\mathcal{O}^{(\uparrow)}), \tag{B41}$$

where

$$-\mathrm{i}\tilde{\Sigma}_{\Delta}^{(3)} \equiv \frac{2\sqrt{2}}{\pi^{3/2}}\alpha_{\mathrm{em}}\Delta\sqrt{|eB|}m\gamma^0. \tag{B42}$$

From the latter, the final expression for $\mathrm{i}\Sigma_{\Delta}(p)$ in the limits $p_0 \to m$, and $\mathbf{p} \to 0$ is

$$\lim_{\substack{p_0\to m\\ \mathbf{p}\to 0}}[-\mathrm{i}\Sigma_{\Delta}(p)]^r = -\mathrm{i}\tilde{\Sigma}_{\Delta}^{(\downarrow)}\mathcal{O}^{(\downarrow)} - \mathrm{i}\tilde{\Sigma}_{\Delta}^{(\uparrow)}\mathcal{O}^{(\uparrow)}, \tag{B43}$$

with

$$-\mathrm{i}\tilde{\Sigma}_{\Delta}^{(\downarrow)} \equiv \frac{\sqrt{2}}{\pi^{3/2}}[3\ln(2) + 8\gamma^0]\alpha_{\mathrm{em}}\Delta\sqrt{|eB|}m \tag{B44a}$$

and

$$-\mathrm{i}\tilde{\Sigma}_{\Delta}^{(\uparrow)} \equiv \frac{\sqrt{2}}{\pi^{3/2}}(3\ln(2) - 2\gamma^0)\alpha_{\mathrm{em}}\Delta\sqrt{|eB|}m. \tag{B44b}$$

Finally, we can define two projections given by

$$\mathcal{P}^{(\pm)} \equiv \frac{1}{2}(\mathbb{1} \pm \gamma^0), \tag{B45}$$

so that

$$\begin{aligned}\mathcal{P}^{(+)} + \mathcal{P}^{(-)} &= \mathbb{1},\\ \mathcal{P}^{(+)} - \mathcal{P}^{(-)} &= \gamma^0,\end{aligned} \tag{B46}$$

in such a way that we can split the self-energy contribution into four subspaces, namely

$$\lim_{\substack{p_0\to m\\ \mathbf{p}\to 0}}[-\mathrm{i}\Sigma_{\Delta}(p)]_r = \sum_{\sigma=\uparrow,\downarrow}\sum_{\lambda=\pm 1}[-\mathrm{i}\tilde{\Sigma}_{\Delta}^{(\sigma,\lambda)}\mathcal{O}^{(\sigma)}\mathcal{P}^{(\lambda)}], \tag{B47}$$

where we defined the coefficients:

$$-\mathrm{i}\tilde{\Sigma}_{\Delta}^{(\downarrow,\pm)} \equiv \frac{\sqrt{2}}{\pi^{3/2}}(3\ln(2) \pm 8)\alpha_{\mathrm{em}}\Delta\sqrt{|eB|}m, \tag{B48a}$$

$$-\mathrm{i}\tilde{\Sigma}_{\Delta}^{(\uparrow,\pm)} \equiv \frac{\sqrt{2}}{\pi^{3/2}}(3\ln(2) \mp 2)\alpha_{\mathrm{em}}\Delta\sqrt{|eB|}m. \tag{B48b}$$

APPENDIX C: DETAILS ON THE CALCULATION OF THE REGULARIZED PHASE-SPACE INTEGRAL

In previous appendixes, we arrived at the regularized integral

$$\mathcal{J}(\tau,\mu) = \int_0^\infty dr\frac{e^{-\mathrm{i}\tau r^2}}{r^2+\mu^2}. \tag{C1}$$

Let us first apply the following integral transformation of the denominator in Eq. (C1)

$$\frac{1}{r^2+\mu^2} = \int_0^\infty dy e^{-y(r^2+\mu^2)}. \tag{C2}$$

Substituting into Eq. (C1), and integrating over r first, we obtain

$$\begin{aligned}\mathcal{J}(\tau,\mu) &= \int_0^\infty dy e^{-\mu^2 y}\int_0^\infty dr e^{-(y+\mathrm{i}\tau)r^2},\\ &= \frac{\sqrt{\pi}}{2}\int_0^\infty dy e^{-\mu^2 y}(y+\mathrm{i}\tau)^{-1/2}.\end{aligned} \tag{C3}$$

Let us shift the integration variable, by defining $z = y + \mathrm{i}\tau$, to arrive at

$$\begin{aligned}\mathcal{J}(\tau,\mu) &= \frac{\sqrt{\pi}}{2}\int_{\mathrm{i}\tau}^\infty dz e^{-\mu^2(z-\mathrm{i}\tau)}z^{-1/2},\\ &= e^{\mathrm{i}\mu^2\tau}\frac{\sqrt{\pi}}{2\mu}\int_{\mathrm{i}\mu^2\tau}^\infty dz e^{-z}z^{-1/2},\end{aligned} \tag{C4}$$

where in the second step we rescaled the integration variable $z \to z/\mu^2$. Let us now define the auxiliary variable

$$z = v^2 \Rightarrow z^{-1/2}dz = v^{-1}\cdot 2v dv = 2dv \tag{C5}$$

and hence the integral becomes

$$\begin{aligned}\mathcal{J}(\tau,\mu) &= e^{\mathrm{i}\mu^2\tau}\frac{\sqrt{\pi}}{\mu}\int_{\mu\sqrt{\mathrm{i}\tau}}^\infty dv e^{-v^2},\\ &= e^{\mathrm{i}\mu^2\tau}\frac{\sqrt{\pi}}{\mu}\left(\int_0^\infty dv e^{-v^2} - \int_0^{\mu\sqrt{\mathrm{i}\tau}} dv e^{-v^2}\right),\\ &= e^{\mathrm{i}\mu^2\tau}\frac{\pi}{2\mu}\left(1 - \frac{2}{\sqrt{\pi}}\int_0^{\mu\sqrt{\mathrm{i}\tau}} dv e^{-v^2}\right),\\ &= e^{\mathrm{i}\mu^2\tau}\frac{\pi}{2\mu}(1-\Phi(\mu\sqrt{\mathrm{i}\tau})),\end{aligned} \tag{C6}$$

where in the last line we applied the integral representation of the probability integral $\Phi(z)$. In particular, when $z = \sqrt{\mathrm{i}}x$ for $x \in \mathbb{R}$, the probability integral is related to the Fresnel integrals $C(z)$ and $S(z)$ by the identity

$$\Phi(\sqrt{\mathrm{i}}x) = \sqrt{2}\mathrm{i}(C(\sqrt{2/\pi}x) - \mathrm{i}S(\sqrt{2/\pi}x)). \tag{C7}$$

Applying this last identity, and using $\sqrt{\mathrm{i}} = (1+\mathrm{i})/\sqrt{2}$, we have

$$\Phi(\mu\sqrt{\mathrm{i}\tau}) = (1+\mathrm{i})C\left(\mu\sqrt{\frac{2\tau}{\pi}}\right) + (1-\mathrm{i})S\left(\mu\sqrt{\frac{2\tau}{\pi}}\right). \tag{C8}$$

Substituting this last relation into Eq. (C6), we finally obtain

$$\mathcal{J}(\tau,\mu) = e^{i\mu^2\tau}\frac{\pi}{2\mu}\left[1-(1+\mathrm{i})C\left(\mu\sqrt{\frac{2\tau}{\pi}}\right)\right. \\ \left. -(1-\mathrm{i})S\left(\mu\sqrt{\frac{2\tau}{\pi}}\right)\right]. \quad (C9)$$

In the vicinity of $\mu = 0$, the integral reduces to the power series

$$\mathcal{J}(\tau,\mu) = \frac{\pi}{2\mu} - (1+\mathrm{i})\sqrt{\frac{\pi\tau}{2}} + \frac{\mathrm{i}\pi\tau\mu}{2} + O(\mu^2). \quad (C10)$$

The regularized integral is thus obtained by subtracting the pole at $\mu \to 0$, such that

$$\mathcal{J}^r(\tau,\mu) \equiv \mathcal{J}(\tau,\mu) - \frac{\pi}{2\mu}. \quad (C11)$$

Therefore, in the self-energy terms, we can finally evaluate this regularized expression at $\mu \to 0$, such that

$$\int_0^\infty dr\frac{e^{-\mathrm{i}\tau r^2}}{r^2} \to \lim_{\mu\to 0}\mathcal{J}^r(\tau,\mu) = -\sqrt{\pi\tau}\frac{(1+\mathrm{i})}{\sqrt{2}}. \quad (C12)$$

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