Stout smearing and Wilson flow in lattice perturbation theory

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We present the expansion of stout smearing and the Wilson flow in lattice perturbation theory to order g_0^3, which is suitable for one-loop calculations. As the Wilson flow is generated by infinitesimal stout smearing steps, the results are related to each other by taking the appropriate limits. We show how to apply perturbative stout smearing or Wilson flow to the Feynman rules of any lattice fermion action and illustrate them by calculating the self-energy of the clover-improved Wilson fermion.

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I. INTRODUCTION

Link smearing is a standard recipe for reducing UV fluctuations in lattice gauge theory calculations, e.g. lattice QCD. It is particularly useful when applied to the covariant derivative in a lattice fermion action. Long ago it was noticed that the taste breaking of staggered fermions gets reduced by link smearing [1–3], and similarly the additive mass shift (taken as a measure for the severity of chiral symmetry breaking) of Wilson/clover fermions gets reduced by link smearing [4–8].

In recent years the "stout smearing" procedure, proposed in Ref. [9], has become very popular, since it yields a smeared link $U_\mu^{(n)}(x)$ which depends in a smooth (differentiable) manner on the unsmeared gauge link $U_\mu(x)$ and its neighbors forming the "staple" around it. This allows one to use smeared fermion actions in the HMC/RHMC algorithm [10–12], which crucially relies on the differentiability with respect to the gauge field.

A related recipe is known as "gradient flow" or "Wilson flow" [13–15]. The flow time t (with the dimension of a length squared) parametrizes the smooth deformation from the unsmeared $U_\mu(x)$ to the smeared/flowed $U_\mu(x,t)$, and the recipe was originally formulated as a differential equation (of the diffusion type) in t. In fact, a sequence of stout smearings whose cumulative smearing parameter matches the flow time is the simplest possible integration scheme ("forward Euler") to implement the gradient flow—see the discussion around Eq. (205) below for details.

The main difference, in contemporary use, between stout smearing and gradient/Wilson flow concerns what is kept fixed in the continuum limit (i.e. if the lattice spacing is sent to zero, $a \to 0$, while the box size L and the renormalized quark masses m_q, all in physical units, remain constant). With stout smearing it is common practice to keep both the smearing parameter ϱ and the number of smearings n_{stout} fixed. With gradient/Wilson flow it is common practice to keep the flow time t (in physical units) constant as the continuum limit is taken, whereupon the flow time in lattice units t/a^2 (dimensionless) grows in proportion to the cutoff squared.

The rationale behind the former choice is that it results in an ultralocal modification of the action. Accordingly, a study with, say, seven stout smearings is guaranteed to yield the same continuum limit as with an unsmeared action (see e.g. Ref. [16] for an example of this philosophy). The rationale behind the latter choice is that $t^{-1/2}$ introduces a second regulator, independent of the cutoff a^{-1}, which persists (and stays finite) in the continuum limit. One ends up with a flow-scale regulated theory (or a perfectly legitimate renormalization scheme and scale), which, contrary to the lattice regulation, shows universal features [17–24]. Hence, in the future, one might calculate a light quark mass or α_s with staggered fermions at the fixed flow scale $t^{-1/2} = 3$ GeV. Upon taking the continuum limit, all memory of the gluon/fermion action combination, which has been used in the computation, is lost, and one ends up with a genuine physics result in the universal flow scheme. Hence, no conversion to the $\overline{\text{MS}}$ scheme is needed any more, the advantage being that the renormalization scheme employed is well defined beyond perturbation theory. There are encouraging signs that the continuum community is getting aware of the advantages of the flow scheme and provides the necessary computational tools [25–30].

So far, we discussed the smoothing of gauge fields. A second vital ingredient in the field-theoretic setup of many

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lattice QCD computations is the concept of Symanzik improvement [31]. Specifically for Wilson fermions this program offers the opportunity of relegating the leading cutoff effects from $O(a)$ to $O(\alpha_s^k a)$ or $O(a^2)$, provided the coefficient c_{SW} in front of the Sheikholeslami-Wohlert term is properly chosen [32–34]. This improvement program may be carried out either perturbatively or nonperturbatively, but even in the latter case the perturbative results are vital for cross-checks and to stabilize the nonperturbative fits [34].

In order to combine the merits of link smoothing and Symanzik improvement, a valid framework for carrying out perturbative calculations with smoothed actions is mandatory. Early steps in this direction have been taken in Refs. [5,8,35,36]. More recently, the mutual benefits which come from combining link smearing with clover improvement for undoubled fermion actions have been emphasized in Ref. [37]. What is still missing is a universal framework that allows for an easy derivation of Feynman rules for fermion actions with arbitrary smearing recipes, i.e. an arbitrary $(\varrho, n_{\mathrm{stout}})$ combination or an arbitrary flow time $t/a^2 \in \mathbb{R}$. In the present paper we try to fill this gap. In particular we try to convince the reader that a "brute force" approach for deriving the Feynman rules (i.e. inserting the perturbative expansion of the smeared/flowed variable into the underlying action) is not recommendable for $n_{\mathrm{stout}} > 1$. We advocate a more elegant method, based on an $SU(N_c)$ expansion, which keeps intermediate expressions much shorter.

The remainder of this article is organized as follows. In Sec. II we work out the perturbative expansion of stout smearing, and in Sec. III we do the same thing for the gradient/Wilson flow. As a by-product the perturbative matching between these recipes is discussed, see Eq. (205). In Sec. IV we discuss the procedure which yields, for any smoothing recipe, the Feynman rules in a manageable form. As an application of this procedure we study in Sec. V the self-energy of a stout smeared clover fermion with variable ϱ and $1 \le n_{\mathrm{stout}} \le 4$ or under the gradient flow. Finally, Sec. VI gives a short summary and an outlook. Some preliminary work on this topic is found in Ref. [38].

II. PERTURBATIVE EXPANSION OF STOUT SMEARING

A. Definition of stout smearing

Stout smearing is a unitary transformation of the link variable $U_\mu(x)$, which depends on the plaquettes containing $U_\mu(x)$. Thereby the configuration of link variables is smoothed. Often multiple smearing steps are performed to increase the smoothing effect. We write $U_\mu^{(n)}(x)$ for the link variable after n smearing steps. The transformation to obtain $U_\mu^{(n+1)}(x)$ is given by

$$U_\mu^{(n+1)}(x) = \exp\{i\varrho Q_\mu^{(n)}(x)\} U_\mu^{(n)}(x) \tag{1}$$

$$U_\mu^{(0)}(x) = U_\mu(x), \tag{2}$$

with the smearing parameter ϱ and the Hermitian operator $Q_\mu^{(n)}(x)$, which is constructed from link variables of the previous smearing step according to

$$Q_\mu^{(n)}(x) = \frac{1}{2i}\left(W_\mu^{(n)}(x) - \frac{1}{N_c}\mathrm{Tr}[W_\mu^{(n)}(x)]\right), \tag{3}$$

$$W_\mu^{(n)}(x) = S_\mu^{(n)}(x) U_\mu^{(n)\dagger}(x) - U_\mu^{(n)}(x) S_\mu^{(n)\dagger}(x), \tag{4}$$

$$S_\mu^{(n)}(x) = \sum_{\nu\neq\mu}\Big(U_\nu^{(n)}(x) U_\mu^{(n)}(x+\hat\nu) U_\nu^{(n)\dagger}(x+\hat\mu) + U_\nu^{(n)\dagger}(x-\hat\nu) U_\mu^{(n)}(x-\hat\nu) U_\nu^{(n)}(x+\hat\mu-\hat\nu)\Big). \tag{5}$$

The superscript (n) denotes quantities that are constructed from link variables after n smearing steps. The quantity $S_\mu^{(n)}(x)$ is referred to as the sum of staples around the link $U_\mu^{(n)}(x)$ and $iQ_\mu^{(n)}(x)$ is constructed as the projection of the product $S_\mu^{(n)}(x) U_\mu^{(n)\dagger}(x)$ onto the algebra $\mathfrak{su}(N_c)$.

B. Stout smearing in perturbation theory

In lattice perturbation theory the group element $U_\mu(x) \in SU(N_c)$ is written as the exponential of a linear combination of generators T^a and gluon fields $A_\mu^a(x)$

$$U_\mu(x) = e^{ig_0 T^a A_\mu^a(x)}, \tag{6}$$

where the summation over repeated adjoint color indices $a, b, c, \dots \in \{1, \dots, N_c^2 - 1\}$ is implied. Expanding the action to the desired order in the bare coupling g_0 then gives the Feynman rules. For one-loop calculations we need to expand up to third order in g_0

$$U_\mu(x) = 1 + ig_0 T^a A_\mu^a(x) - \frac{g_0^2}{2} T^a T^b A_\mu^a(x) A_\mu^b(x) - i\frac{g_0^3}{6} T^a T^b T^c A_\mu^a(x) A_\mu^b(x) A_\mu^c(x) + \mathcal{O}(g_0^4). \tag{7}$$

The Fourier transform of the gluon fields is chosen to be

$$A_\mu^a(x) = \int_{-\pi}^{\pi} \frac{\mathrm{d}^4 k}{(2\pi)^4} e^{i(x+\hat\mu/2)k} A_\mu^a(k). \tag{8}$$

The shift $x + \hat\mu/2$ is needed to eliminate overall phase factors in our results. In order for $U_\mu(x)$ to be unitary, the exponent $ig_0 T^a A_\mu^a(x)$ has to be anti-Hermitian. This means the gluon fields $A_\mu^a(x)$ are real (because the generators are

Hermitian). This implies that the momentum space field $A^a_\mu(k)$ fulfils

$$A^a_\mu(k) = A^a_\mu(-k). \tag{9}$$

As we are going to be working in momentum space, it is helpful to keep in mind that taking the Hermitian conjugate in position space involves the transformation $k \to -k$ in momentum space.

For a link variable after stout smearing things are more complicated. As the smeared link variable $U^{(n)}_\mu(x)$ is again a group element of $SU(N_c)$ [9] we expect it to have an exponential representation similar to the unsmeared one,

$$U^{(n)}_\mu(x) = e^{ig_0 T^a \tilde{A}^{(n)a}_\mu(g_0,x)} \tag{10}$$

$$= 1 + ig_0 T^a \tilde{A}^{(n)a}_\mu(g_0,x) - \frac{g_0^2}{2} T^a T^b \tilde{A}^{(n)a}_\mu(g_0,x)\tilde{A}^{(n)b}_\mu(g_0,x) - i\frac{g_0^3}{6} T^a T^b T^c \tilde{A}^{(n)a}_\mu(g_0,x)\tilde{A}^{(n)b}_\mu(g_0,x)\tilde{A}^{(n)c}_\mu(g_0,x) + \mathcal{O}(T^4). \tag{11}$$

This expansion in terms of generators T does, however, no longer coincide with the expansion in the bare coupling g_0. The new fields $\tilde{A}^{(n)a}_\mu(g_0,x)$ are themselves functions of g_0. By expanding $\tilde{A}^{(n)a}_\mu(g_0,x)$ in powers of g_0 we will be able to determine its coefficients by comparison with the perturbative expansion of the right-hand side of Eq. (1). This also means that, as we will see below, we will be able to infer the general structure of the coefficients of $\tilde{A}^{(n)a}_\mu(g_0,x)$ from the perturbative expansion of only the first smearing step

$$U^{(1)}_\mu(x) = \exp\{i\varrho Q^{(0)}_\mu(x)\} U^{(0)}_\mu(x), \tag{12}$$

which in turn allows us to use the following strategy: Expand Eq. (12) to the desired order in g_0 and replace $A^{(0)}_\mu(x)$ by the expansion of $\tilde{A}^{(n)a}_\mu(g_0,x)$. This will generate new contributions to higher orders in g_0 and give the results for a general smearing step $n+1>1$.

We start by defining expansions of $S_\mu(x)$ and $Q_\mu(x)$ in g_0 [we drop the superscript (0) for now]:

$$S_\mu(x) = 6 + ig_0 T^a S^a_{1\mu}(x) - g_0^2 T^a T^b S^{ab}_{2\mu}(x) - ig_0^3 T^a T^b T^c S^{abc}_{3\mu}(x) + \mathcal{O}(g_0^4), \tag{13}$$

$$Q_\mu(x) = g_0 T^a Q^a_{1\mu}(x) + ig_0^2 (T^a T^b - T^b T^a) Q^{ab}_{2\mu}(x) + g_0^3 \left(T^a T^b T^c + T^c T^b T^a - \frac{1}{N_c} \mathrm{Tr}[T^a T^b T^c + T^c T^b T^a] \right) Q^{abc}_{3\mu}(x) + \mathcal{O}(g_0^4). \tag{14}$$

Using the Baker-Campbell-Hausdorff formula,

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}([X,[X,Y]]+[Y,[Y,X]])+\cdots}, \tag{15}$$

we therefore get

$$\begin{aligned}\exp\{i\varrho Q_\mu(x)\}\exp\{ig_0 T^a A^a_\mu(x)\} = \exp\Big\{ & ig_0 T^a A^a_\mu(x) + i\varrho g_0 T^a Q^a_{1\mu}(x) - \varrho g_0^2 [T^a,T^b] Q^{[ab]}_{2\mu}(x) \\ & + i\varrho g_0^3 \left(T^a T^b T^c + T^c T^b T^a - \frac{1}{N_c}\mathrm{Tr}[T^a T^b T^c + T^c T^b T^a] \right) Q^{abc}_{3\mu}(x) \\ & + \frac{1}{2}\left(-\varrho g_0^2 [T^a,T^b] Q^a_{1\mu}(x) A^b_\mu(x) - i\varrho g_0^3 [[T^a,T^b],T^c] Q^{[ab]}_{2\mu}(x) A^c_\mu(x) \right) \\ & + \frac{1}{12}\left(-i\varrho^2 g_0^3 [T^a,[T^b,T^c]] Q^a_{1\mu}(x) Q^b_{1\mu}(x) A^c_\mu(x) - i\varrho g_0^3 [T^a,[T^b,T^c]] A^a_\mu(x) A^b_\mu(x) Q^c_{1\mu}(x) \right) \\ & + \mathcal{O}(g_0^4) \Big\}\end{aligned} \tag{16}$$

$$
\begin{aligned}
&= 1 + ig_0T^a(A^a_\mu(x) + \varrho Q^a_{1\mu}(x)) - \frac{1}{2}g_0^2T^aT^b(A^a_\mu(x) + \varrho Q^a_{1\mu}(x))(A^b_\mu(x) + \varrho Q^b_{1\mu}(x)) \\
&- \varrho g_0^2[T^a, T^b]\left(\frac{1}{2}Q^a_{1\mu}(x)A^b_\mu(x) + Q^{ab}_{2\mu}(x)\right) \\
&- i\frac{g_0{}^3}{6}T^aT^bT^c\left(A^a_\mu(x) + \varrho Q^a_{1\mu}(x)\right)\left(A^b_\mu(x) + \varrho Q^b_{1\mu}(x)\right)\left(A^c_\mu(x) + \varrho Q^c_{1\mu}(x)\right) \\
&- i\frac{g_0^3}{2}\{T^a, [T^b, T^c]\}\left(A^a_\mu(x) + \varrho Q^a_{1\mu}(x)\right)\left(\frac{1}{2}\varrho Q^b_{1\mu}(x)A^c_\mu(x) + \varrho Q^{bc}_{2\mu}(x)\right) \\
&+ i\frac{g_0^3}{2}[T^a, [T^b, T^c]]A^a_\mu(x)\varrho Q^{[bc]}_{2\mu}(x) \\
&+ \frac{1}{12}\left(-i\varrho^2g_0^3[T^a, [T^b, T^c]]Q^a_{1\mu}(x)Q^b_{1\mu}(x)A^c_\mu(x) - i\varrho g_0^3[T^a, [T^b, T^c]]A^a_\mu(x)A^b_\mu(x)Q^c_{1\mu}(x)\right) \\
&+ i\varrho g_0^3\left(T^aT^bT^c + T^cT^bT^a - \frac{1}{N_c}\mathrm{Tr}[T^aT^bT^c + T^cT^bT^a]\right)Q^{abc}_{3\mu}(x) + \mathcal{O}(g_0^4)
\end{aligned} \tag{17}
$$

where we have used

$$
e^{ax+bx^2+cx^3} = 1 + ax + \left(\frac{1}{2}a^2 + b\right)x^2 + \left(\frac{1}{6}a^3 + \frac{1}{2}(ab+ba) + c\right)x^3 + \mathcal{O}(x^4) \tag{18}
$$

in the second step. Based on this, it is convenient to define fields $A^{(n)a}$, $A^{(n)ab}$, $\bar{A}^{(n)abc}$, and $\hat{A}^{(n)abc}$ at each order in g_0 for a general n such that

$$
\begin{aligned}
U^{(n)}_\mu(x) &= 1 + ig_0T^aA^{(n)a}_\mu(x) - \frac{g_0^2}{2}\left(T^aT^bA^{(n)a}_\mu(x)A^{(n)b}_\mu(x) + [T^a, T^b]A^{(n)ab}_\mu(x)\right) \\
&- i\frac{g_0^3}{6}\left(T^aT^bT^cA^{(n)a}_\mu(x)A^{(n)b}_\mu(x)A^{(n)c}_\mu(x) + \frac{3}{2}\{T^a, [T^b, T^c]\}A^{(n)a}_\mu(x)A^{(n)bc}_\mu(x)\right. \\
&\left.+ [T^a, [T^b, T^c]]\bar{A}^{(n)abc}_\mu(x) + \left(T^aT^bT^c + T^cT^bT^a - \frac{1}{N_c}\mathrm{Tr}[T^aT^bT^c + T^cT^bT^a]\right)\hat{A}^{(n)abc}_\mu(x)\right) + \mathcal{O}(g_0^4).
\end{aligned} \tag{19}
$$

We can now also incorporate $\bar{A}$ into $\hat{A}$ because[1]

$$
[T^a, [T^b, T^c]]\bar{A}^{(n)abc}_\mu(x) = 2[T^a, T^bT^c]\bar{A}^{(n)a[bc]}_\mu(x) = 2(T^aT^bT^c + T^cT^bT^a)\bar{A}^{(n)a[bc]}_\mu(x) \tag{20}
$$

and

$$
\mathrm{Tr}[T^aT^bT^c + T^cT^bT^a]\bar{A}^{(n)a[bc]}_\mu(x) = \mathrm{Tr}[T^aT^bT^c - T^bT^cT^a]\bar{A}^{(n)a[bc]}_\mu(x) = 0. \tag{21}
$$

Thus we can write

$$
\begin{aligned}
U^{(n)}_\mu(x) &= 1 + ig_0T^aA^{(n)a}_\mu(x) - \frac{g_0^2}{2}\left(T^aT^bA^{(n)a}_\mu(x)A^{(n)b}_\mu(x) + [T^a, T^b]A^{(n)ab}_\mu(x)\right) \\
&- i\frac{g_0^3}{6}\left(T^aT^bT^cA^{(n)a}_\mu(x)A^{(n)b}_\mu(x)A^{(n)c}_\mu(x) + \frac{3}{2}\{T^a, [T^b, T^c]\}A^{(n)a}_\mu(x)A^{(n)bc}_\mu(x)\right. \\
&\left.+ \left(T^aT^bT^c + T^cT^bT^a - \frac{1}{N_c}\mathrm{Tr}[T^aT^bT^c + T^cT^bT^a]\right)A^{(n)abc}_\mu(x)\right) + \mathcal{O}(g_0^4)
\end{aligned} \tag{22}
$$

[1]We use the following shorthand notation for symmetrized and antisymmetrized expressions: $A^{\{ab\}} = \frac{1}{2}(A^{ab} + A^{ba})$ and $A^{[ab]} = \frac{1}{2}(A^{ab} - A^{ba})$.

with

$$A_\mu^{(n)abc}(x) = \hat{A}_\mu^{(n)abc}(x) + 2\bar{A}_\mu^{(n)a[bc]}(x) \tag{23}$$

We define their Fourier transforms as

$$A_\mu^{(n)a}(x) = \int_{-\pi}^{\pi} \frac{\mathrm{d}^4 k}{(2\pi)^4} e^{i(x+\hat{\mu}/2)k} A_\mu^{(n)a}(k), \tag{24}$$

$$A_\mu^{(n)ab}(x) = \int_{-\pi}^{\pi} \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{\mathrm{d}^4 k_2}{(2\pi)^4} e^{i(x+\hat{\mu}/2)(k_1+k_2)} A_\mu^{(n)ab}(k_1, k_2), \tag{25}$$

$$A_\mu^{(n)abc}(x) = \int_{-\pi}^{\pi} \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{\mathrm{d}^4 k_2}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{\mathrm{d}^4 k_3}{(2\pi)^4} e^{i(x+\hat{\mu}/2)(k_1+k_2+k_3)} A_\mu^{(n)abc}(k_1, k_2, k_3). \tag{26}$$

These fields are ultimately functions of the original gluon fields

$$A_\mu^{(n)a}(x) = \sum_\nu \sum_y \tilde{g}_{\mu\nu}^{(n)}(\varrho, y) A_\nu^{(0)a}(x+y), \tag{27}$$

$$A_\mu^{(n)ab}(x) = \sum_{\nu\rho} \sum_{yz} \tilde{g}_{\mu\nu\rho}^{(n)}(\varrho, y, z) A_\nu^{(0)a}(x+y) A_\rho^{(0)b}(x+z), \tag{28}$$

$$A_\mu^{(n)abc}(x) = \sum_{\nu\rho\sigma} \sum_{yzr} \tilde{g}_{\mu\nu\rho\sigma}^{(n)}(\varrho, y, z, r) A_\nu^{(0)a}(x+y) A_\rho^{(0)b}(x+z) A_\sigma^{(0)c}(x+r), \tag{29}$$

$$A_\mu^{(n)a}(k) = \sum_\nu \tilde{g}_{\mu\nu}^{(n)}(\varrho, k) A_\nu^{(0)a}(k), \tag{30}$$

$$A_\mu^{(n)ab}(k_1, k_2) = \sum_{\nu\rho} \tilde{g}_{\mu\nu\rho}^{(n)}(\varrho, k_1, k_2) A_\nu^{(0)a}(k_1) A_\rho^{(0)b}(k_2), \tag{31}$$

$$A_\mu^{(n)abc}(k_1, k_2, k_3) = \sum_{\nu\rho\sigma} \tilde{g}_{\mu\nu\rho\sigma}^{(n)}(\varrho, k_1, k_2, k_3) A_\nu^{(0)a}(k_1) A_\rho^{(0)b}(k_2) A_\sigma^{(0)c}(k_3), \tag{32}$$

with some complicated functions simply denoted by $\tilde{g}$ here. Sections II C–II E will be largely concerned with determining the momentum space $\tilde{g}$ explicitly.

We have used *Mathematica* [39] to supplement and double check our analytical derivations as well as to derive Feynman rules and perform the calculations of the fermion self energy presented in Sec. V [40].

1. Connecting the perturbative and $SU(N_c)$ expansions

In order to relate the smeared gluon field $\tilde{A}_\mu^{(n)a}(g_0, x)$ to the perturbative functions $A_\mu^{(n)a}(x)$, $A_\mu^{(n)ab}(x)$, and $A_\mu^{(n)abc}(x)$, we need to expand the products of color matrices such that only terms with a single T^a remain. We use

$$[T^a, T^b] = i f^{abc} T^c, \qquad f^{abc} = -2i \mathrm{Tr}(T^a [T^b, T^c]), \tag{33}$$

$$\{T^a, T^b\} = \frac{1}{N_c} \delta^{ab} + d^{abc} T^c, \qquad d^{abc} = 2\mathrm{Tr}(T^a \{T^b, T^c\}), \tag{34}$$

and

$$T^a T^b = \frac{1}{2N_c} \delta^{ab} + \frac{1}{2}(d^{abc} + i f^{abc}) T^c \tag{35}$$

$$\mathrm{Tr}[T^a T^b T^c] = \frac{1}{2} \mathrm{Tr}[T^a \{T^b, T^c\} + T^a [T^b, T^c]] = \frac{1}{4}(d^{abc} + i f^{abc}). \tag{36}$$

With the above color identities we can now rewrite the following expressions in a way that only one generator T remains:

$$[T^a, T^b]A_\mu^{(n)[ab]}(x) = if^{abc}T^a A_\mu^{(n)[bc]}(x) \tag{37}$$

$$\begin{aligned}&\left(T^aT^bT^c + T^cT^bT^a - \frac{1}{N_c}\mathrm{Tr}[T^aT^bT^c + T^cT^bT^a]\right)A_\mu^{(n)abc}(x)\\ &= \left(\frac{1}{2}\left(\{T^a,\{T^b,T^c\}\} + [T^a,[T^b,T^c]]\right) - \frac{1}{N_c}\mathrm{Tr}[T^aT^bT^c + T^cT^bT^a]\right)A_\mu^{(n)abc}(x)\end{aligned} \tag{38}$$

$$= T^a\left[\frac{1}{N_c}\delta^{ae}\delta^{bc} + \frac{1}{2}(d^{bcd}d^{eda} - f^{bcd}f^{eda})\right]A_\mu^{(n)ebc}(x). \tag{39}$$

Thus we can rearrange our perturbative expansion (19) to look like

$$\begin{aligned}U_\mu^{(n)}(x) = 1 &+ ig_0T^a\left\{A_\mu^{(n)a}(x) - \frac{g_0}{2}f^{abc}A_\mu^{(n)[bc]}(x) - \frac{g_0^2}{6}\left[\frac{1}{N_c}\delta^{ae}\delta^{bc} + \frac{1}{2}(d^{bcd}d^{eda} - f^{bcd}f^{eda})\right]A_\mu^{(n)ebc}(x) + \mathcal{O}(g_0^3)\right\}\\ &- \frac{g_0^2}{2}T^aT^b\left\{A_\mu^{(n)a}(x)A_\mu^{(n)a}(x) - \frac{g_0}{2}\left(A_\mu^{(n)a}(x)f^{bcd} + f^{acd}A_\mu^{(n)b}(x)\right)A_\mu^{(n)[cd]}(x) + \mathcal{O}(g_0^2)\right\}\\ &- i\frac{g_0^3}{6}T^aT^bT^c\left\{A_\mu^{(n)a}(x)A_\mu^{(n)b}(x)A_\mu^{(n)c}(x) + \mathcal{O}(g_0)\right\} + \mathcal{O}(T^4).\end{aligned} \tag{40}$$

We can see that the expressions in the large braces correspond to the first, second, and third order of an expansion of the exponential function, each cut off to give an overall highest order of g_0^3. Hence we can give the first few orders of the smeared gluon field as

$$\tilde{A}_\mu^{(n)a}(g_0, x) = A_\mu^{(n)a}(x) - \frac{g_0}{2}f^{abc}A_\mu^{(n)[bc]}(x) - \frac{g_0^2}{6}\left[\frac{1}{N_c}\delta^{ae}\delta^{bc} + \frac{1}{2}(d^{bcd}d^{eda} - f^{bcd}f^{eda})\right]A_\mu^{(n)ebc}(x) + \mathcal{O}(g_0^3). \tag{41}$$

Thus, we have expressed the new gluon field in terms of the perturbative fields.

C. Leading order

As we have seen, at leading order the perturbative and $SU(N_c)$ expansions are identical: $\tilde{A}_\mu^{(n)a}(g_0, x) = A_\mu^{(n)a}(x) + \mathcal{O}(g_0)$. The sum of staples at leading order is

$$\begin{aligned}S_\mu^{(n)}(x) = 6 + ig_0T^a\Bigg(\sum_{\nu=1}^{4}\Big[&A_\nu^{(n)a}(x) + A_\mu^{(n)a}(x+\hat{\nu}) - A_\nu^{(n)a}(x+\hat{\mu}) - A_\nu^{(n)a}(x-\hat{\nu})\\ &+ A_\mu^{(n)a}(x-\hat{\nu}) + A_\nu^{(n)a}(x+\hat{\mu}-\hat{\nu})\Big] - 2A_\mu^{(n)a}(x)\Bigg) + \mathcal{O}(g_0^2)\end{aligned} \tag{42}$$

where the sum now includes the $\nu = \mu$ term, which is compensated by the subtraction of $2A_\mu^{(n)a}(x)$. Therefore

$$Q_{1\mu}^{(n)a}(x) = \sum_{\nu=1}^{4}\left[A_\nu^{(n)a}(x) + A_\mu^{(n)a}(x+\hat{\nu}) - A_\nu^{(n)a}(x+\hat{\mu}) - A_\nu^{(n)a}(x-\hat{\nu}) + A_\mu^{(n)a}(x-\hat{\nu}) + A_\nu^{(n)a}(x+\hat{\mu}-\hat{\nu}) - 2A_\mu^{(n)a}(x)\right] \tag{43}$$

$$= -6A_\mu^{(n)a}(x) + S_{1\mu}^{(n)a}(x). \tag{44}$$

The smeared gluon field in position space can then be written as a convolution [5]:

$$A_\mu^{(n+1)a}(x) = A_\mu^{(n)a}(x) + \varrho Q_{1\mu}^{(n)a}(x) \tag{45}$$

$$= A_\mu^{(n)a}(x) + \varrho \sum_{\nu=1}^{4} \sum_y g_{\mu\nu}(y) A_\nu^{(n)a}(x+y) \tag{46}$$

$$=: \sum_\nu \sum_y \tilde{g}_{\mu\nu}(\varrho, y) A_\nu^{(n)a}(x+y) \tag{47}$$

with $\tilde{g}_{\mu\nu}(\varrho, y) = \delta_{\mu\nu} + \varrho g_{\mu\nu}(y)$ and

$$g_{\mu\nu}(y) = \delta(y,0) - \delta(y,\hat{\mu}) - \delta(y,-\hat{\nu}) + \delta(y,\hat{\mu}-\hat{\nu}) + \delta_{\mu\nu}\left[\sum_{\tau=1}^{4}(\delta(y,\hat{\tau}) + \delta(y,-\hat{\tau})) - 8\delta(y,0)\right]. \tag{48}$$

Performing a Fourier transformation using $\delta(x,y) = \int_{-\pi}^{\pi} \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{i(x-y)p}$, we get

$$e^{iyp}\left(1 - e^{-ip_\mu} - e^{ip_\nu} + e^{-i(p_\mu - p_\nu)} + \delta_{\mu\nu}\left[-8 + \sum_{\tau=1}^{4}(e^{-ip_\tau} + e^{ip_\tau})\right]\right) = e^{iyp} e^{-\frac{i}{2}(p_\mu - p_\nu)}\left(-\delta_{\mu\nu}\hat{p}^2 + \hat{p}_\mu \hat{p}_\nu\right) \tag{49}$$

with $\hat{p}_\mu = 2\sin(\frac{1}{2}p_\mu)$ and $\hat{p}^2 = \sum_{\mu=1}^4 \hat{p}_\mu^2$. Now we can express the Fourier transform of the first order smeared gluon field as

$$A_\mu^{(n+1)a}(x) = \sum_{\nu,y} \int_{-\pi}^{\pi} \frac{\mathrm{d}^4 k}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{iyp} e^{-\frac{i}{2}(p_\mu - p_\nu)} \left(\delta_{\mu\nu} + \varrho\left(-\delta_{\mu\nu}\hat{p}^2 + \hat{p}_\mu \hat{p}_\nu\right)\right) e^{i(x+y+\hat{\nu}/2)k} A_\nu^{(n)a}(k) \tag{50}$$

$$= \sum_\nu \int_{-\pi}^{\pi} \frac{\mathrm{d}^4 k}{(2\pi)^4} e^{i(x+\hat{\mu}/2)k} \left(\delta_{\mu\nu} + \varrho\left(-\delta_{\mu\nu}\hat{k}^2 + \hat{k}_\mu \hat{k}_\nu\right)\right) A_\nu^{(n)a}(k), \tag{51}$$

where we have used $\sum_x e^{-ix(p-q)} = (2\pi)^4 \delta(p-q)$ and performed the integration over p. Thus at leading order the result in momentum space is

$$A_\mu^{(n+1)a}(k) = A_\mu^{(n)a}(k) + \varrho \sum_\nu g_{\mu\nu}(k) A_\nu^{(n)a}(k) \tag{52}$$

with

$$g_{\mu\nu}(k) = -\delta_{\mu\nu}\hat{k}^2 + \hat{k}_\mu \hat{k}_\nu. \tag{53}$$

Expressing it in terms of the function $f(k) = 1 - \varrho\hat{k}^2$, it becomes easier to iterate

$$A_\mu^{(n+1)a}(k) = \sum_\nu \left(f(k)\delta_{\mu\nu} - (f(k) - 1)\frac{\hat{k}_\mu \hat{k}_\nu}{\hat{k}^2}\right) A_\nu^{(n)a}(k). \tag{54}$$

In this form, the form factor retains the same structure after multiple iterations, only the powers of f increase [5,37]:

$$A_\mu^{(n)a}(k) = \sum_\nu \left(f^n(k)\delta_{\mu\nu} - (f^n(k) - 1)\frac{\hat{k}_\mu \hat{k}_\nu}{\hat{k}^2}\right) A_\nu^{(0)a}(k) \tag{55}$$

$$=: \sum_\nu \tilde{g}_{\mu\nu}^n(\varrho, k) A_\nu^{(0)a}(k). \tag{56}$$

We note that $\tilde{g}_{\mu\nu}^n(\varrho, k)$ is not exactly the Fourier transform of $\tilde{g}_{\mu\nu}^n(\varrho, y)$ but rather fulfils the same purpose in momentum space of relating the newly smeared field to the one of the previous smearing step.

D. Next-to-leading order

At higher orders the expansions deviate, and Eqs. (16) and (17) would need to be modified for a general smearing step $n \to n+1$ to include the higher order perturbative fields $A_\mu^{(n)[ab]}(x)$, $\bar{A}_\mu^{(n)abc}(x)$, and $\hat{A}_\mu^{(n)abc}(x)$. This is why we start again by considering the first smearing step. At order g_0^2, the sum of staples $S_\mu^{(0)}(x)$ is given by

$$
\begin{aligned}
S_{2\mu}^{(0)ab}(x) = \sum_{\nu\neq\mu}\Bigg[&A_\nu^a(x)A_\mu^b(x+\hat\nu) - A_\nu^a(x)A_\nu^b(x+\hat\mu) - A_\mu^a(x+\hat\nu)A_\nu^b(x+\hat\mu)\\
&- A_\nu^a(x-\hat\nu)A_\mu^b(x-\hat\nu) - A_\nu^a(x-\hat\nu)A_\nu^b(x+\hat\mu-\hat\nu) + A_\mu^a(x-\hat\nu)A_\nu^b(x+\hat\mu-\hat\nu)\\
&+\frac{1}{2}\Big\{A_\nu^a(x)A_\nu^b(x) + A_\mu^a(x+\hat\nu)A_\mu^b(x+\hat\nu) + A_\nu^a(x+\hat\mu)A_\nu^b(x+\hat\mu)\\
&+ A_\nu^a(x-\hat\nu)A_\nu^b(x-\hat\nu) + A_\mu^a(x-\hat\nu)A_\mu^b(x-\hat\nu) + A_\nu^a(x+\hat\mu-\hat\nu)A_\nu^b(x+\hat\mu-\hat\nu)\Big\}\Bigg].
\end{aligned}
\tag{57}
$$

The part in braces $\{\cdots\}$ is symmetric in $a\leftrightarrow b$ and will therefore disappear in $A_\mu^{(1)[ab]}(x)$. We get

$$
Q_{2\mu}^{(0)ab}(x) = \frac{1}{2}\left(-S_{1\mu}^{(0)a}(x)A_\mu^{(0)b}(x) + S_{2\mu}^{(0)ab}(x)\right) \tag{58}
$$

which leaves

$$
A_\mu^{(1)[ab]}(x) = 2\varrho\left(\frac{1}{2}Q_{1\mu}^{(0)[a}(x)A_\mu^{(0)b]}(x) + Q_{2\mu}^{(0)[ab]}(x)\right) = \varrho S_{2\mu}^{(0)[ab]}(x), \tag{59}
$$

which we can write as a double convolution:

$$
S_{2\mu}^{(0)[ab]}(x) = \sum_{y,z}\sum_{\nu,\rho=1}^{4} g_{\mu\nu\rho}(y,z)\cdot A_\nu^{[a}(x+y)A_\rho^{b]}(x+z) \tag{60}
$$

$$
= \sum_{y,z}\sum_{\nu,\rho=1}^{4}\frac{1}{2}(g_{\mu\nu\rho}(y,z) - g_{\mu\rho\nu}(z,y))\cdot A_\nu^a(x+y)A_\rho^b(x+z) \tag{61}
$$

with

$$
\begin{aligned}
g_{\mu\nu\rho}(y,z) = {}&\delta_{\mu\nu}[\delta(y,-\hat\rho)\delta(z,\hat\mu-\hat\rho) - \delta(y,\hat\rho)\delta(z,\hat\mu)] + \delta_{\mu\rho}[\delta(y,0)\delta(z,\hat\nu) - \delta(y,-\hat\nu)\delta(z,-\hat\nu)]\\
&- \delta_{\nu\rho}[\delta(y,0)\delta(z,\hat\mu) + \delta(y,-\hat\nu)\delta(z,\hat\mu-\hat\nu)].
\end{aligned}
\tag{62}
$$

Fourier transforming gives us

$$
A_\mu^{(1)[ab]}(k_1,k_2) = \varrho\sum_{\nu,\rho} g_{\mu\nu\rho}(k_1,k_2)A_\nu^a(k_1)A_\rho^b(k_2), \tag{63}
$$

where

$$
g_{\mu\nu\rho}(k_1,k_2) = 2i\Big(\delta_{\mu\rho}c(k_{1\mu})s(2k_{2\nu}+k_{1\nu}) - \delta_{\mu\nu}c(k_{2\mu})s(2k_{1\rho}+k_{2\rho}) + \delta_{\nu\rho}s(k_{1\mu}-k_{2\mu})c(k_{1\nu}+k_{2\nu})\Big). \tag{64}
$$

Here we use the common shorthand for the trigonometric functions

$$
s(k_\mu) = \sin\left(\frac{1}{2}k_\mu\right), \qquad c(k_\mu) = \cos\left(\frac{1}{2}k_\mu\right), \tag{65}
$$

$$
\bar s(k_\mu) = \sin(k_\mu), \qquad \bar c(k_\mu) = \cos(k_\mu), \tag{66}
$$

$$s^2(k) = \sum_\mu s^2(k_\mu). \tag{67}$$

Although $g_{\mu\nu\rho}(k_1, k_2)$ is imaginary [and therefore so is $A_\mu^{(1)[ab]}(k_1, k_2)$], it also fulfils $g_{\mu\nu\rho}(-k_1, -k_2) = -g_{\mu\nu\rho}(k_1, k_2)$, which means that $A_\mu^{(1)[ab]}(x)$ is real.

If we were to repeat the previous calculation for a smearing step $n \to n+1$, we would not only have to replace the $A_\mu^{(0)a}(x)$ by $A_\mu^{(n)a}(x)$ but the expansion of $S_{2\mu}^{(n)ab}(x)$ would also contain terms linear in $A_\mu^{(n)[ab]}(x)$. Instead, it is easier to realize that we can get the same contribution by replacing

$$T^a A_\mu^{(0)a}(x) \to T^a \tilde{A}_\mu^{(n)a}(g_0, x) = T^a A_\mu^{(n)a}(x) - \frac{g_0}{2i}[T^a, T^b] A_\mu^{(n)[ab]}(x) + \mathcal{O}(g_0^2) \tag{68}$$

in the *leading order* calculation. Then we get for a general smearing step

$$A_\mu^{(n+1)[ab]}(k_1, k_2) = \sum_\nu \tilde{g}_{\mu\nu}(\varrho, k_1+k_2) A_\nu^{(n)[ab]}(k_1, k_2) + \varrho \sum_{\nu,\rho} g_{\mu\nu\rho}(k_1, k_2) A_\nu^{(n)a}(k_1) A_\rho^{(n)b}(k_2). \tag{69}$$

By iterating once

$$A_\mu^{(2)[ab]}(k_1, k_2) = \sum_\nu \tilde{g}_{\mu\nu}(\varrho, k_1+k_2) \varrho \sum_{\rho\sigma} g_{\nu\rho\sigma}(k_1, k_2) A_\rho^{(0)a}(k_1) A_\sigma^{(0)b}(k_2) + \varrho \sum_{\nu,\rho} g_{\mu\nu\rho}(k_1, k_2) A_\nu^{(1)a}(k_1) A_\rho^{(1)b}(k_2) \tag{70}$$

and twice

$$\begin{aligned} A_\mu^{(3)[ab]}(k_1, k_2) = {} & \sum_\nu \tilde{g}_{\mu\nu}^2(\varrho, k_1+k_2) \varrho \sum_{\rho\sigma} g_{\nu\rho\sigma}(k_1, k_2) A_\rho^{(0)a}(k_1) A_\sigma^{(0)b}(k_2) \\ & + \sum_\nu \tilde{g}_{\mu\nu}(\varrho, k_1+k_2) \varrho \sum_{\rho\sigma} g_{\nu\rho\sigma}(k_1, k_2) A_\rho^{(1)a}(k_1) A_\sigma^{(1)b}(k_2) + \varrho \sum_{\nu,\rho} g_{\mu\nu\rho}(k_1, k_2) A_\nu^{(2)a}(k_1) A_\rho^{(2)b}(k_2), \end{aligned} \tag{71}$$

we can infer the general form

$$A_\mu^{(n)[ab]}(k_1, k_2) = \sum_{\nu\rho\sigma} \varrho g_{\nu\rho\sigma}(k_1, k_2) \sum_{m=0}^{n-1} \tilde{g}_{\mu\nu}^{n-m-1}(\varrho, k_1+k_2) A_\rho^{(m)a}(k_1) A_\sigma^{(m)b}(k_2). \tag{72}$$

Finally, we can also express the first order fields in the sum in terms of the original unsmeared ones

$$\begin{aligned} A_\mu^{(n)[ab]}(k_1, k_2) &= \sum_{\alpha\beta\gamma\nu\rho} \varrho g_{\alpha\beta\gamma} \left[\sum_{m=0}^{n-1} \tilde{g}_{\mu\alpha}^{n-m-1}(\varrho, k_1+k_2) \tilde{g}_{\beta\nu}^m(\varrho, k_1) \tilde{g}_{\gamma\rho}^m(\varrho, k_2) \right] A_\nu^{(0)a}(k_1) A_\rho^{(0)b}(k_2) \\ &=: \sum_{\nu\rho} \tilde{g}_{\mu\nu\rho}^{(n)}(\varrho, k_1, k_2) A_\nu^{(0)a}(k_1) A_\rho^{(0)b}(k_2). \end{aligned} \tag{73}$$

E. Next-to-next-to-leading order

As before, we start again by considering the first smearing step and generalize later to an arbitrary number of smearing steps. At third order we have defined two perturbative fields $\bar{A}$ and $\hat{A}$ in Eq. (19), where the former can be read off of Eq. (17) to be

$$\bar{A}_\mu^{(1)abc}(x) = -3\varrho A_\mu^a(x) Q_{2\mu}^{[bc]}(x) + \frac{1}{2}\varrho^2 Q_{1\mu}^a(x) Q_{1\mu}^b(x) A_\mu^c(x) + \frac{1}{2}\varrho A_\mu^a(x) A_\mu^b(x) Q_{1\mu}^c(x) \tag{74}$$

$$= -\frac{3}{2}A_\mu^a(x)A_\mu^{(1)[bc]}(x) + \frac{3}{2}\varrho A_\mu^a(x)Q_{1\mu}^b(x)A_\mu^c(x) + \frac{1}{2}\varrho^2 Q_{1\mu}^a(x)Q_{1\mu}^b(x)A_\mu^c(x) + \frac{1}{2}\varrho A_\mu^a(x)A_\mu^b(x)Q_{1\mu}^c(x) \tag{75}$$

$$= -\frac{3}{2}A_\mu^a(x)A_\mu^{(1)[bc]}(x) + \frac{1}{2}A_\mu^a(x)A_\mu^{(1)b}(x)A_\mu^c(x) + \frac{1}{2}A_\mu^{(1)a}(x)A_\mu^{(1)b}(x)A_\mu^c(x). \tag{76}$$

We have dropped all terms that were symmetric in $b \leftrightarrow c$ in the last step. Thus $\bar{A}$ is completely given by already known quantities, which is not surprising as it stems only from the commutator terms of the Baker-Campbell-Hausdorff formula.

From the third order in the expansion of $Q_\mu(x)$ we get

$$\hat{A}_\mu^{(1)abc}(x) = -6\varrho Q_{3\mu}^{abc}(x) \tag{77}$$

with

$$Q_{3\mu}^{abc}(x) = A_\mu^a(x)A_\mu^b(x)A_\mu^c(x) - \frac{1}{2}S_{1\mu}^a(x)A_\mu^b(x)A_\mu^c(x) + S_{2\mu}^{ab}(x)A_\mu^c(x) - \frac{1}{2}S_{3\mu}^{abc}(x) \tag{78}$$

$$= -2A_\mu^a(x)A_\mu^b(x)A_\mu^c(x) - \frac{1}{2}Q_{1\mu}^a(x)A_\mu^b(x)A_\mu^c(x) + S_{2\mu}^{ab}(x)A_\mu^c(x) - S_{3\mu}^{abc}(x). \tag{79}$$

We revisit the sum of staples at second order and define

$$S_{2\mu}^{ab}(x) = s_{1\mu}^{ab}(x) + \frac{1}{2}s_{2\mu}^{ab}(x) \tag{80}$$

with

$$\begin{aligned} s_{1\mu\nu}^{ab}(x) = \sum_\nu \Big[& A_\nu^a(x)A_\mu^b(x+\hat{\nu}) - A_\nu^a(x)A_\nu^b(x+\hat{\mu}) - A_\mu^a(x+\hat{\nu})A_\nu^b(x+\hat{\mu}) \\ & - A_\nu^a(x-\hat{\nu})A_\mu^b(x-\hat{\nu}) - A_\nu^a(x-\hat{\nu})A_\nu^b(x+\hat{\mu}-\hat{\nu}) + A_\mu^a(x-\hat{\nu})A_\nu^b(x+\hat{\mu}-\hat{\nu}) \Big], \end{aligned} \tag{81}$$

$$\begin{aligned} s_{2\mu}^{ab}(x) = \sum_\nu \Big[& A_\nu^a(x)A_\nu^b(x) + A_\mu^a(x+\hat{\nu})A_\mu^b(x+\hat{\nu}) + A_\nu^a(x+\hat{\mu})A_\nu^b(x+\hat{\mu}) \\ & + A_\nu^a(x-\hat{\nu})A_\nu^b(x-\hat{\nu}) + A_\mu^a(x-\hat{\nu})A_\mu^b(x-\hat{\nu}) + A_\nu^a(x+\hat{\mu}-\hat{\nu})A_\nu^b(x+\hat{\mu}-\hat{\nu}) - 2\delta_{\mu\nu}A_\mu^a(x)A_\mu^b(x) \Big]. \end{aligned} \tag{82}$$

Their Fourier transforms are

$$\begin{aligned} s_{1\mu}^{ab}(x) &= \sum_{yz}\sum_{\nu\rho} a_{1\mu\nu\rho}(y,z)A_\nu^a(x+y)A_\rho^b(x+z) \\ &= \sum_{\nu\rho}\int \frac{\mathrm{d}^4k_1\mathrm{d}^4k_2}{(2\pi)^8} e^{-i(x+\frac{1}{2}\hat{\mu})(k_1+k_2)} \cdot a_{1\mu\nu\rho}(k_1,k_2)A_\nu^a(k_1)A_\rho^b(k_2), \end{aligned} \tag{83}$$

$$\begin{aligned} s_{2\mu}^{ab}(x) &= \sum_{y}\sum_{\nu} a_{2\mu\nu}(y)A_\nu^a(x+y)A_\nu^b(x+y) \\ &= \sum_{\nu}\int \frac{\mathrm{d}^4k_1\mathrm{d}^4k_2}{(2\pi)^8} e^{-i(x+\frac{1}{2}\hat{\mu})(k_1+k_2)} \cdot a_{2\mu\nu}(k_1,k_2)A_\nu^a(k_1)A_\nu^b(k_2), \end{aligned} \tag{84}$$

with

$$a_{1\mu\nu\rho}(y,z) = \delta_{\mu\nu}[\delta(y,-\hat\rho)\delta(z,\hat\mu-\hat\rho) - \delta(y,\hat\rho)\delta(z,\hat\mu)] + \delta_{\mu\rho}[\delta(y,0)\delta(z,\hat\nu) - \delta(y,-\hat\nu)\delta(z,-\hat\nu)] - \delta_{\nu\rho}[\delta(y,0)\delta(z,\hat\mu) + \delta(y,-\hat\nu)\delta(z,\hat\mu-\hat\nu)], \tag{85}$$

$$a_{2\mu\nu}(y) = \delta_{\mu\nu}\sum_{\alpha=1}^{4}\left[\delta(y,\hat\alpha) + \delta(y,-\hat\alpha) - 2\delta(y,0)\right] + 6\delta_{\mu\nu}\delta(y,0) + \delta(y,0) + \delta(y,\hat\mu) + \delta(y,-\hat\nu) + \delta(y,\hat\mu-\hat\nu), \tag{86}$$

and

$$a_{1\mu\nu\rho}(k_1,k_2) = -\delta_{\mu\nu}\left[2i\sin\left(k_{1\rho} + \frac{1}{2}k_{2\rho}\right)e^{\frac{i}{2}k_{2\mu}}\right] + \delta_{\mu\rho}\left[2i\sin\left(k_{2\nu} + \frac{1}{2}k_{1\nu}\right)e^{-\frac{i}{2}k_{1\mu}}\right] - \delta_{\nu\rho}\left[2\cos\left(\frac{1}{2}k_{1\nu} + \frac{1}{2}k_{2\nu}\right)e^{-\frac{i}{2}(k_{1\mu}-k_{2\mu})}\right] \tag{87}$$

$$= -h_{\mu\nu\rho}(k_1,k_2) + g_{\mu\nu\rho}(k_1,k_2). \tag{88}$$

Here $h_{\mu\nu\rho}$ and $g_{\mu\nu\rho}$ are essentially the real and imaginary parts of $a_{1\mu\nu\rho}$, i.e.

$$g_{\mu\nu\rho}(k_1,k_2) = 2i\Big(-\delta_{\mu\nu}c(k_{2\mu})s(2k_{1\rho}+k_{2\rho}) + \delta_{\mu\rho}c(k_{1\mu})s(2k_{2\nu}+k_{1\nu}) + \delta_{\nu\rho}s(k_{1\mu}-k_{2\mu})c(k_{1\nu}+k_{2\nu})\Big), \tag{89}$$

$$h_{\mu\nu\rho}(k_1,k_2) = 2\Big(-\delta_{\mu\nu}s(k_{2\mu})s(2k_{1\rho}+k_{2\rho}) - \delta_{\mu\rho}s(k_{1\mu})s(2k_{2\nu}+k_{1\nu}) + \delta_{\nu\rho}c(k_{1\mu}-k_{2\mu})c(k_{1\nu}+k_{2\nu})\Big). \tag{90}$$

We know the antisymmetric $g_{\mu\nu\rho}$ already from the second order.

Furthermore,

$$a_{2\mu\nu}(k_1,k_2) = 6\delta_{\mu\nu} + \delta_{\mu\nu}\sum_{\alpha=1}^{4}\left[2\cos(k_{1\alpha}+k_{2\alpha}) - 2\right] + 4\cos\left(\frac{1}{2}(k_{1\mu}+k_{2\mu})\right)\cos\left(\frac{1}{2}(k_{1\nu}+k_{2\nu})\right) = 6\delta_{\mu\nu} - \delta_{\mu\nu}4s^2(k_1+k_2) + 4c(k_{1\mu}+k_{2\mu})c(k_{1\nu}+k_{2\nu}) \tag{91}$$

$$=: 6\delta_{\mu\nu} - g^{(i)}_{\mu\nu}(k_1+k_2) \tag{92}$$

where we have defined $g^{(i)}_{\mu\nu}(k)$ which is a variation of $g_{\mu\nu}(k)$ with cosines in the second term instead of sines.

For the sum of staples at third order we define

$$S^{abc}_{3\mu}(x) = s^{abc}_{1\mu}(x) + \frac{1}{2}s^{abc}_{2\mu}(x) + \frac{1}{6}s^{abc}_{3\mu}(x) \tag{93}$$

with

$$s^{abc}_{1\mu}(x) = \sum_{\nu}\left[-A^a_\nu(x)A^b_\mu(x+\hat\nu)A^c_\nu(x+\hat\mu) - A^a_\nu(x-\hat\nu)A^b_\mu(x-\hat\nu)A^c_\nu(x+\hat\mu-\hat\nu)\right] \tag{94}$$

$$
\begin{aligned}
s_{2\mu}^{abc}(x) = \sum_{\nu} \Big[& \{ A_\nu^a(x) A_\mu^b(x+\hat{\nu}) A_\mu^c(x+\hat{\nu}) + A_\nu^a(x) A_\nu^b(x+\hat{\mu}) A_\nu^c(x+\hat{\mu}) \\
& + A_\mu^a(x+\hat{\nu}) A_\nu^b(x+\hat{\mu}) A_\nu^c(x+\hat{\mu}) - A_\nu^a(x-\hat{\nu}) A_\mu^b(x-\hat{\nu}) A_\mu^c(x-\hat{\nu}) \\
& - A_\nu^a(x-\hat{\nu}) A_\nu^b(x+\hat{\mu}-\hat{\nu}) A_\nu^c(x+\hat{\mu}-\hat{\nu}) + A_\mu^a(x-\hat{\nu}) A_\nu^b(x+\hat{\mu}-\hat{\nu}) A_\nu^c(x+\hat{\mu}-\hat{\nu}) \} \\
& + \{ -A_\mu^a(x+\hat{\nu}) A_\mu^b(x+\hat{\nu}) A_\nu^c(x+\hat{\mu}) + A_\nu^a(x) A_\nu^b(x) A_\mu^c(x+\hat{\nu}) - A_\nu^a(x) A_\nu^b(x) A_\nu^c(x+\hat{\mu}) \\
& + A_\mu^a(x-\hat{\nu}) A_\mu^b(x-\hat{\nu}) A_\nu^c(x+\hat{\mu}-\hat{\nu}) + A_\nu^a(x-\hat{\nu}) A_\nu^b(x-\hat{\nu}) A_\mu^c(x-\hat{\nu}) \\
& + A_\nu^a(x-\hat{\nu}) A_\nu^b(x-\hat{\nu}) A_\nu^c(x+\hat{\mu}-\hat{\nu}) \} \Big], \qquad (95)
\end{aligned}
$$

$$
\begin{aligned}
s_{3\mu}^{abc}(x) = \sum_{\nu} \Big[& -2\delta_{\mu\nu} A_\mu^a(x) A_\mu^b(x) A_\mu^c(x) + A_\nu^a(x) A_\nu^b(x) A_\nu^c(x) + A_\mu^a(x+\hat{\nu}) A_\mu^b(x+\hat{\nu}) A_\mu^c(x+\hat{\nu}) \\
& - A_\nu^a(x+\hat{\mu}) A_\nu^b(x+\hat{\mu}) A_\nu^c(x+\hat{\mu}) - A_\nu^a(x-\hat{\nu}) A_\nu^b(x-\hat{\nu}) A_\nu^c(x-\hat{\nu}) \\
& + A_\mu^a(x-\hat{\nu}) A_\mu^b(x-\hat{\nu}) A_\mu^c(x-\hat{\nu}) + A_\nu^a(x+\hat{\mu}-\hat{\nu}) A_\nu^b(x+\hat{\mu}-\hat{\nu}) A_\nu^c(x+\hat{\mu}-\hat{\nu}) \Big]. \qquad (96)
\end{aligned}
$$

They can be written as

$$
s_{1\mu}^{abc}(x) = \sum_{y,z,r} \sum_{\nu\rho\sigma} b_{1\mu\nu\rho\sigma}(y,z,r) A_\nu^a(x+y) A_\rho^b(x+z) A_\sigma^c(x+r), \qquad (97)
$$

$$
\begin{aligned}
s_{2\mu}^{abc}(x) = & \sum_{y,z} \sum_{\nu\rho} b_{21\mu\nu\rho}(y,z) A_\nu^a(x+y) A_\rho^b(x+z) A_\rho^c(x+z) \\
& + \sum_{y,z} \sum_{\nu\rho} b_{22\mu\nu\rho}(y,z) A_\nu^a(x+y) A_\nu^b(x+y) A_\rho^c(x+z), \qquad (98)
\end{aligned}
$$

$$
s_{3\mu}^{abc}(x) = \sum_{y} \sum_{\nu} b_{3\mu\nu}(y) A_\nu^a(x+y) A_\nu^b(x+y) A_\nu^c(x+y), \qquad (99)
$$

with

$$
b_{1\mu\nu\rho\sigma}(y,z,r) = -\delta_{\mu\rho}\delta_{\nu\sigma}\Big(\delta(y,0)\delta(z,\hat{\nu})\delta(r,\hat{\mu}) + \delta(y,-\hat{\nu})\delta(z,-\hat{\nu})\delta(r,\hat{\mu}-\hat{\nu})\Big), \qquad (100)
$$

$$
\begin{aligned}
b_{21\mu\nu\rho}(y,z) = & \delta_{\mu\nu}\Big(\delta(y,\hat{\rho})\delta(z,\hat{\mu}) + \delta(y,-\hat{\rho})\delta(z,\hat{\mu}-\hat{\rho})\Big) + \delta_{\mu\rho}\Big(\delta(y,0)\delta(z,\hat{\nu}) - \delta(y,-\hat{\nu})\delta(z,-\hat{\nu})\Big) \\
& + \delta_{\nu\rho}\Big(\delta(y,0)\delta(z,\hat{\mu}) - \delta(y,-\hat{\nu})\delta(z,\hat{\mu}-\hat{\nu})\Big), \qquad (101)
\end{aligned}
$$

$$
\begin{aligned}
b_{22\mu\nu\rho}(y,z) = & \delta_{\mu\nu}\Big(-\delta(y,\hat{\rho})\delta(z,\hat{\mu}) + \delta(y,-\hat{\rho})\delta(z,\hat{\mu}-\hat{\rho})\Big) + \delta_{\mu\rho}\Big(\delta(y,0)\delta(z,\hat{\nu}) + \delta(y,-\hat{\nu})\delta(z,-\hat{\nu})\Big) \\
& + \delta_{\nu\rho}\Big(-\delta(y,0)\delta(z,\hat{\mu}) + \delta(y,-\hat{\nu})\delta(z,\hat{\mu}-\hat{\nu})\Big), \qquad (102)
\end{aligned}
$$

$$
b_{3\mu\nu}(y) = 6\delta_{\mu\nu}\delta(y,0) + g_{\mu\nu}(y). \qquad (103)
$$

Fourier transforming the $s_{i\mu}^{abc}(x)$ leads to

$$
s_{1\mu}^{abc}(x) = \sum_{\nu\rho\sigma} \int \frac{\mathrm{d}^4k_1 \mathrm{d}^4k_2 \mathrm{d}^4k_3}{(2\pi)^{12}} e^{-i(x+\frac{1}{2}\hat{\mu})(k_1+k_2+k_3)} b_{1\mu\nu\rho\sigma}(k_1,k_2,k_3) A_\nu^a(k_1) A_\rho^b(k_2) A_\sigma^c(k_3), \qquad (104)
$$

$$s_{2\mu}^{abc}(x) = \sum_{\nu\rho} \int \frac{d^4k_1 d^4k_2 d^4k_3}{(2\pi)^{12}} e^{-i(x+\frac{1}{2}\hat{\mu})(k_1+k_2+k_3)} b_{21\mu\nu\rho}(k_1, k_2+k_3) A_\nu^a(k_1) A_\rho^b(k_2) A_\rho^c(k_3)$$
$$+ \sum_{\nu\rho} \int \frac{d^4k_1 d^4k_2 d^4k_3}{(2\pi)^{12}} e^{-i(x+\frac{1}{2}\hat{\mu})(k_1+k_2+k_3)} b_{22\mu\nu\rho}(k_1+k_2, k_3) A_\nu^a(k_1) A_\nu^b(k_2) A_\rho^c(k_3), \quad (105)$$

$$s_{3\mu}^{abc}(x) = \sum_{\nu} \int \frac{d^4k_1 d^4k_2 d^4k_3}{(2\pi)^{12}} e^{-i(x+\frac{1}{2}\hat{\mu})(k_1+k_2+k_3)} g_{\mu\nu}(k_1+k_2+k_3) A_\nu^a(k_1) A_\nu^b(k_2) A_\nu^c(k_3), \quad (106)$$

with

$$b_{1\mu\nu\rho\sigma}(k_1,k_2,k_3) = -2\delta_{\mu\rho}\delta_{\nu\sigma}\cos\left(\frac{1}{2}k_{1\nu} + k_{2\nu} + \frac{1}{2}k_{3\nu}\right) e^{\frac{i}{2}(k_{1\mu}-k_{3\mu})}, \quad (107)$$

$$b_{21\mu\nu\rho}(k_1, k_2+k_3) = \delta_{\mu\nu} 2\cos\left(k_{1\rho} + \frac{1}{2}(k_{2\rho}+k_{3\rho})\right) e^{-\frac{i}{2}(k_{2\mu}+k_{3\mu})} + \delta_{\mu\rho}(-2i)\sin\left(\frac{1}{2}k_{1\nu} + k_{2\nu} + k_{3\nu}\right) e^{\frac{i}{2}k_{1\mu}}$$
$$+ \delta_{\nu\rho}(-2i)\sin\left(\frac{1}{2}(k_{1\nu}+k_{2\nu}+k_{3\nu})\right) e^{\frac{i}{2}(k_{1\mu}-k_{2\mu}-k_{3\mu})}, \quad (108)$$

$$b_{22\mu\nu\rho}(k_1+k_2, k_3) = \delta_{\mu\nu} 2i\sin\left(k_{1\rho} + k_{2\rho} + \frac{1}{2}k_{3\rho}\right) e^{-\frac{i}{2}k_{3\mu}} + \delta_{\mu\rho} 2\cos\left(\frac{1}{2}(k_{1\nu}+k_{2\nu}) + k_{3\nu}\right) e^{\frac{i}{2}(k_{1\mu}+k_{2\mu})}$$
$$+ \delta_{\nu\rho} 2i\sin\left(\frac{1}{2}(k_{1\nu}+k_{2\nu}+k_{3\nu})\right) e^{\frac{i}{2}(k_{1\mu}+k_{2\mu}-k_{3\mu})}, \quad (109)$$

$$b_{3\mu\nu}(k_1+k_2+k_3) = 6\delta_{\mu\nu} + g_{\mu\nu}(k_1+k_2+k_3). \quad (110)$$

We can symmetrize these expressions according to

$$(T^aT^bT^c + T^cT^bT^a) v_{i\mu}^{abc} = (T^aT^bT^c + T^cT^bT^a)\frac{1}{2}(v_{i\mu}^{abc} + v_{i\mu}^{cba}) \quad (111)$$

and observe that under the integrals

$$b_{1\mu\nu\rho\sigma}(k_1,k_2,k_3) A_\nu^c(k_1) A_\rho^b(k_2) A_\sigma^a(k_3) = b_{1\mu\sigma\rho\nu}(k_3,k_2,k_1) A_\nu^a(k_1) A_\rho^b(k_2) A_\sigma^c(k_3), \quad (112)$$

$$b_{21\mu\nu\rho}(k_1, k_2+k_3) A_\nu^c(k_1) A_\rho^b(k_2) A_\rho^a(k_3) = b_{21\mu\rho\nu}(k_3, k_1+k_2) A_\nu^a(k_1) A_\nu^b(k_2) A_\rho^c(k_3), \quad (113)$$

$$b_{22\mu\nu\rho}(k_1+k_2, k_3) A_\nu^c(k_1) A_\nu^b(k_2) A_\rho^a(k_3) = b_{22\mu\rho\nu}(k_2+k_3, k_1) A_\nu^a(k_1) A_\rho^b(k_2) A_\rho^c(k_3). \quad (114)$$

Thus we need

$$b_{1\mu\nu\rho\sigma}(k_1,k_2,k_3) + b_{1\mu\sigma\rho\nu}(k_3,k_2,k_1) = -4\delta_{\mu\rho}\delta_{\nu\sigma} c(k_{1\nu} + 2k_{2\nu} + k_{3\nu}) c(k_{1\mu} - k_{3\mu}) \quad (115)$$

$$=: g_{\mu\nu\rho\sigma}^{(iii)}(k_1,k_2,k_3), \quad (116)$$

$$b_{21\mu\nu\rho}(k_1, k_2+k_3) + b_{22\mu\rho\nu}(k_2+k_3, k_1) = 4\delta_{\mu\nu} c(2k_{1\rho} + k_{2\rho} + k_{3\rho}) c(k_{2\mu} + k_{3\mu})$$
$$+ 4\delta_{\mu\rho} s(k_{1\mu}) s(k_{1\nu} + 2(k_{2\nu} + k_{3\nu}))$$
$$+ 4\delta_{\nu\rho} s(k_{1\mu} - k_{2\mu} - k_{3\mu}) s(k_{1\nu} + k_{2\nu} + k_{3\nu}) \quad (117)$$

$$=: g_{\mu\nu\rho}^{(ii)}(k_1, k_2+k_3), \quad (118)$$

$$b_{22\mu\nu\rho}(k_1+k_2, k_3) + b_{21\mu\rho\nu}(k_3, k_1+k_2) = g_{\mu\rho\nu}^{(ii)}(k_3, k_1+k_2). \quad (119)$$

Now we can write

$$\begin{aligned}\hat{A}_\mu^{(1)abc}(k_1,k_2,k_3) = \rho\sum_{\nu\rho\sigma}\bigg(&3(h_{\mu\nu\rho}(k_1,k_2)-g_{\mu\nu\rho}(k_1,k_2))\delta_{\mu\sigma}+\frac{3}{2}g_{\mu\nu}(k_1)\delta_{\mu\rho}\delta_{\mu\sigma}\\ &+\frac{3}{2}\Big(g^{(i)}_{\mu\nu}(k_1,k_2)\delta_{\nu\rho}\delta_{\mu\sigma}+g^{(ii)}_{\mu\nu\rho}(k_1,k_2+k_3)\delta_{\rho\sigma}+g^{(iii)}_{\mu\nu\rho\sigma}(k_1,k_2,k_3)\Big)\\ &+\frac{1}{2}g_{\mu\nu}(k_1+k_2+k_3)\delta_{\nu\rho}\delta_{\nu\sigma}\bigg)A^a_\nu(k_1)A^b_\rho(k_2)A^c_\sigma(k_3).\end{aligned} \tag{120}$$

For a general smearing step $n>1$ we have to take into account that in the previous step $A_\mu^{(n-1)[ab]}\neq 0$, $\bar{A}_\mu^{(n-1)abc}\neq 0$, and $\hat{A}_\mu^{(n-1)abc}\neq 0$. Again we replace the unsmeared gluon field $A^{(0)}$ in the previous orders by the smeared one $\tilde{A}^{(n)}$ and collect all terms of order g_0^3:

$$\begin{aligned}U_\mu^{(n+1)}(k) = 1 + ig_0\sum_\nu\tilde{g}_{\mu\nu}(\varrho,k)\bigg(&T^aA_\nu^{(n)a}(k)-\frac{g_0}{2i}[T^a,T^b]A_\nu^{(n)[ab]}(k_1,k_2)\\ &-\frac{g_0^2}{6}\Big(T^aT^bT^c+T^cT^bT^a-\frac{1}{N_c}\mathrm{Tr}[T^aT^bT^c+T^cT^bT^a]\Big)A_\nu^{(n)abc}(k_3,k_4,k_5)\bigg)\\ &-\frac{g_0^2}{2}\sum_{\nu\rho}\Big(\tilde{g}_{\mu\nu}(\varrho,k_1)\tilde{g}_{\mu\rho}(\varrho,k_2)+2g_{\mu\nu\rho}(k_1,k_2)\Big)\Big(T^aA_\nu^{(n)a}(k_1)-\frac{g_0}{2i}[T^a,T^b]A_\nu^{(n)[ab]}(k_{11},k_{12})\Big)\\ &\times\Big(T^aA_\rho^{(n)a}(k_2)-\frac{g_0}{2i}[T^a,T^b]A_\rho^{(n)[ab]}(k_{21},k_{22})\Big)\\ &-i\frac{g_0^3}{6}\sum_{\nu\rho\sigma}\bigg(T^aT^bT^c\tilde{g}_{\mu\nu}(\varrho,k_1)\tilde{g}_{\mu\rho}(\varrho,k_2)\tilde{g}_{\mu\sigma}(\varrho,k_3)+\frac{3}{2}\{T^a,[T^b,T^c]\}\tilde{g}_{\mu\nu}(\varrho,k_1)\rho g_{\mu\rho\sigma}(k_2,k_3)\\ &+[T^a,[T^b,T^c]]\Big(-\frac{3}{2}\delta_{\mu\nu}\varrho g_{\mu\rho\sigma}(k_2,k_3)+\frac{1}{2}\delta_{\mu\nu}\tilde{g}_{\mu\rho}(\varrho,k_2)\delta_{\mu\sigma}+\frac{1}{2}\tilde{g}_{\mu\nu}(\varrho,k_1)\tilde{g}_{\mu\rho}(\varrho,k_2)\delta_{\mu\sigma}\Big)\\ &+\Big(T^aT^bT^c+T^cT^bT^a-\frac{1}{N_c}\mathrm{Tr}[T^aT^bT^c+T^cT^bT^a]\Big)\hat{G}_{\mu\nu\rho\sigma}(k_1,k_2,k_3)\bigg)A_\nu^{(n)a}(k_1)A_\rho^{(n)b}(k_2)A_\sigma^{(n)c}(k_3)\end{aligned} \tag{121}$$

with $k=k_1+k_2=k_3+k_4+k_5$, $k_1=k_{11}+k_{12}$, and $k_2=k_{21}+k_{22}$. From the third term involving the product $\tilde{A}_\nu\tilde{A}_\rho$ the following terms appear at order g_0^3:

$$\begin{aligned}&-i\frac{g_0^3}{4}\sum_{\nu\rho}\Big((\tilde{g}_{\mu\nu}(\varrho,k_1)\tilde{g}_{\mu\rho}(\varrho,k_2+k_3)+2g_{\mu\nu\rho}(k_1,k_2+k_3))T^a[T^b,T^c]A_\nu^{(n)a}(k_1)A_\rho^{(n)[bc]}(k_2,k_3)\\ &+(\tilde{g}_{\mu\nu}(\varrho,k_1+k_2)\tilde{g}_{\mu\rho}(\varrho,k_3)+2g_{\mu\nu\rho}(k_1+k_2,k_3))[T^a,T^b]T^cA_\nu^{(n)[ab]}(k_1,k_2)A_\rho^{(n)c}(k_3)\Big)\\ &=-i\frac{g_0^3}{6}\bigg(\frac{3}{2}\{T^a,[T^b,T^c]\}\sum_{\nu\rho}\tilde{g}_{\mu\nu}(\varrho,k_1)\tilde{g}_{\mu\rho}(\varrho,k_2+k_3)A_\nu^{(n)a}(k_1)A_\rho^{(n)[bc]}(k_2,k_3)\\ &\quad+3[T^a,[T^b,T^c]]\sum_{\nu\rho}g_{\mu\nu\rho}(k_1,k_2+k_3)A_\nu^{(n)a}(k_1)A_\rho^{(n)[bc]}(k_2,k_3)\bigg).\end{aligned} \tag{122}$$

Then we have for our two third-order fields $\bar{A}$ and $\hat{A}$:

$$\begin{aligned}\bar{A}_\mu^{(n+1)abc}(k_1,k_2,k_3) = &\sum_\nu \tilde{g}_{\mu\nu}(\varrho, k_1+k_2+k_3)\bar{A}_\nu^{(n)abc}(k_1,k_2,k_3) + 3\varrho\sum_{\nu\rho} g_{\mu\nu\rho}(k_1,k_2+k_3)A_\nu^{(n)a}(k_1)A_\rho^{(n)[bc]}(k_2,k_3) \\ &-\frac{3}{2}\varrho\sum_{\nu\rho\sigma}\delta_{\mu\nu}g_{\mu\rho\sigma}(k_2,k_3)A_\nu^{(n)a}(k_1)A_\rho^{(n)b}(k_2)A_\sigma^{(n)c}(k_3) \\ &+\frac{1}{2}A_\mu^{(n)a}(k_1)A_\mu^{(n+1)b}(k_2)A_\mu^{(n)c}(k_3) + \frac{1}{2}A_\mu^{(n+1)a}(k_1)A_\mu^{(n+1)b}(k_2)A_\mu^{(n)c}(k_3),\end{aligned} \tag{123}$$

$$\begin{aligned}\hat{A}_\mu^{(n+1)abc}(k_1,k_2,k_3) = &\sum_\nu \tilde{g}_{\mu\nu}(\varrho, k_1+k_2+k_3)\hat{A}_\nu^{(n)abc}(k_1,k_2,k_3) \\ &\times\varrho\sum_{\nu\rho\sigma}\Bigg(3(h_{\mu\nu\rho}(k_1,k_2)+g_{\mu\nu\rho}(k_1,k_2))\delta_{\mu\sigma} + \frac{3}{2}g_{\mu\nu}(k_1)\delta_{\mu\rho}\delta_{\mu\sigma} \\ &+\frac{3}{2}\left(g_{\mu\nu}^{(i)}(k_1,k_2)\delta_{\nu\rho}\delta_{\mu\sigma} + g_{\mu\nu\rho}^{(ii)}(k_1,k_2+k_3)\delta_{\rho\sigma} + g_{\mu\nu\rho\sigma}^{(iii)}(k_1,k_2,k_3)\right) \\ &+\frac{1}{2}g_{\mu\nu}(k_1+k_2+k_3)\delta_{\nu\rho}\delta_{\nu\sigma}\Bigg)A_\nu^{(n)a}(k_1)A_\rho^{(n)b}(k_2)A_\sigma^{(n)c}(k_3),\end{aligned} \tag{124}$$

and finally

$$A_\mu^{(n+1)abc}(k_1,k_2,k_3) = \hat{A}_\mu^{(n+1)abc}(k_1,k_2,k_3) + 2\bar{A}_\mu^{(n+1)a[bc]}(k_1,k_2,k_3) \tag{125}$$

$$\begin{aligned}= &\sum_\nu \tilde{g}_{\mu\nu}(\varrho, k_1+k_2+k_3)A_\nu^{(n)abc}(k_1,k_2,k_3) + 6\varrho\sum_{\nu\rho} g_{\mu\nu\rho}(k_1,k_2+k_3)A_\nu^{(n)a}(k_1)A_\rho^{(n)[bc]}(k_2,k_3) \\ &+\sum_{\nu\rho\sigma}\left(\frac{1}{2}\varrho^2\left(g_{\mu\nu}(k_1)g_{\mu\rho}(k_2)\delta_{\mu\sigma} - g_{\mu\nu}(k_1)\delta_{\mu\rho}g_{\mu\sigma}(k_3)\right) + \varrho G_{\mu\nu\rho\sigma}(k_1,k_2,k_3)\right) \\ &\times A_\nu^{(n)a}(k_1)A_\rho^{(n)b}(k_2)A_\sigma^{(n)c}(k_3)\end{aligned} \tag{126}$$

with

$$\begin{aligned}G_{\mu\nu\rho\sigma}(k_1,k_2,k_3) = &\frac{1}{2}g_{\mu\nu}(k_1)\delta_{\mu\rho}\delta_{\mu\sigma} + \delta_{\mu\nu}g_{\mu\rho}(k_2)\delta_{\mu\sigma} + 3h_{\mu\nu\rho}(k_1,k_2)\delta_{\mu\sigma} \\ &+\frac{3}{2}\left(g_{\mu\nu}^{(i)}(k_1,k_2)\delta_{\nu\rho}\delta_{\mu\sigma} + g_{\mu\nu\rho}^{(ii)}(k_1,k_2+k_3)\delta_{\rho\sigma} + g_{\mu\nu\rho\sigma}^{(iii)}(k_1,k_2,k_3)\right) + \frac{1}{2}g_{\mu\nu}(k_1+k_2+k_3)\delta_{\nu\rho}\delta_{\nu\sigma}.\end{aligned} \tag{127}$$

After two smearing steps:

$$\begin{aligned}A_\mu^{(2)abc}(k_1,k_2,k_3) = &\sum_\alpha \tilde{g}_{\mu\alpha}(\varrho, k_1+k_2+k_3)\sum_{\nu\rho\sigma} G_{\alpha\nu\rho\sigma}(\varrho,k_1,k_2,k_3)A_\nu^{(0)a}(k_1)A_\rho^{(0)b}(k_2)A_\sigma^{(0)c}(k_3) \\ &+6\varrho\sum_{\nu\rho} g_{\mu\nu\rho}(k_1,k_2+k_3)A_\nu^{(1)a}(k_1)A_\rho^{(1)[bc]}(k_2,k_3) + \sum_{\nu\rho\sigma}\tilde{G}_{\mu\nu\rho}(\varrho,k_1,k_2,k_3)A_\nu^{(1)a}(k_1)A_\rho^{(1)b}(k_2)A_\sigma^{(1)c}(k_3).\end{aligned} \tag{128}$$

After three smearing steps:

$$
\begin{aligned}
A_\mu^{(3)abc}(k_1,k_2,k_3) = &\sum_\alpha \tilde{g}^2_{\mu\alpha}(\varrho,k_1+k_2+k_3)\sum_{\nu\rho\sigma}\tilde{G}_{\alpha\nu\rho\sigma}(\varrho,k_1,k_2,k_3)A_\nu^{(0)a}(k_1)A_\rho^{(0)b}(k_2)A_\sigma^{(0)c}(k_3)\\
&+\sum_\alpha \tilde{g}_{\mu\alpha}(\varrho,k_1+k_2+k_3)\sum_{\nu\rho\sigma}\tilde{G}_{\alpha\nu\rho\sigma}(\varrho,k_1,k_2,k_3)A_\nu^{(1)a}(k_1)A_\rho^{(1)b}(k_2)A_\sigma^{(1)c}(k_3)\\
&+\tilde{G}_{\mu\nu\rho}(\varrho,k_1,k_2,k_3)A_\nu^{(2)a}(k_1)A_\rho^{(2)b}(k_2)A_\sigma^{(2)c}(k_3)\\
&+6\varrho\sum_\alpha \tilde{g}_{\mu\alpha}(\varrho,k_1+k_2+k_3)\sum_{\nu\rho} g_{\alpha\nu\rho}(k_1,k_2+k_3)A_\nu^{(1)a}(k_1)A_\rho^{(1)[bc]}(k_2,k_3)\\
&+6\varrho\sum_{\nu\rho} g_{\mu\nu\rho}(k_1,k_2+k_3)A_\nu^{(2)a}(k_1)A_\rho^{(2)[bc]}(k_2,k_3). \qquad (129)
\end{aligned}
$$

After n smearing steps:

$$
\begin{aligned}
A_\mu^{(n)abc}(k_1,k_2,k_3) = &\,6\varrho\sum_{\alpha\nu\rho} g_{\alpha\nu\rho}(k_1,k_2+k_3)\sum_{m=1}^{n-1}\tilde{g}^{n-m-1}_{\mu\alpha}(\rho,k_1+k_2+k_3)A_\nu^{(m)a}(k_1)A_\rho^{(m)[bc]}(k_2,k_3)\\
&+\sum_{\alpha\nu\rho\sigma}\left(\frac{1}{2}\varrho^2\Big(g_{\alpha\nu}(k_1)g_{\mu\rho}(k_2)\delta_{\mu\sigma}-g_{\mu\nu}(k_1)\delta_{\mu\rho}g_{\mu\sigma}(k_3)\Big)+\varrho G_{\alpha\nu\rho\sigma}(k_1,k_2,k_3)\right)\\
&\times\sum_{m=0}^{n-1}\tilde{g}^{n-m-1}_{\mu\alpha}(\varrho,k_1+k_2+k_3)A_\nu^{(m)a}(k_1)A_\rho^{(m)b}(k_2)A_\sigma^{(m)c}(k_3). \qquad (130)
\end{aligned}
$$

Plugging in the relations from leading order and next-to-leading order, we express the end result as a function of the unsmeared fields:

$$
\begin{aligned}
A_\mu^{(n)abc}(k_1,k_2,k_3) = &\sum_{\nu\rho\sigma}\left(\sum_{\alpha\beta\gamma}6\varrho g_{\alpha\beta\gamma}(k_1,k_2+k_3)\sum_{m=1}^{n-1}\tilde{g}^{n-m-1}_{\mu\alpha}(\varrho,k_1+k_2+k_3)\tilde{g}^m_{\beta\nu}(\varrho,k_1)\tilde{g}^{(m)}_{\gamma\rho\sigma}(\varrho,k_2,k_3)\right.\\
&+\sum_{\alpha\beta\gamma\delta}\left(\frac{1}{2}\varrho^2\Big(g_{\alpha\beta}(k_1)g_{\alpha\gamma}(k_2)\delta_{\alpha\delta}-g_{\alpha\beta}(k_1)\delta_{\alpha\gamma}g_{\alpha\delta}(k_3)\Big)+\varrho G_{\alpha\beta\gamma\delta}(k_1,k_2,k_3)\right)\\
&\left.\times\sum_{m=0}^{n-1}\tilde{g}^{n-m-1}_{\mu\alpha}(\varrho,k_1+k_2+k_3)\tilde{g}^m_{\beta\nu}(\varrho,k_1)\tilde{g}^m_{\gamma\rho}(\varrho,k_2)\tilde{g}^m_{\delta\sigma}(\varrho,k_3)\right)A_\nu^{(0)a}(k_1)A_\rho^{(0)b}(k_2)A_\sigma^{(0)c}(k_3)\\
=:&\sum_{\nu\rho\sigma}\tilde{g}^{(n)}_{\mu\nu\rho\sigma}(\varrho,k_1,k_2,k_3)A_\nu^{(0)a}(k_1)A_\rho^{(0)b}(k_2)A_\sigma^{(0)c}(k_3), \qquad (131)
\end{aligned}
$$

and we have found the last of the three $\tilde{g}$ functions from Eq. (32).

III. PERTURBATIVE EXPANSION OF THE WILSON FLOW

The Wilson flow is defined through

$$\partial_t U_\mu(x,t) = iQ_\mu(x,t)U_\mu(x,t), \qquad (132)$$

$$U_\mu(x,0) = U_\mu(x), \qquad (133)$$

with the same Q_μ as in the definition of stout smearing but made up from the "flowed" fields $U_\mu(x,t)$ instead of the smeared fields. We already know the perturbative expansion of Q_μ,

$$Q_\mu(x,t) = g_0T^aQ^a_{1\mu}(x,t)+ig_0^2[T^a,T^b]Q^{ab}_{2\mu}(x,t)+g_0^3\left(T^aT^bT^c+T^cT^bT^a-\frac{1}{N_c}\mathrm{Tr}[T^aT^bT^c+T^cT^bT^a]\right)Q^{abc}_{3\mu}(x,t). \qquad (134)$$

We define the expansion of the flowed link variable in terms of fields $A_\mu^a(x,t)$, $A_\mu^{[ab]}(x,t)$, and $A_\mu^{abc}(x,t)$ such that similarly to the smeared case

$$\begin{aligned}U_\mu(x,t) &= 1 + ig_0 T^a A_\mu^a(x,t) - \frac{g_0^2}{2}\left(T^aT^bA_\mu^a(x,t)A_\mu^b(x,t) + [T^a,T^b]A_\mu^{[ab]}(x,t)\right)\\ &\quad - i\frac{g_0^3}{6}\left(T^aT^bT^cA_\mu^a(x,t)A_\mu^b(x,t)A_\mu^c(x,t) + \frac{3}{2}\{T^a,[T^b,T^c]\}A_\mu^a(x,t)A_\mu^{[ab]}(x,t)\right.\\ &\quad \left. + \left(T^aT^bT^c + T^cT^bT^a - \frac{1}{N_c}\mathrm{Tr}[T^aT^bT^c + T^cT^bT^a]\right)A_\mu^{abc}(x,t)\right) + \mathcal{O}(g_0^4). \qquad (135)\end{aligned}$$

They fulfil the initial conditions

$$A_\mu^a(x,0) = A_\mu^a(x), \qquad A_\mu^{[ab]}(x,0) = 0, \qquad A_\mu^{abc}(x,0) = 0, \qquad (136)$$

and we define their Fourier transforms consistently:

$$A_\mu^a(x,t) = \int_{-\pi}^{\pi}\frac{\mathrm{d}^4k}{(2\pi)^4}e^{i(x+\hat\mu/2)k}A_\mu^a(k,t), \qquad (137)$$

$$A_\mu^{ab}(x,t) = \int_{-\pi}^{\pi}\frac{\mathrm{d}^4k_1}{(2\pi)^4}\int_{-\pi}^{\pi}\frac{\mathrm{d}^4k_2}{(2\pi)^4}e^{i(x+\hat\mu/2)(k_1+k_2)}A_\mu^{ab}(k_1,k_2,t), \qquad (138)$$

$$A_\mu^{abc}(x,t) = \int_{-\pi}^{\pi}\frac{\mathrm{d}^4k_1}{(2\pi)^4}\int_{-\pi}^{\pi}\frac{\mathrm{d}^4k_2}{(2\pi)^4}\int_{-\pi}^{\pi}\frac{\mathrm{d}^4k_3}{(2\pi)^4}e^{i(x+\hat\mu/2)(k_1+k_2+k_3)}A_\mu^{abc}(k_1,k_2,k_3,t). \qquad (139)$$

As in the stout smearing case, we can connect the perturbative fields to the flowed gluon field $\tilde{A}_\mu^a(g_0,x,t)$ in the $SU(N_c)$ expansion such that

$$U_\mu(x,t) = e^{ig_0T^a\tilde{A}_\mu^a(g_0,x,t)} \qquad (140)$$

and

$$\tilde{A}_\mu^a(g_0,x,t) = A_\mu^a(x,t) - \frac{g_0}{2}f^{abc}A_\mu^{[bc]}(x,t) - \frac{g_0^2}{6}\left[\frac{1}{N_c}\delta^{ae}\delta^{bc} + \frac{1}{2}(d^{bcd}d^{eda} - f^{bcd}f^{eda})\right]A_\mu^{ebc}(x,t) + \mathcal{O}(g_0^3). \qquad (141)$$

A. Leading order

At leading order the differential equation is

$$\partial_t A_\mu^a(x,t) = Q_{1\mu}^a(x,t) = \sum_{\nu,y} g_{\mu\nu}(y)A_\nu^a(x+y,t) \qquad (142)$$

which in momentum space becomes

$$\partial_t A_\mu^a(k,t) = \sum_\nu g_{\mu\nu}(k)A_\nu^a(k,t) \qquad (143)$$

with the same $g_{\mu\nu}$ as defined at leading order of the stout smearing. With

$$g_{\mu\nu}^2(k) = \sum_\alpha g_{\mu\alpha}(k)g_{\alpha\nu}(k) = -\hat{k}^2 g_{\mu\nu}(k), \qquad (144)$$

the matrix exponential of g is

$$\begin{aligned}\exp(g(k)t)_{\mu\nu} &= \delta_{\mu\nu} + g_{\mu\nu}t + \frac{1}{2}g_{\mu\nu}^2t^2 + \cdots\\ &= \delta_{\mu\nu} - \frac{1}{\hat{k}^2}\left(e^{-\hat{k}^2t} - 1\right)g_{\mu\nu}(k). \qquad (145)\end{aligned}$$

With the initial condition for $t = 0$, we arrive at

$$A^a_\mu(k,t) = \sum_\nu \left(e^{-\hat{k}^2 t}\delta_{\mu\nu} - \left(e^{-\hat{k}^2 t} - 1\right)\frac{\hat{k}_\mu \hat{k}_\nu}{\hat{k}^2}\right) A^a_\nu(k,0) \tag{146}$$

$$=: \sum_\nu B_{\mu\nu}(k,t) A^a_\nu(k,0). \tag{147}$$

It is easy to see that for an infinitesimal flow-time $t = \epsilon$, the exponentials become $e^{-\hat{k}^2\epsilon} = 1 - \epsilon\hat{k}^2 + \mathcal{O}(\epsilon^2)$ and the flow transformation becomes the same as a stout smearing with smearing parameter $\varrho = \epsilon$, reiterating the fact that the Wilson flow is generated by infinitesimal stout smearing.

The same result is obtained by starting from n stout smearings and taking the limit $n \to \infty$ while keeping the product $n\varrho =: t$ constant. Then $f(k)^n = (1 - \frac{t}{n}\hat{k}^2)^n \to e^{-\hat{k}^2 t}$ [38]. Moreover, this means for a small flow time $t < 1$:

$$B_{\mu\nu}(k,t) = \tilde{g}_{\mu\nu}(t,k) + \frac{t^2}{2}(\hat{k}^2)^2 g_{\mu\nu}(k) + \mathcal{O}(t^3) \tag{148}$$

$$= \tilde{g}^2_{\mu\nu}(t/2,k) + \frac{t^2}{4}(\hat{k}^2)^2 g_{\mu\nu}(k) + \mathcal{O}(t^3) \tag{149}$$

$$= \tilde{g}^n_{\mu\nu}(t/n,k) + \frac{t^2}{2n}(\hat{k}^2)^2 g_{\mu\nu}(k) + \mathcal{O}(t^3). \tag{150}$$

Thus the error we get by approximating the flowed field by the field which has been smeared n times with smearing parameter $\varrho = t/n$ is always of the order t^2 but decreases with $1/n$.

B. Next-to-leading order

At order g_0^2, the differential equation is

$$T^a T^b \partial_t(A^a_\mu(x,t)A^b_\mu(x,t)) + [T^a,T^b]\partial_t A^{[ab]}_\mu(x) = 2T^a T^b Q^a_{1\mu}(x,t)A^b_\mu(x,t) + 2[T^a,T^b]Q^{ab}_{2\mu}(x,t). \tag{151}$$

With

$$Q^{ab}_{2\mu}(x,t) = \frac{1}{2}\left(-S^a_{1\mu}(x,t)A^b_\mu(x,t) + S^{ab}_{2\mu}(x,t) - 6A^{[ab]}_\mu(x,t)\right) \tag{152}$$

and the result from the leading order, we arrive at

$$\partial_t A^{[ab]}_\mu(x) = S^{[ab]}_{2\mu}(x,t) - 6A^{[ab]}_\mu(x,t) \tag{153}$$

$$= \sum_{\nu,y} g_{\mu\nu}(y)A^{[ab]}_\nu(x+y,t) + \sum_{\nu\rho,yz} g_{\mu\nu\rho}(y,z)A^a_\nu(x+y,t)A^b_\rho(x+z,t). \tag{154}$$

In momentum space, we have

$$\partial_t A^{ab}_\mu(k_1,k_2,t) = \sum_\nu g_{\mu\nu}(k_1+k_2)A^{[ab]}_\nu(k_1,k_2,t) + \sum_{\nu\rho} g_{\mu\nu\rho}(k_1,k_2)A^a_\nu(k_1,t)A^b_\rho(k_2,t) \tag{155}$$

with the same $g_{\mu\nu\rho}$ we know from stout smearing. Now we have an inhomogenous differential equation and make the ansatz

$$A^{[ab]}_\mu(k_1,k_2,t) = \sum_\alpha B_{\mu\alpha}(k_1+k_2,t)C^{[ab]}_\alpha(k_1,k_2,t) \tag{156}$$

as we know that $B_{\mu\nu}$ solves the homogeneous part. For the inhomogeneous part we need to solve

$$\sum_\alpha B_{\mu\alpha}(k_1+k_2,t)\partial_t C^{[ab]}_\alpha(k_1,k_2,t) = \sum_{\nu\rho} g_{\mu\nu\rho}(k_1,k_2)A^a_\nu(k_1,t)A^b_\rho(k_2,t), \tag{157}$$

or equivalently

$$\partial_t C^{[ab]}_\mu(k_1,k_2,t) = \sum_\alpha B_{\mu\alpha}(k_1+k_2,-t)\sum_{\nu\rho} g_{\alpha\nu\rho}(k_1,k_2)A^a_\nu(k_1,t)A^b_\rho(k_2,t), \tag{158}$$

because $(B_{\mu\nu}(k,t))^{-1} = B_{\mu\nu}(k,-t)$. Thus

$$C^{[ab]}_\mu(k_1,k_2,t) = \sum_{\alpha\nu\rho} g_{\alpha\nu\rho}(k_1,k_2)\int_0^t B_{\mu\alpha}(k_1+k_2,-t')A^a_\nu(k_1,t')A^b_\rho(k_2,t')dt' \tag{159}$$

and

$$A_\mu^{[ab]}(k_1,k_2,t)=\sum_{\alpha\beta\nu\rho}B_{\mu\alpha}(k_1+k_2,t)\int_0^t B_{\alpha\beta}(k_1+k_2,-t')g_{\beta\nu\rho}(k_1,k_2)A_\nu^a(k_1,t')A_\rho^b(k_2,t')dt'$$
$$=\sum_{\alpha\nu\rho}g_{\alpha\nu\rho}(k_1,k_2)\int_0^t B_{\mu\alpha}(k_1+k_2,t-t')A_\nu^a(k_1,t')A_\rho^b(k_2,t')dt', \tag{160}$$

where we have used $\sum_\alpha B_{\mu\alpha}(k,t_1)B_{\alpha\nu}(k,t_2)=B_{\mu\nu}(k,t_1+t_2)$. We can also go further and express $A_\mu^{[ab]}$ in terms of the original gluon fields

$$A_\mu^{[ab]}(k_1,k_2,t)=\sum_{\alpha\beta\gamma\nu\rho}g_{\alpha\beta\gamma}(k_1,k_2)\left[\int_0^t B_{\mu\alpha}(k_1+k_2,t-t')B_{\beta\nu}(k_1,t')B_{\gamma\rho}(k_2,t')dt'\right]A_\nu^a(k_1,0)A_\rho^b(k_2,0)$$
$$=:\sum_{\nu\rho}B_{\mu\nu\rho}(k_1,k_2,t)A_\nu^a(k_1,0)A_\rho^b(k_2,0). \tag{161}$$

Again, we can check that we recover $A_\mu^{(1)[ab]}(k_1,k_2)$ for an infinitesimal flow time $t=\epsilon$

$$A_\mu^{[ab]}(k_1,k_2,\epsilon)=\sum_{\alpha\beta\gamma\nu\rho}g_{\alpha\beta\gamma\delta}(k_1,k_2)\left[\epsilon B_{\mu\alpha}(k_1+k_2,0)B_{\beta\nu}(k_1,0)B_{\gamma\rho}(k_2,0)\right]A_\nu^a(k_1,0)A_\rho^b(k_2,0) \tag{162}$$

$$=\sum_{\nu\rho}\epsilon g_{\mu\nu\rho}(k_1,k_2)A_\nu^a(k_1,0)A_\rho^b(k_2,0)=A_\mu^{(1)[ab]}(k_1,k_2)|_{\rho=\epsilon}. \tag{163}$$

If we expand further (for small $t<1$), we get

$$B_{\mu\nu\rho}(k_1,k_2,t)=\sum_{\alpha\beta\gamma}g_{\alpha\beta\gamma}(k_1,k_2)\left(\delta_{\mu\alpha}\delta_{\beta\nu}\delta_{\gamma\rho}t+\frac{1}{2}(g_{\mu\alpha}(k_1+k_2)\delta_{\beta\nu}\delta_{\gamma\rho}+\delta_{\mu\alpha}g_{\beta\nu}(k_1)\delta_{\gamma\rho}+\delta_{\mu\alpha}\delta_{\beta\nu}g_{\gamma\rho}(k_2))t^2+\mathcal{O}(t^3)\right) \tag{164}$$

$$=\tilde{g}^{(1)}_{\mu\nu\rho}(t,k_1,k_2)+\frac{t^2}{2}(\cdots)+\mathcal{O}(t^3) \tag{165}$$

$$=\tilde{g}^{(n)}_{\mu\nu\rho}(t/n,k_1,k_2)+\frac{t^2}{2n}(\cdots)+\mathcal{O}(t^3) \tag{166}$$

which shows the same behavior as the first order fields.

C. Next-to-next-to-leading order

The differential equation at third order is

$$-i\frac{g_0^3}{6}\left[T^aT^bT^c\partial_t(A_\mu^a(x,t)A_\mu^b(x,t)A_\mu^c(x,t))+\frac{3}{2}\{T^a,[T^b,T^c]\}\partial_t(A_\mu^a(x,t)A_\mu^{[bc]}(x,t))\right.$$
$$\left.+\left(T^aT^bT^c+T^cT^bT^a-\frac{1}{N_c}\mathrm{Tr}(T^aT^bT^c+T^cT^bT^a)\right)\partial_tA_\mu^{abc}(x,t)\right]$$
$$=-ig_0^3\left[\frac{1}{2}T^aT^bT^cQ_{1\mu}^a(x,t)A_\mu^b(x,t)A_\mu^c(x,t)+\frac{1}{2}T^a[T^b,T^c]Q_{1\mu}^a(x,t)A_\mu^{[bc]}(x,t)\right.$$
$$\left.+[T^a,T^b]T^cQ_{2\mu}^{ab}(x,t)A_\mu^c(x,t)-\left(T^aT^bT^c+T^cT^bT^a-\frac{1}{N_c}\mathrm{Tr}(T^aT^bT^c+T^cT^bT^a)\right)Q_{3\mu}^{abc}(x,t)\right]. \tag{167}$$

The right-hand side can be rewritten to look like

$$
\begin{aligned}
&- i\frac{g_0^3}{6}\bigg(T^aT^bT^c\partial_t(A_\mu^a(x,t)A_\mu^b(x,t)A_\mu^c(x,t)) + \frac{3}{2}\{T^a[T^b,T^c]\}\partial_t(A_\mu^a(x,t)A_\mu^{[bc]}(x,t))\\
&- \bigg(T^aT^bT^c + T^cT^bT^a - \frac{1}{N_c}\mathrm{Tr}(T^aT^bT^c + T^cT^bT^a)\bigg)\bigg(-\frac{1}{2}Q_{1\mu}^a(x,t)A_\mu^b(x,t)A_\mu^c(x,t)\\
&- A^aQ_{1\mu}^bA_\mu^c(x,t) + 3V_{2\mu}^{\{ab\}}(x,t)A_\mu^c(x,t) - 3V_{3\mu}^{abc}(x,t) - 6A_\mu^a(x,t)A_\mu^b(x,t)A_\mu^c(x,t) + 6A_\mu^{abc}(x,t)\bigg)\bigg). \qquad (168)
\end{aligned}
$$

Thus

$$
\begin{aligned}
\partial_tA_\mu^{abc}(x,t) &= \frac{1}{2}Q_{1\mu}^a(x,t)A_\mu^b(x,t)A_\mu^c(x,t) + A^aQ_{1\mu}^bA_\mu^c(x,t) - 3V_{2\mu}^{\{ab\}}(x,t)A_\mu^c(x,t)\\
&\quad + 3V_{3\mu}^{abc}(x,t) + 6A_\mu^a(x,t)A_\mu^b(x,t)A_\mu^c(x,t) - 6A_\mu^{abc}(x,t). \qquad (169)
\end{aligned}
$$

In momentum space, we get

$$
\begin{aligned}
\partial_tA_\mu^{abc}(k_1,k_2,k_3,t) &= \sum_\nu g_{\mu\nu}(k_1+k_2+k_3)A_\nu^{abc}(k_1,k_2,k_3,t) + 6\sum_{\nu\rho}g_{\mu\nu\rho}(k_1,k_2+k_3)A_\nu^a(k_1,t)A_\rho^{[bc]}(k_2,k_3,t)\\
&\quad + \sum_{\nu\rho\sigma}G_{\mu\nu\rho\sigma}(k_1,k_2,k_3)A_\nu^a(k_1,t)A_\rho^b(k_2,t)A_\sigma^c(k_3,t). \qquad (170)
\end{aligned}
$$

We make again the following ansatz

$$
A_\mu^{abc}(k_1,k_2,k_3,t) = \sum_\alpha B_{\mu\alpha}(k_1+k_2+k_3,t)C_\alpha^{abc}(k_1,k_2,k_3,t) \qquad (171)
$$

as we know from the leading order that $B_{\mu\nu}(k_1+k_2+k_3,t)$ satisfies the homogeneous part of the differential equation. Then we are left with the inhomogeneous part

$$
\begin{aligned}
\sum_\alpha B_{\mu\alpha}(k_1+k_2+k_3,t)\partial_tC_\alpha^{abc}(k_1,k_2,k_3,t) &= 6\sum_{\nu\rho}g_{\mu\nu\rho}(k_1,k_2+k_3)A_\nu^a(k_1,t)A_\rho^{[bc]}(k_2,k_3,t)\\
&\quad + \sum_{\nu\rho\sigma}G_{\mu\nu\rho\sigma}(k_1,k_2,k_3)A_\nu^a(k_1,t)A_\rho^b(k_2,t)A_\sigma^c(k_3,t), \qquad (172)
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
\partial_tC_\alpha^{abc}(k_1,k_2,k_3,t) &= 6\sum_{\alpha\nu\rho}B_{\mu\alpha}(k_1+k_2+k_3,-t)g_{\alpha\nu\rho}(k_1,k_2+k_3)A_\nu^a(k_1,t)A_\rho^{[bc]}(k_2,k_3,t)\\
&\quad + \sum_{\alpha\nu\rho\sigma}B_{\mu\alpha}(k_1+k_2+k_3,-t)G_{\alpha\nu\rho\sigma}(k_1,k_2,k_3)A_\nu^a(k_1,t)A_\rho^b(k_2,t)A_\sigma^c(k_3,t). \qquad (173)
\end{aligned}
$$

Integrating this and putting everything together, we get

$$
\begin{aligned}
A_\mu^{abc}(k_1,k_2,k_3,t) &= \sum_{\alpha\nu\rho}g_{\alpha\nu\rho}(k_1,k_2+k_3)\int_0^t B_{\mu\alpha}(k_1+k_2+k_3,t-t')A_\nu^a(k_1,t')A_\rho^{[bc]}(k_2,k_3,t')dt'\\
&\quad + \sum_{\alpha\nu\rho\sigma}G_{\alpha\nu\rho\sigma}(k_1,k_2,k_3)\int_0^t B_{\mu\alpha}(k_1+k_2+k_3,t-t')A_\nu^a(k_1,t')A_\rho^b(k_2,t')A_\sigma^c(k_3,t')dt' \qquad (174)
\end{aligned}
$$

and we can define a function $B_{\mu\nu\rho\sigma}(k_1,k_2,k_3,t)$ for the third order of the Wilson flow:

$$
\begin{aligned}
A^{abc}_{\mu}(k_1,k_2,k_3,t) = \sum_{\nu\rho\sigma}\Bigg(&\sum_{\alpha\beta\gamma} g_{\alpha\beta\gamma}(k_1,k_2+k_3)\int_0^t B_{\mu\alpha}(k_1+k_2+k_3,t-t')B_{\beta\nu}(k_1,t')B_{\gamma\rho\sigma}(k_2,k_3,t')dt' \\
&+\sum_{\alpha\beta\gamma\delta} G_{\alpha\beta\gamma\delta}(k_1,k_2,k_3)\int_0^t B_{\mu\alpha}(k_1+k_2+k_3,t-t')B_{\beta\nu}(k_1,t')B_{\gamma\rho}(k_2,t')B_{\delta_\sigma}(k_3,t')dt'\Bigg) \\
&\times A^a_\nu(k_1,0)A^b_\rho(k_2,0)A^c_\sigma(k_3,0) \\
=: \sum_{\nu\rho\sigma} & B_{\mu\nu\rho\sigma}(k_1,k_2,k_3,t)A^a_\nu(k_1,0)A^b_\rho(k_2,0)A^c_\sigma(k_3,0).
\end{aligned} \tag{175}
$$

Again, we can check that the error one makes by approximating the Wilson flow by stout smearing is of the order $t^2/(2n)$:

$$
B_{\mu\nu\rho\sigma}(k_1,k_2,k_3,t) = \tilde{g}^{(n)}_{\mu\nu\rho\sigma}(t/n,k_1,k_2,k_3) + \frac{t^2}{2n}(\cdots). \tag{176}
$$

Thus we have found three B functions that are the Wilson flow equivalent to the $\tilde{g}$ functions from the expansion of stout smearing.

D. Continuum limit

In the continuum limit $a \to 0$, we replace $\cos(k_\mu) \to 1$ and $\sin(k_\mu) \to k_\mu$, and expand to first order in the momenta. We get

$$
B_{\mu\nu}(k,t) \to \sum_\nu \left(e^{-k^2t}\delta_{\mu\nu} - \left(e^{-k^2t}-1\right)\frac{k_\mu k_\nu}{k^2}\right), \tag{177}
$$

$$
g_{\mu\nu\rho}(k_1,k_2) \to i\Big(-\delta_{\beta\nu}(2k_{1\rho}+k_{2\rho}) + \delta_{\beta\rho}(2k_{2\nu}+k_{1\nu}) + \delta_{\nu\rho}(k_{1\mu}-k_{2\mu})\Big), \tag{178}
$$

$$
G_{\mu\nu\rho\sigma}(k_1,k_2,k_3) \to 6(\delta_{\mu\nu}\delta_{\rho\sigma} - \delta_{\mu\rho}\delta_{\nu\sigma}), \tag{179}
$$

and the flowed field $\tilde{A}_\mu$ from Eq. (141) in this limit fulfils the differential equation of the continuum gradient flow

$$
\partial_t\tilde{A}_\mu(x,t) = \sum_\nu \tilde{D}_\nu\tilde{F}_{\nu\mu}, \qquad \tilde{A}_\mu(x,0) = A_\mu(x) \tag{180}
$$

with

$$
\begin{aligned}
\tilde{D}_\mu &= \partial_\mu + [\tilde{A}_\mu(x,t),\cdot], \\
\tilde{F}_{\mu\nu} &= \partial_\mu\tilde{A}_\nu(x,t) - \partial_\nu\tilde{A}_\mu(x,t) + [\tilde{A}_\mu(x,t),\tilde{A}_\nu(x,t)].
\end{aligned} \tag{181}
$$

These results agree with the ones given in Refs. [13,14].

IV. FEYNMAN RULES OF FERMION ACTIONS WITH STOUT SMEARING OR WILSON FLOW

The Feynman rules are obtained by inserting the expansion of the link variables (182) into the Fermion action. This results in expressions for antiquark-quark-n-gluon vertices at each order g_0^n. These vertices are proportional to a^{n-1} such that only the $\bar{q}qg$ vertex survives in the continuum limit $a \to 0$, as is expected from continuum perturbation theory. For the purpose of one-loop calculations, we will need the $\bar{q}qg$-, $\bar{q}qgg$- and $\bar{q}qggg$-vertex, i.e. anything up to order g_0^3.

For deriving the Feynman rules of the fermion action including stout smearing or Wilson flow, there are two possible strategies. One is to simply insert the perturbative expansion of the smeared or flowed link variable into the action and expand to the needed order in g_0. This "brute force" approach leads to very large intermediate expressions and involves many steps where indices have to be renamed etc. This is not feasible beyond one step of stout smearing. The second much more effective method is based on the $SU(N_c)$ expansion of the link variable. One simply derives the Feynman rules for generic link variables of the form

$$
\begin{aligned}
U_\mu(x) = 1 + ig_0T^aA^a_\mu(x) - \frac{g_0^2}{2}T^aT^bA^a_\mu(x)A^b_\mu(x) \\
- i\frac{g_0^3}{6}T^aT^bT^cA^a_\mu(x)A^b_\mu(x)A^c_\mu(x)
\end{aligned} \tag{182}
$$

which results in fairly simple expressions. One then couples the stout or flow "form factors" to these generic Feynman rules to obtain the complete results. This also practically allows us to easily split lengthy expressions into smaller parts for *Mathematica* to work on. A similar procedure in the context of Feynman rules for the HISQ action has been described in Ref. [41]. The following sections show the derivations of the generic Feynman rules and how to couple them to stout smearing or the Wilson flow.

We start with the Wilson quark action

$$S_W = \sum_{x,y} \sum_{\mu=\pm 1}^{\pm 4} \bar{\psi}(x) \frac{1}{2} \left[U_\mu(x) \delta_{x+\hat{\mu},y} \gamma_\mu - r\left(U_\mu(x) \delta_{x+\hat{\mu},y} - \delta_{x,y} \right) \right] \psi(y). \tag{183}$$

Links with negative indices are given by $U_{-\mu}(x) = U_\mu^\dagger(x-\hat{\mu})$ and $\gamma_{-\mu} = -\gamma_\mu$. Up to order g_0^3 we get the well-known three vertices of a quark antiquark pair coupling to one, two, and three gluons respectively,

$$V^a_{1W\mu}(p,q) = -g_0 T^a \Big(i\gamma_\mu c(p_\mu + q_\mu) + rs(p_\mu + q_\mu) \Big), \tag{184}$$

$$V^{ab}_{2W\mu\nu}(p,q) = \frac{g_0^2}{2} T^a T^b \delta_{\mu\nu} \Big(i\gamma_\mu s(p_\mu + q_\mu) - rc(p_\mu + q_\mu) \Big), \tag{185}$$

$$V^{abc}_{3W\mu\nu\rho}(p,q) = \frac{g_0^3}{6} T^a T^b T^c \delta_{\mu\nu} \delta_{\mu\rho} \Big(i\gamma_\mu c(p_\mu + q_\mu) + rs(p_\mu + q_\mu) \Big). \tag{186}$$

These appear in the expansion of the action contracted with the (original) gluon fields

$$S_W \sim \sum_\mu \bar{\psi}(q) T^a V_{1\mu}(p,q) A^a_\mu(q-p) \psi(p) + \sum_{\mu\nu} \bar{\psi}(q) T^a T^b V_{2\mu\nu}(p,q) A^a_\mu(k_1) A^b_\nu(k_2) \delta(q-p-k_1-k_2) \psi(p) + \sum_{\mu\nu\rho} \bar{\psi}(q) T^a T^b T^c V_{3\mu\nu\rho}(p,q) A^a_\mu(k_1) A^b_\nu(k_2) A^c_\rho(k_3) \delta(q-p-k_1-k_2-k_3) \psi(p). \tag{187}$$

In order to derive the Feynman rules with smearing or gradient flow, we replace the gluon fields A^a_μ by $\tilde{A}^{(n)a}_\mu$ or $\tilde{A}^a_\mu(t)$ respectively. Thereby new terms are added at higher orders of g_0. For example, replacing the A^a_μ which couples to $V^a_{1\mu}$ by a smeared or flowed field provides the new $\bar{q}qg$ vertex with a form factor and adds terms to the $\bar{q}qgg$ and $\bar{q}qggg$ vertices proportional to $A^{[ab]}_\mu$ and A^{abc}_μ respectively. Specifically, the new stout smeared vertices look like

$$V^{(n)a}_{1\mu}(p,q) = T^a \sum_\nu \tilde{g}^n_{\mu\nu}(\varrho, q-p) V_{1\nu}(p,q), \tag{188}$$

$$V^{(n)ab}_{2\mu\nu}(p,q,k_1,k_2) = T^a T^b \sum_{\rho\sigma} \tilde{g}^n_{\mu\rho}(\varrho,k_1) \tilde{g}^n_{\nu\sigma}(\varrho,k_2) V_{2\rho\sigma}(p,q,k_1,k_2) - \frac{g_0}{2i} [T^a, T^b] \sum_\rho \tilde{g}^{(n)}_{\rho\mu\nu}(k_1,k_2) V_{1\rho}(p,q), \tag{189}$$

$$\begin{aligned} V^{(n)abc}_{3\mu\nu\rho}(p,q,k_1,k_2,k_3) = {} & T^a T^b T^c \sum_{\alpha\beta\gamma} \tilde{g}^n_{\mu\alpha}(\varrho,k_1) \tilde{g}^n_{\nu\beta}(\varrho,k_2) \tilde{g}^n_{\rho\gamma}(\varrho,k_3) V_{3\alpha\beta\gamma}(p,q,k_1,k_2,k_3) \\ & - \frac{g_0}{2i} \{T^a, [T^b, T^c]\} \sum_{\alpha\beta} \tilde{g}^n_{\mu\alpha}(\varrho,k_1) \tilde{g}^{(n)}_{\beta\nu\rho}(\varrho,k_2,k_3) V_{2\alpha\beta}(p,q,k_1,k_2+k_3) \\ & - \frac{g_0^2}{6} \left[T^a T^b T^c + T^c T^b T^a - \frac{1}{N_c} \mathrm{Tr}(T^a T^b T^c + T^c T^b T^a) \right] \sum_\alpha \tilde{g}^{(n)}_{\alpha\mu\nu\rho}(\varrho,k_1,k_2,k_3) V_{1\alpha}(p,q), \end{aligned} \tag{190}$$

with the $\tilde{g}$ given in Eqs. (56), (73), and (131). The vertices with Wilson flow are analogously

$$V^a_{1\mu}(p,q,t) = T^a \sum_\nu B_{\mu\nu}(q-p,t) V_{1\nu}(p,q), \tag{191}$$

$$V^{ab}_{2\mu\nu}(p,q,k_1,k_2,t) = T^a T^b \sum_{\rho\sigma} B_{\mu\rho}(k_1,t) B_{\nu\sigma}(k_2,t) V_{2\rho\sigma}(p,q,k_1,k_2) - \frac{g_0}{2i} [T^a, T^b] \sum_\rho B_{\rho\mu\nu}(k_1,k_2,t) V_{1\rho}(p,q) \tag{192}$$

$$V^{abc}_{3\mu\nu\rho}(p,q,k_1,k_2,k_3,t) = T^aT^bT^c\sum_{\alpha\beta\gamma}B_{\mu\alpha}(k_1,t)B_{\nu\beta}(k_2,t)B_{\rho\gamma}(k_3,t)V_{3\alpha\beta\gamma}(p,q,k_1,k_2,k_3)$$
$$-\frac{g_0}{2i}\{T^a,[T^b,T^c]\}\sum_{\alpha\beta}B_{\mu\alpha}(k_1,t)B_{\beta\nu\rho}(k_2,k_3,t)V_{2\alpha\beta}(p,q,k_1,k_2+k_3)$$
$$-\frac{g_0^2}{6}\left[T^aT^bT^c+T^cT^bT^a-\frac{1}{N_c}\mathrm{Tr}(T^aT^bT^c+T^cT^bT^a)\right]\sum_{\alpha}B_{\alpha\mu\nu\rho}(k_1,k_2,k_3,t)V_{1\alpha}(p,q), \quad (193)$$

with the B given in Eqs. (147), (161), and (175).

For the calculation of the self-energy below we need the $\bar{q}qg$- and $\bar{q}qgg$-vertices and we use the Feynman rules of the Wilson clover action

$$V_{1\mu}(p,q) = V_{1W\mu}(p,q) + V_{1C\mu}(p,q), \quad (194)$$

$$V_{2\mu\nu}(p,q,k_1,k_2) = V_{2W\mu\nu}(p,q) + V_{2C\mu\nu}(p,q,k_1,k_2) \quad (195)$$

with the clover vertices

$$V_{1C\mu}(p,q) = i\frac{g_0}{2}c_{\mathrm{SW}}\sum_{\nu}\sigma_{\mu\nu}c(p_\mu-q_\mu)\bar{s}(p_\nu-q_\nu), \quad (196)$$

$$V_{2C\mu\nu}(p,q,k_1,k_2) = ig_0^2c_{\mathrm{SW}}\left(\frac{1}{4}\delta_{\mu\nu}\sum_{\rho}\sigma_{\mu\rho}s(q_\mu-p_\mu)(\bar{s}(k_{2\rho})-\bar{s}(k_{1\rho}))\right.$$
$$\left.+\sigma_{\mu\nu}\left(c(k_{1\nu})c(k_{2\mu})c(q_\mu-p_\mu)c(p_\nu-p_\nu)-\frac{1}{2}c(k_{1\mu})c(k_{2\nu})\right)\right), \quad (197)$$

where $\sigma_{\mu\nu}=\frac{i}{2}[\gamma_\mu,\gamma_\nu]$ and the tree-level value $c_{\mathrm{SW}}=r$.

V. FERMION SELF-ENERGY WITH STOUT SMEARING OR WILSON FLOW

To illustrate the use of the smeared Feynman rules derived in the previous section, we give in this section the result of the self-energy of a clover fermion coupled to a gluon background through stout-smeared or gradient-flowed vertices. The smoothing thus affects both the covariant derivative (in the unimproved part of the fermion action) and the field strength tensor (in the part proportional to c_{SW}).

At the one-loop order the fermion self-energy Σ receives contributions from the "tadpole" and the "sunset" diagrams shown in Fig. 1. We define Σ_0 through the linearly divergent part and Σ_1 as the finite piece in an expansion in the lattice spacing [42–44]

$$\Sigma = \frac{g_0^2C_F}{16\pi^2}\left(\frac{\Sigma_0}{a}+\Sigma_1 i\not{p}+\mathcal{O}(a)\right), \quad (198)$$

where p_μ is the momentum of the external fermion (see Fig. 1), and the additive mass shift follows as

$$am_{\mathrm{crit}} = \frac{g_0^2C_F}{16\pi^2}\Sigma_0. \quad (199)$$

The contributions to Σ from the two diagrams read

$$\Sigma^{(\mathrm{tadpole})} = \int_{-\pi}^{\pi}\frac{\mathrm{d}^4k}{(2\pi)^4}\sum_{\mu,\nu,a}G_{\mu\nu}(k)V^{aa}_{2\mu\nu}(p,p,k,-k), \quad (200)$$

$$\Sigma^{(\mathrm{sunset})} = \int_{-\pi}^{\pi}\frac{\mathrm{d}^4k}{(2\pi)^4}\sum_{\mu,\nu,a}V^a_{1\mu}(p,k)G_{\mu\nu}(p-k)S(k)V^a_{1\nu}(k,p), \quad (201)$$

respectively, and from the sum we obtain Σ_0 and Σ_1 by

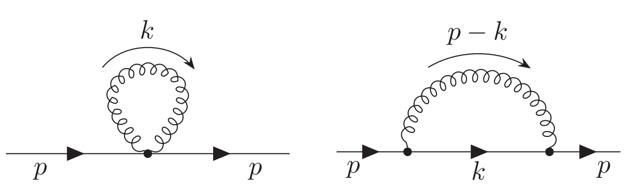

FIG. 1. Tadpole (left) and sunset (right) diagrams of the quark self-energy.

$$\frac{g_0^2 C_F}{16\pi^2}\frac{\Sigma_0}{a} = \Sigma|_{p=0}, \qquad (202)$$

$$\frac{g_0^2 C_F}{16\pi^2}\Sigma_1 = \frac{1}{4}\mathrm{Tr}\left[\frac{\partial}{\partial p_\mu}\Sigma\cdot\gamma_\mu\right]_{p=0}, \qquad (203)$$

where μ is kept fixed, i.e. not summed over. While Σ_0 is gauge independent, Σ_1 is part of the fermion field renormalization and does depend on the gauge. We use Feynman gauge and all results for Σ_1 are given in this gauge. We refer the reader to Ref. [45] for the explicit form of the gluon and fermion propagators in this gauge. Furthermore, for the gluon propagator we employ both the Wilson "plaquette" gauge action and the tree-level "Symanzik improved" gluon action, since these are the most popular choices in contemporary nonperturbative studies. These results will be labeled as "plaq" and "sym," respectively, in the figures and tables below. A somewhat technical point is that in Feynman gauge Σ_1 depends logarithmically on an external momentum $p \to 0$

$$\Sigma_1 = \log(p^2) + \Sigma_{10} \qquad (204)$$

and this is why we shall plot or quote the constant part Σ_{10}.

Besides the choice of the gauge action also the numerical values of the Wilson parameter r and the clover coefficient c_{SW} matter and, of course, the parameter $(\varrho, n_{\mathrm{stout}})$ or the gradient flow time t/a^2. To enhance verbosity, in this section n_{stout} is used to denote the number of stout steps. It goes without saying that we cannot represent all information in a compact form. This is why we restrict ourselves to a quick overview on how Σ varies as a function of ρ at fixed $c_{\mathrm{SW}} = r$, and an overview on how it depends on $(c_{\mathrm{SW}} = r)$ c_{SW} and r for a few choices of $(\varrho, n_{\mathrm{stout}})$ or the flow time t/a^2. Some additional information is shifted to the Appendix, in the hope that the selection made is broad enough to allow the user to make an informed decision when it comes to choosing a lattice fermion action for future nonperturbative studies.

In Ref. [37] results for Σ_0 with up to three steps of stout smearing at the specific value $\varrho = 0.1$ (labeled there as $\alpha = 0.6$, due to the perturbative matching between stout and APE smearing) were given, which agree with our results (see our Appendix for details). In addition, results for Σ_0 and Σ_1 with one step of SLINC smearing for plaquette and Lüscher-Weisz glue were given in Ref. [46]. In the SLINC action only the links entering the Wilson part of the fermion action are subject to one step of stout smearing (with variable ϱ), while the clover part stays unsmeared. Reproducing these results[2] served as a cross-check for our code.

Calculating Σ and extracting Σ_0 and Σ_{10} from it becomes more challenging as the number of stout steps (or the overall flow time) increases. We managed to get reasonably accurate results for up to $n_{\mathrm{stout}} = 4$ smearing steps.

Figure 2 displays how the piece Σ_0 of the self-energy of a clover fermion ($c_{\mathrm{SW}} = r$) depends on ϱ for up to four stout steps. Regardless of the gauge background adopted (left versus right column), choosing a reasonable ϱ is found to reduce the additive mass shift (bottom row) quite drastically, while (in Feynman gauge) the "tadpole" and "sunset" contributions, inspected individually, do not show any noteworthy feature (top and middle panels). In the physical result (bottom row) the initial slope is proportional to n_{stout} (see below). Compared to the full curve the SLINC curve (wide-spaced dots) seems to be an improvement for small values of ϱ, but for a value $\varrho \sim 0.08$ the critical mass of SLINC fermions changes sign. On the other hand, the "overall smearing" advocated in Ref. [37] seems to be rather benevolent; the more smearing steps (with $0 < \varrho < 0.12$) are taken, the more Σ_0 tends to zero. Moreover, regardless of n_{stout} and the chosen gluon background, the maximum of Σ_0 is always near $\varrho \sim 0.12$, see Table I for details. This is consistent with the observation made in Ref. [37] that ϱ in the stout recipe should be kept below 0.125 in order to have a (first order) smearing form factor smaller than one throughout the Brillouin zone.

Figure 3 displays how Σ_{10} depends on ϱ for up to four stout steps (again for $c_{\mathrm{SW}} = r$). Iterated stout smearings with a parameter in the vicinity of $\varrho \simeq 0.1$ seem to bring this quantity to a value near -6. Again the extremal points are found to be near $\varrho \sim 0.12$, see Table I for details.

It is interesting to plot the results for iterated stout smearing versus $n_{\mathrm{stout}} \cdot \varrho$ rather than versus ϱ; this is done in Figs. 4 and 5. With a bit of intuition one can see that for any fixed value of $n_{\mathrm{stout}} \cdot \varrho$ these results converge to a universal curve, which we indicate through black crosses. Given our results in Sec. III, this should not come as a surprise. The black curves in Figs. 4 and 5 indicate the results for the Wilson variety of the gradient flow. Evidently, what we see is a graphical illustration of the correspondence[3]

$$t/a^2 = n_{\mathrm{stout}} \cdot \varrho \qquad (205)$$

that links the dimensionless flow time to the cumulative parameter of the iterated stout recipe. Note that this identification becomes exact under $n_{\mathrm{stout}} \to \infty$ and $\varrho \to 0$ with the product $n_{\mathrm{stout}} \cdot \varrho$ held constant. This matching has been discussed in Refs. [47,48], too.

The thin vertical lines in Figs. 4 and 5 indicate integer multiples of 0.125. They are meant to separate each curve into a part where it gives a reasonable approximation to the

[2]There is a typo in Eq. (53) of Ref. [46], the number 2235.407087 should read 2335.407087.

[3]Corrections to (205) start at order $t^2/(a^4 n_{\mathrm{stout}})$, see Eqs. (150), (166), and (176). Hence, the left-hand side of (205) is to be dressed with a factor $(1 + \mathcal{O}(t/(a^2 n_{\mathrm{stout}})))$.

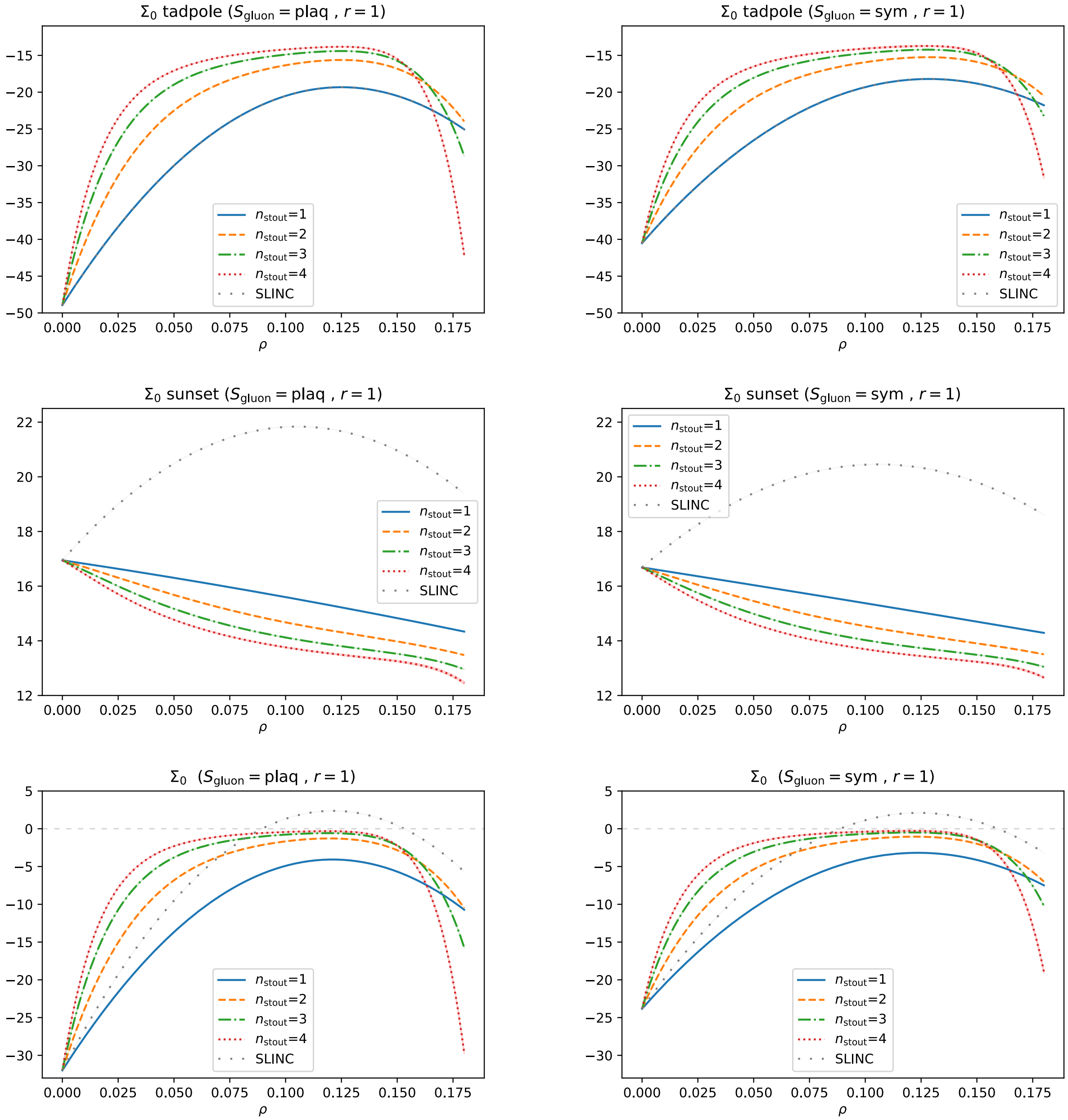

FIG. 2. Contributions to the divergent piece Σ_0 of the clover ($c_{SW} = r = 1$) fermion self-energy from the tadpole (top) and sunset (middle) diagrams and their sum (bottom). Results are shown as a function of the smearing parameter ϱ, with plaquette gluon background (left) and tree-level Symanzik-improved gluon background (right). In the top row the results for one step of overall stout smearing ($n_{\mathrm{stout}} = 1$) coincide with those for SLINC fermions [46].

TABLE I. Position of the extreme points in the bottom panels of Figs. 2 and 3.

	$n = 1$	$n = 2$	$n = 3$	$n = 4$
$\varrho_{\max}$ of Σ_0 (plaq)	0.12096871(3)	0.11973422(9)	0.1189366(3)	0.1187(4)
$\varrho_{\max}$ of Σ_0 (sym)	0.12357712(3)	0.1220287(3)	0.120999(5)	0.1204(9)
$\varrho_{\min}$ of Σ_{10} (plaq)	0.13078397(5)	0.1285409(2)	0.1271189(9)	0.1263(6)
$\varrho_{\min}$ of Σ_{10} (sym)	0.13417728(6)	0.1315944(2)	0.129866(6)	0.1287(6)

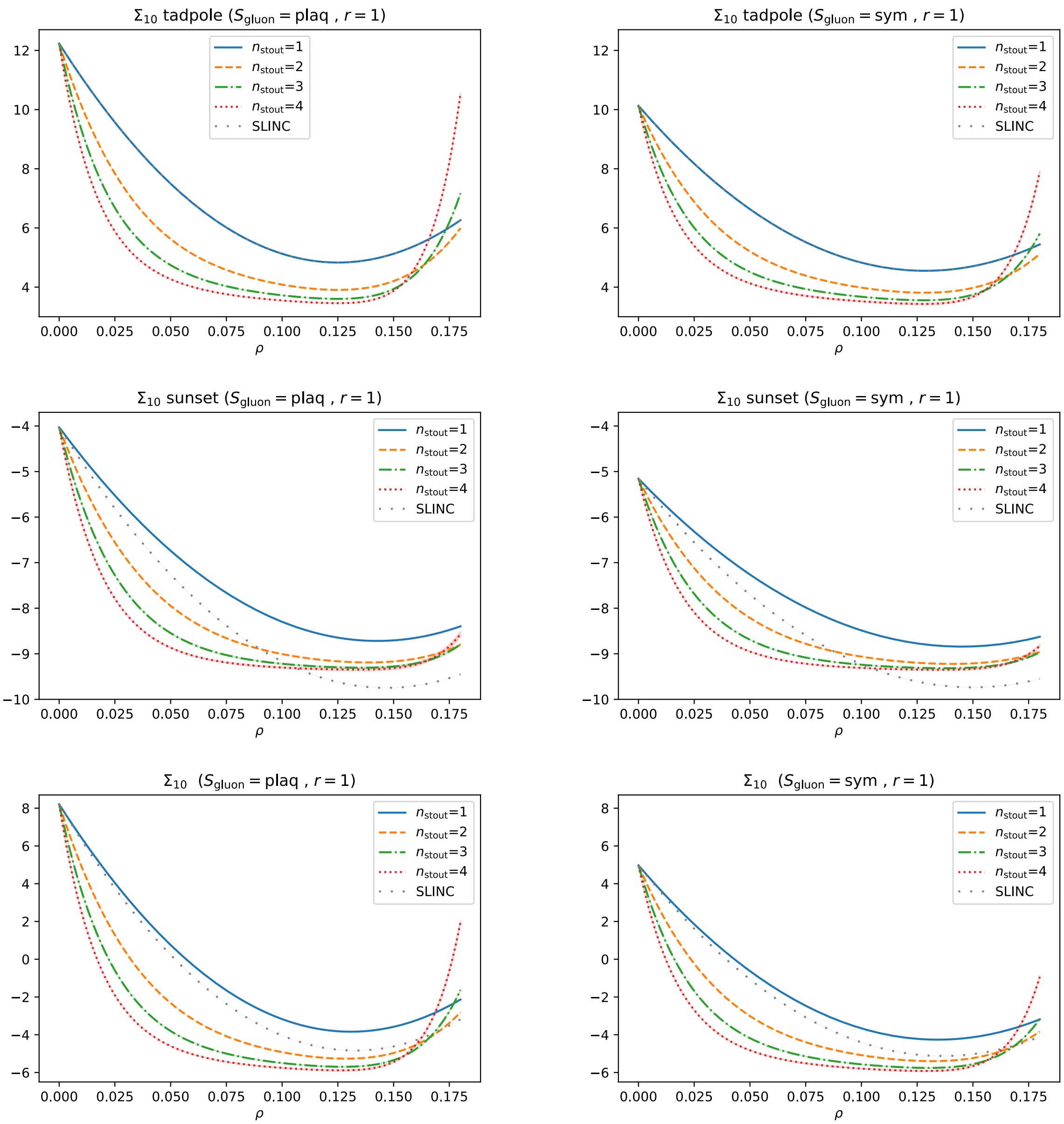

FIG. 3. Same as Fig. 2, but for the finite piece Σ_{10}; see Eq. (204).

gradient flow (to the left) and a part where it does not (to the right). For instance the mark at $3 \cdot 0.125 = 0.375$ separates the dash-dotted line ($n_{\rm stout} = 3$) into an ascending part (where it approximates the crosses quite well), and a descending part (where it does not). In the terminology of numerical mathematics one would say that multiple stout smearings implement a simple integration scheme (with an integration error in the flow time t/a^2 which scales like $1/n_{\rm stout}$), and the caveat of Ref. [37] is meant to limit the integration error.

Let us consider the behavior of the self-energy as a function of the Wilson parameter r (with $c_{\rm SW} = r$) as shown in Figs. 6 and 7. Even though the self-energy at one-loop order is not a linear function in r analytically (since r appears in the denominator of the fermion propagator), Σ_0 and Σ_{10} seem almost linear over the plotted range. The former quantity decreases and the latter one increases with increasing r. For ϱ approaching 0.125 and/or more smearing steps the behavior becomes gradually flatter. In practical terms this means that the Wilson parameter r is not a useful knob to engineer the self-energy of tree-level clover fermions.

In Figs. 8 and 9 we finally shift to results with the gradient flow recipe. For $r = 1$ (green dots) the "tadpole" and the "sunset" contribution seem to tend to unsuspicious finite values, as t/a^2 is taken large, and the physical sum tends to zero. Varying r (and synchronously $c_{\rm SW} = r$) seems to affect only the splitting between the tadpole and

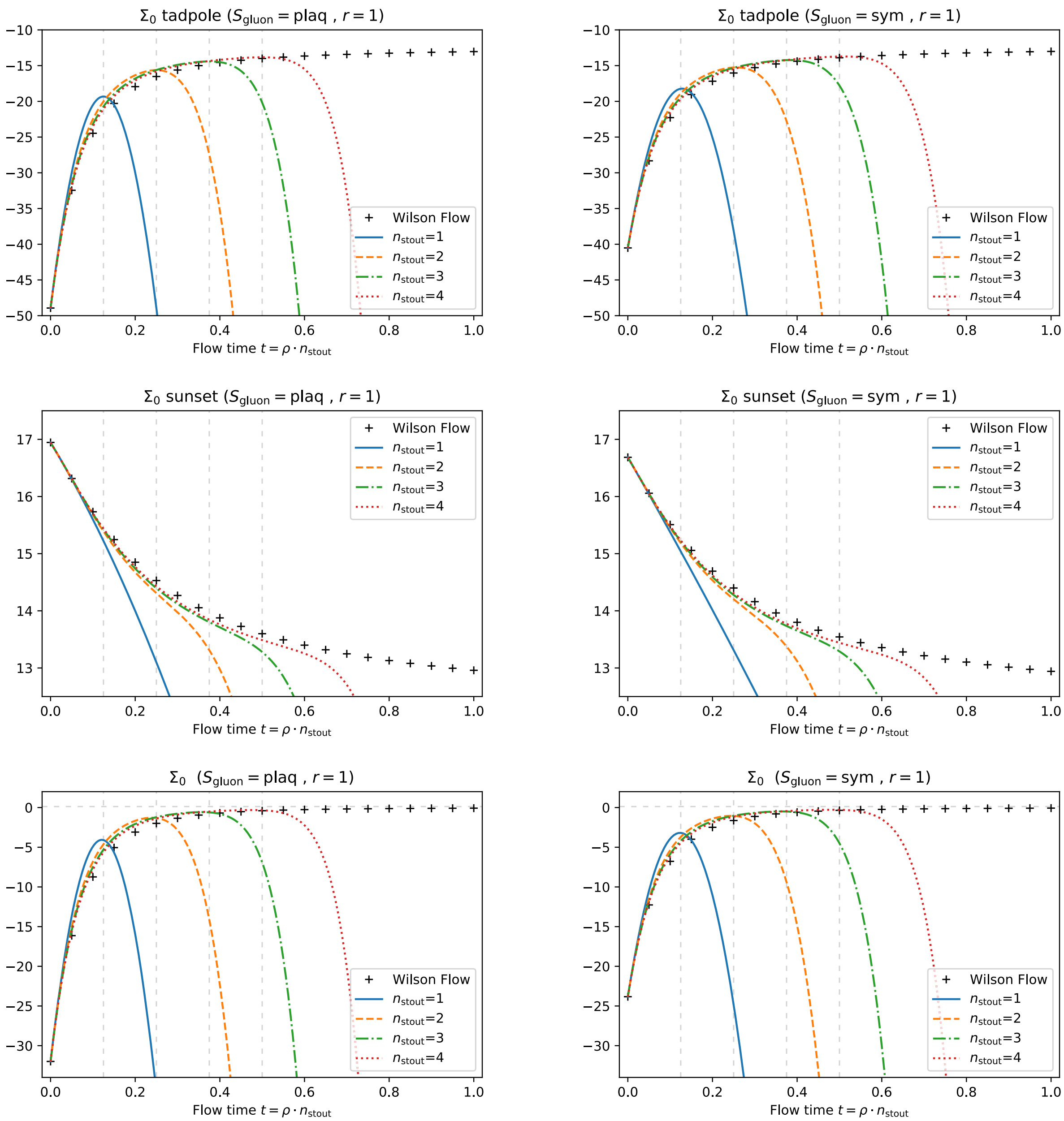

FIG. 4. Same information as in Fig. 2, but plotted versus the effective flow time $t = \varrho n_{\mathrm{stout}}$. The vertical dashed lines indicate integer multiples of $t = 0.125$ (in lattice units). The curve of the Wilson flow (to which these results converge under $n_{\mathrm{stout}} \to \infty$) is indicated with black crosses.

the sunset contributions, but barely the final (physical) result. And the former observation is only pronounced for Σ_0; for Σ_{10} it is rather mild, here the tadpole contribution does not depend on r at all. For completeness we show in Fig. 10 how Σ_0 and Σ_{10} depend on r for a few Wilson flow times t/a^2. Any dependence that is still present at tiny flow times $(t/a^2 \simeq 0.1)$ is found to quickly disappear for moderate flow times $(t/a^2 \simeq 1)$.

In our view, the takeaway message from this section is that there is a perturbative backing of the well-known benefit that comes from multiple stout smearing. One step reduces the absolute value of Σ_0 by about 85%, two steps by 95%, three by 98%, and four steps by 99%.

VI. CONCLUSIONS

In this paper we have worked out a systematic approach for deriving Feynman rules for lattice fermion actions with smeared gauge links (to be used interchangeably in the covariant derivative, in the species-separating covariant Laplacian and/or a possible clover term).

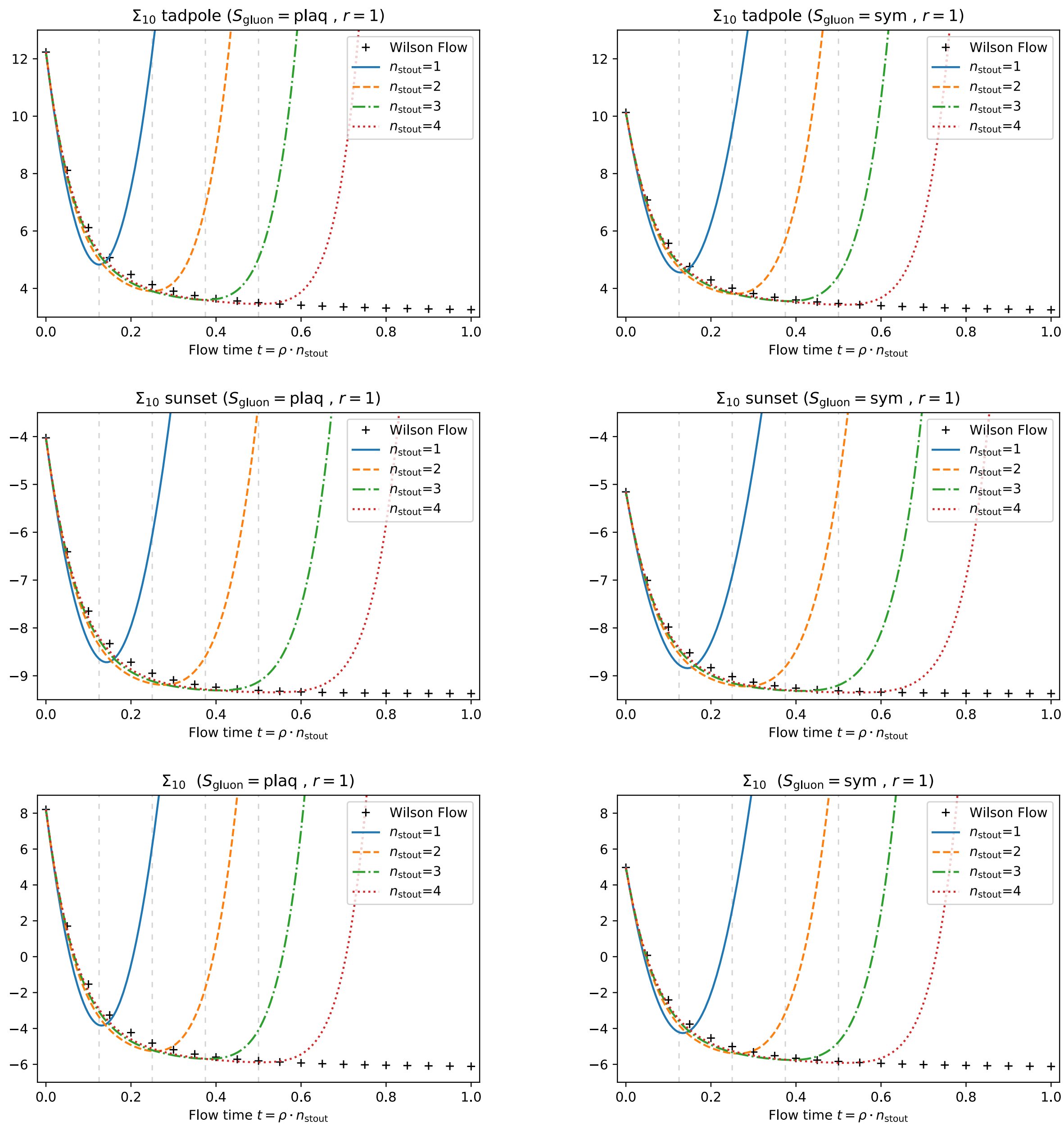

FIG. 5. Same as Fig. 4, but for the finite piece Σ_{10}; see Eq. (204).

The basis for such calculations is provided by a perturbative expansion of the stout smearing recipe of Ref. [9] and the Wilson variety of the gradient flow introduced in Refs. [13–15]; these expansions have been presented in Secs. II and III, respectively.

With these results in hand, one may derive the Feynman rules for a fermion action with an arbitrary parameter combination $(\varrho, n_{\text{stout}})$ in case of stout smearing or an arbitrary flow time $t/a^2 \in \mathbb{R}$ in case of the gradient flow. In Sec. IV we argue that the "brute force" approach, i.e. inserting the perturbative expansion of the smeared/flowed variable into the underlying action, is not recommendable, since it leads to excessively long intermediate expressions. We present an approach based on the $SU(N_c)$ expansion of the original ("unsmeared") links $U_\mu(x)$ to the required order in g_0 (typically g_0^3 for one-loop calculations), which keeps intermediate expressions much shorter. This holds true for both stout smearing and the gradient flow, and confirms, as a by-product, the perturbative matching (205) between these recipes.

To give an immediate application of our recipe, we derive in Sec. IV the Feynman rules of a Wilson fermion with clover improvement for arbitrary Wilson parameter r and Sheikholeslami-Wohlert parameter c_{SW}. With these

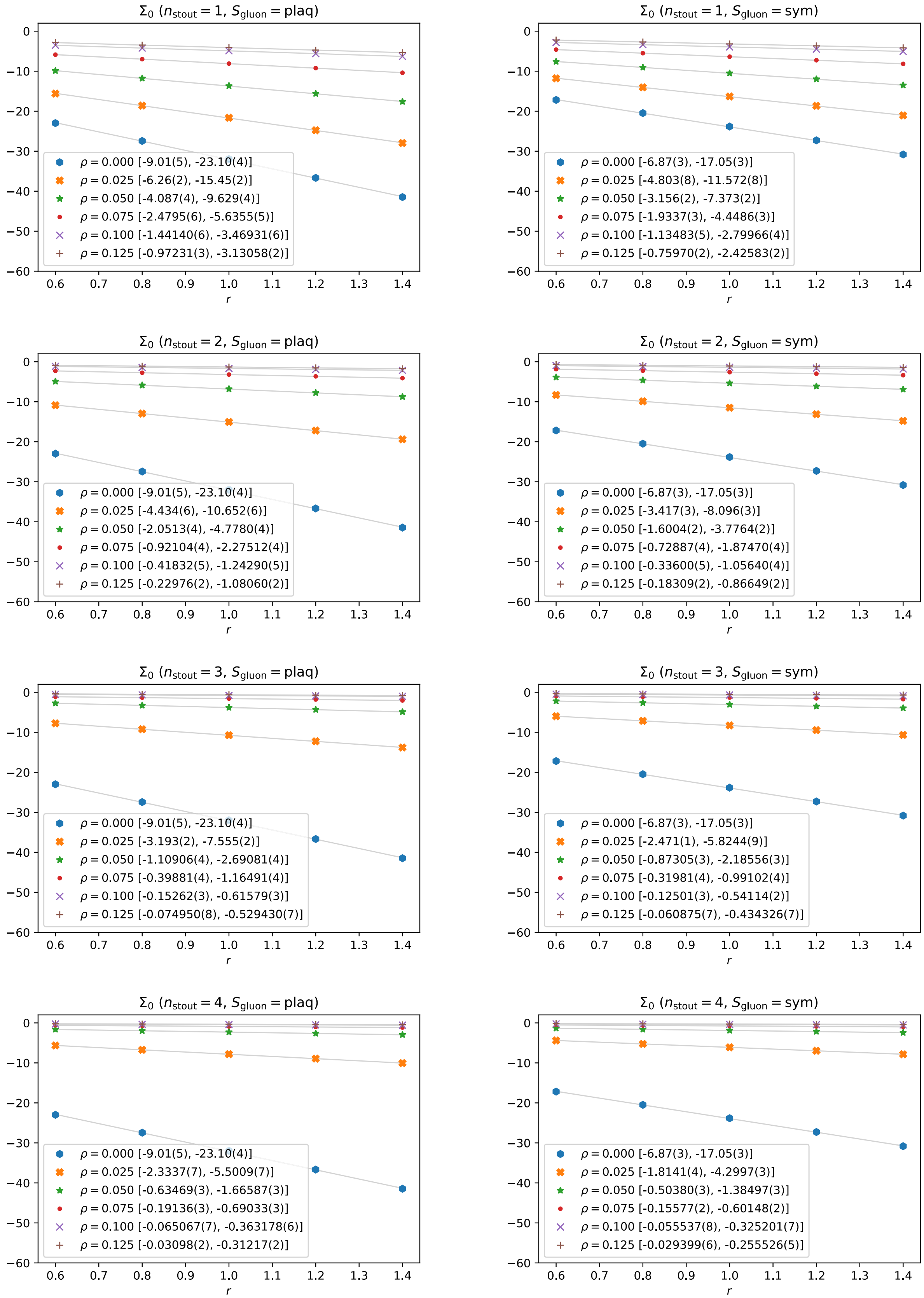

FIG. 6. The divergent piece Σ_0 of the clover fermion ($c_{\mathrm{SW}} = r$) self-energy as a function of r, with 1 to 4 stout steps (top to bottom). The background uses plaquette glue (left) or tree-level Symanzik improved glue (right). The legends give the coefficients $[c_0, c_1]$ of linear least-squares fits of the form $c_0 + c_1 \cdot r$. The entry for $\varrho = 0$ is the unsmeared case.

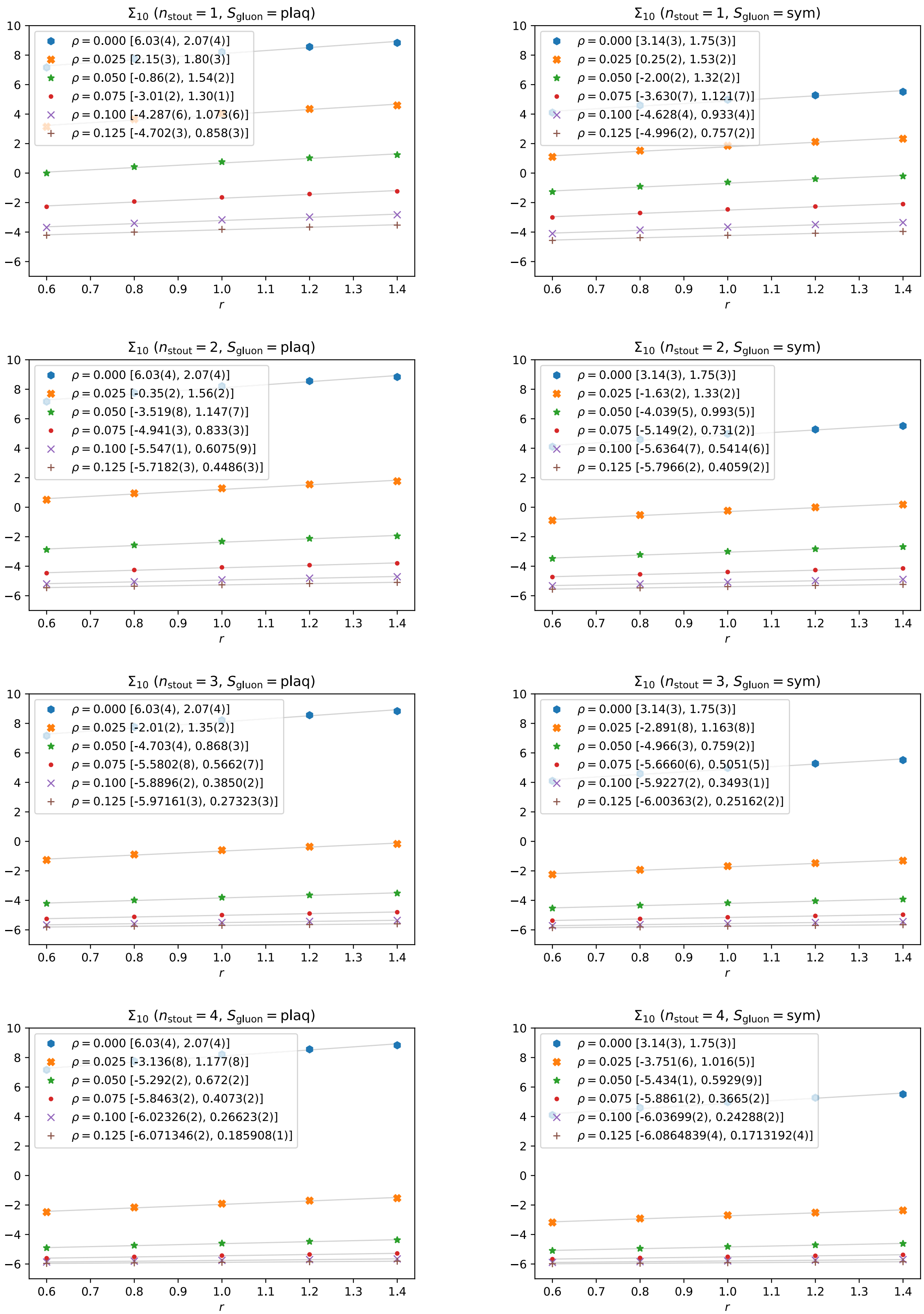

FIG. 7. Same as Fig. 6, but for the finite piece Σ_{10}; see Eq. (204).

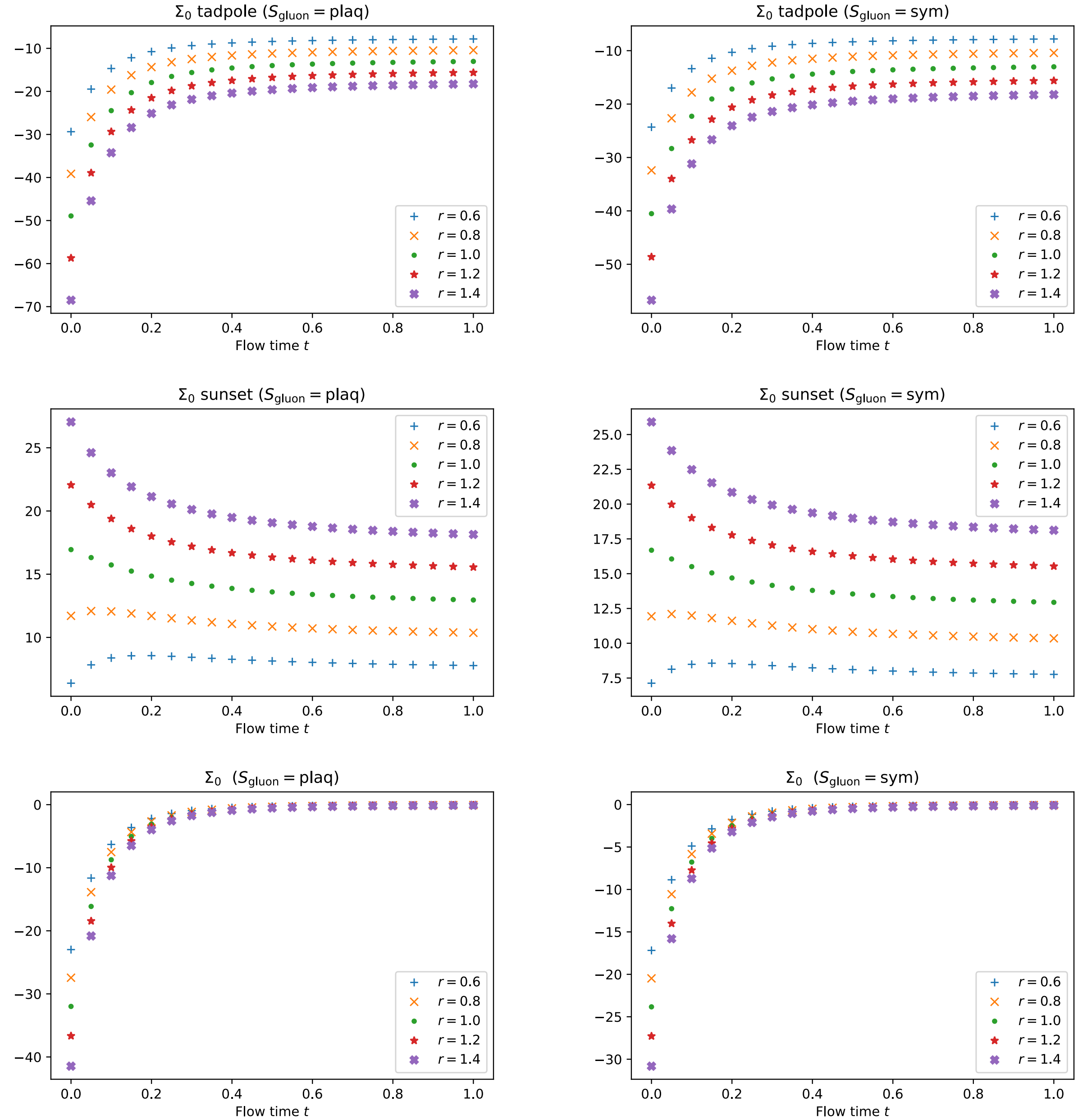

FIG. 8. Contributions to the divergent piece Σ_0 of the clover ($c_{\rm SW} = r = 1$) fermion self-energy from the tadpole (top) and sunset (middle) diagrams and their sum (bottom). Results for several values of r are given as a function of the gradient-flow time t in lattice units.

rules in hand, we calculate in Sec. V the self-energy of the clover fermion for $c_{\rm SW} = r$, for variable ϱ and $1 \le n_{\rm stout} \le 4$ in case of stout smearing and the range $0 \le t/a^2 \le 1$ in case of gradient flowing.

The cross-check with Ref. [37], where numerical results obtained in the "brute force" approach were presented, proves particularly enlightening. Of course, reproducing these results felt like a relief, but the difference is in the details. In Ref. [37] results for Σ_0 were given for a single stout parameter ($\varrho = 0.1$ in our terminology), restricted to $n_{\rm stout} \le 3$ due to the numerical burden. In our approach we managed to go up to $n_{\rm stout} = 4$ and to evaluate the generic result for arbitrary ϱ (using rather modest computational resources). This illustrates that the more elegant approach implies not just aesthetic pleasures but actual savings compared to the brute force approach. Still, the bottom line is an affirmation of the finding of Ref. [37] that combining tree-level clover improvement with stout smearing or gradient flowing with a cumulative flow time of the order $t/a^2 \simeq 1$ yields an undoubled fermion action with surprisingly good chiral properties (see the discussion at the end of our Sec. V).

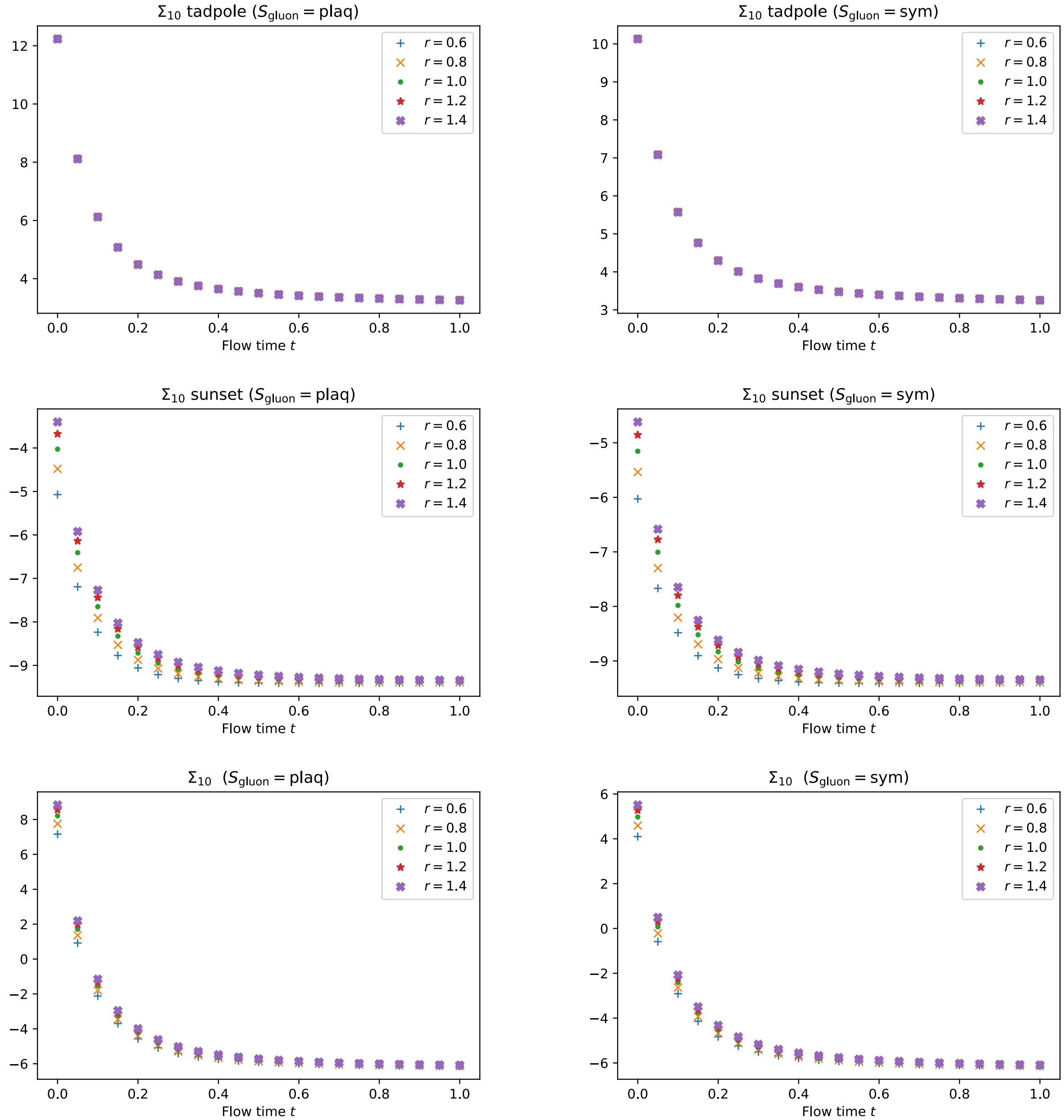

FIG. 9. Same as Fig. 8, but for the finite piece Σ_{10}; see Eq. (204).

A logical next step would be to work out the one-loop values of c_{SW} for Wilson and/or Brillouin fermions with a limited number of stout steps (hopefully again up to four iterations) and/or a few values of the flow time t/a^2, for instance $t/a^2 \in \{0.1, 0.2, 0.4, 0.7, 1.0\}$. This amounts to combining the work we did in Ref. [45] with the framework for smeared/flowed Feynman rules presented in this paper, and we hope to present such results in the not so distant future. In a further perspective, one might also work out the improvement factors $Z_{S,P,V,A}$ to one-loop (and perhaps even two-loop) order in stouted/flowed perturbation theory.

The alert reader might interject that a fully nonperturbative improvement strategy is more desirable than a perturbative one, and the link smearing may imply only mild technical complications in the nonperturbative approach. However, as mentioned in the Introduction and detailed in Ref. [34], the nonperturbative approach builds, in practical terms, on existing perturbative results to one-loop (or even better: two-loop) order, as such results provide essential constraints in the final step of summarizing the results through a rational function in g_0^2. To us this indicates that perturbative work continues to be useful in lattice gauge theory.

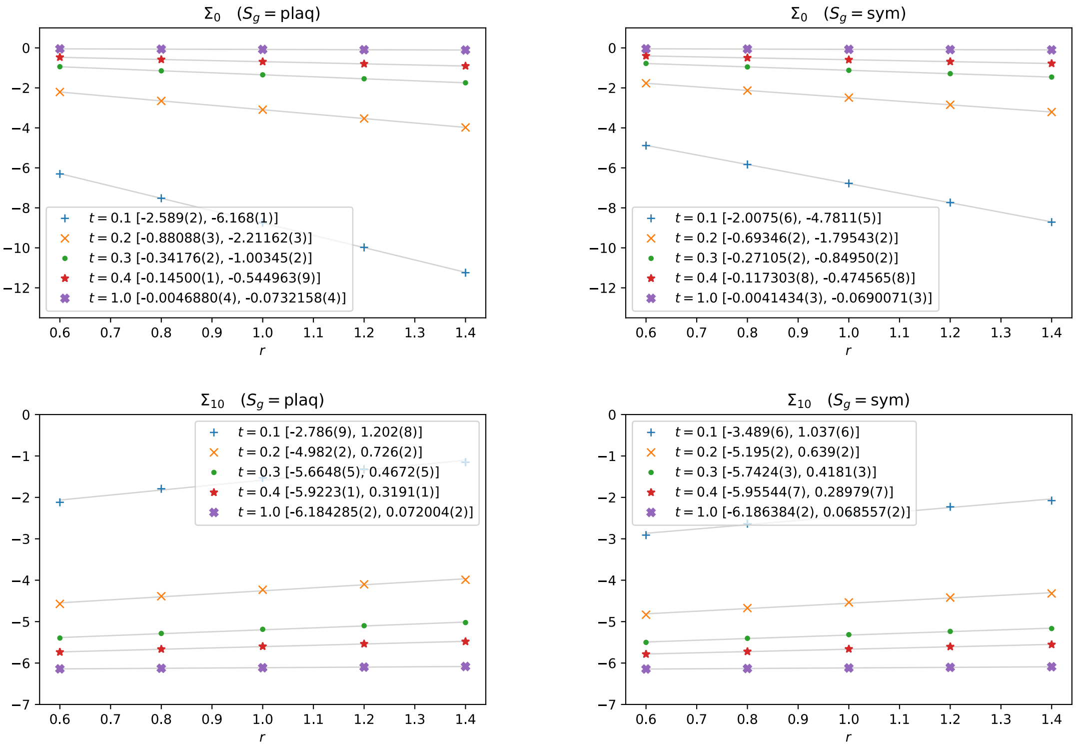

FIG. 10. The dependence on the Wilson paramter r of Σ_0 (top) and Σ_{10} (bottom) at different flow times t of the Wilson flow. The coefficients $[c_0, c_1]$ of linear least-square fits of the form $c_0 + c_1 \cdot r$ are given in brackets.

APPENDIX: CLOVER FERMION SELF-ENERGY DETAILS

In this appendix we give more details for the self-energy of a Wilson fermion (with and without a clover term) in the presence of stout smearing or gradient flow.

We start with the tree-level improved case ($c_{\mathrm{SW}} = r = 1$). The pieces Σ_0 and Σ_{10}, see Eqs. (198) and (204), for up to four stout steps are listed in Table II for a selection of ϱ values. We refer to Ref. [37] for an argument why choosing ϱ larger than 0.125 should be avoided. We believe that a spline interpolation will suffice for practical purposes. In the event the reader requests higher precision, Tables III and IV hold the necessary information to reconstruct Σ_0 and Σ_{10} for arbitrary ϱ. The first respective lines (i.e. those where the entry does not depend on n) hold the unsmeared values for comparison.

The results with Wilson flow are more cumbersome to communicate. Figure 11 shows Σ_0 and Σ_{10} for plaquette and Lüscher-Weisz glue as a function of the flow time in lattice units. It turns out that a two-exponential fit represents the data with reasonable accuracy over the range shown. The plots contain this information in the legends. Note that these results reflect the tree-level choice $c_{\mathrm{SW}} = r = 1$.

Sometimes the reader may choose another value of c_{SW}, e.g. no improvement at all ($c_{\mathrm{SW}} = 0$). Since (to one-loop order) Σ_0 and Σ_{10} are quadratic in c_{SW}, we split them according to

$$\Sigma_0 = \Sigma_0^{(0)} + c_{\mathrm{SW}}\Sigma_0^{(1)} + c_{\mathrm{SW}}^2\Sigma_0^{(2)}, \tag{A1}$$

$$\Sigma_{10} = \Sigma_{10}^{(0)} + c_{\mathrm{SW}}\Sigma_{10}^{(1)} + c_{\mathrm{SW}}^2\Sigma_{10}^{(2)}. \tag{A2}$$

and give $\Sigma_0^{(0)}$, $\Sigma_0^{(1)}$, and $\Sigma_0^{(2)}$ in Table V and $\Sigma_{10}^{(0)}$, $\Sigma_{10}^{(1)}$, and $\Sigma_{10}^{(2)}$ in Table VI for the same selection of ϱ values as before (the parameter in the Wilson term is always $r = 1$). Again, we expect that a simple spline interpolation will suffice to obtain useful numbers for a different choice of ϱ. In the event this expectation is wrong, Tables VII–XII hold the necessary information to reconstruct Σ_0 and Σ_{10} for arbitrary ϱ.

The results with Wilson flow are represented in Figs. 12 and 13 in the same style as before. Again a two-exponential fit is found to reproduce the data well over the range shown.

Physics wise, the most impressive lesson to be learned from this appendix comes from comparing the top panels of Fig. 11 to Fig. 12. With tree-level improvement (Fig. 11) the additive mass shift (which is proportional to Σ_0) approaches zero quite quickly with increasing flow time. For unimproved clover fermions this is not the case (top row in Fig. 12); in fact the nice feature seen in Fig. 11 stems from a cancellation between $\Sigma_0^{(0)}$, $\Sigma_0^{(1)}$, and $\Sigma_0^{(2)}$ of Fig. 12.

TABLE II. Values of Σ_0 and Σ_{10} for some values of the smearing parameter ϱ and smearing steps $n \in [1, 2, 3, 4]$ with plaquette gauge action (plaq) and Lüscher-Weisz gauge action (sym). All information refers to the choice $c_{\mathrm{SW}} = r = 1$.

	ϱ	$n = 1$	$n = 2$	$n = 3$	$n = 4$
Σ_0 (plaq)	0.05	−13.67830(4)	−6.81841(4)	−3.79814(4)	−2.30(1)
	0.09	−5.89958(4)	−2.12071(5)	−1.00136(6)	−0.56(2)
	0.11	−4.29949(4)	−1.37888(5)	−0.62450(6)	−0.34(2)
	0.12	−4.07175(4)	−1.27800(6)	−0.57513(6)	−0.32(3)
	0.125	−4.10097(4)	−1.31229(6)	−0.60582(7)	−0.34(3)
	0.13	−4.22557(4)	−1.41647(6)	−0.69334(7)	−0.42(3)
Σ_0 (sym)	0.05	−10.50283(3)	−5.36991(3)	−3.05827(4)	−1.89(4)
	0.09	−4.70711(3)	−1.76141(5)	−0.85802(8)	−0.49(6)
	0.11	−3.43187(3)	−1.15565(5)	−0.5431(2)	−0.31(7)
	0.12	−3.19992(3)	−1.04722(6)	−0.4871(2)	−0.28(8)
	0.125	−3.18535(3)	−1.05160(6)	−0.4966(2)	−0.29(8)
	0.13	−3.23840(3)	−1.10355(6)	−0.5435(3)	−0.33(9)
Σ_{10} (plaq)	0.05	0.75710(4)	−2.32322(4)	−3.80405(4)	−4.599(7)
	0.09	−2.66716(4)	−4.65582(5)	−5.34360(5)	−5.66(2)
	0.11	−3.53426(4)	−5.11323(5)	−5.60877(5)	−5.83(2)
	0.12	−3.75656(4)	−5.23165(5)	−5.67941(6)	−5.88(2)
	0.125	−3.81490(4)	−5.26045(5)	−5.69546(6)	−5.89(2)
	0.13	−3.83802(4)	−5.26563(5)	−5.69366(7)	−5.88(3)
Σ_{10} (sym)	0.05	−0.625159(8)	−3.00765(1)	−4.18067(2)	−4.824(9)
	0.09	−3.25805(1)	−4.86118(2)	−5.43657(5)	−5.71(2)
	0.11	−3.95912(2)	−5.24318(2)	−5.66454(9)	−5.86(2)
	0.12	−4.15581(2)	−5.35110(2)	−5.7305(2)	−5.90(3)
	0.125	−4.21569(2)	−5.38307(2)	−5.7497(2)	−5.92(3)
	0.13	−4.24993(2)	−5.39832(2)	−5.7566(2)	−5.92(3)

TABLE III. Coefficients of the polynomials in the smearing parameter ϱ for Σ_0 for smearing steps $n \in [1, 2, 3, 4]$ with plaquette gauge action (plaq) and Lüscher-Weisz gauge action (sym).

		Σ_0			
		$n = 1$	$n = 2$	$n = 3$	$n = 4$
ϱ^0	plaq	−31.98644(3)	−31.98644(3)	−31.98644(3)	−31.98644(3)
	sym	−23.83234(3)	−23.83234(3)	−23.83234(3)	−23.83234(3)
ϱ^1	plaq	461.5488(1)	923.0975(2)	1384.645(4)	1846.19(6)
	sym	334.1999(1)	668.3997(1)	1002.5996(2)	1336.8(6)
ϱ^2	plaq	−1907.719658(3)	−11446.31795(2)	−28615.795(2)	−53416.150(3)
	sym	−1352.19157(4)	−8113.149(2)	−20282.874(1)	−37861.370(2)
ϱ^3	plaq		69587.70312(7)	347938.5(8)	974228(2)
	sym		48476.67898(2)	242383.40(2)	678673.4(2)
ϱ^4	plaq		−171121.8788(3)	−2566828(2)	−11978531(7)
	sym		−117482.6334(5)	−1762239.50(1)	−8223785(5)
ϱ^5	plaq			10728080(5)	100128744(41)
	sym			7273681(5)	67887689(29)
ϱ^6	plaq			−19627720(22)	−549575266(290)
	sym			−13163619.12(3)	−368581513(347)
ϱ^7	plaq				1795933838(5106)
	sym				1193026157(2578)
ϱ^8	plaq				−2658101497(19216)
	sym				−1750940862(13091)

TABLE IV. Coefficients of the polynomials in the smearing parameter ϱ for Σ_{10} for smearing steps $n \in [1, 2, 3, 4]$ with plaquette gauge action (plaq) and Lüscher-Weisz gauge action (sym).

		Σ_{10}			
		$n = 1$	$n = 2$	$n = 3$	$n = 4$
ϱ^0	plaq	8.20627(3)	8.20627(3)	8.20627(3)	8.20627(3)
	sym	4.973633(5)	4.973633(5)	4.973633(5)	4.973633(5)
ϱ^1	plaq	−184.19270(6)	−368.3854(2)	−552.58(5)	−736.77(7))
	sym	−137.61668(5)	−275.2334(1)	−412.85002(9)	−550.5(2)
ϱ^2	plaq	704.18682(5)	4225.1209(3)	10562.8(2)	19717.2(3)
	sym	512.81660(2)	3076.8996(2)	7692.2490(3)	14358.9(2)
ϱ^3	plaq		−24227.54502(9)	−121137.7(5)	−339186(1)
	sym		−17289.6263(2)	−86448.13(4)	−242054.9(4)
ϱ^4	plaq		56866.2480(2)	852993.70(3)	3980637(6)
	sym		39893.8762(1)	598408.143(2)	2792571.48(4)
ϱ^5	plaq			−3431931(1)	−32031358(9)
	sym			−2372221(1)	−22140728(2)
ϱ^6	plaq			6083759(7)	170345240(173)
	sym			4150857.326(4)	116224020(336)
ϱ^7	plaq				−542124208(1774)
	sym				−365651055(67)
ϱ^8	plaq				784643279(12269)
	sym				523847362(2021)

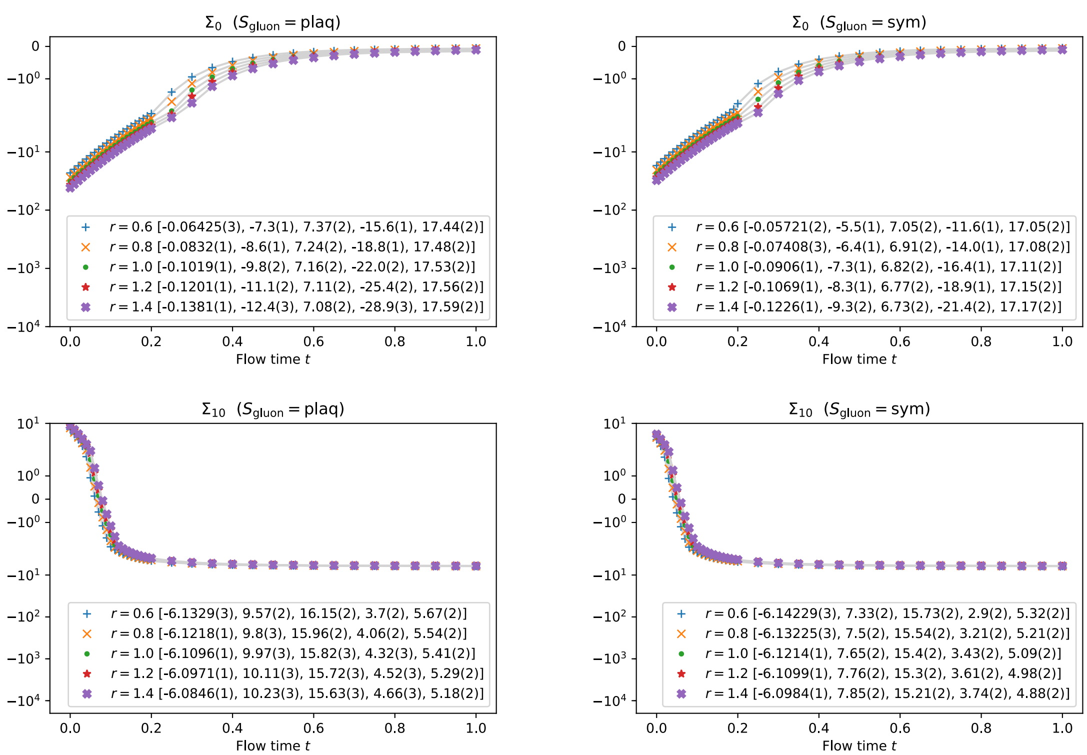

FIG. 11. Double exponential fits of the form $c_0 + c_1 e^{-c_2 t} + c_3 e^{-c_4 t}$ of Σ_0 (top) and Σ_{10} (bottom) with Wilson flow for different values of r (with $c_{SW} = r$) plotted on a logarithmic y axis. The fit coeffcients $[c_0, c_1, c_2, c_3, c_4]$ are given in brackets.

TABLE V. Same as Table II but for $\Sigma_0^{(0)}$, $\Sigma_0^{(1)}$, and $\Sigma_0^{(2)}$ from Eq. (A1).

	ϱ	$n=1$	$n=2$	$n=3$	$n=4$
$\Sigma_0^{(0)}$ (plaq)	0.05	−26.65808(5)	−16.28605(5)	−11.14760(7)	− 8.268(9)
	0.09	−15.24425(5)	−8.17389(7)	−5.4550(2)	−4.08(2)
	0.11	−12.33982(5)	−6.44793(8)	−4.3213(2)	−3.26(2)
	0.12	−11.58822(6)	−5.96517(9)	−3.9821(2)	−3.00(2)
	0.125	−11.38758(6)	−5.83167(9)	−3.8853(3)	−2.92(2)
	0.13	−11.30371(6)	−5.7827(1)	−3.8560(3)	−2.90(2)
$\Sigma_0^{(0)}$ (sym)	0.05	−21.88153(4)	−13.84082(5)	−9.73971(9)	−7.38(4)
	0.09	−13.10032(4)	−7.34394(7)	−5.0335(2)	−3.83(6)
	0.11	−10.73243(5)	−5.88197(8)	−4.0428(3)	−3.09(7)
	0.12	−10.05417(5)	−5.43642(9)	−3.7247(4)	−2.85(8)
	0.125	−9.84145(5)	−5.29131(9)	−3.6179(4)	−2.76(8)
	0.13	−9.71302(5)	− 5.2058(1)	−3.5577(5)	−2.72(8)
$\Sigma_0^{(1)}$ (plaq)	0.05	9.7541(2)	7.4558(4)	5.9965(6)	5.001(4)
	0.09	7.4250(4)	5.0954(7)	3.882(1)	3.138(6)
	0.11	6.5452(4)	4.3673(8)	3.288(2)	2.641(8)
	0.12	6.1764(5)	4.0752(9)	3.054(2)	2.45(1)
	0.125	6.0098(5)	3.9442(9)	2.950(2)	2.36(2)
	0.13	5.8551(5)	3.8227(9)	2.853(2)	2.28(2)
$\Sigma_0^{(1)}$ (sym)	0.05	8.6852(2)	6.7573(3)	5.5094(4)	4.644(2)
	0.09	6.7474(3)	4.7368(5)	3.6600(7)	2.987(3)
	0.11	6.0027(3)	4.0976(6)	3.1260(9)	2.532(4)
	0.12	5.6865(4)	3.8381(7)	2.913(1)	2.353(5)
	0.125	5.5423(4)	3.7209(7)	2.818(1)	2.273(5)
	0.13	5.4075(4)	3.6115(7)	2.729(1)	2.198(6)
$\Sigma_0^{(2)}$ (plaq)	0.05	3.2256(3)	2.0118(3)	1.3529(3)	0.9637(9)
	0.09	1.9196(3)	0.9578(3)	0.5719(3)	0.380(2)
	0.11	1.4952(3)	0.7018(3)	0.4088(3)	0.267(3)
	0.12	1.3400(3)	0.6120(3)	0.3527(3)	0.229(4)
	0.125	1.2768(3)	0.5751(3)	0.3296(3)	0.213(5)
	0.13	1.2230(3)	0.5436(3)	0.3097(3)	0.199(5)
$\Sigma_0^{(2)}$ (sym)	0.05	2.6935(2)	1.7136(2)	1.1721(2)	0.8470(3)
	0.09	1.6458(2)	0.8457(2)	0.5155(2)	0.348(2)
	0.11	1.2978(2)	0.6287(2)	0.3736(2)	0.248(3)
	0.12	1.1678(2)	0.5511(2)	0.3240(2)	0.214(5)
	0.125	1.1138(2)	0.5188(2)	0.3034(2)	0.200(6)
	0.13	1.0671(2)	0.4907(2)	0.2853(2)	0.187(7)

TABLE VI. Same as Table II but for $\Sigma_{10}^{(0)}$, $\Sigma_{10}^{(1)}$, and $\Sigma_{10}^{(2)}$ from Eq. (A2).

	ϱ	$n=1$	$n=2$	$n=3$	$n=4$
$\Sigma_{10}^{(0)}$ (plaq)	0.05	3.60121(2)	−0.01039(3)	−1.86437(3)	−2.934(4)
	0.09	−0.33688(3)	−2.95019(5)	−4.00186(5)	−4.551(6)
	0.11	−1.41830(3)	−3.61776(5)	−4.45331(7)	−4.886(7)
	0.12	−1.73711(3)	−3.82460(6)	−4.59974(9)	−4.999(9)
	0.125	−1.84104(3)	−3.89407(6)	−4.65020(9)	−5.039(9)
	0.13	−1.90798(3)	−3.93775(7)	−4.6807(1)	−5.06(1)
$\Sigma_{10}^{(0)}$ (sym)	0.05	1.956497(6)	−0.882122(8)	−2.38041(2)	−3.266(8)
	0.09	−1.114339(7)	−3.26513(2)	−4.16661(3)	−4.65(2)
	0.11	−2.000100(8)	−3.83374(2)	−4.56328(5)	−4.95(2)
	0.12	−2.280566(8)	−4.02084(2)	−4.69847(6)	−5.06(2)
	0.125	−2.380196(9)	−4.08938(2)	−4.74919(7)	−5.10(2)
	0.13	−2.452757(9)	−4.13934(2)	−4.78580(8)	−5.13(2)
$\Sigma_{10}^{(0)}$ (plaq)	0.05	3.60121(2)	−0.01039(3)	−1.86437(3)	−2.934(4)
	0.09	−0.33688(3)	−2.95019(5)	−4.00186(5)	−4.551(6)
	0.11	−1.41830(3)	−3.61776(5)	−4.45331(7)	−4.886(7)
	0.12	−1.73711(3)	−3.82460(6)	−4.59974(9)	−4.999(9)
	0.125	−1.84104(3)	−3.89407(6)	−4.65020(9)	−5.039(9)
	0.13	−1.90798(3)	−3.93775(7)	−4.6807(1)	−5.06(1)
$\Sigma_{10}^{(0)}$ (sym)	0.05	1.956497(6)	−0.882122(8)	−2.38041(2)	−3.266(8)
	0.09	−1.114339(7)	−3.26513(2)	−4.16661(3)	−4.65(2)
	0.11	−2.000100(8)	−3.83374(2)	−4.56328(5)	−4.95(2)
	0.12	−2.280566(8)	−4.02084(2)	−4.69847(6)	−5.06(2)
	0.125	−2.380196(9)	−4.08938(2)	−4.74919(7)	−5.10(2)
	0.13	−2.452757(9)	−4.13934(2)	−4.78580(8)	−5.13(2)
$\Sigma_{10}^{(2)}$ (plaq)	0.05	−1.09264(2)	−0.88380(2)	−0.73530(2)	−0.6262(7)
	0.09	−0.89206(2)	−0.64168(2)	−0.49704(2)	−0.404(2)
	0.11	−0.80614(2)	−0.55755(2)	−0.42364(2)	−0.341(2)
	0.12	−0.76678(2)	−0.52223(2)	−0.39401(2)	−0.316(3)
	0.125	−0.74800(2)	−0.50598(2)	−0.38060(3)	−0.305(3)
	0.13	−0.72981(2)	−0.49059(2)	−0.36803(3)	−0.295(3)
$\Sigma_{10}^{(2)}$ (sym)	0.05	−0.986186(4)	−0.807676(5)	−0.678888(6)	−0.5831(4)
	0.09	−0.815953(5)	−0.597649(7)	−0.468699(8)	−0.3850(8)
	0.11	−0.742363(6)	−0.523351(7)	−0.40253(1)	−0.327(2)
	0.12	−0.708450(6)	−0.491894(8)	−0.37558(1)	−0.304(2)
	0.125	−0.692213(6)	−0.477374(8)	−0.36334(1)	−0.294(2)
	0.13	−0.676458(6)	−0.463584(8)	−0.35183(2)	−0.284(2)

TABLE VII. Same as Table III but for $\Sigma_0^{(0)}$ from Eq. (A1).

		$\Sigma_0^{(0)}$			
		$n=1$	$n=2$	$n=3$	$n=4$
ϱ^0	plaq	−51.43471(4)	−51.43471(4)	−51.43471(4)	−51.435(8)
	sym	−40.44323(3)	−40.44323(3)	−40.44323(3)	−40.443(7)
ϱ^1	plaq	612.3029(1)	1224.6058(2)	1836.9087(3)	2449.21(3)
	sym	455.5139(1)	911.0278(2)	1366.5417(9)	1822.1(6)
ϱ^2	plaq	−2335.4071(3)	−14012.442(2)	−35031.106(4)	−65391.6(2)
	sym	−1685.5973(3)	−10113.584(3)	−25283.960(4)	−47196.889(4)
ϱ^3	plaq		81277.32696(7)	406386.6344(6)	1137883.43(5)
	sym		57399.24192(3)	286996.21(2)	803589.90(8)
ϱ^4	plaq		−193630.40(3)	−2904456.0(4)	−13554132(2)
	sym		−134388.50(2)	−2015827.5(3)	−9407196.9(4)
ϱ^5	plaq			11863189.778(7)	110723094(4)
	sym			8114869(5)	75738794(14)
ϱ^6	plaq			−21329904.75(2)	−597236493(336)
	sym			−14410242.41(3)	−403487007(319)
ϱ^7	plaq				1925422531(5277)
	sym				1286849516(2631)
ϱ^8	plaq				−2819282113(19240)
	sym				−1866581334(13091)

TABLE VIII. Same as Table III but for $\Sigma_0^{(1)}$ from Eq. (A1).

		$\Sigma_0^{(1)}$			
		$n=1$	$n=2$	$n=3$	$n=4$
ϱ^0	plaq	13.73313(4)	13.73313(4)	13.73313(4)	13.732(2)
	sym	11.94822(3)	11.94822(3)	11.94822(3)	11.947(2)
ϱ^1	plaq	−91.442(4)	−182.884(7)	−274.33(1)	−365.76(3)
	sym	−74.603(3)	−149.206(5)	−223.808(8)	−298.403(4)
ϱ^2	plaq	237.247540(3)	1423.48523(3)	3558.71310(5)	6642.94(9)
	sym	186.845029(2)	1121.070182(3)	2802.6754(3)	5231.66(3)
ϱ^3	plaq		−6094.07640(6)	−30470.3820(4)	−85317(2)
	sym		−4689.94425(4)	−23449.721432(8)	−65659.2(8)
ϱ^4	plaq		11191.09479(3)	167866.4219(4)	783374(2)
	sym		8465.23735(4)	126978.5602(6)	592567(2)
ϱ^5	plaq			−543236.523(8)	−5070208(26)
	sym			−405150.278(4)	−3781420.66(5)
ϱ^6	plaq			789161.8462(9)	22096532(104)
	sym			581383.242(7)	16278817(190)
ϱ^7	plaq				−58433357(1223)
	sym				−42575391.8(3)
ϱ^8	plaq				71058490(163)
	sym				51248188(2163)

TABLE IX. Same as Table III but for $\Sigma_0^{(2)}$ from Eq. (A1).

		$\Sigma_0^{(2)}$			
		$n=1$	$n=2$	$n=3$	$n=4$
ϱ^0	plaq	5.7151(3)	5.7151(3)	5.7151(3)	5.7150(5)
	sym	4.6627(2)	4.6627(2)	4.6627(2)	4.66255(5)
ϱ^1	plaq	−59.3118850(8)	−118.623770(2)	−177.935654(4)	−237.248(4)
	sym	−46.7112574(3)	−93.4225147(6)	−140.1337727(3)	−186.845(1)
ϱ^2	plaq	190.439888(2)	1142.639327(9)	2856.59832(3)	5332.32(7)
	sym	146.560758(2)	879.36454(1)	2198.41137(2)	4103.70(5)
ϱ^3	plaq		−5595.54740(2)	−27977.73695(9)	−78337.66(7)
	sym		−4232.61867(2)	−21163.09330(3)	−59256.7(2)
ϱ^4	plaq		11317.4275(3)	169761.412(4)	792220(5)
	sym		8440.6307(2)	126609.461(2)	590846.979(8)
ϱ^5	plaq			−591871.37(2)	−5524133(26)
	sym			−436037.438(2)	−4069704(48)
ϱ^6	plaq			913021.209(7)	25564594(2)
	sym			665240.039(6)	18626734(383)
ϱ^7	plaq				−71058490(163)
	sym				−51248188(2163)
ϱ^8	plaq				90125153(5113)
	sym				64387254(3655)

TABLE X. Same as Table IV but for $\Sigma_{10}^{(0)}$ from Eq. (A2).

		$\Sigma_{10}^{(0)}$			
		$n=1$	$n=2$	$n=3$	$n=4$
ϱ^0	plaq	11.852396(8)	11.852396(8)	11.852396(8)	11.852(2)
	sym	8.231254(4)	8.231254(4)	8.231254(4)	8.231(2)
ϱ^1	plaq	−202.0079(2)	−404.0158(3)	−606.0237(2)	−808.03(4)
	sym	−152.56416(3)	−305.12832(6)	−457.6925(2)	−610.3(2)
ϱ^2	plaq	739.6836(2)	4438.102(1)	11095.255(3)	20711.14(3)
	sym	541.38051(4)	3248.2831(3)	8120.7077(3)	15158.66(4)
ϱ^3	plaq		−24956.6947(4)	−124783.4729(5)	−349393.7(5)
	sym		−17858.09191(3)	−89290.4593(2)	−250013.3(2)
ϱ^4	plaq		57973.619(4)	869604.29(5)	4058153.0(5)
	sym		40735.013(3)	611025.19(5)	2851450.82(9)
ϱ^5	plaq			−3477653.965(2)	−32458091(8)
	sym			−2406174(1)	−22457620(2)
ϱ^6	plaq			6141663(3)	171966439(99)
	sym			4192996.284(4)	117403911(55)
ϱ^7	plaq				−545947087(1804)
	sym				−368383101(142)
ϱ^8	plaq				788877857(5503)
	sym				526824035(1817)

TABLE XI. Same as Table IV but for $\Sigma_{10}^{(1)}$ from Eq. (A2).

		$\Sigma_{10}^{(1)}$			
		$n=1$	$n=2$	$n=3$	$n=4$
ϱ^0	plaq	−2.248869(6)	−2.248869(6)	−2.248869(6)	−2.249(2)
	sym	−2.015426(5)	−2.015426(5)	−2.015426(5)	−2.01543(5)
ϱ^1	plaq	11.123836(8)	22.24767(2)	33.37151(3)	44.497(3)
	sym	9.346879(5)	18.693758(9)	28.040642(8)	37.388(3)
ϱ^2	plaq	−23.518665(6)	−141.11198(3)	−352.77996(7)	−658.52(8)
	sym	−18.958350(2)	−113.75010(2)	−284.37525(2)	−530.83(5)
ϱ^3	plaq		525.716776(9)	2628.58386(7)	7360.035(6)
	sym		411.66550(4)	2058.3275(2)	5763.32(2)
ϱ^4	plaq		−878.20974(7)	−13173.1458(7)	−61475(3)
	sym		−672.98584(5)	−10094.788(2)	−47107.302(7)
ϱ^5	plaq			39917.11(2)	372559.6(2)
	sym			30072.058(6)	280672.57(9)
ϱ^6	plaq			−55331.94(5)	−1549295(216)
	sym			−41099.241(6)	−1150778.8(7)
ϱ^7	plaq				3959099.6(6)
	sym				2904907(3)
ϱ^8	plaq				−4693170(2)
	sym				−3406008(4)

TABLE XII. Same as Table IV but for $\Sigma_{10}^{(2)}$ from Eq. (A2).

		$\Sigma_{10}^{(2)}$			
		$n=1$	$n=2$	$n=3$	$n=4$
ϱ^0	plaq	−1.397261(9)	−1.397261(9)	−1.397261(9)	−1.3973(3)
	sym	−1.242203(4)	−1.242203(4)	−1.242203(4)	−1.24223(9)
ϱ^1	plaq	6.69138(4)	13.38277(7)	20.07415(9)	26.762(8)
	sym	5.60061(2)	11.20122(4)	16.80183(6)	22.399(5)
ϱ^2	plaq	−11.978147(2)	−71.868869(6)	−179.67220(2)	−335.388(8)
	sym	−9.6055598(7)	−57.633356(1)	−144.083397(9)	−268.980(2)
ϱ^3	plaq		203.43279(2)	1017.16394(6)	2848.1(5)
	sym		156.799846(8)	783.99926(3)	2195.2(4)
ϱ^4	plaq		−229.16130(5)	−3437.4195(4)	−16041.293(2)
	sym		−168.15048(2)	−2522.2572(4)	−11770.532(2)
ϱ^5	plaq			5804.3207(6)	54173.62(4)
	sym			3880.9479(9)	36222.19(2)
ϱ^6	plaq			−2569.25(2)	−71939.0(4)
	sym			−1039.729(6)	−29112.3(3)
ϱ^7	plaq				−136222(83)
	sym				−172862(90)
ϱ^8	plaq				458914(149)
	sym				428921(211)

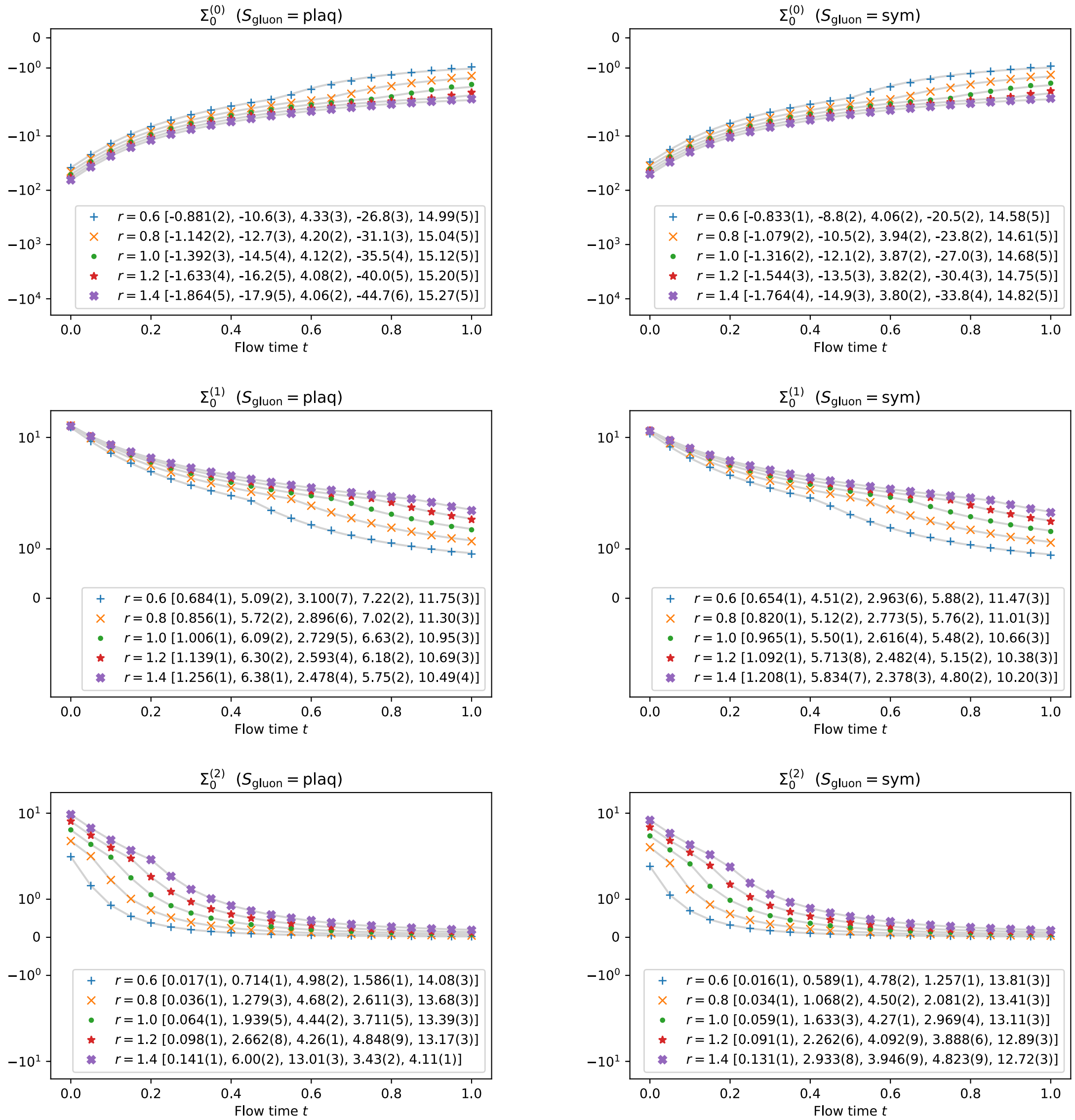

FIG. 12. Double exponential fits of the form $c_0 + c_1 e^{-c_2 t} + c_3 e^{-c_4 t}$ of $\Sigma_0^{(0)}$, $\Sigma_0^{(1)}$, and $\Sigma_0^{(2)}$ (top to bottom) with Wilson flow for different values of r plotted on a logarithmic y axis. The fit coefficients $[c_0, c_1, c_2, c_3, c_4]$ are given in brackets.

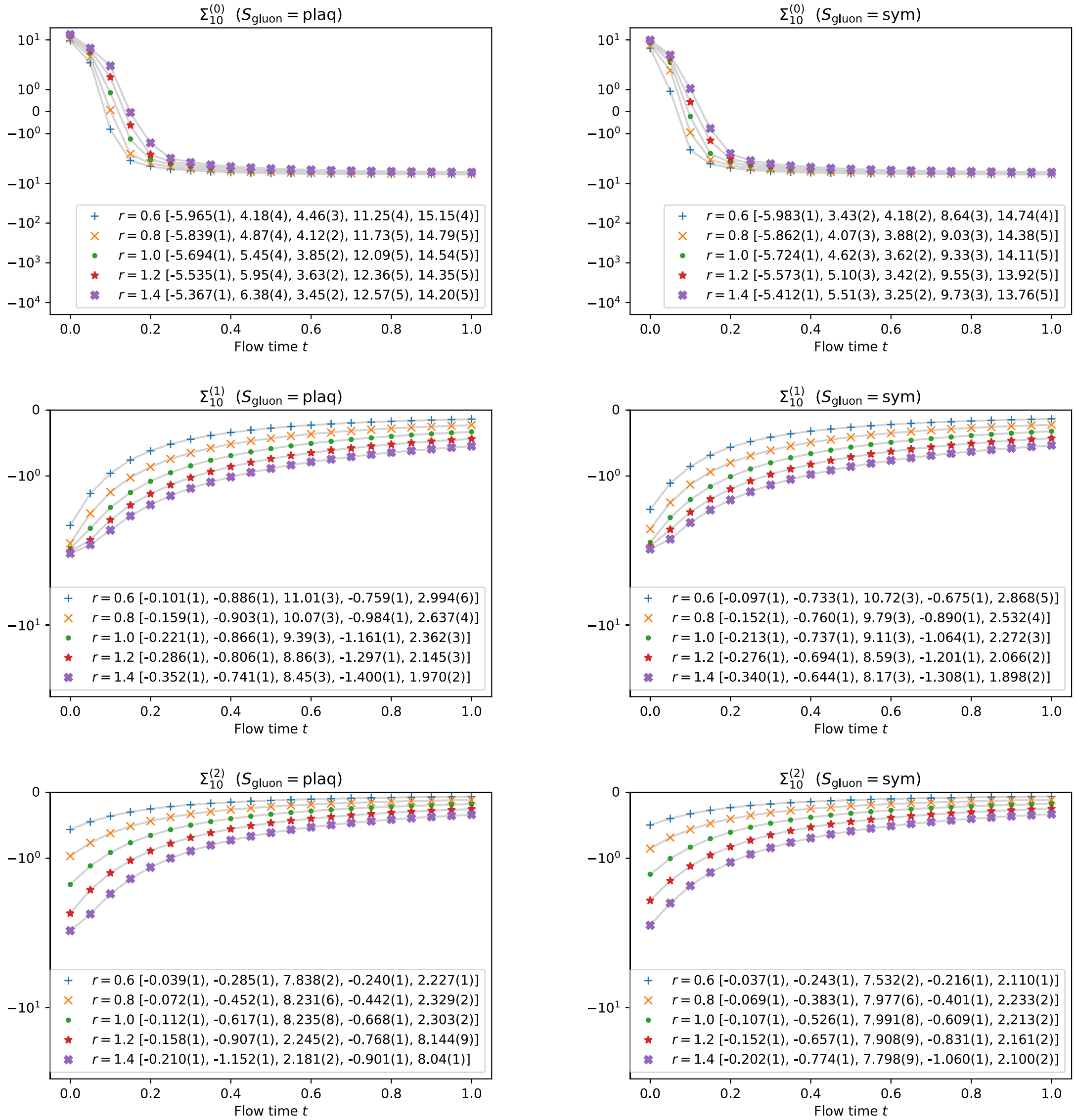

FIG. 13. Same as Fig. 12 but for $\Sigma_{10}^{(0)}$, $\Sigma_{10}^{(1)}$, and $\Sigma_{10}^{(2)}$ (top to bottom).

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