CP-odd window into long distance dynamics in rare semileptonic *B* decays

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We consider the combined measurements of CP-averaged decay rates and direct CP asymmetries of $B^\pm \to K^\pm \ell^+\ell^-$ and $B^\pm \to \pi^\pm \ell^+\ell^-$ to probe (nonlocal) four-quark operator matrix element contributions to rare semileptonic B meson decays. We also explore how their effects could be in principle disentangled from possible local new physics effects using U-spin relations. To this end, we construct a ratio of CP-odd decay rate differences which are exactly predicted within the standard model in the U-spin limit, while the leading U-spin breaking effects can also be systematically calculated. Our results motivate binned measurements of the direct CP asymmetry in $B^\pm \to \pi^\pm \ell^+\ell^-$ as well as dedicated theoretical estimates of U-spin breaking both in local form factors as well as in four-quark matrix elements.

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I. INTRODUCTION

Over the past decade, the LHCb experiment has produced several intriguing results on rare semileptonic decays of b-flavored hadrons [1–12]. In particular, the measurements of several decay rates as well as angular observables in $B \to K^{(*)}\mu^+\mu^-$ exhibit persistent tensions with current theoretical estimates with the standard model (SM) [13,14].

The theory of rare semileptonic decays unfortunately suffers from substantial hadronic uncertainties. The dominant contributions to the decay amplitudes can be divided into matrix elements of local (quark field bilinear) operators and nonlocal matrix elements of four-quark operators contracted with the electromagnetic current [15,16]. There has been tremendous progress in precision evaluations of the former using lattice QCD methods [17]. On the other hand, a robust theoretical estimation of the latter is still beyond reach, despite enduring efforts [15,18–26].

In the next decade, large datasets of both LHCb [27] as well as Belle II [28] experiments are expected to provide more detailed and precise measurements of rates, spectra as well as angular observables and CP asymmetries in both $b \to s$ and $b \to d$ semileptonic transitions. If the current intriguing results are confirmed and strengthened, it will be imperative to disentangle possible explanations in terms of unaccounted for hadronic effects from possible signals of physics beyond the standard model (BSM), see, e.g., Refs. [24,25,29–35] for some recent proposals.

In the present work we explore how the measurements of (direct) CP asymmetries in $B^\pm \to K^\pm \ell^+\ell^-$ and $B^\pm \to \pi^\pm \ell^+\ell^-$ could play a crucial role in this endeavor. In particular, direct CP violation arises from the interference of decay amplitudes with different CP-odd as well as CP-even phases. While local (short distance) SM as well as possible BSM amplitudes in these $B \to K\ell\ell$ and $B \to \pi\ell\ell$ processes can carry different CP-odd phases, there can be no significant CP-even phase differences between them.[1] Any signal of direct CP violation is thus necessarily proportional to the absorptive parts of some long-distance rescattering contributions. In the SM, these are dominated by nonlocal four-quark operator matrix elements and have precisely predicted CP-odd phases relative to the computable short distance amplitudes.

In the following we demonstrate how future combined measurements of CP-averaged decay rates and direct

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[1]Possibly observable electromagnetic rescattering effects in presence of large BSM $b \to s\tau^+\tau^-$ amplitudes have recently been considered in Ref. [36].

CP-asymmetries of $B^\pm \to K^\pm \ell^+\ell^-$ and $B^\pm \to \pi^\pm \ell^+\ell^-$ transitions can be used to learn about the sizes of the four-quark operator matrix elements and how their effects could be in principle disentangled from possible local BSM effects. As a byproduct, we construct a ratio of CP-odd decay rate differences of $B^\pm \to K^\pm \ell^+\ell^-$ and $B^\pm \to \pi^\pm \ell^+\ell^-$ which is exactly predicted in the U-spin limit and within a certain kinematical regime. We estimate this ratio in presence of know U-spin breaking effects due to kinematics and differences in local operator matrix elements (form factors).

The rest of the paper is structured as follows: In Sec. II we decompose the CP-even and CP-odd $B \to P\ell^+\ell^-$ rates in terms of local and long-distance amplitudes. We apply this decomposition to specific $b \to s$ and $b \to d$ transitions in Sec. III and discuss the Cabibbo-Kobayashi-Maskawa (CKM) and kinematics induced hierarchies of different contributions. Next, in Sec. IV we discuss how to combine information from both flavor modes using U spin and project the resulting sensitivity of possible future LHCb and Belle II measurements to long-distance effects, while in Sec. V we investigate possible sensitivity to short distance effects beyond the SM. Finally, we summarize our results in Sec. VI.

II. DECOMPOSITION OF CP STRUCTURE IN $B \to P\ell^+\ell^-$

The weak effective Lagrangian mediating $b \to q'\ell^+\ell^-$ processes reads

$$\mathcal{L}_{\rm eff}^{(q')} = \frac{4G_F\lambda_t^{(q')}}{\sqrt{2}}\sum_{i=3}^{10}\mathcal{C}_i\mathcal{O}_i^{(q')} + \frac{4G_F}{\sqrt{2}}\sum_{p=u,c}\lambda_p^{(q')}\sum_{i=1,2}\mathcal{C}_i\mathcal{O}_{i,p}^{(q')}, \tag{1}$$

where we have introduced CKM factors $\lambda_p^{(q')} = V_{pb}V_{pq'}*$ and separated short-distance top-quark contribution from the charged current operators involving u and c quarks. Within the SM we have the following semileptonic operators:

$$\mathcal{O}_7^{(q')} = \frac{em_b}{(4\pi)^2}\bar{q}'_L\sigma_{\mu\nu}b_RF^{\mu\nu}, \tag{2}$$

$$\mathcal{O}_9^{(q')} = \frac{\alpha}{4\pi}(\bar{q}'_L\gamma_\mu b_L)(\bar{\ell}\gamma^\mu\ell), \tag{3}$$

$$\mathcal{O}_{10}^{(q')} = \frac{\alpha}{4\pi}(\bar{q}'_L\gamma_\mu b_L)(\bar{\ell}\gamma^\mu\gamma^5\ell). \tag{4}$$

Here G_F and α are the Fermi and the fine structure constant, respectively. In addition, there are QCD penguin operators $\mathcal{O}_{3,\ldots,6}^{(q')}$ [37] as well as charged-current four-quark operators

$$\mathcal{O}_{1,p}^{(q')} = (\bar{q}'_{L\alpha}\gamma^\mu p_{L\beta})(\bar{p}_{L\beta}\gamma_\mu b_{L\alpha}), \tag{5}$$

$$\mathcal{O}_{2,p}^{(q')} = (\bar{q}'_L\gamma^\mu p_L)(\bar{p}_L\gamma_\mu b_L), \tag{6}$$

where α, β are color indices. The differential decay rate of $B \to P\ell^+\ell^-$, mediated by the above $b \to q'\ell^+\ell^-$ effective Lagrangian, with $\ell = e, \mu$ and $q' = s(d)$ for $P = K(\pi)$, can be written as [38]

$$\frac{d\Gamma_P}{dq^2} = \mathcal{N}_P(f_+^{(P)})^2(|\mathcal{C}_{10}|^2 + |\mathcal{C}_9^{\rm eff} + \tilde{f}_T^{(P)}\mathcal{C}_7|^2). \tag{7}$$

Here $q^2 \equiv (p_{\ell^+} + p_{\ell^-})^2$ and we have neglected terms of $\mathcal{O}(m_\ell^2)$.[2] In the above expression, $f_+^{(P)}(q^2)$ is the form factor for the vector-current $B \to P$ matrix element while $\tilde{f}_T^{(P)} \equiv 2f_T^{(P)}(q^2)(m_b + m_{q'})/f_+^{(P)}(q^2)(m_B + m_P)$ is a ratio of tensor-to-vector form factors. We use the conventional definition of the form factors (as in, e.g., [18]) and employ lattice QCD results for $B \to \pi$ [39] and $B \to K$ [26] transitions. Correspondingly, $\mathcal{C}_i$ are the relevant local operator Wilson coefficients with short-distance SM values $\mathcal{C}_{10}^{\rm SM} = -4.31$, $\mathcal{C}_7^{\rm SM} = -0.292$ and $\mathcal{C}_9^{\rm SM} = 4.07$ at the scale $\mu_b = 4.8$ GeV [40]. We assume that short distance $\mathcal{C}_{7,9,10}$ are real throughout the paper unless explicitly stated otherwise. Finally, the q^2-dependent normalization factor $\mathcal{N}_P$ reads

$$\mathcal{N}_P = \frac{G_F^2\alpha^2|\lambda_t^{(q')}|^2}{2^9 \times 3\pi^5 m_B^3}\lambda_P^{3/2}(q^2), \tag{8}$$

with $\lambda_P(q^2) = (m_B^2 + m_P^2 + q^2)^2 - 2(m_B^2m_P^2 + m_B^2q^2 + m_P^2q^2)$. Contributions from four-quark operators can be taken into account by effectively modifying $\mathcal{C}_9$ as follows

$$\mathcal{C}_9^{\rm eff}(q^2) = \mathcal{C}_9 - \tilde{\lambda}_c^{(q')}Y_{c\bar{c}}(q^2) - \tilde{\lambda}_u^{(q')}Y_{u\bar{u}}(q^2) + Y_{d\bar{d}}(q^2) + Y_{s\bar{s}}(q^2), \tag{9}$$

where $Y_{p\bar{p}}(q^2)$ parametrize the effects due to $p\bar{p}$ quark rescattering amplitudes. In particular, they represent both local and nonlocal contributions of effective four-quark operators $\mathcal{O}_{i,p}^{(q')}$ and $\mathcal{O}_{3,\ldots,6}^{(q')}$ contracted with the electromagnetic vertices $eQ_p\bar{p}\not{A}p$ to the amplitude [21]. Explicitly, for $p = u$ or c

$$f_+^{(P)}Y_{p\bar{p}}(q^2) \equiv \frac{(8\pi)^2Q_p}{\lambda_P(q^2)}\int d^4xe^{iq\cdot x}\langle P(k)| \times \mathcal{T}\left\{\bar{p}\not{k}p(x), \sum_{i=1,2}\mathcal{C}_i\mathcal{O}_{i,p}^{(q')}(0)\right\}|\bar{B}(q+k)\rangle, \tag{10}$$

where $\mathcal{T}$ is the time-ordering operator. For $p = d, s$ only the QCD penguin operators $\mathcal{O}_{3\cdots,6}^{(q')}$ contribute in the above time-ordered product. Due to their tiny Wilson coefficients [37] we

[2] We consistently set $m_\ell = 0$ in all subsequent expressions.

can safely neglect them and set $Y_{d\bar{d}}, Y_{s\bar{s}} \to 0$. In general $Y_{p\bar{p}}(q^2)$ depend on the p flavor as well as the final state meson P with light flavor q'. We leave these dependencies implicit and return to this point in Sec. IV. In the above expressions we have introduced $\tilde{\lambda}_p^{(q')} = \lambda_p^{(q')}/\lambda_t^{(q')}$ and we will repeatedly employ the unitarity of the CKM, $\tilde{\lambda}_u^{(q')} + \tilde{\lambda}_c^{(q')} + 1 = 0$, in the following. Short distance contributions to $Y_{q\bar{q}}(q^2)$ can be computed perturbatively [37]; however in the following we do not factorize the short and long distance (LD) effects. Then the CP-averaged decay rate and the CP-odd rate difference read

$$\frac{(d\Gamma_P + d\bar{\Gamma}_P)/2}{dq^2} = N_P (f_+^{(P)})^2 [\mathcal{C}_{10}^2 + (\mathcal{C}_9 + \tilde{f}_T^{(P)}\mathcal{C}_7)^2 - 2(\mathcal{C}_9 + \tilde{f}_T^{(P)}\mathcal{C}_7)\{\mathrm{Re}(\tilde{\lambda}_c^{(q')})(\mathrm{Re}Y_{c\bar{c}} - \mathrm{Re}Y_{u\bar{u}}) - \mathrm{Re}Y_{u\bar{u}}\} \\ -2(|\tilde{\lambda}_c^{(q')}|^2 + \mathrm{Re}\tilde{\lambda}_c^{(q')})\mathrm{Re}(Y_{c\bar{c}}Y_{u\bar{u}}^*) + |\tilde{\lambda}_c^{(q')}|^2 |Y_{c\bar{c}}|^2 + |\tilde{\lambda}_u^{(q')}|^2 |Y_{u\bar{u}}|^2], \tag{11}$$

$$\frac{d\Gamma_P - d\bar{\Gamma}_P}{dq^2} = 4\mathcal{N}_P (f_+^{(P)})^2 \mathrm{Im}\tilde{\lambda}_c^{(q')}[(\mathcal{C}_9 + \tilde{f}_T^{(P)}\mathcal{C}_7)(\mathrm{Im}Y_{c\bar{c}} - \mathrm{Im}Y_{u\bar{u}}) + \mathrm{Im}(Y_{c\bar{c}}Y_{u\bar{u}}^*)]. \tag{12}$$

Here we denote by Γ_P ($\bar{\Gamma}_P$) the decay rate of $B^{-(+)} \to P^{-(+)}\ell^+\ell^-$. We immediately observe that the expression for the CP-odd rate difference is proportional to the absorptive amplitude and thus uniquely probes imaginary parts of $Y_{c\bar{c}}$ and $Y_{u\bar{u}}$. Further simplifications to the above expressions can arise due to specific CKM ($\tilde{\lambda}_p^{(q')}$) hierarchies and in specific kinematical (q^2) regions where the absorptive amplitudes are constrained. We study these effects for specific cases of $B \to K$ and $B \to \pi$ transitions in the next section.

III. $B \to K\ell\ell$ VS $B \to \pi\ell\ell$

In the case of $b \to s$ transition the overall CKM factor using the Wolfenstein expansion up to $\mathcal{O}(\lambda^4)$ is $\lambda_t^{(s)} = -A\lambda^2 + A\lambda^4(1/2 + i\eta - \rho)$, whereas $Y_{c\bar{c}}$ and $Y_{u\bar{u}}$ enter the amplitude with relative CKM factors

$$\tilde{\lambda}_c^{(s)} = -1 + \lambda^2(\rho - i\eta) + \mathcal{O}(\lambda^4), \\ \tilde{\lambda}_u^{(s)} = -\lambda^2(\rho - i\eta) + \mathcal{O}(\lambda^4). \tag{13}$$

The CKM hierarchy suggests that the CP-averaged rate is linearly sensitive to $\mathrm{Re}Y_{c\bar{c}}$. Indeed, we find that the rate is only quadratically sensitive to $\mathrm{Im}Y_{c\bar{c}}$ and that $Y_{u\bar{u}}$ does not contribute up to $\mathcal{O}(\lambda^4)$, thus

$$\frac{(d\Gamma_K + d\bar{\Gamma}_K)/2}{dq^2} = \mathcal{N}_K (f_+^{(K)})^2 [\mathcal{C}_{10}^2 + (\mathcal{C}_9 + \tilde{f}_T^{(K)}\mathcal{C}_7)^2 \\ + 2(\mathcal{C}_9 + \tilde{f}_T^{(K)}\mathcal{C}_7)\mathrm{Re}Y_{c\bar{c}} + |Y_{c\bar{c}}|^2 + \mathcal{O}(\lambda^2)]. \tag{14}$$

Although we are considering low $q^2 \lesssim 6\ \mathrm{GeV}^2$ region below the $c\bar{c}$ threshold, we cannot discard $\mathrm{Im}Y_{c\bar{c}}$ since the $\bar{c}c\bar{s}b$ operators can generate absorptive contributions at any q^2 via the intermediate on-shell DD_s^* and similar states [21,41,42].

The CP-odd rate difference, being proportional to $\mathrm{Im}\tilde{\lambda}_c^{(s)}$, is $\eta\lambda^2$ suppressed compared to the CP-averaged rate:

$$\frac{d\Gamma_K - d\bar{\Gamma}_K}{dq^2} = 4\mathcal{N}_K (f_+^{(K)})^2 \eta\lambda^2 [1 + \mathcal{O}(\lambda^2)] \\ \times [(\mathcal{C}_9 + \tilde{f}_T^{(K)}\mathcal{C}_7)\mathrm{Im}(Y_{u\bar{u}} - Y_{c\bar{c}}) - \mathrm{Im}(Y_{c\bar{c}}Y_{u\bar{u}}^*)]. \tag{15}$$

It is interesting to note that the CP-odd rate difference in $B \to K\ell\ell$ is mostly driven by relative strong phase between $Y_{u\bar{u}} - Y_{c\bar{c}}$ and the short-distance local contribution, while the long-distance strong phase $\mathrm{Im}(Y_{c\bar{c}}Y_{u\bar{u}}^*)$ is subleading. Currently observed deviations of experimental rates from predictions based on the known SM contributions allow for additional contributions from $Y_{c\bar{c}}$ compatible with $\frac{Y_{c\bar{c}}}{\mathcal{C}_9^{\mathrm{SM}}} \sim 15$–$45\%$. For instance, Refs. [13,35] consider mode dependent lepton universal contributions to $\mathcal{C}_9$, leading to 31% and 44% corrections, respectively, for the $B \to K$ mode. While in Ref. [33] they obtain mode independent contributions of order 15%. This suggests a dominance of the $\mathrm{Re}Y_{c\bar{c}}$ term over the $|Y_{c\bar{c}}|^2$ term in the CP-averaged rate and of the $\mathrm{Im}(Y_{u\bar{u}} - Y_{c\bar{c}})$ term over the $\mathrm{Im}(Y_{c\bar{c}}Y_{u\bar{u}}^*)$ in the CP-odd rate. Consequently, in this region and assuming only SM short distance contributions, combined measurements of CP-averaged rates and CP-odd rate differences can determine the contributing long distance amplitudes, namely $\mathrm{Re}Y_{c\bar{c}}$ and $\mathrm{Im}(Y_{u\bar{u}} - Y_{c\bar{c}})$.[3] More generally however, the presence of two independent long-distance quantities in two observables

[3]In principle, $\mathrm{Im}(Y_{u\bar{u}} - Y_{c\bar{c}})$ can be extracted from a measurement of CP-odd decay width difference only if we neglect $\mathrm{Im}(Y_{c\bar{c}}Y_{u\bar{u}}^*)$ in Eq. (15). Validity of this assumption can be checked *a posteriori*.

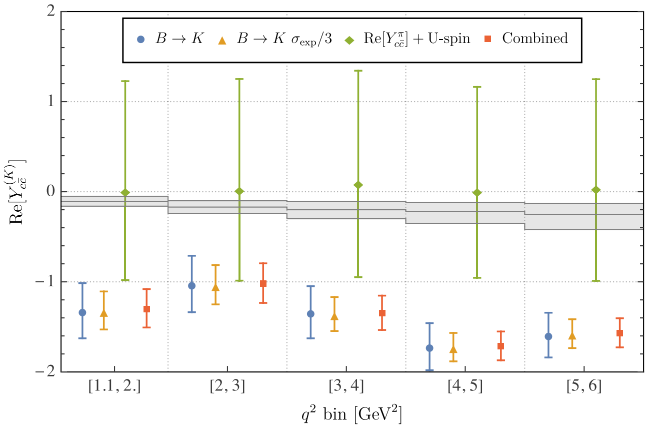

FIG. 1. Current (LHCb) and projected $B \to K\mu\mu$ and $B \to \pi\mu\mu$ 1σ constraints on $\mathrm{Re}[Y^{(K)}_{c\bar{c}}]$ by q^2 bins from 1.1 to 6 GeV^2 following Table I. In blue (circles) we show the current $B \to K\mu\mu$ LHCb constrains [1], while in yellow (triangles), we show projected uncertainties. In green (diamonds) we show projected constraints from the $B \to \pi\mu\mu$ mode, where we marginalize over the U-spin breaking parameter ϵ_c defined in Sec. IV assuming a uniform prior with support $[-0.3, 0.3]$. In orange (squares), we show the combination of both $B \to K\mu\mu$ and $B \to \pi\mu\mu$ projected constraints. For comparison the predictions of $\mathrm{Re}[Y^{(K)}_{c\bar{c}}]$ for $B \to K\mu\mu$ from Ref. [21] are shown as a gray band.

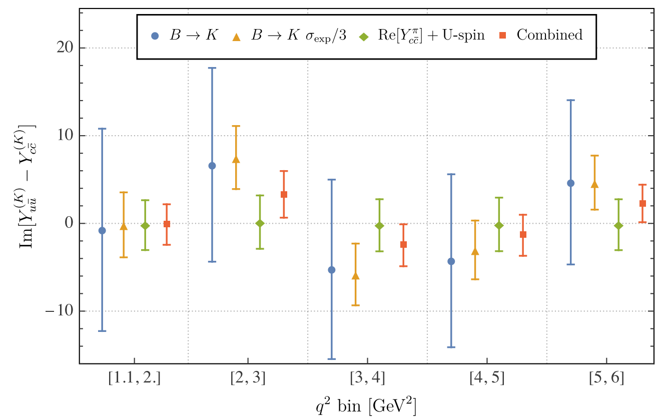

FIG. 2. Current (LHCb) [47] and projected $B \to K\mu\mu$ and $B \to \pi\mu\mu 1\sigma$ constraints on $\mathrm{Im}[Y^{(K)}_{u\bar{u}} - Y^{(K)}_{c\bar{c}}]$ by q^2 bins from 1.1 to 6 GeV^2 following Table I. Color coding is the same as Fig. 1. To obtain the constraint from the $B \to \pi\mu\mu$ mode we marginalize over the U-spin breaking parameter ϵ_{uc} defined in Sec. IV assuming a uniform prior with support $[-0.3, 0.3]$.

implies that BSM effects in $\mathcal{C}_7$ and/or $\mathcal{C}_9$ cannot be disentangled from effects in $\mathrm{Im}(Y_{u\bar{u}} - Y_{c\bar{c}})$ and $\mathrm{Re}Y_{c\bar{c}}$ without additional theoretical input for the latter (for a recent attempt in this direction see Refs. [25,43,44]).

To illustrate the sensitivity of current LHCb measurements, namely the binned CP-averaged rate and the direct CP asymmetry [defined as $\mathcal{A}_{CP} = (\Gamma_K - \bar{\Gamma}_K)/(\Gamma_K + \bar{\Gamma}_K)$] in $B \to K\mu^+\mu^-$ [1,45–47], we plot in Figs. 1, 2 in blue and yellow the resulting allowed parameter space in the plane of $\mathrm{Re}Y_{c\bar{c}}$ and $\mathrm{Im}(Y_{u\bar{u}} - Y_{c\bar{c}})$,[4] for the q^2 bins ranging from $q^2 \in [1.1, 2]$ to $q^2 \in [5, 6]$ GeV^2. To derive these constraints we have used the most recent evaluation of the relevant form factors including their (correlated) uncertainties from Ref. [26], while the CKM and EW parameters as well as quark and meson masses are taken from PDG [48]. The currently allowed ranges from experimental data can be compared to theoretical estimates for $Y_{c\bar{c}}$ and $Y_{u\bar{u}}$ based on QCD factorization and light cone sum rules (dubbed light cone operator product expansion) [21,23], shown in gray.

In the case of $b \to d$ transition there is no hierarchy between the CKM factors

$$\tilde{\lambda}_c^{(d)} = \frac{\rho - 1 + i\eta}{(1-\rho)^2 + \eta^2} + \mathcal{O}(\lambda^2),$$
$$\tilde{\lambda}_u^{(d)} = \frac{\rho(1-\rho) - \eta^2 - i\eta}{(1-\rho)^2 + \eta^2} + \mathcal{O}(\lambda^2), \qquad (16)$$

[4]More precisely we plot their values averaged over the corresponding q^2 bins.

however, the numerical values $\tilde{\lambda}_c^{(d)} = -0.98 + 0.40i$, $\tilde{\lambda}_u^{(d)} = -0.020 - 0.40i$ reveal an accidental cancellation in the real part of $\tilde{\lambda}_u^{(d)}$, which is related to the smallness of $\xi \equiv \rho(1-\rho) - \eta^2 = -0.022$ (equivalently, unitarity angle $\alpha = \phi_3 \approx \pi/2$). Nonetheless, the CP-averaged rate depends in general on both $Y_{c\bar{c}}$, $Y_{u\bar{u}}$:

$$\begin{aligned}\frac{(d\Gamma_\pi + d\bar{\Gamma}_\pi)/2}{dq^2} = {} & N_\pi (f_+^{(\pi)})^2 [\mathcal{C}_{10}^2 + (\mathcal{C}_9 + \tilde{f}_T^{(\pi)} \mathcal{C}_7)^2 \\ & + 2(\mathcal{C}_9 + \tilde{f}_T^{(\pi)} \mathcal{C}_7)\mathrm{Re}Y_{c\bar{c}} + |Y_{c\bar{c}}|^2 \\ & + (\mathrm{Im}\tilde{\lambda}_u^{(d)})^2 |Y_{u\bar{u}} - Y_{c\bar{c}}|^2 + \mathcal{O}(\xi)]. \qquad (17)\end{aligned}$$

The leading dependence is again on $\mathrm{Re}Y_{c\bar{c}}$, followed, in decreasing order of numerical importance, by $|Y_{c\bar{c}}|^2$ and then $(\mathrm{Im}\tilde{\lambda}_u^{(d)})^2 |Y_{u\bar{u}} - Y_{c\bar{c}}|^2$. The latter term is rendered unimportant by a prefactor $(\mathrm{Im}\tilde{\lambda}_u^{(d)})^2 = 0.16$. In our numerical analysis we have kept only the linear $\mathrm{Re}Y_{c\bar{c}}$ term. Note that for theoretically preferred values of $Y_{c\bar{c}}$ and $Y_{u\bar{u}}$ [22], the error we are making with this approximation is at most ~2%.[5] The $B \to \pi CP$-odd rate difference reads

$$\begin{aligned}\frac{d\Gamma_\pi - d\bar{\Gamma}_\pi}{dq^2} = {} & 4\mathcal{N}_\pi (f_+^{(\pi)})^2 \frac{(-\eta)[1 + \mathcal{O}(\lambda^2)]}{(1-\rho)^2 + \eta^2} \\ & \times [(\mathcal{C}_9 + \tilde{f}_T^{(\pi)} \mathcal{C}_7)\mathrm{Im}(Y_{u\bar{u}} - Y_{c\bar{c}}) - \mathrm{Im}(Y_{c\bar{c}} Y_{u\bar{u}}^*)]. \qquad (18)\end{aligned}$$

Since currently there is no available determination of the $B \to \pi\mu\mu CP$ asymmetry in the nonresonant q^2 regions, we show in Figs. 1, 2 projected constrains of $\mathrm{Re}[Y_{c\bar{c}}]$ and

[5]This error corresponds to $|Y_{c\bar{c}}| \sim 1$.

TABLE I. Current LHCb measurements [1,47] and projected measurements used to obtain Figs. 1 and 2. The projections for the $B \to \pi\mu\mu$ mode assume a 20% uncertainty for the CP-averaged rate $\frac{d(\Gamma+\bar{\Gamma})/2}{dq^2}$ and the direct CP-asymmetry ($\mathcal{A}_{CP}$) measurements. In the case of the $B \to K\mu\mu$ we assume a reduction of a factor of 3 of uncertainties.

	Observable	Bin				
		[1.1, 2.0] GeV^2	[2.0, 3.0] GeV^2	[3.0, 4.0] GeV^2	[4.0, 5.0] GeV^2	[5.0, 6.0] GeV^2
Current	$\mathrm{Br}(B \to K\mu\mu) \times 10^8$ [1]	2.33 ± 0.19	2.82 ± 0.21	2.54 ± 0.20	2.21 ± 0.18	2.31 ± 0.18
	$\mathcal{A}_{CP}(B \to K\mu\mu)$ [47]	-0.004 ± 0.068	0.042 ± 0.059	-0.034 ± 0.063	-0.021 ± 0.064	0.031 ± 0.062
Projected	$\mathrm{Br}(B \to K\mu\mu) \times 10^8$	2.33 ± 0.06	2.82 ± 0.07	2.54 ± 0.06	2.21 ± 0.05	2.31 ± 0.06
	$\mathcal{A}_{CP}(B \to K\mu\mu)$	-0.004 ± 0.023	0.042 ± 0.020	-0.034 ± 0.021	-0.021 ± 0.021	0.031 ± 0.021
	$\mathrm{Br}(B \to \pi\mu\mu) \times 10^{10}$	7.6 ± 1.5	8.5 ± 1.7	8.5 ± 1.7	8.4 ± 1.7	8.4 ± 1.7
	$\mathcal{A}_{CP}(B \to \pi\mu\mu)$	0.0 ± 0.2	0.0 ± 0.2	0.0 ± 0.2	0.0 ± 0.2	0.0 ± 0.2

$\mathrm{Im}[Y_{u\bar{u}} - Y_{c\bar{c}}]$ assuming a 20% uncertainty for both the CP-averaged rate and the direct CP asymmetry. Again we need to neglect the term $\mathrm{Im}(Y_{c\bar{c}}Y^*_{u\bar{u}})$ to extract the dominant $\mathrm{Im}(Y_{c\bar{c}} - Y_{u\bar{u}})$ contribution in Eq. (18).

IV. COMBINING $B \to K$ AND $B \to \pi$ CP DIFFERENCES

Given the similarity between the theoretical expressions for the CP-odd decay rate differences in $B \to K$ [Eq. (15)] and $B \to \pi$ [Eq. (18)] transitions, it is interesting to consider their ratio

$$R^{CP}_{K/\pi} \equiv -\frac{(d\Gamma_K - d\bar{\Gamma}_K)/dq^2}{(d\Gamma_\pi - d\bar{\Gamma}_\pi)/dq^2}. \tag{19}$$

In the U-spin limit of $Y_{q\bar{q}}$ functions, namely that $Y_{u\bar{u}}(q^2)$ and $Y_{c\bar{c}}(q^2)$ are equal for the $B \to K$ and $B \to \pi$ decays,[6] the $R^{CP}_{K/\pi}$ ratio is predictable within the SM. We do incorporate well-known sources of U-spin breaking in the form factors and kinematics. The ratio is $\mathcal{O}(1)$, since both rate differences are of Cabibbo order $A^2\lambda^6$, even though CP-averaged rates and CP asymmetries are very different for the two modes.

Now we put explicit P label on $Y^{(P)}_{q\bar{q}}$ functions and introduce three U-spin breaking parameters: ϵ_c, ϵ_{uc}, and ω_{uc}. In particular, we use $Y^{(K)}_{u\bar{u}}$ and $Y^{(K)}_{c\bar{c}}$ as four-quark contributions in $B \to K\ell\ell$ whereas the corresponding $B \to \pi\ell\ell$ contributions are given as

$$\begin{aligned}
\mathrm{Re}Y^{(\pi)}_{c\bar{c}} &= (1+\epsilon_c)\mathrm{Re}Y^{(K)}_{c\bar{c}},\\
\mathrm{Im}(Y^{(\pi)}_{u\bar{u}} - Y^{(\pi)}_{c\bar{c}}) &= (1+\epsilon_{uc})\mathrm{Im}(Y^{(K)}_{u\bar{u}} - Y^{(K)}_{c\bar{c}}),\\
\mathrm{Im}(Y^{(\pi)}_{c\bar{c}}Y^{(\pi)*}_{u\bar{u}}) &= (1+\omega_{uc})\mathrm{Im}(Y^{(K)}_{c\bar{c}}Y^{(K)*}_{u\bar{u}}).
\end{aligned} \tag{20}$$

Combining the expressions (15) and (18) while keeping the quadratic terms in $Y^{(P)}_{c\bar{c}}$ and $Y^{(P)}_{u\bar{u}}$, we obtain the SM prediction:

$$\begin{aligned}
R^{CP}_{K/\pi}|_{\mathrm{SM}} &= \left(\frac{\lambda_K}{\lambda_\pi}\right)^{3/2}\left(\frac{f^{(K)}_+}{f^{(\pi)}_+}\right)^2(1+\mathcal{O}(\lambda^4))\\
&\quad\times\left\{1+\frac{\mathrm{Im}(Y^{(K)}_{u\bar{u}} - Y^{(K)}_{c\bar{c}})[\mathcal{C}^{\mathrm{SM}}_7(\tilde{f}^{(\pi)}_T - \tilde{f}^{(K)}_T) + \epsilon_{uc}(\mathcal{C}^{\mathrm{SM}}_9 + \mathcal{C}^{\mathrm{SM}}_7\tilde{f}^{(\pi)}_T)] - \omega_{uc}\mathrm{Im}(Y^{(K)}_{c\bar{c}}Y^{(K)*}_{u\bar{u}})}{(\mathcal{C}^{\mathrm{SM}}_9 + \mathcal{C}^{\mathrm{SM}}_7\tilde{f}^{(K)}_T)\mathrm{Im}(Y^{(K)}_{u\bar{u}} - Y^{(K)}_{c\bar{c}}) - \mathrm{Im}(Y^{(K)}_{c\bar{c}}Y^{(K)*}_{u\bar{u}})}\right\}^{-1}.
\end{aligned} \tag{21}$$

In the exact U-spin limit this ratio becomes 1. Well known sources of U-spin breaking are contained in kinematics and in different form factors for $B \to K$ and $B \to \pi$ transitions and when included $R^{CP}_{K/\pi}$ takes the values shown in Table II and Fig. 3. The unknown U-spin breaking in $Y^{(P)}_{q\bar{q}}$ is parameterized by ε_{uc} and ω_{uc} that can be up to $\sim$30%, a value supported by the experimental data on branching fractions of $B^+ \to J/\psi K^+$ and $B^+ \to J/\psi\pi^+$. Namely, the amplitudes of those decays are proportional to $Y^{(K)}_{c\bar{c}}(q^2 = m^2_{J/\psi})$ and $Y^{(\pi)}_{c\bar{c}}(q^2 = m^2_{J/\psi})$, respectively. We extract their ratio from the measured decay widths [48], and while correcting for differences in CKM factors and final state momenta $|\boldsymbol{k}_P|$, we find that the U-spin breaking is indeed within the assumed limits:

$$\begin{aligned}
\left|\frac{Y^{(K)}_{c\bar{c}}}{Y^{(\pi)}_{c\bar{c}}}\right|_{q^2=m^2_{J/\psi}} &= \left|\frac{\lambda^{(d)}_c}{\lambda^{(s)}_c}\right|\sqrt{\frac{|\boldsymbol{k}_\pi|}{|\boldsymbol{k}_K|}\frac{\Gamma(B^+\to J/\psi K^+)}{\Gamma(B^+\to J/\psi\pi^+)}},\\
&= 1.2.
\end{aligned} \tag{22}$$

We are thus allowed to expand the second line of Eq. (21) and get, in the leading U-spin breaking approximation:

[6] $Y^{(K)}_{u\bar{u}}(q^2) = Y^{(\pi)}_{u\bar{u}}(q^2)$ and $Y^{(K)}_{c\bar{c}}(q^2) = Y^{(\pi)}_{c\bar{c}}(q^2)$.

TABLE II. Binned SM prediction for the $R^{CP}_{K/\pi}$ ratio in the U-spin limit for the hadronic corrections ($\Delta_U = 0$).

q^2[GeV] bin	$R^{CP}_{K/\pi}$
[1.1, 2.0]	1.92 ± 0.32
[2.0, 3.0]	1.90 ± 0.31
[3.0, 4.0]	1.88 ± 0.30
[4.0, 5.0]	1.86 ± 0.29
[5.0, 6.0]	1.84 ± 0.28

$$R^{CP}_{K/\pi}|_{\mathrm{SM}} = \left(\frac{\lambda_K}{\lambda_\pi}\right)^{3/2}\left(\frac{f_+^{(K)}}{f_+^{(\pi)}}\right)^2 (1+\mathcal{O}(\lambda^4)) \times\left[1-\frac{\mathcal{C}_7^{\mathrm{SM}}(\tilde{f}_T^{(\pi)}-\tilde{f}_T^{(K)})}{\mathcal{C}_9^{\mathrm{SM}}+\mathcal{C}_7^{\mathrm{SM}}\tilde{f}_T^{(K)}}-\epsilon_{uc}+\mathcal{O}(\Delta_U^2)\right]. \quad (23)$$

Here we have expanded to linear order in parameter Δ_U, which stands for any of the following small quantities:

$$\Delta_U \in \left\{\epsilon_{uc}, \omega_{uc}, \frac{\mathcal{C}_7^{\mathrm{SM}}(\tilde{f}_T^{(\pi)}-\tilde{f}_T^{(K)})}{\mathcal{C}_9^{\mathrm{SM}}+\mathcal{C}_7^{\mathrm{SM}}\tilde{f}_T^{(K)}}\right\}. \quad (24)$$

We have also neglected corrections of the order

$$\frac{\Delta_U}{\mathcal{C}_9^{\mathrm{SM}}+\mathcal{C}_7^{\mathrm{SM}}\tilde{f}_T^{(K)}}\frac{\mathrm{Im}(Y_{c\bar{c}}^{(K)}Y_{u\bar{u}}^{(K)*})}{\mathrm{Im}(Y_{u\bar{u}}^{(K)}-Y_{c\bar{c}}^{(K)})}. \quad (25)$$

Note that $\mathrm{Im}Y_{c\bar{c}}^{(K)}$ and $\mathrm{Im}Y_{u\bar{u}}^{(K)}$ are expected to be significantly different functions of q^2 and thus the second factor in the above expression is expected to be well bounded. Thus the uncertainty on $R^{CP}_{K/\pi}|_{\mathrm{SM}}$ is determined by U-spin breaking while the values of $Y_{q\bar{q}}$ play no important role.

The square bracket in Eq. (23) can be different from 1 mostly due to ϵ_{uc} while the known U-spin breaking is smaller due to $\mathcal{C}_7^{\mathrm{SM}}/\mathcal{C}_9^{\mathrm{SM}}$ suppression and smallness of $\tilde{f}_T^{(K)}-\tilde{f}_T^{(\pi)}$. Note also that the dependence on CKM cancels out in Eq. (23).

V. SENSITIVITY TO NEW PHYSICS

A. *CP*-even new physics

The expression (23) remains valid in the presence of minimal flavour violating new physics where the contributions to $b\to s\ell\ell$ and $b\to d\ell\ell$ are aligned with the CKM factors of the SM [49]. Specifically, the contribution of new physics to $\mathcal{C}_9$ and $\mathcal{C}_7$ should be real and U-spin symmetric in order for (23) to apply, with real shifts $\mathcal{C}_i^{\mathrm{SM}} \to \mathcal{C}_i^{\mathrm{SM}} + \delta\mathcal{C}_i$, for both transitions ($b\to s\ell\ell$ or $b\to d\ell\ell$) alike.

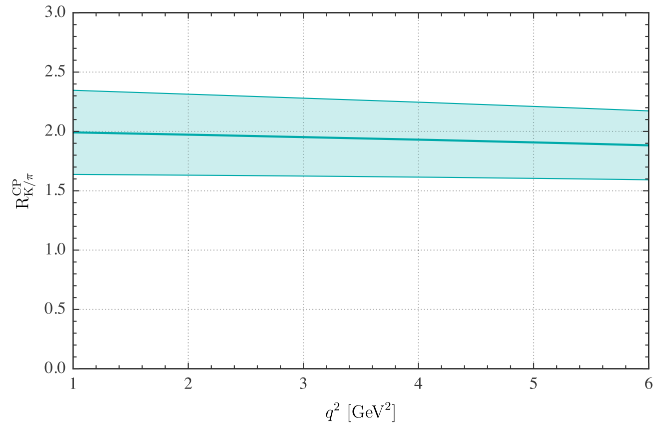

FIG. 3. SM prediction for the $R^{CP}_{K/\pi}$ ratio in the U-spin limit for the hadronic corrections ($\epsilon_{uc} = 0$).

In order to incorporate more general scenarios, such as having NP contribution only in $b\to s\ell\ell$ transitions, we have to modify the $R^{CP}_{K/\pi}$ expression accordingly. Here we introduce NP as a real modification of Wilson coefficients for $b\to s\ell\ell$: $\mathcal{C}_{7,9}^{(s)} = \mathcal{C}_{7,9}^{\mathrm{SM}} + \delta\mathcal{C}_{7,9}^{(s)}$. For such U-spin breaking NP contributions $R^{CP}_{K/\pi}$ becomes

$$R^{CP}_{K/\pi}\Big|_{\mathrm{ReNP}} = R^{CP}_{K/\pi}|_{\mathrm{SM}}\left(1+\frac{\delta\mathcal{C}_9^{(s)}+\delta\mathcal{C}_7^{(s)}\tilde{f}_T^{(K)}}{\mathcal{C}_9^{\mathrm{SM}}+\mathcal{C}_7^{\mathrm{SM}}\tilde{f}_T^{(K)}}\right) \times\left[1+\mathcal{O}\left(\frac{\delta\mathcal{C}_i^{(s)}}{(\mathcal{C}_9^{\mathrm{SM}})^2}\frac{\mathrm{Im}(Y_{c\bar{c}}^{(K)}Y_{u\bar{u}}^{(K)*})}{\mathrm{Im}(Y_{u\bar{u}}^{(K)}-Y_{c\bar{c}}^{(K)})}\right)\right]. \quad (26)$$

The correction in square brackets is negligible in comparison to leading uncertainties in $R^{CP}_{K/\pi}|_{\mathrm{SM}}$. In order to discern $\delta\mathcal{C}_9^{(s)}$ effect from the U-spin breaking the relative modification of $\delta\mathcal{C}_9^{(s)}/\mathcal{C}_9^{\mathrm{SM}}$ should be larger than ϵ_{uc} and form factors' uncertainty combined. We note that in each bin $R^{(CP)}_{K/\pi}$ could resolve accidental cancellations between CP-even NP and $Y_{q\bar{q}}$ in CP-averaged rate $B\to K\ell\ell$, since it probes a different combination of $\delta\mathcal{C}_i$ and $Y_{q\bar{q}}$.

B. *CP*-violating new physics

Let us discuss now what happens if we instead introduce CP violating (CPV) NP contributions to Wilson coefficients, $\delta\mathcal{C}_i = i\mathrm{Im}\delta\mathcal{C}_i$. The CP-odd differences of $B\to K$ rates and $B\to\pi$ get modified by additional contributions, on top of those in (15) and (18). Relative shift with respect to the SM values are

$$\frac{(d\Gamma_P-d\bar{\Gamma}_P)_{\mathrm{ImNP}}}{(d\Gamma_P-d\bar{\Gamma}_P)_{\mathrm{SM}}} = \frac{\mathrm{Im}\delta\mathcal{C}_9^{(q')}+\tilde{f}_T^{(P)}\mathrm{Im}\delta\mathcal{C}_7^{(q')}}{\mathrm{Im}\tilde{\lambda}_c^{(q')}}\frac{\mathrm{Im}Y_{c\bar{c}}^{(P)}}{(\mathcal{C}_9^{\mathrm{SM}}+\tilde{f}_T^{(P)}\mathcal{C}_7^{\mathrm{SM}})(\mathrm{Im}Y_{c\bar{c}}^{(P)}-\mathrm{Im}Y_{u\bar{u}}^{(P)})+\mathrm{Im}(Y_{c\bar{c}}^{(P)}Y_{u\bar{u}}^{(P)*})}, \quad (27)$$

where $P = \pi, K$ corresponds to $q' = d,\ s$, respectively. These contributions are proportional to *CP*-even real parts $\mathrm{Re}\tilde{\lambda}_{u,c}^{(q')}$ whose Cabibbo hierarchy selects $c\bar{c}$ as the dominant contribution [see Eqs. (13), (16)]. Accordingly $\mathrm{Im}Y_{c\bar{c}}^{(P)}$ is always a dominant source of the strong phase, once we neglect $\mathcal{O}(\lambda^2)$ and $\mathcal{O}(\xi)$ CKM suppressed terms. The presence of $\mathrm{Im}Y_{c\bar{c}}$ as a new type of absorptive part makes the $R_{K/\pi}^{CP}$ ratio less informative due to proliferation of *U*-spin breaking parameters. It is important to point out that the *CP*-odd difference of $B \to K\ell^+\ell^-$ can be drastically enhanced due to small weak phase in the SM: $1/\mathrm{Im}\tilde{\lambda}_c^{(s)} = -54$ [50]; however the prediction also depends on an unknown strong phase $\mathrm{Im}Y_{c\bar{c}}^{(K)}$.

VI. DISCUSSION AND CONCLUSIONS

Effects of four-quark operators in rare semileptonic *B* meson decays have important implications both for our understanding of QCD dynamics as well as for physics BSM. In this work we have shown how combined measurements of *CP*-averaged decay rates and direct *CP* asymmetries of $B^\pm \to K^\pm\ell^+\ell^-$ and $B^\pm \to \pi^\pm\ell^+\ell^-$ transitions can be used to learn about the sizes of the four-quark operator matrix elements and how their contributions interplay with possible local BSM effects.

The relative importance of dispersive and absorptive amplitudes involving $\bar{b}q'\bar{c}c$ and $\bar{b}q'\bar{u}u$ operators in the SM, dictated by the hierarchies of the CKM, leads to an interesting interplay of *CP*-even and *CP*-odd observables in $B \to K$ and $B \to \pi$ transitions. In particular, our results motivate dedicated binned measurements of the direct *CP* asymmetry (or the corresponding *CP*-odd decay rate difference) in $B \to \pi\ell^+\ell^-$. As shown in Fig. 2 such measurements could help significantly reduce the current uncertainty on the absorptive long distance amplitudes entering $B \to K\ell^+\ell^-$. The corresponding improvement in precision of dispersive amplitudes is projected to be modest, with one caveat: in our projections we have assumed that the experimental precision on $B \to K$ modes will remain an order of magnitude better compared to measurements with pion final states. Within our approach, current measurements are in mild tension with existing theoretical estimates [21] of the dominant dispersive long distance SM contributions to $B \to K\ell^+\ell^-$ rates. A similar comparison for absorptive amplitudes would require a dedicated theoretical estimation of $\bar{b}q'\bar{u}u$ operator effects, which is currently not available in the literature and beyond the scope of this work. We also note that the extraction of absorptive LD amplitudes from the *CP*-odd rate difference measurements will remain dominated by experimental uncertainties even for our HL-LHCb projections and thus insensitive to theoretical form factor uncertainties.

Motivated by these results, we have also constructed a ratio $R_{K/\pi}^{CP}$ of *CP*-odd decay rate differences of $B^\pm \to K^\pm\ell^+\ell^-$ and $B^\pm \to \pi^\pm\ell^+\ell^-$ which can be computed exactly in the *U*-spin limit of the SM, see Eq. (23). We have estimated $R_{K/\pi}^{CP}$ in presence of known *U*-spin breaking effects due to kinematics and differences in form factors. On the other hand, the explicit dependence on LD amplitudes is suppressed by *U*-spin breaking and $|\mathcal{C}_7/\mathcal{C}_9^2| \lesssim 0.02$. The current theoretical uncertainties on $R_{K/\pi}^{CP}|_{\mathrm{SM}}$ are currently dominated by our knowledge of the relevant form factors, mainly because we have to treat their uncertainties as completely uncorrelated. Conversely, a correlated extraction of $B \to K$ and $B \to \pi$ form factors on the lattice could potentially significantly reduce this error. Furthermore, the remaining *U*-spin breaking effects in LD amplitudes could in principle be estimated, for example using light-cone sum rules techniques [21,22]. Thus, $R_{K/\pi}^{CP}$ has the potential to become one of the theoretically cleanest observables related to rare semileptonic *B* meson decays within the SM. The proposed approach could be exploited to its full potential if future experimental data of $B \to K$ and $B \to \pi$ were analyzed with same binning in q^2.

We have also explored the interplay between SM four-quark contributions and possible short-distance NP effects in *CP*-odd $B \to P\ell^+\ell^-$ decay rate differences. In presence of *CP* conserving NP affecting only the kaon mode, one could in principle disentangle it from SM LD effects using the pion mode measurements. In particular, the extraction of absorptive amplitudes from both modes in this case would disagree (i.e., green and yellow bars in Fig. 2). Obviously, such a discrimination is in practice observable only if the relevant NP effects are bigger than uncertainties related to *U*-spin breaking. In case of CPV NP, larger relative effects are expected in the $B \to K$ mode, due to the CKM suppression of CPV within the SM. Unfortunately, quantitative predictions of such NP effects on $R_{K/\pi}^{CP}$ would require independent theoretical estimations of the absorptive $\bar{b}q'\bar{c}c$ four-quark amplitudes (i.e., they cannot be extracted from measurements of the $B \to \pi$ transition).

We conclude by noting that, while this work focused on $B \to P\ell^+\ell^-$ transitions, the same analysis can be applied to individual helicity amplitudes in rare semileptonic decays of B mesons to vector meson final states (e.g., $B \to \rho$ vs $B \to K^*$ or $B_s \to \phi$ vs $B_s \to K^*$). We leave a dedicated analysis of such transitions for future work.

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