Bell nonlocality and entanglement in $e^+e^- \to Y\bar{Y}$ at BESIII

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The Bell nonlocality and entanglement are two kinds of quantum correlations in quantum systems. Due to the recent upgrade in the Beijing Spectrometer III (BESIII) experiment, it is possible to explore the nonlocality and entanglement in hyperon-antihyperon systems produced in electron-positron annihilation with high precision data. We provide a systematic method for studying quantum correlations in spin-1/2 hyperon-antihyperon systems through the measures for the nonlocality and entanglement. We find that with nonvanishing polarizations of the hyperon and its antihyperon, the kinematic region of nonlocality in the hyperon-antihyperon system is more restricted than the $\tau^+\tau^-$ system in which polarizations of τ leptons are vanishing. We also present an experimental proposal to probe the nonlocality and entanglement in hyperon-antihyperon systems at BSEIII.

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I. INTRODUCTION

Quantum mechanics, as a foundational pillar for modern physics, governs the properties of fundamental particles and their interactions. In this context, quantum information properties of fundamental particles can offer a novel perspective on understanding quantum mechanics. The Bell nonlocality, characterized by the violation of Bell-type inequalities [1–3], is a distinctive quantum property with significant implications for quantum mechanics. Closely related to the Bell nonlocality, the quantum entanglement is an invisible link between two particles that allows one to instantly affect the other regardless of their distance. The entanglement has practical applications in quantum information processing, including quantum computing [4], quantum metrology [5], and quantum communication [6]. In the research area of quantum information theory, theoretical details of the Bell nonlocality and entanglement have been thoroughly discussed (see, e.g., Refs. [7,8] for recent reviews). Historically, the Bell nonlocality and entanglement have been widely studied in photonic and atomic systems [9,10].

High-energy colliders provide an alternative test ground for the nonlocality and entanglement [11]. The significant improvement in collider and detector technology has led to a large collection of high precision data, thereby enabling the possibility of observing the quantum correlation in high energy processes. Recently quantum correlations in elementary particle systems, e.g., top quark pairs at Large Hadron Collider (LHC) [12–16], leptons pairs [17,18], gauge bosons from Higgs decay [19–22], have been investigated.

In contrast to elementary particles, the use of hadronic final states to test quantum correlations has a relatively long history, dating back to early 1980s [23]. Subsequent studies came up in the past decades aiming at probing quantum correlations in hyperon systems [24–30]. The hyperon's weak decay can serve as its own polarimeter and make it possible to extract spin observable in the hyperon-antihyperon system, including polarization and correlation, in experiments. With the recent upgrade of Beijing Spectrometer III (BESIII) at Beijing electron-positron collider, there is considerable potential to explore quantum correlations in hyperon-antihyperon production processes in electron-positron annihilation [31–33].

In this paper, we investigate the Bell nonlocality and entanglement in $e^+e^- \to \gamma^*/\psi \to Y\bar{Y}$ processes at BESIII, where Y and $\bar{Y}$ denote the spin-1/2 hyperon and its antihyperon respectively. Our study is based on the two-qubit density operator [34,35] for $Y\bar{Y}$. Unlike elementary particle systems such as $\tau^+\tau^-$ at Belle II and $t\bar{t}$ at LHC, the

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existence of electromagnetic form factors (EMFFs) in a polarized $Y\bar{Y}$ state at BESIII [36] makes the $Y\bar{Y}$ system different from elementary particle systems [15,18]. Recognizing that the final $Y\bar{Y}$ state is local unitary equivalent to the two-qubit X state, we will derive the analytical expressions of nonlocality and entanglement for $Y\bar{Y}$. At the end of this paper, we will discuss the effect of EMFFs in quantum correlation and also give a proposal to probe the nonlocality and entanglement at BESIII.

This paper is organized as follows. We will introduce the two-qubit density operator for $Y\bar{Y}$ produced in electron-positron annihilation in Sec. II. In Sec. III, we will discuss the two-qubit X state and investigate the Bell nonlocality for $Y\bar{Y}$. The quantum entanglement in $Y\bar{Y}$ will be addressed in Sec. IV. The relation between the Bell nonlocality and entanglement will be discussed in Sec. V. In Sec. VI, we will give a proposal to probe the nonlocality and entanglement at BESIII. The final section, Sec. VII, presents a summary of main results and outlook for future directions of study.

II. PRELIMINARIES

Hyperon-antihyperon pairs can be produced in electron-positron annihilation either through the virtual photon exchange $e^+e^- \to \gamma^* \to Y\bar{Y}$ or through vector charmonium decays, e.g., $e^+e^- \to J/\psi \to Y\bar{Y}$, where Y denotes a ground-state octet hyperon Λ, Σ^+, Ξ^-, or Ξ^0. In BESIII experiments, a huge number of events for vector charmonia J/ψ and $\psi(2S)$ have been collected. These vector charmonia can decay into hyperon-antihyperon pairs. A $Y\bar{Y}$ pair made of two spin-1/2 particles forms a massive two-qubit system. Due to momentum conservation, in the center of mass (CM) frame, the outgoing hyperon and antihyperon are back-to-back in momentum. Their spin states can be characterized by a two-qubit density operator

$$\rho_{Y\bar{Y}} = \frac{1}{4}\left(\mathbb{1}\otimes\mathbb{1} + \mathbf{P}^+\cdot\boldsymbol{\sigma}\otimes\mathbb{1} + \mathbb{1}\otimes\mathbf{P}^-\cdot\boldsymbol{\sigma} + \sum_{i,j} C_{ij}\sigma_i\otimes\sigma_j\right), \tag{1}$$

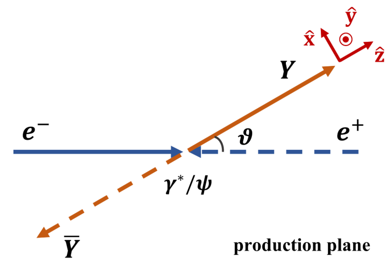

FIG. 1. The coordinate system used in the analysis with $\{\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}\}$ being three directions in the rest frame of Y as well as that of $\bar{Y}$.

with $\boldsymbol{\sigma} = (\sigma_1,\sigma_2,\sigma_3)$ being Pauli matrices, $\mathbf{P}^\pm$ the polarization or Bloch vectors of hyperon/antihyperon, and C_{ij} their correlation matrix. The two-qubit density operator Eq. (1) can also be put into a more compact form: $\rho_{Y\bar{Y}} = (1/4)\Theta_{\mu\nu}\sigma_\mu\otimes\sigma_\nu$ with $\Theta_{00}=1$, $\Theta_{i0}=P_i^+$, $\Theta_{0j}=P_j^-$, and $\Theta_{ij}=C_{ij}$. Here, σ_0 is defined as the 2×2 identity matrix $\mathbb{1}$. In $\rho_{Y\bar{Y}}$ there are 15 real parameters for the spin configuration of the $Y\bar{Y}$ pair.

The 4×4 matrix $\Theta_{\mu\nu}$ is frame-dependent. For the hyperon Y, we choose its helicity rest frame as

$$\hat{\mathbf{y}} = \frac{\hat{\mathbf{p}}_e\times\hat{\mathbf{p}}_Y}{|\hat{\mathbf{p}}_e\times\hat{\mathbf{p}}_Y|}, \qquad \hat{\mathbf{z}} = \hat{\mathbf{p}}_Y, \qquad \hat{\mathbf{x}} = \hat{\mathbf{y}}\times\hat{\mathbf{z}}, \tag{2}$$

which is shown in Fig. 1. While for the antihyperon $\bar{Y}$, we also adopt its rest frame, but three axes are chosen to be the same as the hyperon's: $\{\hat{\mathbf{x}}_{\bar{Y}},\hat{\mathbf{y}}_{\bar{Y}},\hat{\mathbf{z}}_{\bar{Y}}\} = \{\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}\}$. The three axes we choose are different from Refs. [34,35], resulting in slightly different entries of $\Theta_{\mu\nu}$. Adopting this coordinate system is convenient since the rest frames of Y and $\bar{Y}$ differ only by a pure boost along their momenta without rotation.

In the rest frames of Y and $\bar{Y}$ with three axes in Eq. (2), through virtual photon exchange $\Theta_{\mu\nu}$ has the form [34,35]

$$\Theta_{\mu\nu} = \frac{1}{1+\alpha_\psi\cos^2\vartheta}\left[\begin{array}{c|ccc} 1+\alpha_\psi\cos^2\vartheta & 0 & \beta_\psi\sin\vartheta\cos\vartheta & 0 \\ \hline 0 & \sin^2\vartheta & 0 & \gamma_\psi\sin\vartheta\cos\vartheta \\ \beta_\psi\sin\vartheta\cos\vartheta & 0 & -\alpha_\psi\sin^2\vartheta & 0 \\ 0 & \gamma_\psi\sin\vartheta\cos\vartheta & 0 & \alpha_\psi+\cos^2\vartheta \end{array}\right], \tag{3}$$

where ϑ is the angle between the incoming electron's and outgoing hyperon's momenta with $\cos\vartheta = \hat{\mathbf{p}}_e\cdot\hat{\mathbf{p}}_Y$. Here $\hat{\mathbf{p}}_e$ and $\hat{\mathbf{p}}_Y$ are momentum directions of the electron and hyperon respectively. In Eq. (3), $\alpha_\psi\in[-1,+1]$ is the decay parameter of the vector charmonium $\psi(c\bar{c})$, and β_ψ and γ_ψ are defined as

$$\beta_\psi = \sqrt{1-\alpha_\psi^2}\sin(\Delta\Phi), \qquad \gamma_\psi = \sqrt{1-\alpha_\psi^2}\cos(\Delta\Phi), \quad (4)$$

where $\Delta\Phi \in (-\pi, +\pi]$ is the relative form factor phase.

The polarization and correlation can be read out from $\Theta_{\mu\nu}$ in Eq. (3)

$$P_y^+ = P_y^- = \frac{\sqrt{1-\alpha_\psi^2}\sin(\Delta\Phi)\sin\vartheta\cos\vartheta}{1+\alpha_\psi\cos^2\vartheta}, \quad (5)$$

and

$$C_{xx} = \frac{\sin^2\vartheta}{1+\alpha_\psi\cos^2\vartheta}, \qquad C_{yy} = \frac{-\alpha_\psi\sin^2\vartheta}{1+\alpha_\psi\cos^2\vartheta},$$
$$C_{zz} = \frac{\alpha_\psi + \cos^2\vartheta}{1+\alpha_\psi\cos^2\vartheta},$$
$$C_{xz} = C_{zx} = \frac{\sqrt{1-\alpha_\psi^2}\cos(\Delta\Phi)\sin\vartheta\cos\vartheta}{1+\alpha_\psi\cos^2\vartheta}. \quad (6)$$

Here P_y^+ and P_y^- are the polarization of Y and $\bar{Y}$ along the direction $\hat{\mathbf{y}}$ (the normal direction of the production plane), respectively. The symmetry property of the polarization and correlation arises from the invariance under parity transformation and charge conjugation. We do not consider $\mathcal{CP}$ violation in our analysis.

III. BELL NONLOCALITY

In this section, we will use the hyperon-antihyperon spin density operator to investigate Bell nonlocality in the $Y\bar{Y}$ system.

A. Local unitary equivalence and X states

Before our investigation of Bell nonlocality, it is convenient to transform the two-qubit state in Eqs. (1) and (3) to the X state. First, we swap the $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ axes in Y and $\bar{Y}$'s rest frame. Then we diagonalize C_{ij} for Y and $\bar{Y}$. The transformed spin density operator can be written in terms of Pauli matrices as

$$\rho_{Y\bar{Y}}^X = \frac{1}{4}\left(\mathbb{1}\otimes\mathbb{1} + a\sigma_z\otimes\mathbb{1} + \mathbb{1}\otimes a\sigma_z + \sum_i t_i\sigma_i\otimes\sigma_i\right), \quad (7)$$

which is in the standard form of a symmetric *two-qubit X state* [37]. Thus we place the superscript "X" to $\rho_{Y\bar{Y}}$. The corresponding $\Theta_{\mu\nu}$ becomes

$$\Theta_{\mu\nu}^X = \left[\begin{array}{c|ccc} 1 & 0 & 0 & a \\ \hline 0 & t_1 & 0 & 0 \\ 0 & 0 & t_2 & 0 \\ a & 0 & 0 & t_3 \end{array}\right], \quad (8)$$

where the elements a and t_i $(i = 1, 2, 3)$ are given by

$$a = \frac{\beta_\psi\sin\vartheta\cos\vartheta}{1+\alpha_\psi\cos^2\vartheta},$$
$$t_{1,2} = \frac{1+\alpha_\psi \pm \sqrt{(1+\alpha_\psi\cos 2\vartheta)^2 - \beta_\psi^2\sin^2 2\vartheta}}{2(1+\alpha_\psi\cos^2\vartheta)},$$
$$t_3 = \frac{-\alpha_\psi\sin^2\vartheta}{1+\alpha_\psi\cos^2\vartheta}. \quad (9)$$

We note that $a = P_y^\pm$, $t_3 = C_{yy}$, and $t_{1,2}$ come from diagonalizing the block matrix of C_{ij} with $i, j = x, z$ in $\Theta_{\mu\nu}$.

We note that the swapping of $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ axes and diagonalizing C_{ij} can be obtained by a local unitary transformation:

$$\rho_{Y\bar{Y}}^X = (U_Y \otimes U_{\bar{Y}})\rho_{Y\bar{Y}}(U_Y\otimes U_{\bar{Y}})^\dagger, \quad (10)$$

where U_Y and $U_{\bar{Y}}$ are two unitary operators acting independently in Y and $\bar{Y}$'s Hilbert space respectively [38]. The states described by $\rho_{Y\bar{Y}}$ and $\rho_{Y\bar{Y}}^X$ are said to be *local unitary equivalent* in the sense that they have same quantum correlation properties such as Bell nonlocality and entanglement [39]. In the remainder of this paper, all analyses are based on the X state in Eqs. (7) and (8).

B. Bell nonlocality

The nonlocal property in a quantum entangled system can be tested by the violation of Bell inequality [1]. The most widely used Bell-type inequality is the CHSH inequality [40]

$$|\langle A_1\otimes B_1\rangle + \langle A_1\otimes B_2\rangle + \langle A_2\otimes B_1\rangle - \langle A_2\otimes B_2\rangle| \le 2, \quad (11)$$

where $A_i = \mathbf{a}_i\cdot\boldsymbol{\sigma}$, $B_i = \mathbf{b}_i\cdot\boldsymbol{\sigma}$, and $\langle A_i\otimes B_j\rangle \equiv \mathrm{Tr}[\rho(\mathbf{a}_i\cdot\boldsymbol{\sigma}\otimes\mathbf{b}_j\cdot\boldsymbol{\sigma})]$ with $i, j = 1, 2$. Here $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{b}_1$, and $\mathbf{b}_2$ are four directions (unit vectors) along which the spin polarization is measured. Then the inequality can be rewritten in a simpler form

$$|\mathbf{a}_1^\mathrm{T}C(\mathbf{b}_1+\mathbf{b}_2) + \mathbf{a}_2^\mathrm{T}C(\mathbf{b}_1-\mathbf{b}_2)| \le 2, \quad (12)$$

with C being the correlation matrix C_{ij} in Eq. (1). Those quantum states that violate the CHSH inequality are called

TABLE I. Some parameters in $e^+e^- \to J/\psi \to Y\bar{Y}$, where $Y\bar{Y}$ is a pair of ground-state octet hyperons.

	$\mathcal{B}(\times 10^{-4})$	α_ψ	$\Delta\Phi$/rad	References
$\Lambda\bar{\Lambda}$	19.43(33)	0.475(4)	0.752(8)	[31,42]
$\Sigma^+\bar{\Sigma}^-$	15.0(24)	−0.508(7)	−0.270(15)	[43,44]
$\Xi^-\bar{\Xi}^+$	9.7(8)	0.586(16)	1.213(49)	[32,45]
$\Xi^0\bar{\Xi}^0$	11.65(4)	0.514(16)	1.168(26)	[46,47]

Bell nonlocal states. The maximum of the left-hand side of Eq. (12) can be obtained by tuning $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{b}_1$, and $\mathbf{b}_2$ as

$$\mathcal{B}[\rho] \equiv \max_{\mathbf{a}_1,\mathbf{a}_2,\mathbf{b}_1,\mathbf{b}_2} |\mathbf{a}_1^{\mathrm{T}} C(\mathbf{b}_1 + \mathbf{b}_2) + \mathbf{a}_2^{\mathrm{T}} C(\mathbf{b}_1 - \mathbf{b}_2)| = 2\sqrt{m_1 + m_2}, \tag{13}$$

where m_1 and m_2 are two largest eigenvalues of $C^{\mathrm{T}}C$ [41]. Therefore, the CHSH inequality can be violated iff (if and only if) $m_1 + m_2 > 1$, and the maximum possible violation of the CHSH inequality is the upper bound value $2\sqrt{2}$. For convenience, we define a function of two-qubit density operator $\mathfrak{m}_{12}[\rho] \equiv m_1 + m_2 \in [0, 2]$ to be a measure of the Bell nonlocality [16,18].

Since we have put the density operator into the X form in (8), the correlation matrix is diagonal: $\mathbf{t} = \mathrm{diag}\{t_1, t_2, t_3\}$. The three eigenvalues of $C^{\mathrm{T}}C$ or $\mathbf{t}^{\mathrm{T}}\mathbf{t}$ are t_1^2, t_2^2, and t_3^2. Then, according to Eq. (13), one needs to select the largest two values among them.

As we can see from Eq. (9) that $t_{1,2,3}$ are functions of three parameters α_ψ, $\Delta\Phi$ and ϑ. Since $t_1^2 \geq t_2^2$ always holds for any values of α_ψ, $\Delta\Phi$ and ϑ, t_1^2 should not be the smallest one. Then, one needs to compare t_2^2 and t_3^2. If $\alpha_\psi \geq 0$, we always have $t_2^2 \geq t_3^2$. Therefore, the measure of nonlocality becomes $\mathfrak{m}_{12}[\rho^X_{Y\bar{Y}}] = t_1^2 + t_2^2$. However, for $\alpha_\psi < 0$, one cannot judge which is larger t_2^2 or t_3^2, since it depends on the specific values of three parameters. In this case the measure of nonlocality can be expressed as $\mathfrak{m}_{12}[\rho^X_{Y\bar{Y}}] = \max\{t_1^2 + t_2^2, t_1^2 + t_3^2\}$. In summary, the measure of the Bell nonlocality reads

$$\mathfrak{m}_{12}[\rho^X_{Y\bar{Y}}] = \begin{cases} t_1^2 + t_2^2, & \alpha_\psi \geq 0 \\ \max\{t_1^2 + t_2^2, t_1^2 + t_3^2\}, & \alpha_\psi < 0 \end{cases} \tag{14}$$

where $t_1^2 + t_2^2$ and $t_1^2 + t_3^2$ are given by

$$t_1^2 + t_2^2 = 1 + \left(\frac{\alpha_\psi \sin^2\vartheta}{1 + \alpha_\psi \cos^2\vartheta}\right)^2 - 2\left(\frac{\beta_\psi \sin\vartheta\cos\vartheta}{1 + \alpha_\psi \cos^2\vartheta}\right)^2, \tag{15}$$

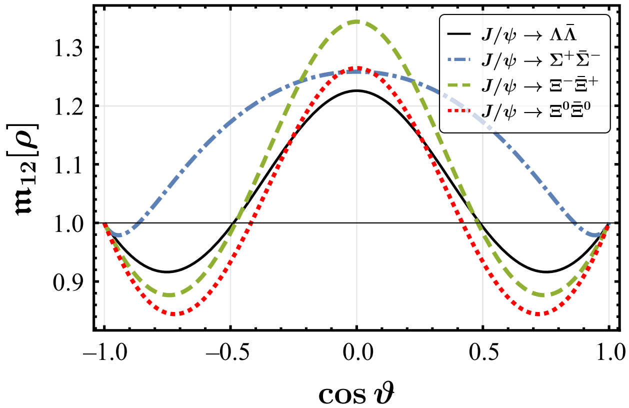

FIG. 2. The measure of the Bell nonlocality $\mathfrak{m}_{12}[\rho^X_{Y\bar{Y}}]$ as functions of $\cos\vartheta$ (ϑ is the scattering angle) in $e^+e^- \to J/\psi \to Y\bar{Y}$ with $Y = \Lambda, \Sigma^+, \Xi^-$ and Ξ^0 corresponding to curves in black solid, blue dash-dotted, green dashed, and red dotted lines respectively. The black horizontal line is the nonlocality bound $\mathfrak{m}_{12} = 1$. The CHSH inequality is violated iff $\mathfrak{m}_{12} > 1$.

$$t_1^2 + t_3^2 = \left(\frac{1 + \alpha_\psi + \sqrt{(1 + \alpha_\psi \cos 2\vartheta)^2 - \beta_\psi^2 \sin^2 2\vartheta}}{2(1 + \alpha_\psi \cos^2\vartheta)}\right)^2 + \frac{\alpha_\psi^2 (1 - \cos 2\vartheta)^2}{4(1 + \alpha_\psi \cos^2\vartheta)^2}. \tag{16}$$

In Table I are listed the values of α_ψ and $\Delta\Phi$ for J/ψ's decays into a pair of octet hyperons in electron-positron annihilation. According to these parameters, we plot $\mathfrak{m}_{12}$ as a function of the scattering angle ϑ in Fig. 2 for different decay channels. From Fig. 2, we find that $\mathfrak{m}_{12}$ is a symmetric function of ϑ relative to $\vartheta = \pi/2$ in the range $\vartheta \in [0, \pi]$, and it reaches the maximum value $1 + \alpha_\psi^2$ at $\vartheta = \pi/2$. Thus we obtain

$$\max_\vartheta \mathcal{B}[\rho^X_{Y\bar{Y}}] = 2\sqrt{1 + \alpha_\psi^2}. \tag{17}$$

By solving $\mathfrak{m}_{12} > 1$ in Eq. (14) with fixed α_ψ and $\Delta\Phi$, we obtain the nonlocality range of the scattering angle as $(\vartheta^*, \pi - \vartheta^*)$. For $\alpha_\psi \geq 0$, we can have an analytical expression for the critical angle ϑ^*

$$\vartheta^* = \arctan\left|\sqrt{2 - 2\alpha_\psi^2}\,\frac{\sin\Delta\Phi}{\alpha_\psi}\right|, \quad \text{for } \alpha_\psi \geq 0. \tag{18}$$

TABLE II. The maximum violation $\mathcal{B}_{\max}$ in Eq. (17) and critical angles ϑ^* for the CHSH inequality in $e^+e^- \to J/\psi \to Y\bar{Y}$.

	$\Lambda\bar{\Lambda}$	$\Sigma^+\bar{\Sigma}^-$	$\Xi^-\bar{\Xi}^+$	$\Xi^0\bar{\Xi}^0$
$\mathcal{B}_{\max}$	2.214	2.243	2.318	2.249
ϑ^*	60.81°	30.29°	61.37°	65.27°

The maximum violation in Eq. (17) and critical angles in different decay channels are listed in Table II.

IV. QUANTUM ENTANGLEMENT

In this section we will discuss the quantum entanglement in the $Y\bar{Y}$ system and its relation to the Bell nonlocality.

A. Entanglement measure and concurrence

For a bipartite quantum system living in the combined Hilbert space $\rho_{AB} \in \mathscr{H}_A \otimes \mathscr{H}_B$, the state is said to be *separable* iff the following decomposition holds

$$\rho_{AB} = \sum_k p_k \rho_A^k \otimes \rho_B^k, \tag{19}$$

where $p_k \geq 0$ and $\sum_k p_k = 1$, and ρ_A^k and ρ_B^k are the density operator of the corresponding subsystem A and B, respectively. A state that cannot be decomposed into the above form is called *nonseparable* or *entangled*.

For two-qubit and qubit-qutrit systems (2×2 and 2×3 respectively), the Peres-Horodecki criterion provides a sufficient and necessary condition for separability [48,49]: a state ρ_{AB} is separable iff its partial transpose $\rho_{AB}^{\mathrm{T}_B}$ with respect to the second subsystem is positive semi-definite. The Peres-Horodecki criterion is also called Positive Partial Transpose (PPT) criterion.

The *concurrence* is an entanglement monotone, and it has a direct relationship with *entanglement of formation* [50]. In this work, we utilize the concurrence as a measure of the entanglement. In Ref. [51], Wootters derived the two-qubit concurrence as

$$\mathcal{C}[\rho] \equiv \max\{0, \mu_1 - \mu_2 - \mu_3 - \mu_4\}, \tag{20}$$

where μ_i with $i = 1, 2, 3, 4$ are the eigenvalues of the Hermitian matrix $\sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$ with $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$ in the decreasing order, and ρ^* denotes the complex conjugate of ρ in the spin basis of σ_z. Wootters' concurrence is a function in the range [0, 1]. A state is separable for $\mathcal{C}[\rho] = 0$ and is entangled for $\mathcal{C}[\rho] > 0$. When $\mathcal{C}[\rho] = 1$, the state is said to be maximally entangled.

We rewrite the spin density operator for the hyperon-antihyperon system in the σ_z basis

$$\rho_{Y\bar{Y}}^X = \frac{1}{4}\begin{bmatrix} 1+2a+t_3 & 0 & 0 & t_1 - t_2 \\ 0 & 1 - t_3 & t_1 + t_2 & 0 \\ 0 & t_1 + t_2 & 1 - t_3 & 0 \\ t_1 - t_2 & 0 & 0 & 1 - 2a + t_3 \end{bmatrix}, \tag{21}$$

where a and $t_{1,2,3}$ are defined in Eq. (9). The above expression can be directly obtained by expanding Pauli operators in Eq. (7) into a 2×2 matrix form. The name X state comes from its resemblance to the letter X.

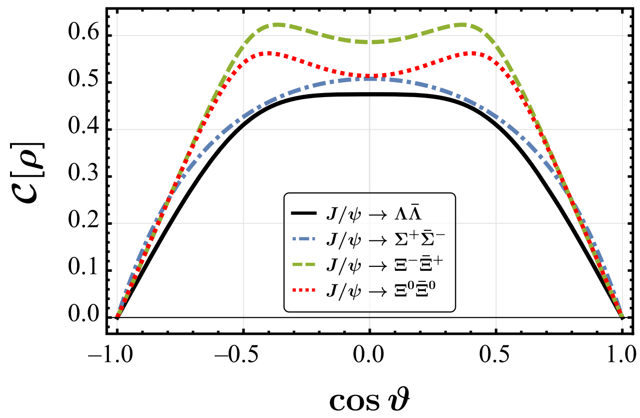

FIG. 3. Wootters' concurrence $\mathcal{C}[\rho_{Y\bar{Y}}^X]$ as functions of $\cos\vartheta$ (ϑ is the scattering angle), where $Y = \Lambda, \Sigma^+, \Xi^-$, and Ξ^0 corresponding to curves in black solid, blue dash-dotted, green dashed, and red dotted lines, respectively. The black horizontal line is the entanglement bound. The $Y\bar{Y}$ system is entangled iff $\mathcal{C} > 0$.

The Peres-Horodecki criterion for a general X state claims that the state is entangled iff either $\rho_{22}^X\rho_{33}^X < |\rho_{14}^X|^2$ or $\rho_{11}^X\rho_{44}^X < |\rho_{23}^X|^2$ holds [52], but both conditions cannot be satisfied simultaneously [53]. The Wootters' concurrence for the X state is given by [37]

$$\mathcal{C}[\rho^X] = 2\max\left\{0, |\rho_{14}^X| - \sqrt{\rho_{22}^X\rho_{33}^X}, |\rho_{23}^X| - \sqrt{\rho_{11}^X\rho_{44}^X}\right\}, \tag{22}$$

with ρ_{ij}^X being given in (21). We see that the Peres-Horodecki criterion for the X state is compatible with the concurrence.

With Eqs. (21) and (22), we derive the concurrence for the hyperon-antihyperon system as

$$\mathcal{C}[\rho_{Y\bar{Y}}^X] = |t_2| = \frac{\left|1 + \alpha_\psi - \sqrt{(1 + \alpha_\psi\cos 2\vartheta)^2 - \beta_\psi^2\sin^2 2\vartheta}\right|}{2(1 + \alpha_\psi\cos^2\vartheta)}. \tag{23}$$

The results for the concurrence as functions of ϑ for octet hyperons listed in Table I are shown in Fig. 3. We see that the entanglement of $Y\bar{Y}$ pairs exists in the whole range of the scattering angle ϑ except at two collinear limits $\vartheta = 0$ or π. However, unlike the Bell nonlocality, the maximum value of the concurrence (or maximum entanglement) does not necessarily take place at $\vartheta = \pi/2$. Instead, it can occur at other angles such as the ones for $\Xi^-\bar{\Xi}^+$ and $\Xi^0\bar{\Xi}^0$ shown in Table III.

In summary, the outgoing hyperon-antihyperon pairs are entangled in the full range of the scattering angle except at two boundaries.

TABLE III. The maximum concurrence $\mathcal{C}_{\max}$ in Eq. (23) and their corresponding angles $\vartheta_{\max}$ in $e^+e^- \to J/\psi \to Y\bar{Y}$.

	$\Lambda\bar{\Lambda}$	$\Sigma^+\bar{\Sigma}^-$	$\Xi^-\bar{\Xi}^+$	$\Xi^0\bar{\Xi}^0$
$\mathcal{C}_{\max}$	0.475	0.508	0.623	0.562
$\vartheta_{\max}$	90°	90°	68.60°, 111.40°	66.26°, 113.74°

V. DISCUSSIONS ON BELL NONLOCALITY AND ENTANGLEMENT

In this section, we will discuss the relation between Bell nonlocality and entanglement, the eigenvalue decomposition of the spin density matrix, the role of electromagnetic form factors in quantum correlation of the hyperon-antihyperon system.

A. Bell nonlocality versus entanglement

Given that both Bell nonlocality and quantum entanglement characterize quantum properties of a system, we try to look for the relationship between them.

For a two-qubit density operator ρ with Wootters' concurrence $\mathcal{C}[\rho]$, the maximum violation of the CHSH inequality $\mathcal{B}[\rho]$ has an upper bound [54]

$$\mathcal{B}[\rho] \leq 2\sqrt{1+\mathcal{C}^2[\rho]}, \tag{24}$$

with $\mathcal{B}[\rho] \equiv 2\sqrt{\mathfrak{m}_{12}}$ defined in Eq. (13). In Fig. 4 we plot $\mathcal{B}$ and $2\sqrt{1+\mathcal{C}^2}$ as functions of $\cos\vartheta$. We see in Fig. 4 that the inequality (24) is always satisfied and the equality $\mathcal{B} = 2\sqrt{1+\mathcal{C}^2}$ (or equivalently $\mathfrak{m}_{12} = 1+\mathcal{C}^2$) holds at $\vartheta = \pi/2$. At this transverse scattering angle, Y's and $\bar{Y}$'s polarizations vanish from Eq. (5), then the spin density operator $\rho^X_{Y\bar{Y}}$ reduces to a very special subclass of the X state: *T state* or *Bell diagonal state (BDS)*. The upper bound of $\mathcal{B}$ in (24) is attained for rank-2 BDSs [54].

From Fig. 4, both measures for the Bell nonlocality and entanglement are symmetric with respect to $\vartheta = \pi/2$. However, even if hyperon-antihyperon pairs are entangled in the full range of scattering angle except at $\vartheta = 0$ or π, the Bell nonlocality only appears in the range $\vartheta \in (\vartheta^*, \pi - \vartheta^*)$. This corresponds to the range where orange dot-dashed lines lie above the black line in Fig. 4. This indicates the relation between the Bell nonlocality and entanglement in the hierarchy of quantumness

$$\text{Bell nonlocality} \subset \text{entanglement}.$$

Any nonlocal state must be entangled, but not all entangled states can have nonlocal correlation [55].

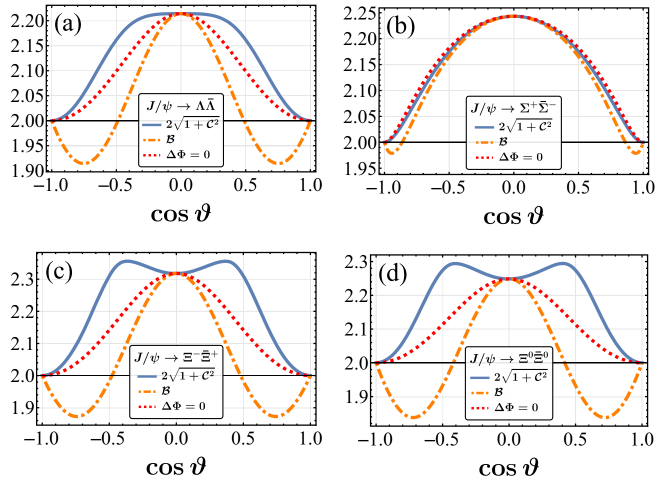

FIG. 4. The measures $\mathcal{B} = 2\sqrt{\mathfrak{m}_{12}}$ and $2\sqrt{1+\mathcal{C}^2}$ for the Bell nonlocality and quantum entanglement as functions of $\cos\vartheta$ (ϑ is the scattering angle). The four panels (a)–(d) correspond to four decay channels of J/ψ to $Y\bar{Y}$ with $Y = \Lambda$, Σ^+, Ξ^-, and Ξ^0, respectively. Solid blue lines are curves of $2\sqrt{1+\mathcal{C}^2}$ for the entanglement, while orange dot-dashed lines are curves of $\mathcal{B}$. The red dotted lines are curves by setting $\Delta\Phi = 0$ which will be explained in Sec. V C. The black solid horizontal line is the value 2. The $Y\bar{Y}$ system is nonlocal or entangled iff $\mathcal{B} > 2$ or $2\sqrt{1+\mathcal{C}^2} > 2$.

Another interesting behavior of the entanglement and nonlocality appears in the panels (c) and (d) in Fig. 4 for $\Xi^-\bar{\Xi}^+$ and $\Xi^0\bar{\Xi}^0$: the entanglement in the range from the maximum concurrence angle $\vartheta_{\max}$ (see Table III) to $\pi/2$ shows a decreasing trend while the Bell nonlocality is still increasing. This phenomenon, where less entanglement corresponds to more nonlocality, sometimes referred to as an *anomaly of nonlocality* [56]. It can be explained by the quantum resource theory that the entanglement and nonlocality may be inequivalent resources [57]. The subtle relationship between the entanglement and nonlocality is still an active topic in this field [58].

B. Eigenvalue decomposition

Any two-qubit density operator can be decomposed as $\rho = \sum_{i=1}^{4} \lambda_i |\lambda_i\rangle\langle\lambda_i|$, with λ_i being the eigenvalue and $|\lambda_i\rangle$ the corresponding eigenstate. According to Eq. (21), the spin density operator has only two nonzero eigenvalues

$$\lambda_{1,2} = \frac{1}{2}\left(1 \mp \frac{\alpha_\psi \sin^2\vartheta}{1+\alpha_\psi \cos^2\vartheta}\right), \tag{25}$$

for the corresponding eigenstates

$$|\lambda_1\rangle = \sqrt{\frac{1+\alpha_\psi\cos 2\vartheta + \beta_\psi \sin 2\vartheta}{2(1+\alpha_\psi\cos 2\vartheta)}}|00\rangle + \sqrt{\frac{1+\alpha_\psi\cos 2\vartheta - \beta_\psi \sin 2\vartheta}{2(1+\alpha_\psi\cos 2\vartheta)}}|11\rangle,$$

$$|\lambda_2\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \tag{26}$$

where we adopt the notation for spin states: $|0\rangle \equiv |\uparrow_z\rangle$, $|1\rangle \equiv |\downarrow_z\rangle$. Through the eigenvalue decomposition, the spin configuration can be clearly shown in Eqs. (25) and (26) that $\rho^X_{Y\bar{Y}}$ can be treated as an ensemble of two pure states $\{|\lambda_1\rangle, |\lambda_2\rangle\}$ with probabilities $\{\lambda_1, \lambda_2\}$.

The eigenstate $|\lambda_1\rangle$ is a superposition of two spin triplet states: $|00\rangle = |S=1, S_z=1\rangle$ and $|11\rangle = |S=1, S_z=-1\rangle$, and $|\lambda_2\rangle$ is another spin triplet state: $|S=1, S_z=0\rangle$. We see that there is no spin singlet component in the $Y\bar{Y}$ system. This is the result of the angular momentum conversation in J/ψ's decay, and it coincides with the partial wave analysis that the outgoing $Y\bar{Y}$ only has contributions from ${}^3S_1$ and ${}^3D_1$ waves [59].

The lack of spin singlet component can also be seen by imposing the spin projection operator $F_S = (1 - \boldsymbol{\sigma}_Y \cdot \boldsymbol{\sigma}_{\bar{Y}})/4$ on the spin density matrix [60] as

$$\mathrm{Tr}\{\rho^X_{Y\bar{Y}} F_S\} = \mathrm{Tr}\mathbf{t} = \mathrm{Tr}C = 0, \tag{27}$$

with $\mathbf{t}$ and C being the correlation matrix in Eqs. (3) and (7) respectively.

C. Electromagnetic form factors

In this subsection, we will look into the timelike electromagnetic form factors (EMFFs) in $e^+e^- \to Y\bar{Y}$ and investigate their effects on nonlocality and entanglement.

The electromagnetic current of the spin-1/2 hyperon can be expressed in terms of the Dirac form factor F_1 and Pauli form factor F_2 as [36]

$$\Gamma^\mu = \gamma^\mu F_1(q^2) + i\frac{\sigma^{\mu\nu} q_\nu}{2M} F_2(q^2), \tag{28}$$

where $q = p_1 + p_2$ with p_1 and p_2 being the four-momentum of the hyperon and antihyperon respectively, and M is the hyperon mass. With $s = q^2$, the electric and magnetic form factors G_E and G_M are related to F_1 and F_2 by

$$G_E(s) = F_1 + \frac{s}{4M^2}F_2, \qquad G_M(s) = F_1 + F_2. \tag{29}$$

Two parameters α_ψ and $\Delta\Phi$ in the process $e^+e^- \to Y\bar{Y}$ are related to G_E and G_M by

$$\alpha_\psi = \frac{s - 4M^2|G_E/G_M|^2}{s + 4M^2|G_E/G_M|^2} \in [-1, 1],$$

$$\Delta\Phi = \arg\{G_E/G_M\} \in (-\pi, \pi]. \tag{30}$$

From Eq. (5), nonvanishing polarizations of Y and $\bar{Y}$ produced in annihilation of unpolarized electron and positron require $\Delta\Phi \neq 0$ and π. At the limit $\Delta\Phi = 0$ or π, however, there is only the spin correlation part in $\rho^X_{Y\bar{Y}}$ and without polarizations part from Eq. (7). This indicates that $\rho_{Y\bar{Y}}$ is reduced to a BDS form as

$$\rho^{\mathrm{BDS}}_{Y\bar{Y}} = \frac{1}{4}\left(\mathbb{1}\otimes\mathbb{1} + \sum_i t_i \sigma_i \otimes \sigma_i\right), \tag{31}$$

where $t_2^2 = t_3^2$. We note that a BDS is also a X state but without polarization.

Following Eqs. (14) and (15), the measure of the Bell nonlocality becomes

$$\mathfrak{m}_{12}[\rho^{\mathrm{BDS}}_{Y\bar{Y}}] = 1 + \left(\frac{\alpha_\psi \sin^2\vartheta}{1+\alpha_\psi\cos^2\vartheta}\right)^2 \geq 1. \tag{32}$$

We see in this circumstance the violation of the CHSH inequality occurs in the full range of the scattering angle $\vartheta \in (0, \pi)$ for any $\alpha_\psi \neq 0$. This result is different from what we discussed in Sec. III, where the Bell nonlocality is violated in a restricted angle range $(\vartheta^*, \pi - \vartheta^*)$. However, the maximal violation of the CHSH inequality also takes place at $\vartheta = \pi/2$ with the value in (17).

The concurrence in Eq. (23) for a BDS is reduced to

$$\mathcal{C}[\rho^{\mathrm{BDS}}_{Y\bar{Y}}] = \frac{|\alpha_\psi|\sin^2\vartheta}{1+\alpha_\psi\cos^2\vartheta}. \tag{33}$$

Comparing Eq. (32) and (33), one can see that the inequality in Eq. (24) becomes an equality $\mathcal{B} = 2\sqrt{1+\mathcal{C}^2}$ (or equivalently $\mathfrak{m}_{12} = 1 + \mathcal{C}^2$) in the whole range of the scattering angle (not only at $\vartheta = \pi/2$). This effect is shown in Fig. 4, where the nonlocality $\mathcal{B}$ (orange dash-dotted line) and $2\sqrt{1+\mathcal{C}^2}$ (blue solid line) converge to the red dotted line in the limit $\Delta\Phi = 0$. It is not a surprise since the property $\mathcal{B} = 2\sqrt{1+\mathcal{C}^2}$ is valid for any rank-2 BDS [54] with the fact that both $|\lambda_1\rangle$ and $|\lambda_2\rangle$ become two Bell states $(|00\rangle + |11\rangle)/\sqrt{2}$ and $(|01\rangle + |10\rangle)/\sqrt{2}$ with $\beta_\psi = 0$.

In other words, nonzero relative phase $\Delta\Phi$ leads to a splitting of $\mathcal{B}$ and $2\sqrt{1+\mathcal{C}^2}$. For Λ, Ξ^- and Ξ^0, Bell nonlocality is suppressed by nonzero $\Delta\Phi$, while the entanglement is enhanced. However, for Σ^+, both the nonlocality and entanglement are slightly suppressed.

From Eq. (30) we see that $\Delta\Phi$ is the relative phase between G_E and G_M. Let us take an example for the limit case $\Delta\Phi = 0, \pi$ by assuming $G_E = \pm G_M$. As a

TABLE IV. Decay parameters for ground-state octet hyperons. In our analysis, we neglect the $\mathcal{CP}$ violation effect so we have $\alpha_{\bar{Y}} = -\alpha_Y$.

Y	$\mathcal{B}(\%)$	α_Y	References
$\Lambda \to p\pi^-$	064	0.755(3)	[32,69]
$\Sigma^+ \to p\pi^0$	052	−0.994(4)	[44]
$\Xi^- \to \Lambda\pi^-$	100	−0.379(4)	[32,45]
$\Xi^0 \to \Lambda\pi^0$	96	−0.375(3)	[45,47]

consequence, the measures for the nonlocality and Wootters' concurrence are given by

$$\mathfrak{m}_{12} = 1 + \left[\frac{(s-4M^2)\sin^2\vartheta}{4M^2\sin^2\vartheta + s(\cos^2\vartheta+1)}\right]^2,$$
$$\mathcal{C} = \frac{(s-4M^2)\sin^2\vartheta}{4M^2\sin^2\vartheta + s(\cos^2\vartheta+1)}. \quad (34)$$

The above expressions coincide with Eqs. (3.4) and (3.7) in Ref. [18] for $e^+e^- \to \tau^+\tau^-$. This is reasonable since the vertex Eq. (28) in $e^+e^- \to \tau^+\tau^-$ is simply γ^μ indicating $G_E = G_M$.

In the process $e^+e^- \to Y\bar{Y}$, the existence of EMFFs manifests in a polarized final state, even if the colliding beams are unpolarized [61]. And this polarization effect leads to the $Y\bar{Y}$ spin correlation different from that in processes $e^+e^- \to \tau^+\tau^-$ and $pp \to t\bar{t}$ pairs [13,15,18].

VI. QUANTUM TOMOGRAPHY IN EXPERIMENTS

The spin polarization of the hyperon and antihyperon can be measured through their weak decays [32,62,63] $Y \to BM$ and $\bar{Y} \to \bar{B}\bar{M}$. The spin correlation in $Y\bar{Y}$ can also be extracted from the joint decay $Y\bar{Y} \to B\bar{B}(M\bar{M})$ through the joint angular distribution of $B\bar{B}$ [64]

$$I(\vartheta;\theta,\bar{\theta}) = \frac{1}{(4\pi)^2}\left[1 + \alpha_Y \sum_i P_i^+(\vartheta)\cos\theta_i + \alpha_{\bar{Y}}\sum_j P_j^-(\vartheta)\cos\bar{\theta}_j + \alpha_Y\alpha_{\bar{Y}}\sum_{i,j} C_{ij}(\vartheta)\cos\theta_i\cos\bar{\theta}_j\right], \quad (35)$$

where $i, j = 1, 2, 3$ or x, y, z denote three directions in the rest frame of Y and $\bar{Y}$ respectively, $\cos\theta_i$ and $\cos\bar{\theta}_j$ are projections of B and $\bar{B}$'s momentum directions onto the axis i and j respectively, and α_Y and $\alpha_{\bar{Y}}$ are the decay parameters in $Y \to BM$ and $\bar{Y} \to \bar{B}\bar{M}$ respectively, see Table IV for their values.

By adopting the idea of the quantum tomography [13,65] and the method of moments, the spin polarization and correlation in the hyperon-antihyperon system can be extracted from the joint distribution (35) as

$$P_i^+(\vartheta) = \frac{3}{\alpha_Y}\langle\cos\theta_i\rangle, \qquad P_j^- = \frac{3}{\alpha_{\bar{Y}}}\langle\cos\bar{\theta}_j\rangle,$$
$$C_{ij}(\vartheta) = \frac{9}{\alpha_Y\alpha_{\bar{Y}}}\langle\cos\theta_i\cos\bar{\theta}_j\rangle. \quad (36)$$

In this way, 15 real parameters $\mathbf{P}^\pm$ and C_{ij} in $\rho_{Y\bar{Y}}$ in Eq. (1) can be constructed from experiment data.

Furthermore, due to parity and charge conjugation invariance, these 15 parameters are not all independent: the only nonzero polarization is perpendicular to the production plane (i.e., in $\hat{\mathbf{y}}$ direction) and $P_y^+ = P_y^-$. The correlation is a symmetric matrix $C_{ij} = C_{ji}$ with $C_{xy} = C_{yz} = 0$. Then the 4×4 matrix $\Theta_{\mu\nu}$ reads

$$\Theta_{\mu\nu}(\vartheta) = \left[\begin{array}{c|ccc} 1 & 0 & P_y & 0 \\ \hline 0 & C_{xx} & 0 & C_{xz} \\ P_y & 0 & C_{yy} & 0 \\ 0 & C_{xz} & 0 & C_{zz} \end{array}\right], \quad (37)$$

where all elements are functions of the scattering angle ϑ. Obviously, Eq. (37) is local unitary equivalent to the standard X state.

The Bell nonlocality measure $\mathfrak{m}_{12}$ is given by the sum of two largest eigenvalues $C^{\mathrm{T}}C$ whose three eigenvalues of $C^{\mathrm{T}}C$ are

$$C_{yy}^2, \frac{1}{4}\left[C_{xx} + C_{zz} \pm \sqrt{4C_{xz}^2 + (C_{xx}-C_{zz})^2}\right]^2. \quad (38)$$

The concurrence $\mathcal{C}$ is given by

$$\mathcal{C} = \frac{1}{2}\max\left\{0, \sqrt{4C_{xz}^2 + (C_{xx}-C_{zz})^2} - |1-C_{yy}|, |C_{xx}+C_{zz}| - \sqrt{(1+C_{yy})^2 - 4P_y^2}\right\}. \quad (39)$$

Since P_y and C_{xx}, C_{yy}, C_{zz} and C_{xz} can all be constructed from data, the Bell nonlocality and entanglement can be tested in experiments.

The above probe to quantum correlation in e^+e^- annihilation at BESIII can also be extended to $p\bar{p} \to Y\bar{Y}$ at PANDA [66], in which the spin-parity of the intermediate resonance is not necessarily 1^-.

VII. SUMMARY AND OUTLOOK

In this work, we present the study of the Bell nonlocality and entanglement in $e^+e^- \to Y\bar{Y}$, with Y being the spin-1/2 octet hyperon. We begin with the spin density operator for $Y\bar{Y}$ and convert it into that for the standard two-qubit X state. Using properties of X states, we derive analytical formulas for the Bell nonlocality and entanglement in various $Y\bar{Y}$ systems, based on two intrinsic parameters, α_ψ and $\Delta\Phi$, along with a kinematic variable, the scattering angle ϑ. We explore the relation between the Bell nonlocality and entanglement and present the experimental proposal to test the nonlocality and entanglement at BESIII.

In $e^+e^- \to Y\bar{Y}$, the relative phase between the electric and magnetic form factors of hyperons lead to their polarizations in the spin density operator. With nonvanishing polarizations of Y and $\bar{Y}$, the kinematic region of nonlocality in the $Y\bar{Y}$ system is more restricted than $\tau^+\tau^-$ [17,18] and $t\bar{t}$ systems [13–16] where polarizations of tau leptons and top quarks are vanishing. The entanglement in the $Y\bar{Y}$ system can also be influenced by the polarization effect in comparison with $\tau^+\tau^-$ and $t\bar{t}$ systems. This is the main result of our work.

Our work offers a theoretical framework for probing the nonlocality and entanglement in hyperon-antihyperon systems at BESIII. Our method can also be applied to other collision processes with X-form final states such as $p\bar{p} \to Y\bar{Y}$ at PANDA [66]. A modified CHSH inequality and related entanglement measures were proposed to quantify the quantum entanglement and spin correlation of $\Lambda\bar{\Lambda}$ in string fragmentation [29]. Our method can also be generalized to describe the nonlocality and entanglement of such hyperon-antihyperon systems in many-body states.

Note added in the second version of this paper: we noticed that a new article appeared in the arXiv addressing similar problems but in a different angle [67]. We also noticed that a new article was posted by the BESIII collaboration about $J/\psi \to \Sigma^0\bar{\Sigma}^0$ [68]. There are no experimental data available for $J/\psi \to \Sigma^-\bar{\Sigma}^+$ yet. We will address the nonlocality and entanglement of $\Sigma^0\bar{\Sigma}^0$ and $\Sigma^-\bar{\Sigma}^+$ in a future study.

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