Z_b states as the mixture of the molecular and diquark-anti-diquark components within the effective field theory

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In this study, we reconsider the states $Z_b(10610)$ and $Z_b(10650)$ by investigating the presence of diquark-anti-diquark components as well as the hadronic molecule components in the framework of effective field theory. The different masses of pseudoscalar mesons such as π^0, η_8, and η_0, as well as vector mesons like ρ^0 and ω violate the Okubo-Zweig-Iizuka (OZI) rule that is well depicted under the $[U(3)_L \otimes U(3)_R]_{\rm global} \otimes [U(3)_V]_{\rm local}$ symmetry. To account for the contribution of intermediate bosons of heavy masses within the one-boson-exchange model, we introduce an exponential form factor instead of the commonly used monopole form factor in the past. By solving the coupled-channel Schrödinger equation with the Gaussian expansion method, our numerical results indicate that the $Z_b(10610)$ and $Z_b(10650)$ states can be explained as hadronic molecules slightly mixing with diquark-anti-diquark states.

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I. INTRODUCTION

In the past decades, a series of quarkonium-like states was discovered. In the $b\bar{b}$ sector, Belle Collaboration reported two charged bottomonium-like states which are known as $Z_b(10610)$ and $Z_b(10650)$ [1] in 2011. Both of them were observed in $\Upsilon(5S) \to \pi^\pm h_b(mP)$ ($m = 1, 2$) and $\Upsilon(5S) \to \pi^\pm \Upsilon(nS)$ ($n = 1, 2, 3$), respectively. Later, Belle confirmed their observations [2,3]. The next year after the first discovery, the neutral state $Z_b^0(10610)$ was found in the $\Upsilon(5S) \to \Upsilon(2S, 3S)\pi^0\pi^0$ decay [4]. The masses and widths of these states listed in PDG (Particle Data Group) [5] are shown below:

$$M_{Z_b^\pm} = 10607.2 \pm 2.0\ \text{MeV}, \quad \Gamma_{Z_b^\pm} = 18.4 \pm 2.4\ \text{MeV},$$
$$M_{Z_b^0} = 10609 \pm 4.0 \pm 4\ \text{MeV},$$
$$M_{Z_b'^\pm} = 10652.2 \pm 1.5\ \text{MeV}, \quad \Gamma_{Z_b'^\pm} = 11.5 \pm 2.2\ \text{MeV},$$

with the quantum numbers $I^G(J^P) = 1^+(1^+)$. For simplicity, here we label the two states $Z_b(10610)$ and $Z_b(10650)$ by Z_b and Z_b', respectively.

Theoretical research had already been performed before the observations of the Z_b states. The authors of Refs. [6,7] indicated that there may exist a loosely bound S-wave $B\bar{B}^*/B^*\bar{B}$ molecular state.

After the observation, the explanations of the nature of the Z_b states were proposed through different assumptions and theoretical methods. Since the masses of the $Z_b(10610)$ and the $Z_b(10650)$ are close to the $B\bar{B}^*$ and $B^*\bar{B}^*$ thresholds, they are good candidates for $B\bar{B}^*$ and $B^*\bar{B}^*$ molecular states [8–42]. However, tetraquark interpretations including diquark-anti-diquark explanation cannot be ruled out [21,33,43–49]. As a consequence, in this work we study these two states in the picture of the mixture of molecular and diquark-anti-diquark components, which is used to investigate the nature of the Z_{cs} states observed by LHCb in our previous work [50].

The hadronic molecule has been proposed based on the study of a deuteron composed of a proton and a neutron. And, this kind of topic has been widely discussed [51–54] within different methods, especially after the observation of $X(3872)$ in 2003 [55]. On the other hand, the concept of the diquark-anti-diquark state was proposed for the first time by Maiani *etal.* [56,57] following the revitalization of interest in the σ meson. In this work, we use both the

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molecular state and the diquark-anti-diquark state, and make the calculation in the framework of the effective field theory. In this way, the mesons and the diquarks are viewed as pointlike particles, which finally form the color singlet system. The forces between these clusters are provided by exchanging pseudoscalar and vector mesons as well as scalar and axial-vector diquarks.

In order to calculate the effective potentials in the coordinate space, form factors for each vertex are needed, such that the high-momentum contributions are suppressed. In the works related to the one-boson-exchange (OBE) model in the past, the monopole form factor is introduced (see the review [51]). However, in some of our cases, since the exchanged particles' masses are not so small, the monopole form factor does not work very well for it to suppress the corresponding potentials by its numerator. So, we introduce the exponentially parametrized form factor and obtain the analytical expressions of the potential in the coordinate space. Considering both the S- and D-wave contributions, we solve the coupled-channel Schrödinger equation to see the existence of the composite particles.

The structure of this paper is organized as follows. After the Introduction, the theoretical framework is presented in Sec. II. The results and discussion are shown in Sec. III. Finally, a brief summary is given in Sec. IV.

II. FORMALISM

A. Wave functions

We give here the flavor wave functions of the negative and neutral $B\bar{B}^*/B^*\bar{B}$ and $B^*\bar{B}^*$ systems constructed in Ref. [13]:

$$\begin{aligned}
|Z^-_{B\bar{B}^*/B^*\bar{B}}\rangle &= \frac{1}{\sqrt{2}}(|B^{*-}B^0\rangle + c|B^-B^{*0}\rangle),\\
|Z^0_{B\bar{B}^*/B^*\bar{B}}\rangle &= \frac{1}{2}[|B^{*+}B^-\rangle - |B^{*0}\bar{B}^0\rangle + c(|B^+B^{*-}\rangle - |B^0\bar{B}^{*0}\rangle)],\\
|Z^-_{B^*\bar{B}^*}\rangle &= |B^{*-}B^{*0}\rangle,\\
|Z^0_{B^*\bar{B}^*}\rangle &= \frac{1}{\sqrt{2}}(|B^{*+}B^{*-}\rangle - |B^{*0}\bar{B}^{*0}\rangle).
\end{aligned} \tag{1}$$

The flavor wave functions of diquark and anti-diquark systems are constructed in analogy to the meson-meson systems by swapping b quark and $\bar{b}$ quark,

$$\begin{aligned}
|Z^-_{S\bar{A}/A\bar{S}}\rangle &= \frac{1}{\sqrt{2}}(|\bar{A}_{bu}S_{bd}\rangle + c|\bar{S}_{bu}A_{bd}\rangle),\\
|Z^0_{S\bar{A}/A\bar{S}}\rangle &= \frac{1}{2}[|A_{bu}\bar{S}_{bu}\rangle - |A_{bd}\bar{S}_{bd}\rangle + c(|S_{bu}\bar{A}_{bu}\rangle - |S_{bd}\bar{A}_{bd}\rangle)],\\
|Z^-_{A\bar{A}}\rangle &= |A_{bd}\bar{A}_{bu}\rangle,\\
|Z^0_{A\bar{A}}\rangle &= \frac{1}{\sqrt{2}}(|A_{bu}\bar{A}_{bu}\rangle - |A_{bd}\bar{A}_{bd}\rangle).
\end{aligned} \tag{2}$$

The value of c depends on the G parity, i.e., $c = \pm 1$ corresponds to $G = \pm 1$. Here, we only pay attention to the situation of $c = +1$, since the G parity of the considered systems are all $+1$.

In this work, both S- and D-wave interactions between the composed particles are considered. In general, the Z_b and Z'_b states can be expressed as

$$|Z_b\rangle = \begin{pmatrix} Z^-_{B\bar{B}^*/B^*\bar{B}}(^3S_1)\\ Z^-_{B\bar{B}^*/B^*\bar{B}}(^3D_1)\\ Z^-_{B^*\bar{B}^*}(^3S_1)\\ Z^-_{B^*\bar{B}^*}(^3D_1)\\ Z^-_{S\bar{A}/A\bar{S}}(^3S_1)\\ Z^-_{S\bar{A}/A\bar{S}}(^3D_1)\\ Z^-_{A\bar{A}}(^3S_1)\\ Z^-_{A\bar{A}}(^3D_1) \end{pmatrix}, \quad |Z'_b\rangle = \begin{pmatrix} Z^-_{B^*\bar{B}^*}(^3S_1)\\ Z^-_{B^*\bar{B}^*}(^3D_1)\\ Z^-_{S\bar{A}/A\bar{S}}(^3S_1)\\ Z^-_{S\bar{A}/A\bar{S}}(^3D_1)\\ Z^-_{A\bar{A}}(^3S_1)\\ Z^-_{A\bar{A}}(^3D_1) \end{pmatrix}. \tag{3}$$

Note that for Z'_b, $|^5D_1\rangle$ state is forbidden due to its G-even parity.

B. The effective Lagrangians and coupling constants

Next, we introduce the interactions of meson-meson and diquark-anti-diquark by constructing the corresponding Lagrangians.

As we all know, each quark is a color triplet resulting in a diquark being a color antitriplet or sextet. The interaction between the two quarks of an antitriplet is attractive, while that of a sextet is repulsive. Consequently, we only consider the effective Lagrangian containing antitriplet. The diquark fields are depicted as

$$S^a = \begin{pmatrix} 0 & S_{ud} & S_{us}\\ -S_{ud} & 0 & S_{ds}\\ -S_{us} & -S_{ds} & 0 \end{pmatrix}^a, \tag{4}$$

$$A^a_\mu = \begin{pmatrix} A_{uu} & \frac{1}{\sqrt{2}}A_{ud} & \frac{1}{\sqrt{2}}A_{us}\\ \frac{1}{\sqrt{2}}A_{ud} & A_{dd} & \frac{1}{\sqrt{2}}A_{ds}\\ \frac{1}{\sqrt{2}}A_{us} & \frac{1}{\sqrt{2}}A_{ds} & A_{ss} \end{pmatrix}^a_\mu, \tag{5}$$

$$S^a_b = (S_{bu} \quad S_{bd} \quad S_{bs})^a, \tag{6}$$

$$A^a_{b\mu} = (A_{bu} \quad A_{bd} \quad A_{bs})^a_\mu, \tag{7}$$

where S^a is the light scalar diquark, A^a_μ the light axial vector diquark, S^a_b the bottomed scalar diquark, and $A^a_{b\mu}$ the bottomed axial vector diquark. The superscript $a = 1$, 2, 3 is the color index. The meson fields read

$$\Phi = \begin{pmatrix} \frac{\sqrt{3}\pi^0+\eta_8+\sqrt{2}\eta_0}{\sqrt{3}} & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & \frac{-\sqrt{3}\pi^0+\eta_8+\sqrt{2}\eta_0}{\sqrt{3}} & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & \frac{-2\eta_8+\sqrt{2}\eta_0}{\sqrt{3}} \end{pmatrix}, \quad (8)$$

$$V_\mu = \frac{g_V}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}}(\rho^0+\omega) & \rho^+ & K^{*+} \\ \rho^- & -\frac{1}{\sqrt{2}}(\rho^0-\omega) & K^{*0} \\ K^{*-} & \bar{K}^{*0} & \phi \end{pmatrix}_\mu, \quad (9)$$

$$P = (B^-, \bar{B}^0, \bar{B}_s^0), \quad (10)$$

$$P_\tau^* = (B^{*-}, \bar{B}^{*0}, \bar{B}_s^{*0})_\tau, \quad (11)$$

with, respectively, Φ and V_μ the light pseudoscalar and vector, and P and P_τ^* the bottomed pseudoscalar and vector.

Considering the $[U(3)_L \otimes U(3)_R]_{\mathrm{global}} \otimes [U(3)]_{\mathrm{local}}$ symmetry [58,59], parity, and charge conjugation, the Lagrangians containing mesons and diquarks are shown as follows:

$$\begin{aligned}\mathcal{L}_1 &= a_1(iP\hat{\alpha}_{\parallel\mu}D^\mu P^\dagger + \mathrm{H.c.}) + a_2(iP\hat{\alpha}_{\perp\mu}P^{*\mu\dagger} + \mathrm{H.c.}) \\ &\quad + a_3(\xi^{\mu\nu\alpha\beta}P_\nu^*\hat{\alpha}_{\perp\alpha}D_\mu P_\beta^{*\dagger} + \mathrm{H.c.}) \\ &\quad + a_4(iP_\nu^*\hat{\alpha}_\parallel^\mu D_\mu P^{*\nu\dagger} + \mathrm{H.c.}),\end{aligned} \quad (12)$$

$$\begin{aligned}\mathcal{L}_2 &= e_1(iPD_\mu S^a A_b^{a\mu\dagger} - iA_b^{a\mu}D_\mu S^{a\dagger}P^\dagger) + e_2(iPA_\mu^a D^\mu S_b^{a\dagger} - iD^\mu S_b^a A_\mu^{a\dagger}P^\dagger) + e_3(\epsilon^{\mu\nu\alpha\beta}PA_{\mu\nu}^a A_{b\alpha\beta}^{a\dagger} + \epsilon^{\mu\nu\alpha\beta}A_{b\alpha\beta}^a A_{\mu\nu}^{a\dagger}P^\dagger) \\ &\quad + e_4(iP_\mu^* D^\mu S^a S_b^{a\dagger} - iS_b^a D^\mu S^{a\dagger}P_\mu^{*\dagger}) + e_5(\epsilon^{\mu\nu\alpha\beta}P_\mu^* D_\nu S^a A_{b\alpha\beta}^{a\dagger} + \epsilon^{\mu\nu\alpha\beta}A_{b\alpha\beta}^a D_\nu S^{a\dagger}P_\mu^{*\dagger}) \\ &\quad + e_6(\epsilon^{\mu\nu\alpha\beta}P_\mu^* A_{\nu\alpha}^a D_\beta S_b^{a\dagger} + \epsilon^{\mu\nu\alpha\beta}D_\beta S_b^a A_{\nu\alpha}^{a\dagger}P_\mu^{*\dagger}) + e_7(iP_\mu^* A^{a\mu\nu}A_{b\nu}^{a\dagger} - iA_{b\nu}^a A^{a\mu\nu\dagger}P_\mu^{*\dagger}) \\ &\quad + e_8(iP_\mu^* A_\nu^a A_b^{a\mu\nu\dagger} - iA_b^{a\mu\nu}A_\nu^{a\dagger}P_\mu^{*\dagger}) + e_9(iP_{\mu\nu}^* A^{a\mu}A_b^{a\nu\dagger} - iA_b^{a\nu}A^{a\mu\dagger}P_{\mu\nu}^{*\dagger}),\end{aligned} \quad (13)$$

$$\begin{aligned}\mathcal{L}_3 &= h_1(iS_b^a\hat{\alpha}_\parallel^{\mu T}D_\mu S_b^{a\dagger} - iD_\mu S_b^a\hat{\alpha}_\parallel^{\mu T}S_b^{a\dagger}) + h_2(\epsilon^{\mu\nu\alpha\beta}A_{b\mu\nu}^a\hat{\alpha}_{\parallel\alpha}^{\mu T}D_\beta S_b^{a\dagger} + \epsilon^{\mu\nu\alpha\beta}D_\beta S_b^a\hat{\alpha}_{\parallel\alpha}^{\mu T}A_{b\mu\nu}^{a\dagger}) + h_3(iA_{b\mu}^a\hat{\alpha}_\perp^{\mu T}S_b^{a\dagger} - iS_b^a\hat{\alpha}_\perp^{\mu T}A_{b\mu}^{a\dagger}) \\ &\quad + h_4(iA_{b\mu}^a\hat{\alpha}_{\parallel\nu}^T A_b^{a\mu\nu\dagger} - iA_b^{a\mu\nu}\hat{\alpha}_{\parallel\nu}^T A_{b\mu}^{a\dagger}) + h_5(\epsilon^{\mu\nu\alpha\beta}A_{b\mu}^a\hat{\alpha}_{\perp\nu}^T A_{b\alpha\beta}^{a\dagger} + \epsilon^{\mu\nu\alpha\beta}A_{b\alpha\beta}^a\hat{\alpha}_{\perp\nu}^T A_{b\alpha\mu}^{a\dagger}),\end{aligned} \quad (14)$$

where

$$D_\mu P = \partial_\mu P + iP\alpha_{\parallel\mu}^\dagger = \partial_\mu P + iP\alpha_{\parallel\mu}, \quad (15)$$

$$D_\mu P_\tau^* = \partial_\mu P_\tau^* + iP_\tau^*\alpha_{\parallel\mu}^\dagger = \partial_\mu P_\tau^* + iP_\tau^*\alpha_{\parallel\mu}, \quad (16)$$

$$\alpha_{\perp\mu} = (\partial_\mu\xi_R\xi_R^\dagger - \partial_\mu\xi_L\xi_L^\dagger)/(2i), \quad (17)$$

$$\alpha_{\parallel\mu} = (\partial_\mu\xi_R\xi_R^\dagger + \partial_\mu\xi_L\xi_L^\dagger)/(2i), \quad (18)$$

$$\hat{\alpha}_{\perp\mu} = (D_\mu\xi_R\xi_R^\dagger - D_\mu\xi_L\xi_L^\dagger)/(2i), \quad (19)$$

$$\hat{\alpha}_{\parallel\mu} = (D_\mu\xi_R\xi_R^\dagger + D_\mu\xi_L\xi_L^\dagger)/(2i), \quad (20)$$

$$\xi_L = e^{i\sigma/F_\sigma}e^{-i\Phi/(2F_\pi)}, \quad (21)$$

$$\xi_R = e^{i\sigma/F_\sigma}e^{i\Phi/(2F_\pi)}, \quad (22)$$

$$A_{\mu\nu}^a = D_\mu A_\nu^a - D_\nu A_\mu^a, \quad (23)$$

$$A_{b\mu\nu}^a = D_\mu A_{b\nu}^a - D_\nu A_{b\mu}^a, \quad (24)$$

$$D_\mu A_\nu^a = \partial_\mu A_\nu^a - iV_\mu A_\nu^a - iA_\nu^a V_\mu^T, \quad (25)$$

$$D_\mu S^a = \partial_\mu S^a - iV_\mu S^a - iS^a V_\mu^T, \quad (26)$$

$$D_\mu A_{b\nu}^a = \partial_\mu A_{b\nu}^a - iA_{b\nu}^a\alpha_{\parallel\mu}^T, \quad (27)$$

$$D_\mu S_b^a = \partial_\mu S_b^a - iS_b^a\alpha_{\parallel\mu}^T. \quad (28)$$

In Eqs. (13) and (14), the Einstein summation convention is used, i.e., the repeated superscripts "a" mean the summation over them. Comparing $\mathcal{L}_1$ with the Lagrangian in Ref. [60], we have

$$a_1 = -\frac{\beta}{m_P}, \quad a_2 = -2g, \quad a_3 = -\frac{g}{m_{P^*}}, \quad a_4 = \frac{\beta}{m_{P^*}}. \quad (29)$$

For g, we take the full width of D^{*+} from PDG [5], and get $g = 0.58 \pm 0.01$. The parameter β is determined by the vector meson dominance [60,61], i.e., $\beta \approx 0.85$. For $\mathcal{L}_2$ and $\mathcal{L}_3$, there are two set of coupling constants $e_i (i = 1, 2, \ldots, 9)$ and $h_j (j = 1, 2, \ldots, 5)$ whose values are still unknown. In this work, we naively use the ${}^3P_0$ model to determine them. Their values are listed in Table I. Note that we cannot fix the sign of e_3, e_5, and e_6 because the relative phase between the amplitudes obtained from the Lagrangian and ${}^3P_0$ model cannot be determined.

TABLE I. The values of the low-energy constants in the Lagrangians containing diquarks.

e_1 (GeV^{-1})	e_2 (GeV^{-1})	e_3 (GeV^{-2})	e_4 (GeV^{-1})	e_5 (GeV^{-2})
−6.353	1.657	±0.555	4.885	±0.566
e_6 (GeV^{-2})	e_7 (GeV^{-1})	e_8 (GeV^{-1})	e_9 (GeV^{-1})	
±1.005	0.909	13.348	−11.530	
h_1 (GeV^{-1})	h_2 (GeV^{-1})	h_3 (GeV^{-1})	h_4 (GeV^{-1})	h_5 (GeV^{-1})
0.084	±0.130	0.266	1.457	±0.011

For e_7, e_8, and e_9, we use the phase in Ref. [50] which explains the Z_{cs} well and get the values of them. In Eqs. (21) and (22), we choose $\sigma = 0$ according to the unitary gauge [59].

C. Effective potentials with the exponential form factor

Making use of the Breit approximation, we obtain the effective potentials in the momentum space,

$$\mathcal{V}^{H_1H_2\to H_3H_4}(\boldsymbol{q}) = \frac{\mathcal{M}^{H_1H_2\to H_3H_4}(\boldsymbol{q})}{\sqrt{\prod_i 2m_i \prod_f 2m_f}}, \tag{30}$$

where m_i ($i = 1, 2, 3, 4$) denotes the mass of the particle labeled by i. By performing the Fourier transformation, we get the effective potential in the coordinate space,

$$\mathcal{V}^{H_1H_2\to H_3H_4}(\boldsymbol{r}) = \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} e^{i\boldsymbol{q}\cdot\boldsymbol{r}} \mathcal{V}^{H_1H_2\to H_3H_4}(\boldsymbol{q}) F^2(q^2). \tag{31}$$

Here, $F(\vec{q}^2)$ is the form factor which suppresses the contribution of high momenta, i.e., small distance. And, the presence of such a form factor is dictated by the extended (quark) structure of the hadrons. In this work, we adopt the exponentially parametrized form factor

$$F(q^2) = e^{q^2/\Lambda^2} = e^{(q_0^2-\vec{q}^2)/\Lambda^2}, \tag{32}$$

with Λ the cutoff.

Another option of the form factor is monopole expression, i.e.,

$$F_M(q^2) = \frac{\Lambda^2 - m_E^2}{\Lambda^2 - q^2}, \tag{33}$$

with m_E the mass of the exchanged particle. If the exchanged meson's mass is large, for instance in the case of ϕ- or diquark-exchange, $F_M(q^2)$ is highly suppressed by the numerator $\Lambda^2 - m_E^2$, which would lead to unreasonable results. Consequently, we choose the exponentially parametrized form factor in this work.

In Eqs. (34)–(51), we list the specific expressions of the nonzero effective subpotentials which are isospin independent:

$$V_v^{\bar{B}^*B\to\bar{B}^*B/\bar{B}B^*\to\bar{B}B^*} = C_v \frac{(\beta g_V)^2 m_{B^*} m_B}{4M_P M_{P^*}} (\boldsymbol{\epsilon_{1/2}} \cdot \boldsymbol{\epsilon}^\dagger_{\boldsymbol{3/4}}) \times Y(\Lambda, m_v, r), \tag{34}$$

$$V_p^{\bar{B}B^*\to\bar{B}^*B} = C_p \left(\frac{g}{2F_\pi}\right)^2 [(\boldsymbol{\epsilon_2} \cdot \boldsymbol{\epsilon}^\dagger_{\boldsymbol{3}}) Z(\Lambda, \tilde{m}_p, r) + S(\hat{\boldsymbol{r}}, \boldsymbol{\epsilon_2}, \boldsymbol{\epsilon}^\dagger_{\boldsymbol{3}}) T(\Lambda, \tilde{m}_p, r)], \tag{35}$$

$$V_p^{\bar{B}^*B/\bar{B}B^*\to\bar{B}^*B^*} = \delta_{B\bar{B}^*/B^*\bar{B}} C_p \frac{g^2 m_{B^*}}{4M_{P^*}F_\pi^2} \times [\boldsymbol{\epsilon}^\dagger_{\boldsymbol{4/3}} \cdot (\boldsymbol{\epsilon_{1/2}} \times \boldsymbol{\epsilon}^\dagger_{\boldsymbol{3/4}}) Z(\Lambda, \tilde{m}_p, r) + S(\hat{\boldsymbol{r}}, \boldsymbol{\epsilon}^\dagger_{\boldsymbol{4/3}}, \boldsymbol{\epsilon_{1/2}} \times \boldsymbol{\epsilon}^\dagger_{\boldsymbol{3/4}}) T(\Lambda, \tilde{m}_p, r)], \tag{36}$$

$$V_{S_{ud}}^{\bar{B}^*B\to S\bar{A}/\bar{B}B^*\to A\bar{S}} = -\frac{\sqrt{3}}{12} e_1 e_4 [(\boldsymbol{\epsilon_{1/2}} \cdot \boldsymbol{\epsilon}^\dagger_{\boldsymbol{4/3}}) Z(\Lambda, \tilde{m}_{S_{ud}}, r) + S(\hat{\boldsymbol{r}}, \boldsymbol{\epsilon_{1/2}}, \boldsymbol{\epsilon}^\dagger_{\boldsymbol{4/3}}) T(\Lambda, \tilde{m}_{S_{ud}}, r)], \tag{37}$$

$$V_{A_{ud}}^{\bar{B}^*B\to S\bar{A}/\bar{B}B^*\to A\bar{S}} = \sqrt{3} e_3 e_6 m_{S_{bq}} m_{A_{bq}} \left[\frac{2}{3} (\boldsymbol{\epsilon_{1/2}} \cdot \boldsymbol{\epsilon}^\dagger_{\boldsymbol{4/3}}) \times Z(\Lambda, \tilde{m}_{A_{ud}}, r) - \frac{1}{3} S(\hat{\boldsymbol{r}}, \boldsymbol{\epsilon_{1/2}}, \boldsymbol{\epsilon}^\dagger_{\boldsymbol{4/3}}) \times T(\Lambda, \tilde{m}_{A_{ud}}, r)\right], \tag{38}$$

$$V_{A_{ud}}^{\bar{B}^*B\to A\bar{S}/\bar{B}B^*\to S\bar{A}} = -\frac{\sqrt{3}}{8} e_2 (e_8 m_{A_{bq}} + e_9 m_{B^*}) m_{S_{bq}} \times \left(\frac{\tilde{m}_{A_{ud}}}{m_{A_{ud}}}\right)^2 (\boldsymbol{\epsilon_{1/2}} \cdot \boldsymbol{\epsilon}^\dagger_{\boldsymbol{3/4}}) Y(\Lambda, \tilde{m}_{A_{ud}}, r), \tag{39}$$

$$V_{S_{ud}}^{\bar{B}^*B/\bar{B}B^*\to A\bar{A}} = \delta_{B\bar{B}^*/B^*\bar{B}} \frac{1}{2\sqrt{3}} e_1 e_5 m_{A_{bq}} \times [\boldsymbol{\epsilon}^\dagger_{\boldsymbol{4/3}} \cdot (\boldsymbol{\epsilon_{1/2}} \times \boldsymbol{\epsilon}^\dagger_{\boldsymbol{3/4}}) Z(\Lambda, \tilde{m}_{S_{ud}}, r) + S(\hat{\boldsymbol{r}}, \boldsymbol{\epsilon}^\dagger_{\boldsymbol{4/3}}, \boldsymbol{\epsilon_{1/2}} \times \boldsymbol{\epsilon}^\dagger_{\boldsymbol{3/4}}) T(\Lambda, \tilde{m}_{S_{ud}}, r)], \tag{40}$$

$$V_{A_{ud}}^{\bar{B}^*B/\bar{B}B^*\to A\bar{A}} = \delta_{B\bar{B}^*/B^*\bar{B}} \sqrt{\frac{1}{3}} e_3 e_7 m_{A_{bq}} \times [\boldsymbol{\epsilon_{1/2}} \cdot (\boldsymbol{\epsilon}^\dagger_{\boldsymbol{3/4}} \times \boldsymbol{\epsilon}^\dagger_{\boldsymbol{4/3}}) Z(\Lambda, \tilde{m}_{A_{ud}}, r) - \frac{1}{2} S(\hat{\boldsymbol{r}}, \boldsymbol{\epsilon}^\dagger_{\boldsymbol{3/4}}, \boldsymbol{\epsilon_{1/2}} \times \boldsymbol{\epsilon}^\dagger_{\boldsymbol{4/3}}) T(\Lambda, \tilde{m}_{A_{ud}}, r) + \frac{1}{2} S(\hat{\boldsymbol{r}}, \boldsymbol{\epsilon_{1/2}}, \boldsymbol{\epsilon}^\dagger_{\boldsymbol{3/4}} \times \boldsymbol{\epsilon}^\dagger_{\boldsymbol{4/3}}) T(\Lambda, \tilde{m}_{A_{ud}}, r)], \tag{41}$$

$$V_v^{\bar{B}^*B^*\to\bar{B}^*B^*} = C_v\left(\frac{\beta g_V m_{B^*}}{2M_{P^*}}\right)^2(\boldsymbol{\epsilon_1}\cdot\boldsymbol{\epsilon_3^\dagger})(\boldsymbol{\epsilon_2}\cdot\boldsymbol{\epsilon_4^\dagger})Y(\Lambda,m_v,r), \tag{42}$$

$$V_p^{\bar{B}^*B^*\to\bar{B}^*B^*} = C_p\left(\frac{g m_{B^*}}{2m_{P^*F_\pi}}\right)^2[(\boldsymbol{\epsilon_1}\times\boldsymbol{\epsilon_3^\dagger})\cdot(\boldsymbol{\epsilon_2}\times\boldsymbol{\epsilon_4^\dagger}) \times Z(\Lambda,m_p,r) + S(\hat{\boldsymbol{r}},\boldsymbol{\epsilon_1}\times\boldsymbol{\epsilon_3^\dagger},\boldsymbol{\epsilon_2}\times\boldsymbol{\epsilon_4^\dagger}) \times T(\Lambda,m_p,r)], \tag{43}$$

$$V_{S_{ud}}^{\bar{B}^*B^*\to S\bar{A}/\bar{S}A} = -\frac{\delta_{S\bar{A}/A\bar{S}}}{2\sqrt{3}}e_4e_5m_{A_{bq}}[\boldsymbol{\epsilon_{1/2}}\cdot(\boldsymbol{\epsilon_{2/1}}\times\boldsymbol{\epsilon_{4/3}^\dagger}) \times Z(\Lambda,\tilde{m}_{S_{ud}},r) + S(\hat{\boldsymbol{r}},\boldsymbol{\epsilon_{1/2}},\boldsymbol{\epsilon_{2/1}}\times\boldsymbol{\epsilon_{4/3}^\dagger}) \times T(\Lambda,\tilde{m}_{S_{ud}},r), \tag{44}$$

$$V_{A_{ud}}^{\bar{B}^*B^*\to S\bar{A}/A\bar{S}} = -\delta_{S\bar{A}/A\bar{S}}\frac{1}{2\sqrt{3}}e_6e_7m_{S_{bq}} \times[\boldsymbol{\epsilon_{2/1}}\cdot(\boldsymbol{\epsilon_{1/2}}\times\boldsymbol{\epsilon_{4/3}^\dagger})Z(\Lambda,\tilde{m}_{A_{ud}},r) + \frac{1}{2}S(\hat{\boldsymbol{r}},\boldsymbol{\epsilon_{2/1}},\boldsymbol{\epsilon_{1/2}}\times\boldsymbol{\epsilon_{4/3}^\dagger})T(\Lambda,\tilde{m}_{A_{ud}},r) - \frac{1}{2}S(\hat{\boldsymbol{r}},\boldsymbol{\epsilon_{4/3}^\dagger},\boldsymbol{\epsilon_{1/2}}\times\boldsymbol{\epsilon_{2/1}})T(\Lambda,\tilde{m}_{A_{ud}},r)], \tag{45}$$

$$V_{S_{ud}}^{\bar{B}^*B^*\to A\bar{A}} = \frac{e_5^2m_{A_{bq}}^2}{\sqrt{3}}[(\boldsymbol{\epsilon_1}\times\boldsymbol{\epsilon_3^\dagger})\cdot(\boldsymbol{\epsilon_2}\times\boldsymbol{\epsilon_4^\dagger})Z(\Lambda,\tilde{m}_{S_{ud}},r) + S(\boldsymbol{r},\boldsymbol{\epsilon_1}\times\boldsymbol{\epsilon_3^\dagger},\boldsymbol{\epsilon_2}\times\boldsymbol{\epsilon_4^\dagger})T(\Lambda,\tilde{m}_{S_{ud}},r)], \tag{46}$$

$$V_{A_{ud}}^{\bar{B}^*B^*\to A\bar{A}} = -\frac{\sqrt{3}e_7^2}{24}[2(\boldsymbol{\epsilon_2}\cdot\boldsymbol{\epsilon_3^\dagger})(\boldsymbol{\epsilon_1}\cdot\boldsymbol{\epsilon_4^\dagger})Z(\Lambda,\tilde{m}_{A_{ud}},r) - 2(\boldsymbol{\epsilon_1}\cdot\boldsymbol{\epsilon_2})(\boldsymbol{\epsilon_3^\dagger}\cdot\boldsymbol{\epsilon_4^\dagger})Z(\Lambda,\tilde{m}_{A_{ud}},r) + (\boldsymbol{\epsilon_2}\cdot\boldsymbol{\epsilon_3^\dagger})S(\boldsymbol{r},\boldsymbol{\epsilon_1},\boldsymbol{\epsilon_4^\dagger})T(\Lambda,\tilde{m}_{A_{ud}},r) - (\boldsymbol{\epsilon_3^\dagger}\cdot\boldsymbol{\epsilon_4^\dagger})S(\hat{\boldsymbol{r}},\boldsymbol{\epsilon_1},\boldsymbol{\epsilon_2})T(\Lambda,\tilde{m}_{A_{ud}},r) + (\boldsymbol{\epsilon_1}\cdot\boldsymbol{\epsilon_4^\dagger})S(\hat{\boldsymbol{r}},\boldsymbol{\epsilon_2},\boldsymbol{\epsilon_3^\dagger})T(\Lambda,\tilde{m}_{A_{ud}},r) - (\boldsymbol{\epsilon_1}\cdot\boldsymbol{\epsilon_2})S(\hat{\boldsymbol{r}},\boldsymbol{\epsilon_3^\dagger},\boldsymbol{\epsilon_4^\dagger})T(\Lambda,\tilde{m}_{A_{ud}},r)] - \frac{\sqrt{3}}{8}(e_8m_{A_{bq}}+e_9m_{B^*})^2\left(\frac{\tilde{m}_{A_{ud}}}{m_{A_{ud}}}\right)^2(\boldsymbol{\epsilon_1}\cdot\boldsymbol{\epsilon_3^\dagger})(\boldsymbol{\epsilon_2}\cdot\boldsymbol{\epsilon_4^\dagger})Y(\Lambda,\tilde{m}_{A_{ud}},r), \tag{47}$$

$$V_v^{S\bar{A}\to S\bar{A}/A\bar{S}\to A\bar{S}} = C_v\left(\frac{g_V}{2}\right)^2h_1h_4m_{A_{bq}}m_{S_{bq}}(\boldsymbol{\epsilon_{2/1}}\cdot\boldsymbol{\epsilon_{4/3}^\dagger})Y(\Lambda,m_v,r), \tag{48}$$

$$V_p^{S\bar{A}\to A\bar{S}/A\bar{S}\to S\bar{A}} = C_p\left(\frac{h_3}{4F_\pi}\right)^2[(\boldsymbol{\epsilon_{1/2}}\cdot\boldsymbol{\epsilon_{4/3}^\dagger})Z(\Lambda,\tilde{m}_p,r) + S(\hat{\boldsymbol{r}},\boldsymbol{\epsilon_{1/2}},\boldsymbol{\epsilon_{4/3}^\dagger})T(\Lambda,\tilde{m}_p,r)], \tag{49}$$

$$V_p^{S\bar{A}/A\bar{S}\to A\bar{A}} = \delta_{S\bar{A}/A\bar{S}}C_p\frac{h_3h_5}{4F_\pi^2}m_{A_{bq}}[\boldsymbol{\epsilon_{3/4}^\dagger}\cdot(\boldsymbol{\epsilon_{2/1}}\times\boldsymbol{\epsilon_{4/3}^\dagger})Z(\Lambda,\tilde{m}_p,r) + S(\hat{\boldsymbol{r}},\boldsymbol{\epsilon_{3/4}^\dagger},\boldsymbol{\epsilon_{2/1}}\times\boldsymbol{\epsilon_{4/3}^\dagger})T(\Lambda,\tilde{m}_p,r)], \tag{50}$$

$$V_v^{A\bar{A}\to A\bar{A}} = C_v\frac{g_V^2h_4^2m_{A_{bq}}^2}{4}(\boldsymbol{\epsilon_1}\cdot\boldsymbol{\epsilon_3^\dagger})(\boldsymbol{\epsilon_2}\cdot\boldsymbol{\epsilon_4^\dagger})Y(\Lambda,m_v,r),$$
$$V_p^{A\bar{A}\to A\bar{A}} = C_p\left(\frac{h_5m_{A_{bq}}}{F_\pi}\right)^2[(\boldsymbol{\epsilon_1}\times\boldsymbol{\epsilon_3^\dagger})\cdot(\boldsymbol{\epsilon_2}\times\boldsymbol{\epsilon_4^\dagger})Z(\Lambda,m_p,r) + S(\boldsymbol{r},\boldsymbol{\epsilon_1}\times\boldsymbol{\epsilon_3^\dagger},\boldsymbol{\epsilon_2}\times\boldsymbol{\epsilon_4^\dagger})T(\Lambda,m_p,r)]. \tag{51}$$

In the above equations, the subscript p denotes the light pseudoscalar mesons π, η_8,and η_0, and v denotes the light vectors ρ and ω. The coefficients $C_{\pi^0}=\frac{1}{3}, C_{\eta_8}=-\frac{1}{9}$, $C_{\eta_0}=-\frac{2}{9}$, $C_\rho=1$, $C_\omega=-1$, $\delta_{B\bar{B}^*}=1$, $\delta_{B^*\bar{B}}=-1$, $\delta_{S\bar{A}}=1$, and $\delta_{A\bar{S}}=-1$. And, $S(\hat{\boldsymbol{r}},\boldsymbol{a},\boldsymbol{b})=3(\hat{\boldsymbol{r}}\cdot\boldsymbol{a})(\hat{\boldsymbol{r}}\cdot\boldsymbol{b})-\boldsymbol{a}\cdot\boldsymbol{b}$, $q_0=\frac{m_2^2-m_1^2+m_3^2-m_4^2}{2(m_3+m_4)}$, $\tilde{m}_E^2=m_E^2-q_0^2$ with E the exchanged particle. The functions $Y(\Lambda,m,r)$, $Z(\Lambda,m,r)$, and $T(\Lambda,m,r)$ are defined as

$$Y(\Lambda,m,r) = \int\frac{d^3q}{(2\pi)^3}e^{i\vec{q}\cdot\vec{r}}\frac{1}{\vec{q}^2+m^2-i\epsilon}e^{2(q_0^2-\vec{q}^2)/\Lambda^2}, \tag{52}$$

$$Z(\Lambda,m,r) = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}Y(\Lambda,m,r), \tag{53}$$

$$T(\Lambda,m,r) = r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}Y(\Lambda,m,r). \tag{54}$$

In the Appendix, we will show the calculation method of the integral $Y(\Lambda,m,r)$.

Taking into account S- and D-wave functions, the products of the polarization vectors in the subpotentials are presented below:

$$\left.\begin{array}{c}\boldsymbol{\epsilon_1}\cdot\boldsymbol{\epsilon}^\dagger_{3/4}\\ \boldsymbol{\epsilon_2}\cdot\boldsymbol{\epsilon}^\dagger_{3/4}\\ (\boldsymbol{\epsilon_1}\cdot\boldsymbol{\epsilon}^\dagger_3)(\boldsymbol{\epsilon_2}\cdot\boldsymbol{\epsilon}^\dagger_4)\\ -(\boldsymbol{\epsilon_1}\cdot\boldsymbol{\epsilon}^\dagger_4)(\boldsymbol{\epsilon_2}\cdot\boldsymbol{\epsilon}^\dagger_3)\\ (\boldsymbol{\epsilon_1}\times\boldsymbol{\epsilon}^\dagger_3)\cdot(\boldsymbol{\epsilon_2}\times\boldsymbol{\epsilon}^\dagger_4)\end{array}\right\}\longrightarrow\begin{pmatrix}1&0\\0&1\end{pmatrix},$$

$$\left.\begin{array}{c}S(\hat{\boldsymbol{r}},\boldsymbol{\epsilon_1},\boldsymbol{\epsilon_4}^\dagger)\\ S(\hat{\boldsymbol{r}},\boldsymbol{\epsilon_2},\boldsymbol{\epsilon_3}^\dagger)\\ S(\hat{\boldsymbol{r}},\boldsymbol{\epsilon_1}\times\boldsymbol{\epsilon}^\dagger_3,\boldsymbol{\epsilon_2}\times\boldsymbol{\epsilon}^\dagger_4)\\ 2(\boldsymbol{\epsilon_2}\cdot\boldsymbol{\epsilon}^\dagger_3)S(\hat{\boldsymbol{r}},\boldsymbol{\epsilon_1},\boldsymbol{\epsilon}^\dagger_4)\\ 2(\boldsymbol{\epsilon_1}\cdot\boldsymbol{\epsilon}^\dagger_4)S(\hat{\boldsymbol{r}},\boldsymbol{\epsilon_2},\boldsymbol{\epsilon}^\dagger_3)\end{array}\right\}\longrightarrow\begin{pmatrix}0&-\sqrt{2}\\-\sqrt{2}&1\end{pmatrix},$$

$$\left.\begin{array}{c}\boldsymbol{\epsilon_{1/2}}\cdot(\boldsymbol{\epsilon}^\dagger_4\times\boldsymbol{\epsilon}^\dagger_3)\\ \boldsymbol{\epsilon_1}\cdot(\boldsymbol{\epsilon_2}\times\boldsymbol{\epsilon}^\dagger_{3/4})\end{array}\right\}\longrightarrow\begin{pmatrix}i\sqrt{2}&0\\0&i\sqrt{2}\end{pmatrix},$$

$$\left.\begin{array}{c}2S(\hat{\boldsymbol{r}},\boldsymbol{\epsilon}^\dagger_4,\boldsymbol{\epsilon}^\dagger_3\times\boldsymbol{\epsilon_{1/2}})\\ 2S(\hat{\boldsymbol{r}},\boldsymbol{\epsilon}^\dagger_3,\boldsymbol{\epsilon_{1/2}}\times\boldsymbol{\epsilon}^\dagger_4)\\ 2S(\hat{\boldsymbol{r}},\boldsymbol{\epsilon_1},\boldsymbol{\epsilon_2}\times\boldsymbol{\epsilon}^\dagger_{3/4})\\ 2S(\hat{\boldsymbol{r}},\boldsymbol{\epsilon_2},\boldsymbol{\epsilon}^\dagger_{3/4}\times\boldsymbol{\epsilon_1})\\ S(\hat{\boldsymbol{r}},\boldsymbol{\epsilon_{1/2}},\boldsymbol{\epsilon}^\dagger_3\times\boldsymbol{\epsilon}^\dagger_4)\\ S(\hat{\boldsymbol{r}},\boldsymbol{\epsilon}^\dagger_{3/4},\boldsymbol{\epsilon_2}\times\boldsymbol{\epsilon_1})\end{array}\right\}\longrightarrow\begin{pmatrix}0&2i\\2i&-i\sqrt{2}\end{pmatrix},$$

$$\left.\begin{array}{c}\boldsymbol{\epsilon_1}\cdot\boldsymbol{\epsilon_2},\boldsymbol{\epsilon}^\dagger_3\cdot\boldsymbol{\epsilon}^\dagger_4\\ (\boldsymbol{\epsilon}^\dagger_3\cdot\boldsymbol{\epsilon}^\dagger_4)S(\hat{\boldsymbol{r}},\boldsymbol{\epsilon_1},\boldsymbol{\epsilon_2})\\ (\boldsymbol{\epsilon_1}\cdot\boldsymbol{\epsilon_2})S(\hat{\boldsymbol{r}},\boldsymbol{\epsilon}^\dagger_3,\boldsymbol{\epsilon}^\dagger_4)\end{array}\right\}\longrightarrow\begin{pmatrix}0&0\\0&0\end{pmatrix}.$$

Hereafter, we label the channels $B\bar{B}^*/B^*\bar{B}$, $B^*\bar{B}^*$, $S\bar{A}/A\overline{S}$, and $A\bar{A}$ by CH_1, CH_2, CH_3, and CH_4, respectively. The elements of the total potential matrix $\hat{V}$ is

$$V^{CH_1\to CH_1}=\frac{1}{2}\left(V^{\bar{B}B^*\to\bar{B}B^*}_{\rho,\omega}+2V^{\bar{B}B^*\to\bar{B}^*B}_{\pi,\eta_8,\eta_0}+V^{\bar{B}^*B\to\bar{B}^*B}_{\rho,\omega}\right),\tag{55}$$

$$V^{CH_1\to CH_2}=\frac{1}{\sqrt{2}}\left(V^{\bar{B}^*B\to\bar{B}^*B^*}_{\pi,\eta_8,\eta_0}+V^{\bar{B}B^*\to\bar{B}^*B^*}_{\pi,\eta_8,\eta_0}\right),\tag{56}$$

$$V^{CH_1\to CH_3}=\frac{1}{2}\left(V^{\bar{B}^*B\to A\bar{S}}_{A_{ud}}+V^{\bar{B}B^*\to A\bar{S}}_{S_{ud},A_{ud}}+V^{\bar{B}^*B\to S\bar{A}}_{S_{ud},A_{ud}}+V^{\bar{B}^*B\to A\bar{S}}_{A_{ud}}\right),\tag{57}$$

$$V^{CH_1\to CH_4}=\frac{1}{\sqrt{2}}\left(V^{\bar{B}^*B\to A\bar{A}}_{S_{ud},A_{ud}}+V^{\bar{B}B^*\to A\bar{A}}_{S_{ud},A_{ud}}\right),\tag{58}$$

$$V^{CH_2\to CH_2}=V^{\bar{B}^*B^*\to\bar{B}^*B^*}_{\rho,\omega}+V^{\bar{B}^*B^*\to\bar{B}^*B^*}_{\pi,\eta_8,\eta_0},\tag{59}$$

$$V^{CH_2\to CH_3}=\frac{1}{\sqrt{2}}\left(V^{\bar{B}^*B^*\to A\bar{S}}_{S_{ud},A_{ud}}+V^{\bar{B}^*B^*\to S\bar{A}}_{S_{ud},A_{ud}}\right),\tag{60}$$

$$V^{CH_2\to CH_4}=V^{\bar{B}^*B^*\to A\bar{A}}_{S_{ud},A_{ud}},\tag{61}$$

$$V^{CH_3\to CH_3}=\frac{1}{2}\left(V^{S\bar{A}\to S\bar{A}}_{\rho,\omega}+2V^{S\bar{A}\to A\bar{S}}_{\pi,\eta_8,\eta_0}+V^{A\bar{S}\to A\bar{S}}_{\rho,\omega}\right),\tag{62}$$

$$V^{CH_3\to CH_4}=\frac{1}{\sqrt{2}}\left(V^{S\bar{A}\to A\bar{A}}_{\pi,\eta_8,\eta_0}+V^{A\bar{S}\to A\bar{A}}_{\pi,\eta_8,\eta_0}\right),\tag{63}$$

$$V^{CH_4\to CH_4}=V^{A\bar{A}\to A\bar{A}}_{\rho,\omega}+V^{A\bar{A}\to A\bar{A}}_{\pi,\eta_8,\eta_0}.\tag{64}$$

The subscripts mean the following summation:

$$V^{CH_i\to CH_j}_{E_1,E_2,\cdots}=\sum_{a=E_1,E_2,\cdots}V^{CH_i\to CH_j}_{a}.\tag{65}$$

III. RESULTS AND DISCUSSION

With the preparation above, adopting the Gaussian expansion method (GEM) [62], we solve the coupled-channel Schrödinger equation to find the boundstate solutions,

$$(\hat{K}+\hat{M}+\hat{V})\Psi=E\Psi.\tag{66}$$

Here, $\hat{K}=\mathrm{diag}(-\frac{\tilde{\Delta}}{2\mu_1},-\frac{\tilde{\Delta}}{2\mu_2},\cdots)$, $\hat{M}=\mathrm{diag}(0,M_2-M_1,M_3-M_1,\cdots)$, and $\tilde{\Delta}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r})-\frac{l(l+1)}{r^2}$. The coupled-channel Schrödinger equation (66) is symmetric under the following transformation:

$$U(\hat{K}+\hat{M}+\hat{V})\Psi=UE\Psi,\tag{67}$$

$$\Rightarrow U(\hat{K}+\hat{M}+\hat{V})U^{-1}U\Psi=EU\Psi,\tag{68}$$

$$\Rightarrow(\hat{K}+\hat{M}+U\hat{V}U^{-1})\tilde{\Psi}=E\tilde{\Psi},\tag{69}$$

where $U=\mathrm{diag}(e^{i\theta_1},e^{i\theta_2},\cdots)$, $\tilde{\Psi}=U\Psi$. Here, the word "symmetric" means the energy E does not change under the transformation of the Schrödinger equation. All the U form a reducible Lie group $U(1)\otimes U(1)\otimes\cdots\otimes U(1)$.

In this work, some of the off-diagonal elements of the potential matrix are imaginary. So, we cannot numerically solve the Schrödinger equation by our program, which needs a real potential matrix. If we perform the

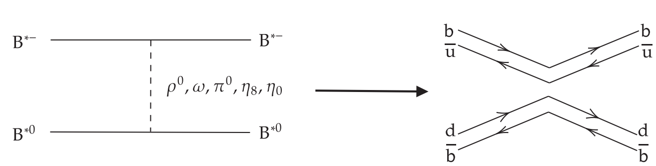

FIG. 1. The diagram of $B^{*-}B^{*0}\to B^{*-}B^{*0}$ on quark and hadron levels.

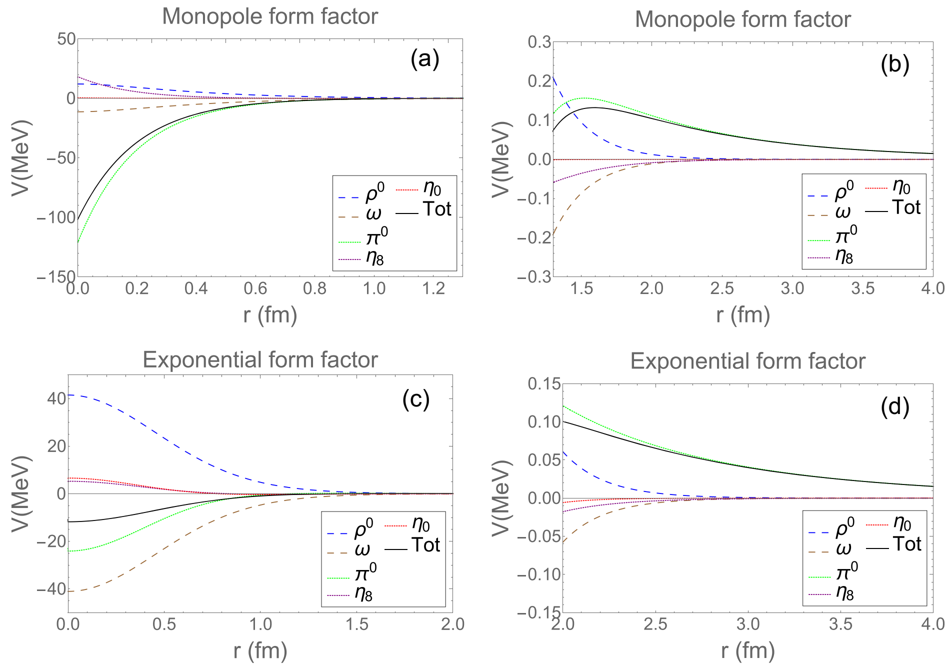

FIG. 2. The effective potential for S-wave $B^*\bar{B}^*$ to S-wave $B^*\bar{B}^*$ with different form factors.

transformation mentioned above, the problem is solved. which means that the potential matrix becomes real. The corresponding parameters are chosen as $\theta_1 = \theta_2 = \frac{\pi}{2}$, $\theta_3 = \theta_4 = 0$, $\theta_5 = \theta_6 = \frac{\pi}{2}$, $\cdots$.

It is also interesting to discuss the OZI suppressed processes on both the quark and hadron levels, which was already pointed out in Ref. [27]. We take the $B^{*-}\bar{B}^{*0} \to B^{*-}\bar{B}^{*0}$ process as an example. On the quark level, we can see from Fig. 1 that the diagram is nonconnected, which means it is OZI suppressed. On the hadron level, the exchanged particle could be the pseudoscalar mesons π, η_8, η_0 and the vector mesons ρ, ω. Since the masses of ρ and ω are almost close to each other, their contributions to the total potential cancel out, which can be seen from Eqs. (42) and (59). Under the $[U(3)_L \otimes U(3)_R]_{\text{global}} \otimes [U(3)]_{\text{local}}$ symmetry, the pseudoscalar mesons appearing in Φ [see Eq. (8)] have the same mass. In this case, the contributions of π, η_8, and η_0 to the total potential also cancel out. This conclusion coincides with the OZI rule. However, the real masses of π, η_8, and η_0 are different, so the total potential is nonzero, which depicts the phenomenon of the violation of the OZI rule.

As we mentioned above, in this work we use the exponentially parametrized form factor in our calculation. For comparison, we take the $B^*\bar{B}^* \to B^*\bar{B}^*$ process as an example, and plot in Fig. 2 the potentials with both the exponential form factor and the monopole one. We notice that

(i) item 1 $V_{\rho^0}^{B^*\bar{B}^*\to B^*\bar{B}^*}$ and $V_{\omega}^{B^*\bar{B}^*\to B^*\bar{B}^*}$ have almost the same absolute value, but different signs, i.e., the contribution of $\rho-$and $\omega-$exchange is approximately zero. So, $V^{B^*\bar{B}^*\to B^*\bar{B}^*}$ is mainly contributed by $\pi-$, η_8-, and η_0-exchange;

(ii) the monopole form factor suppresses the contribution from the exchanged particle with heavier mass, which can be clearly seen from the situation of η_0-exchange, since $m_{\eta_0} \simeq 0.96$ GeV;

(iii) in the long interaction range, the pion-exchange subpotential is dominant with both monopole and exponential form factors;

(iv) in the short and medium range, the vector-exchange subpotentials are compatibly larger than that of the pion exchange with exponential form factor. However, for the monopole form factor, the pion-exchange contribution is still dominant.

The values of the masses of diquarks taken from Ref. [63] are listed in Table II, and the meson masses are taken from PDG. Then, we solve the Schrödinger equation, and the numerical results are presented in Table III. For $B\bar{B}^*/B^*\bar{B}^*/S\bar{A}/A\bar{A}$ and $B^*\bar{B}^*/S\bar{A}/A\bar{A}$ systems, loosely bound states exist when the cutoff is reasonably chosen as $\Lambda \sim 1$ GeV. If the value of Λ increases, the binding energies increase as well, while the root-mean-square radii decrease. For both of these two systems, the D-wave contribution is much smaller than that of the S wave.

TABLE II. The diquark masses we use in effective potentials from Ref. [63]. q indicates u or d quark.

$m_{S_{bq}}$ (GeV)	$m_{A_{bq}}$ (GeV)	$m_{S_{qq}}$ (GeV)	$m_{A_{qq}}$ (GeV)
5.451	5.465	0.691	0.840

TABLE III. The obtained bound-state solutions (binding energy E and root-mean-square radius $r_{\rm RMS}$) for the charged Z_b system. P_M is the probability of the channels $B\bar{B}^*/B^*\bar{B}(^3S_1)$, $B\bar{B}^*/B^*\bar{B}(^3D_1)$, $B^*\bar{B}^*(^3S_1)$, $B^*\bar{B}^*(^3D_1)$ for $Z_b(10610)$, and $B^*\bar{B}^*(^3S_1)$, $B^*\bar{B}^*(^3D_1)$ for $Z_b(10650)$, respectively. P_D is the probability of the channels $S\bar{A}/A\bar{S}(^3S_1)$, $S\bar{A}/A\bar{S}(^3D_1)$, $A\bar{A}(^3S_1)$, $A\bar{A}(^3D_1)$.

State	Λ (GeV)	E (MeV)	$r_{\rm RMS}$ (fm)	P_M (%)	P_D (%)
$Z_b(10610)$	1.10	−1.10	2.02	90.32/0.30/6.94/0.03	1.27/0.06/1.07/0.01
	1.15	−3.78	1.15	78.40/0.22/16.24/0.02	2.24/0.09/2.78/0.01
	1.20	−8.66	0.80	64.11/0.12/27.39/0.01	3.03/0.10/5.23/0.01
$Z_b(10650)$	1.00	−2.76	1.41	95.91/0.29	0.10/0.01/3.67/0.02
	1.05	−5.60	1.06	94.04/0.27	0.18/0.01/5.47/0.03
	1.10	−9.58	0.86	92.02/0.26	0.29/0.01/7.39/0.04

Besides, the meson-meson component is much larger than the diquark-anti-diquark component. That is to say, both $Z_b(10610)$ and $Z_b(10650)$ can be explained as hadronic molecules mixing with a few diquark-anti-diquark components.

It is worth mentioning that the masses of the Z_b and Z_b' states extracted by Belle Collaboration are slightly above the $B\bar{B}^*$ and $B^*\bar{B}^*$ thresholds. However, in Ref. [9], the authors demonstrated that if the Z_b states are indeed generated from nonperturbative $B\bar{B}^* +$ c.c. and $B^*\bar{B}^*$ dynamics, the data should not be analyzed using a Breit-Wigner parametrization. They also showed that the data of Belle experiment are consistent with the assumption that the main components of the lower- and higher Z_b states are S-wave $B\bar{B}^* +$ c.c. and $B^*\bar{B}^*$ bound states, whose masses are located below $B\bar{B}^*$ and $B^*\bar{B}^*$ thresholds. This discussion is also applicable to our case, which means that our study in the picture of bound states does not contradict the experimental data of Belle Collaboration.

IV. SUMMARY

In this work, we extend the $[U(3)_L \otimes U(3)_R]_{\rm global} \otimes [U(3)]_{\rm local}$ symmetry to the diquark sector and construct the Lagrangians containing diquarks. In this way, we can introduce the diquark-exchange interactions and study the nature of the Z_b states as the mixture of hadronic molecule and diquark-anti-diquark state. In order to make the small distance interaction suppressed, we introduce the exponential form factor instead of the monopole form factor for each vertex, and calculate the effective potentials in the coordinate space analytically. We see that in the short- and medium ranges, $\rho-$or $\omega-$exchange contribution is larger than that of pion exchange, while the pion exchange is dominant in the long range. Besides, $B^{(*)-}\bar{B}^{(*)0} \to B^{(*)-}\bar{B}^{(*)0}$ processes are OZI suppressed, which can be easily seen on the quark level. This is well depicted under the $[U(3)_L \otimes U(3)_R]_{\rm global} \otimes [U(3)]_{\rm local}$ symmetry, since the $\pi-$, η_8-, η_0-, $\rho-$, and $\omega-$exchange contributions cancel out. If taking the real masses of the exchanged particles which are different, the potential is nonzero corresponding to OZI violation.

Taking into account S- and D-wave contributions, we solve the coupled-channel Schrödinger equation, which obeys the $U(1) \otimes U(1) \otimes \ldots \otimes U(1)$ symmetry. The numerical results show that both $Z_b(10610)$ and $Z_b(10650)$ can be explained as hadronic molecules slightly mixing with diquark-anti-diquark states. If the cutoff is chosen around $\Lambda = 1$ GeV for both $Z_b(10610)$ and $Z_b(10650)$, the probabilities of molecular components are about 90%, while those of diquark-anti-diquark components are less than 10%. Besides, the S-wave contribution is much larger than that of the D wave. Our work can help to understand the nature of the Z_b states.

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APPENDIX: CALCULATION OF THE FUNCTIONS $Y(\Lambda, m, r)$ AND $U(\Lambda, m, r)$

The definition of $Y(\Lambda, m, r)$ is shown in Eq. (52), i.e.,

$$Y(\Lambda, m, r) = \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{1}{\vec{q}^2 + m^2 - i\epsilon} e^{2(q_0^2 - \vec{q}^2)/\Lambda^2}. \quad \text{(A1)}$$

After integrating out the azimuth and polar angle, we have

$$Y(\Lambda, m, r) = -\frac{e^{2q_0^2/\Lambda^2}}{(2\pi^2 r)} \frac{\partial}{\partial r} F(r, 2/\Lambda^2), \quad \text{(A2)}$$

where

$$F(r, 2/\Lambda^2) = \int_{-\infty}^{\infty} dq \frac{1}{q^2 + m^2 - i\epsilon} e^{-2q^2/\Lambda^2} e^{iqr}. \quad \text{(A3)}$$

One can prove that the function $F(r, 2/\Lambda^2)$ satisfies the following first-order differential equation:

$$\frac{\partial}{\partial\alpha} F(r,\alpha) - m^2 F(r,\alpha) = -\int_{-\infty}^{\infty} dq e^{-q^2\alpha} e^{iqr} = -\sqrt{\frac{\pi}{\alpha}} e^{-\frac{r^2}{4\alpha}}, \quad \text{(A4)}$$

with $\alpha = 2/\Lambda^2$. Besides, we calculate $F(r, 0)$ by using the residue theorem, i.e.,

$$F(r, 0) = \frac{\pi}{m} e^{-mr}, \quad \text{(A5)}$$

which can be treated as the initial condition of the differential equation (A4). Through solving this initial value problem, we have

$$Y(\Lambda, m, r) = -\frac{e^{2q_0^2/\Lambda^2}}{(2\pi)^2 r} \frac{\partial}{\partial r} \left\{ e^{2m^2/\Lambda^2} \frac{\pi}{2m} \left[e^{mr} + e^{-mr} - e^{mr} \mathrm{erf}\left(\frac{\Lambda r}{2\sqrt{2}} + \frac{\sqrt{2}m}{\Lambda} \right) + e^{-mr} \mathrm{erf}\left(\frac{\Lambda r}{2\sqrt{2}} - \frac{\sqrt{2}m}{\Lambda} \right) \right] \right\}. \quad \text{(A6)}$$

Using the same method, as a product, we also calculate the following function:

$$U(\Lambda, m, r) = \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{1}{\vec{q}^2 - m^2 - i\epsilon} e^{2(q_0^2 - \vec{q}^2)/\Lambda^2} = \frac{e^{2q_0^2/\Lambda^2}}{(2\pi)^2 r} \frac{\partial}{\partial r} \left\{ \pi \left[-\frac{1}{2im} \left(e^{-imr} \mathrm{erf}\left(\frac{r\Lambda}{2\sqrt{2}} - \frac{\sqrt{2}im}{\Lambda} \right) - e^{imr} \mathrm{erf}\left(\frac{r\Lambda}{2\sqrt{2}} + \frac{\sqrt{2}im}{\Lambda} \right) \right) - \frac{i}{m} \cos(mr) \right] e^{-2m^2/\Lambda^2} \right\}. \quad \text{(A7)}$$

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