$B \to X_{c(u)}\ell\bar{\nu}_\ell\gamma$ and determination of nonperturbative parameters

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This paper explores the calculation of nonperturbative parameters present in the matrix element of semileptonic inclusive B decays. These are important for the inclusive determination of the Cabbibo-Kobayashi-Maskawa parameters in the Standard Model. We focus on calculating the rate for radiative inclusive decay, $B \to X_c\ell\bar{\nu}_\ell\gamma$, where γ is hard. Both the radiative and nonradiative ($B \to X_c\ell\bar{\nu}_\ell$) modes are found to require the same nonperturbative parameters. We propose forming ratios from $B \to X_c\ell\bar{\nu}_\ell\gamma$ and $B \to X_c\ell\bar{\nu}_\ell$ widths in different lepton energy ranges. Our results provide an efficient way to unambiguously calculate the nonperturbative parameters, especially when combined with total width calculations of $B \to X_c\ell\bar{\nu}_\ell$. This is shown explicitly by working to $\mathcal{O}(1/m_b)$ and determining λ_1 and λ_2.

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I. INTRODUCTION

B meson decays are complex processes that provide an ideal platform for precision studies of the Standard Model (SM) as they involve various physical scales. Inclusive decay modes (where the final state mesons are summed over) are considered to be theoretically cleaner than exclusive ones (where the final state particles are explicitly identified) as the latter involve transition form factors, which are rather difficult to evaluate. The inclusive B meson decays are theoretically cleaner as they take advantage of hard scale, m_b, which is much larger than $\Lambda_{\rm QCD}$. The presence of the hard scale enables the use of the heavy quark expansion (HQE) [1,2] and local operator product expansion (OPE) [3,4], and hence a systematic theoretical treatment of the decay rates.

HQE is a fundamental tool in the study of heavy quark physics. It provides a series expansion in $\frac{1}{m_Q}$, where Q is the heavy quark [5]. Though the mass of b quark is heavy enough to use the expansion in $\frac{1}{m_b}$, corrections to the heavy quark limit are significant for achieving high precision. OPE is a powerful tool that enables a systematic treatment of nonperturbative quantum chromodynamics (QCD) effects by separating the effects originating at large and small distances [6,7]. For a heavy quark system, it turns out to be a series expansion in powers of $\frac{\Lambda_{\rm QCD}}{m_b}$ [8,9] when combined with the theory of heavy quark expansion.

The corrections to the leading term ($m_b \to \infty$) in the heavy quark expansion are expected to be small in the end region of the phase space. This part of the phase space allows the contribution of several hadronic states which satisfy the condition: $m_X^2 \to m_q^2 + \#\Lambda_{\rm QCD} m_b$. An observable, like the decay rate, which averages over these hadronic states, can then be predicted reliably. To $\mathcal{O}(\frac{1}{m_b})$ in HQE, the major uncertainty in these predictions arises from the values of nonperturbative parameters, λ_1 and λ_2. Throughout the paper, we consistently work to $\mathcal{O}(\frac{1}{m_b})$ in HQE. These parameters are defined as [10,11]

$$\lambda_1 = \frac{1}{2m_{H_Q}}\langle H_Q|\bar{Q}(i\mathbf{D})^2 Q|H_Q\rangle, \quad \text{and}$$
$$3\lambda_2 = \frac{1}{2m_{H_Q}}\langle H_Q|\bar{Q}\frac{i}{2}\sigma_{\mu\nu}G^{\mu\nu}Q|H_Q\rangle, \tag{1}$$

where D is the covariant derivative and $|H_Q\rangle$ represents the states of hadrons containing one heavy quark and light cloud (for b hadrons $H_Q = B, \Lambda_b$, etc.). Physically, λ_1 provides information about the average spatial momentum squared of the heavy quark, while λ_2 represents the amount of the color magnetic field produced by the light cloud at the position of the heavy quark [12,13].

In the past, the parameters λ_1 and λ_2 have been determined using QCD models [14–17], and also have been extracted by fitting to the experimental data [18–22].

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Reference [23] determines μ_π^2 and μ_G^2 [1] in quenched lattice QCD using the non-relativistic quantum chromodynamics action including $\mathcal{O}(\frac{1}{m_Q})$ terms for the heavy quark. They compute λ_2 explicitly but not λ_1, and rather calculate the difference of matrix elements using two different methods. Also, these two different methods of computation have followed different central values. Consequently, having an unambiguous prediction for these parameters becomes important.

It should be noted that both $B \to X_c \ell \bar{\nu}_\ell$ and $B \to X_u \ell \bar{\nu}_\ell$ exhibit similar decay signatures characterized by a high-momentum lepton, a hadronic system, and undetected neutrino energy. Thus, distinguishing between these two processes is challenging. Further, close to the lepton energy end point regions, nonperturbative shape functions enter the description of the decay kinematics, making predictions for the decay rates dependent on the precise modeling of these shape functions.[2] Instead, we are exploring the possibility of determining the nonperturbative parameters in an efficient manner. Considering modes with similar cuts such as $B \to X_{c(u)} \ell \bar{\nu}_\ell$ together with $B \to X_{c(u)} \ell \bar{\nu}_\ell \gamma$ may provide complementary information allowing one to extract the nonperturbative parameters. Further, the inclusion of a hard photon in the decay process introduces additional degrees of freedom, such as the angle between the lepton and the photon. As an example, the forward-backward symmetry has been calculated (for details, see Sec. V). The complete angular analysis is left for future work.

In this paper, we explore the experimental determination of the decay rate for the $B \to X_{c(u)} \ell \bar{\nu}_\ell \gamma$ mode, in conjunction with the $B \to X_{c(u)} \ell \bar{\nu}_\ell$ mode. It is worth reiterating that the emitted photon is a hard photon, and the process should not be thought of as soft photon correction to $B \to X_{c(u)} \ell \bar{\nu}_\ell$. To this order in $\frac{1}{m_b}$, the decay widths of both these modes, i.e., $B \to X_{c(u)} \ell \bar{\nu}_\ell$ and $B \to X_{c(u)} \ell \bar{\nu}_\ell \gamma$, have linear dependence on λ_1 and λ_2:

$$\text{decay width} \sim A + B\lambda_1 + C\lambda_2, \qquad (2)$$

for different values of A, B, C for the two modes. Therefore, knowing (or experimentally measuring) one of the sides of these equations will provide a simultaneous set of linear equations, which can then be solved to get unambiguous determinations of λ_1 and λ_2. While we are including terms $\mathcal{O}(\frac{1}{m_b})$ in HQE, it is straightforward to include higher order terms. To avoid the uncertainties which may arise due to the presence of the Cabibbo-Kobayashi-Maskawa (CKM) element (V_{ub} or V_{cb}), we propose to consider the ratio of the decay widths in different ranges of the leptonic energy instead of directly working with the decay width (see Sec. V for details). Such ratios are defined as (for radiative width, Γ_γ, and nonradiative, Γ)

$$R_1 = \frac{\int_0^{0.2} dy \frac{d\Gamma_\gamma}{dy}}{\int_0^{0.5} dy \frac{d\Gamma_\gamma}{dy}} \quad \text{and} \quad R_2 = \frac{\int_0^{0.5} dy \frac{d\Gamma}{dy}}{\int_0^{1} dy \frac{d\Gamma}{dy}}, \qquad (3)$$

where y is the lepton energy expressed in dimensionless units. Further, the considered cuts are representative cuts and can be chosen in accordance with the experimentally suitable ones. Here, we are exploring an avenue which could provide complementary information in a simple way.

The rest of the paper is organized as follows: in Sec. II, as a warm-up, we first calculate the decay width for the nonradiative process $B \to X_{c(u)} \ell \bar{\nu}_\ell$ mode. This is achieved by directly using the Cutkosky method applied to the amplitude in contrast with the usual approach of writing down the hadronic tensor in terms of invariant quantities and employing analytical properties. Both approaches are actually equivalent, and we explicitly verify that the result matches with [4,11] as it should. In Sec. III, we provide the details of the calculation of the decay width for the $B \to X_{c(u)} \ell \bar{\nu}_\ell \gamma$ mode. In Sec. V, we discuss our results for the differential decay rate for $B \to X_{c(u)} \ell \bar{\nu}_\ell \gamma$ and its sensitivity to the energy of the hard photon. Also, we provide a complementary method to extract nonperturbative parameters λ_1 and λ_2. Finally, we conclude in Sec. VI and discuss the implications of the decay rate of $B \to X_c \ell \bar{\nu}_\ell \gamma$ for the calculation of λ_1 and λ_2.

II. DECAY RATE OF $B \to X_q \ell \bar{\nu}_\ell$ ($q = c/u$)

The weak Hamiltonian density for the inclusive semi-leptonic B meson decays to final state containing a c or u quark ($B \to X_q \ell \bar{\nu}_\ell$) is given by

$$\mathcal{H}_{\text{weak}} = \frac{4G_F}{\sqrt{2}} V_{qb} (\bar{q}\gamma_\mu P_L b)(\bar{\ell}\gamma^\mu P_L \nu_\ell), \qquad (4)$$

where G_F and V_{qb} are the Fermi constant and CKM element, respectively. The process of $B \to X_q \ell \bar{\nu}_\ell$ involves the $b \to q$ transition and sum over the final state mesons containing the q quark. To calculate the decay width for the inclusive process, the forward scattering matrix element is a useful quantity. Figure 1 shows the Feynman diagram for the forward scattering matrix element of the inclusive semi-leptonic $B \to X_c \ell \bar{\nu}_\ell$ decay. The imaginary part of the forward scattering matrix element called the transition matrix element, $\langle B|\hat{T}|B\rangle$, is related to the decay to an inclusive final state through the optical theorem. It is given by

$$\Gamma \propto \frac{1}{2m_B} \text{Im}\{\langle B|\hat{T}(b \to q \to b)|B\rangle\}, \qquad (5)$$

[1] In this study, we are specifically considering the $\lambda_{1(2)}$ parameters rather than $\mu^2_{\pi(G)}$.

[2] We briefly touch upon the shape functions in Sec. VI.

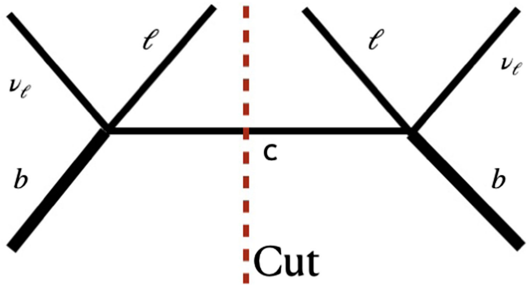


FIG. 1. Representative diagram of forward scattering for $b \to c\ell\nu_\ell$.

where the operator sandwiched between the hadronic states is called the forward scattering operator (or transition operator). The explicit form of the transition operator reads as

$$\hat{T}^{\mu\nu}(b \to q \to b) = i \int d^4x e^{-iq.x} \frac{1}{2m_B} T\{J_w^{\mu\dagger}(x) J_w^\nu(0)\}, \tag{6}$$

where T denotes the time ordered product and $J_w^\mu (= \bar{q}\gamma^\mu P_L b)$ is the weak current. Further, the differential decay rate for $B \to X_q \ell \bar{\nu}_\ell$ process is given by

$$\begin{aligned} \frac{d^3\Gamma}{dq^2 dE_\ell dE_\nu} &= \frac{1}{4} \sum_{X_q} \sum_{\text{spins}} \frac{1}{2m_B} |\langle X_q \ell \bar{\nu}_\ell | \mathcal{H}_{\text{weak}} | B \rangle|^2 \\ &\quad \times \delta^4(p_B - p_X - p_\ell - p_\nu), \\ &= 2G_F^2 |V_{ub}|^2 \mathcal{M}_{\mu\nu} \mathcal{L}^{\mu\nu}, \end{aligned} \tag{7}$$

where $q^2 = (p_\ell + p_\nu)^2$, E_ℓ is the lepton energy, and E_ν is the neutrino energy. Sum over spins indicates sum over lepton spins. Furthermore, the leptonic tensor ($\mathcal{L}_{\mu\nu}$) is defined directly from the electroweak Lagrangian,

$$\mathcal{L}_{\mu\nu} = \sum_{\text{spins}} (\bar{\nu}_\ell \gamma_\mu (1-\gamma_5)\ell)(\bar{\ell}\gamma_\nu(1-\gamma_5)\nu), \tag{8}$$

whereas the hadronic tensor, $\mathcal{M}_{\mu\nu}$, is expressed in terms of matrix elements of electroweak currents and is given by

$$\mathcal{M}_{\mu\nu} = \frac{1}{2m_B} \sum_{X_q} \langle B | J_\mu | X_q \rangle \langle | X_q | J_\nu | B \rangle \delta^4(p_B - p_X - p_\ell - p_\nu). \tag{9}$$

The hadronic tensor ($\mathcal{M}_{\mu\nu}$) is defined as the absorptive part of the matrix element of the transition operator [i.e., Eq. (6)]. Explicitly,

$$\mathcal{M}_{\mu\nu} = -i\,\text{Disc}(\langle B | \hat{T}_{\mu\nu} | B \rangle). \tag{10}$$

Instead of using the general parametrization for the hadronic tensor and relying on the analytical properties, we directly evaluate the transition operator ($\hat{T}_{\mu\nu}$) along with the leptonic tensor ($\mathcal{L}_{\mu\nu}$) to obtain the matrix element involved in the inclusive semileptonic $B \to X_q \ell \bar{\nu}_\ell$ decay. At the lowest order in perturbation theory (i.e., in α_s expansion), the quark level amplitude ($\mathcal{M}_{\text{NR}}$) for Fig. 1 is given by

$$\begin{aligned} \mathcal{M}_{\text{NR}} &= \frac{i}{(p_b + \Pi - q)^2 - m_q^2} \bar{b}\gamma_\nu \\ &\quad \times P_L(\not{p}_b + \not{\Pi} - \not{q} + m_b)\gamma_\mu P_L b \mathcal{L}^{\mu\nu}. \end{aligned} \tag{11}$$

The subscript "NR" refers to a nonradiative process, i.e., the process without a photon in the final state. Further, $p_b + \Pi$ is the effective momentum of b quark, where $p_b = m_b v$ is heavy quark momentum while Π is residual momentum of heavy quark. Also, $q (= p_\ell + p_\nu) \sim \mathcal{O}(m_b)$ while $\Pi \sim \mathcal{O}(\Lambda_{\text{QCD}})$. Hence, the expansion of $\mathcal{M}_{\text{NR}}$ in powers of Π produces an expansion in powers of $\frac{\Lambda_{\text{QCD}}}{m_b}$. Expanding the denominator to $(\frac{\Lambda_{\text{QCD}}}{m_b})^2$,

$$\begin{aligned} \frac{1}{(p_b + \Pi - q)^2 - m_q^2} &= \frac{1}{((p_b - q)^2 - m_q^2)} \\ &\quad \times \left[1 - \frac{2(p_b - q).\Pi + \Pi^2}{((p_b - q)^2 - m_q^2)} \right. \\ &\quad \left. + \frac{2((p_b - \Pi).\Pi)^2}{((p_b - q)^2 - m_q^2))^2}\right]. \end{aligned} \tag{12}$$

The hadronic part of $\mathcal{M}_{\text{NR}}$ is then sandwiched between the B meson states. It is to be noted that since the OPE involved here is in the expansion of the inverse power of m_b, therefore the b-quark field in QCD is converted to those in heavy quark effective theory at order $1/m_b$ through the relation

$$b(x) = e^{-im_b v.x}\left(1 + \frac{i\not{D}}{2m_b}\right) b_v(x). \tag{13}$$

Now, the obtained matrix elements are simplified as [11]

$$\begin{aligned} \langle B(v) | \bar{b}_v \gamma^\mu b_v | B(v) \rangle &= 2p_B^\mu, \\ \langle B(v) | \bar{b}_v \gamma_\mu D_\tau b_v | B(v) \rangle &= \frac{\lambda_1 + 3\lambda_2}{3m_b}(2g_{\mu\tau} - 5v_\mu v_\tau), \\ \langle B(v) | \bar{b}_v \gamma_\mu D_{(\alpha} D_{\beta)} b_v | B(v) \rangle &= \frac{2\lambda_1}{3m_b}(g_{\alpha\beta} - v_\alpha v_\beta) v_\mu, \\ \langle B(v) | \bar{b}_v \gamma_\mu D^2 b_v | B(v) \rangle &= \frac{2\lambda_1}{m_b} v_\mu. \end{aligned} \tag{14}$$

Next, we calculate the imaginary part of the denominator,

$$\frac{1}{((p_b - q)^2 - m_q^2)} \to (-2\pi i)\delta((p_b - q)^2 - m_q^2)\Theta((p_b^0 - q^0)). \tag{15}$$

Integrating over the neutrino energy, the double differential decay rate for the $B \to X_c \ell \bar{\nu}_\ell$ mode is calculated as

$$\begin{aligned}\frac{d^2\Gamma}{dy d\widehat{q^2}} = \frac{G_F^2 |V_{ub}|^2 m_b^5}{96\pi^3} y \Bigg[& 6\left(1 - \frac{\widehat{q^2}}{y}\right)(1 - \rho - y + \widehat{q^2}) + \lambda_1 \Bigg(-3 + 3\rho + 4\frac{\hat{q}^2}{y} - 4\rho\frac{\widehat{q^2}}{y} - 6\widehat{q^2} + 4\frac{(\widehat{q^2})^2}{y} \\ & - \delta(z)\Bigg(1 - 2\rho + \rho^2 - 3y(1-\rho) - 3\frac{\widehat{q^2}}{y} + 2\rho\frac{\widehat{q^2}}{y} + \rho^2\frac{\widehat{q^2}}{y} + 11\widehat{q^2} - 3\rho\overline{q^2} - 3x\widehat{q^2} - 6\frac{(\widehat{q^2})^2}{y} \\ & - 2\rho\frac{(\widehat{q^2})^2}{y} + 2(\widehat{q^2})^2 + \frac{(\widehat{q^2})^3}{y}\Bigg) + \delta'(z)\left(1 - \frac{\widehat{q^2}}{y}\right)(1 - \rho - y + \widehat{q^2})(1 - 2\rho + \rho^2 - 2\widehat{q^2} - 2\rho\widehat{q^2} + (\widehat{q^2})^2)\Bigg) \\ & + 3\lambda_2\Bigg(1 - 5\rho + 2\frac{\widehat{q^2}}{y} + 10\rho\frac{\widehat{q^2}}{y} + 10\widehat{q^2}\left(1 - \frac{\widehat{q^2}}{y}\right) - \delta(z)\Bigg(-1 + 6\rho - 5\rho^2 + y(1-5\rho) + \frac{\widehat{q^2}}{y}(1-2\rho) \\ & + 5\rho^2\frac{\widehat{q^2}}{y} + \widehat{q^2}(1 + 15\rho) + 5y\widehat{q^2} - 2\frac{(\widehat{q^2})^2}{y} - 10(1+y)\frac{(\widehat{q^2})^2}{y} + \frac{(\widehat{q^2})^3}{y}\Bigg)\Bigg)\Bigg], \end{aligned} \tag{16}$$

where $y = \frac{2E_\ell}{m_b}$, $q^2 = (p_\ell + p_\nu)^2$, $\widehat{q^2} = \frac{q^2}{m_b^2}$, $\rho = \frac{m_c^2}{m_b^2}$ and $z = 1 - y - \frac{\widehat{q^2}}{y} + \widehat{q^2} - \rho$. Equation (16) is in perfect agreement with [4,11]. Integrating over $\hat{q}^2$, the lepton spectrum for $B \to X_c \ell \bar{\nu}_\ell$ is obtained as

$$\begin{aligned}\frac{d\Gamma}{dy} = 2\Gamma_0 y^2 \Bigg[& (3 - 2y) - 3\rho - \frac{3\rho^2}{(1-y)^2} + \frac{(3-y)\rho^3}{(1-y)^3} - \frac{\lambda_1}{m_b^2}\left(-\frac{5}{3}y - \frac{y(5-2y)\rho^2}{(1-y)^4} + \frac{2y(10 - 5y + y^2)\rho^3}{3(1-y)^5}\right) \\ & - \frac{3\lambda_2}{m_b^2}\left(-\frac{(6+5y)}{3} + \frac{2(3-2y)\rho}{(1-y)^2} + \frac{3(2-y)\rho^2}{(1-y)^3} - \frac{5(6 - 4y + y^2)\rho^3}{3(1-y)^4}\right)\Bigg]. \end{aligned} \tag{17}$$

Further, in the limit $\rho \to 0$ the lepton spectrum for $B \to X_u \ell \bar{\nu}_\ell$ can be obtained as

$$\frac{d\Gamma}{dy} = 2\Gamma_0\left[y^2(3-2y) - \frac{\lambda_1}{m_b^2}\left(-\frac{5}{3}y^3 + \frac{1}{6}\delta(1-y) + \frac{1}{6}\delta'(1-y)\right) - \frac{\lambda_2}{m_b^2}\left(-(6+5y)y^2 + \frac{11}{2}\delta(1-y)\right)\right], \tag{18}$$

where $\Gamma_0 = \frac{G_F^2 |V_{c(u)b}|^2 m_b^5}{192\pi^3}$. It is important to note that the contribution from the parton model, which is proportional to $2y^2(3-2y)$, does not vanish at the end point. This leads to the presence of delta functions and their derivatives in the lepton spectrum. After integrating over the lepton energy, the total decay rate for $B \to X_c \ell \bar{\nu}_\ell$ is obtained as

$$\begin{aligned}\Gamma = \Gamma_0 & \left(1 + \frac{\lambda_1}{2m_b^2} + \frac{3\lambda_2}{2m_b^2}\left(2\rho\frac{d}{d\rho} - 3\right)\right) \\ & \times (1 - 8\rho + 8\rho^3 - \rho^4 - 12\rho^2 \ln\rho), \end{aligned} \tag{19}$$

which has the same form as shown in Eq. (2), and matches the result obtained in [4,11].

III. DIFFERENTIAL RATE OF $B \to X_c \ell \bar{\nu}_\ell \gamma$

In this section, we calculate the differential rate for the $B \to X_c \ell \bar{\nu}_\ell \gamma$ mode. This decay mode is more likely to be measured, compared to the case of the $B \to X_u$ mode which is additionally strongly suppressed by $|V_{ub}/V_{cb}|^2$. Figure 2 shows all the Feynman diagrams[3] contributing to the decay width of $B \to X_c \ell \bar{\nu}_\ell \gamma$ at leading order in perturbation theory. At the leading order ($m_b \to \infty$), the decay width for the process $B \to X_c \ell \bar{\nu}_\ell \gamma$ is obtained by the partonic result and further the preasymptotic effects, i.e., the subleading contributions in heavy quark expansion are obtained in the powers of $\frac{\Lambda_{\rm QCD}}{m_b}$. Similar to Sec. II, our focus is on the $B \to B$ forward scattering matrix element instead of the amplitude for $B \to X_c \ell \bar{\nu}_\ell \gamma$ itself. The imaginary part of the amplitude shown in Fig. 3 is related to the inclusive rate for the $B \to X_c \ell \bar{\nu}_\ell \gamma$ transition, as dictated by the optical theorem. However, the process $B \to X_c \ell \bar{\nu}_\ell \gamma$ is highly nontrivial compared to the $B \to X_c \ell \bar{\nu}_\ell$ mode due to the presence of a photon line between the charged quarks and leptons, as

[3]We have considered only those diagrams which after cutting the photon and c-quark lines lead to $B \to X_c \ell \bar{\nu}_\ell \gamma$.

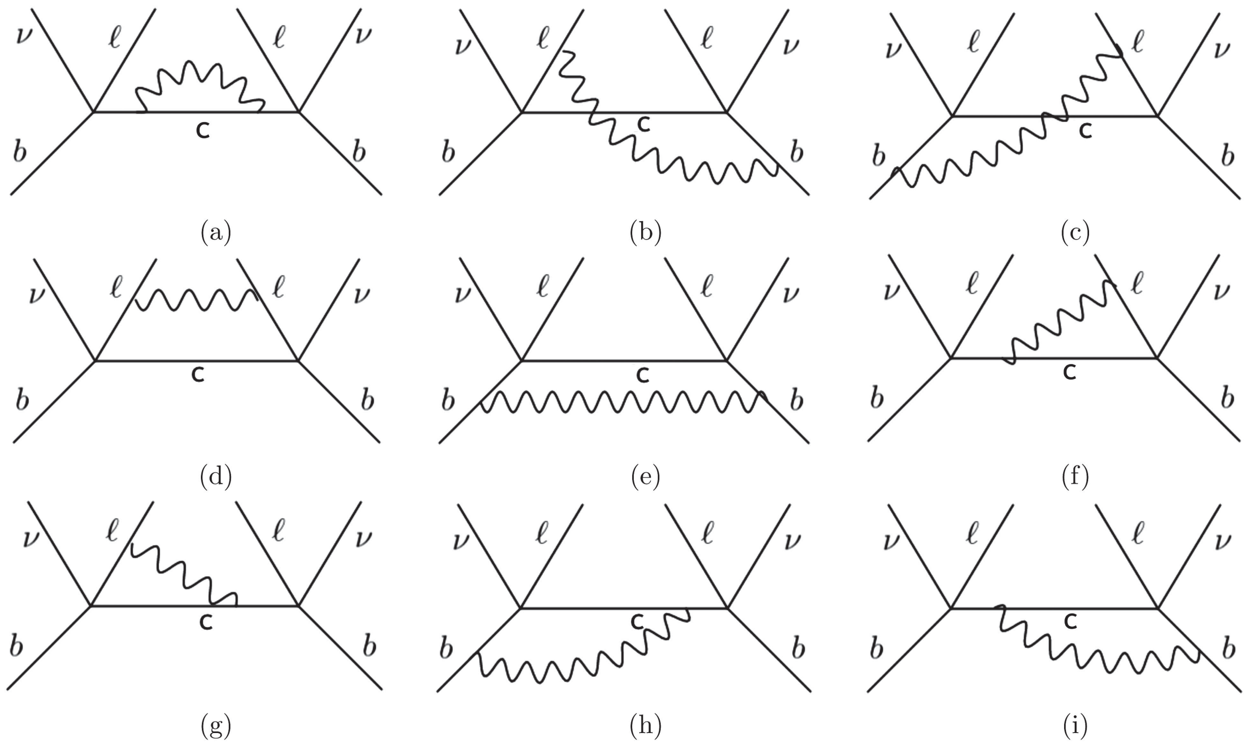


FIG. 2. Feynman diagrams for $B \to X_c\ell\bar{\nu}_\ell\gamma$.

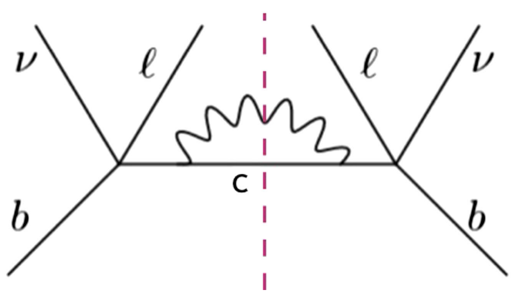


FIG. 3. Representative diagram with explicit cut.

shown in Fig. 2. This means that some diagrams, such as Figs. 2(b) and 2(c), do not easily separate into leptonic and hadronic parts like in the $B \to X_c\ell\bar{\nu}_\ell$ mode. Since the hadronic part in some of the diagrams communicates with the leptonic part through the photon, the calculation of the matrix element is more complex when expressed in terms of invariant tensors and using analytic properties of the transition operator. Also, in the present case, the transition tensor will be a four index object, two for the weak currents and two for the electromagnetic currents representing the photon emission. Therefore, it is more straightforward to use the Cutkosky method directly to compute the matrix element. As we verified in the last section, this method reproduces correct results for the decay rate of the $B \to X_q\ell\bar{\nu}_\ell$ mode.

Now, the decay rate for the semileptonic inclusive process $B \to X_c\ell\bar{\nu}_\ell\gamma$ is given by

$$\Gamma_\gamma = \left(\frac{4G_F}{\sqrt{2}}\right)^2 |V_{ub}|^2 \frac{1}{2m_B} \int \frac{d^3p_l}{(2\pi)^3 2E_l} \int \frac{d^3p_\nu}{(2\pi)^3 2E_\nu} \int \frac{d^4k}{(2\pi)^4} \mathrm{Im}[\langle B|I\mathcal{M}_{\mu\nu}\mathcal{L}^{\mu\nu}|B\rangle], \tag{20}$$

with

$$I\mathcal{M}_{\mu\nu}\mathcal{L}^{\mu\nu} = \sum_{m=1}^{9} I_m \mathcal{M}_{\mu\nu}^{(m)} \mathcal{L}^{\mu\nu(m)}, \tag{21}$$

where $\mathcal{M}_{\mu\nu}^{(m)}$ and $\mathcal{L}_{\mu\nu}^{(m)}$ contain the Dirac structure for the quark and leptonic part respectively, and I_m contains the denominator part of the propagator. Also, $m = 1, \ldots 9$ corresponds to Figs. 2(a)–2(i). k is the photon four momentum, and $p_{\ell(\nu)}$ is the lepton (neutrino) four momentum. The explicit calculation of forward scattering operator for Fig. 2(a) is presented below. All other diagrams can be calculated analogously. Calculating the hadronic and leptonic tensors requires the computation of $\mathcal{M}_{\mu\nu}^{(m)}$, $\mathcal{L}_{\mu\nu}^{(m)}$, and I_m. At the leading order in α_s, the explicit forms of $\mathcal{M}_{\mu\nu}^{(1)}$, $\mathcal{L}_{\mu\nu}^{(1)}$, and I_1 are

$$\mathcal{M}^{(1)}_{\mu\nu} = 2(-ig^{\alpha\beta})\bar{b}\gamma^{\nu}(1-\gamma^5)i(\not{p}_b+\not{\Pi}-\not{q}+m_c)(-ieQ_u)\gamma^{\alpha}i(\not{p}_b+\not{\Pi}-\not{k}-\not{q}+m_c)(-ieQ_u)\gamma^{\beta}i(\not{p}_b+\not{\Pi}-\not{q}+m_c)\gamma^{\mu}(1-\gamma^5)b,$$
$$= 2(e^2Q_u^2g^{\alpha\beta})[\bar{b}\gamma^{\nu}(1-\gamma^5)(\not{p}_b+\not{\Pi}-\not{q})\gamma^{\alpha}(\not{p}_b+\not{\Pi}-\not{k}-\not{q})\gamma^{\beta}(\not{p}_b+\not{\Pi}-\not{q})\gamma^{\mu}(1-\gamma^5)b$$
$$+z^2m_b^2(\bar{b}\gamma^{\nu}(\not{p}_b+\not{\Pi}-\not{q})\gamma^{\alpha}(\not{p}_b+\not{\Pi}-\not{k}-\not{q})\gamma^{\beta}(\not{p}_b+\not{\Pi}-\not{q})\gamma^{\mu}(1-\gamma^5)b)], \quad (22)$$

$$\mathcal{L}^{(1)}_{\mu\nu} = (\bar{\ell}\gamma^{\mu}(1-\gamma^5)\nu_{\ell})(\bar{\nu}_{\ell}\gamma^{\nu}(1-\gamma^5)\ell), \quad (23)$$

and

$$I_1 = \frac{1}{(k^2+i\epsilon)((p_b+\Pi-q)^2-m_c^2+i\epsilon)((p_b+\Pi-q-k)^2-m_c^2+i\epsilon)((p_b+\Pi-q)^2-m_c^2+i\epsilon)}, \quad (24)$$

respectively. As before, the effective momentum of b quark is given by $p_b+\Pi$. Expanding I_1 in powers of Π yields an expansion in terms of $\frac{\Lambda_{\rm QCD}}{m_b}$, similar to the $B\to X_c\ell\bar{\nu}_{\ell}$ mode. Further, the matrix element is also expanded in the powers of z to order z^3, where $z=m_c/m_b$. Now, the explicit form of I_1 up to $\mathcal{O}(\Pi^2)$ is obtained from Eq. (24) and is given by

$$I_1 = \frac{1}{k^2((p_b-q-k)^2-m_c^2)}\left[\frac{1}{(p_c.k)^2}-\frac{2(p_b-q).\Pi}{(p_c.k)^3}-\frac{\Pi^2}{(p_c.k)^3}+\frac{2((p_b-q).\Pi)^2}{(p_c.k)^4}\right]$$
$$-\frac{1}{k^2((p_b-q-k)^2-m_c^2)^2}\left[\frac{2p_c.\Pi}{(p_c.k)^2}-\frac{4(p_c.\Pi)(p_b-q).\Pi}{(p_c.k)^3}-\frac{\Pi^2}{(p_c.k)^2}\right]$$
$$+\frac{1}{k^2((p_b-q-k)^2-m_c^2)^3}\left[\frac{2(p_c.\Pi)^2}{(p_c.k)^2}\right]. \quad (25)$$

Similar to Sec. II, the Cutkosky method is exploited (see Fig. 3) to calculate the imaginary part of the matrix element. Mathematically, this essentially replaces the cut propagator with a product of the delta function and theta function, enforcing the positive energy condition. For example

$$\frac{1}{((p_b-q-k)^2-m_c^2+i\epsilon)} \to (-2\pi i)\delta((p_b-q-k)^2-m_c^2)\Theta((p_b^0-q^0-k^0)). \quad (26)$$

More generally, one has the identity

$$-\frac{1}{\pi}\mathrm{Im}\left(\frac{1}{(p_b-q-k)^2-m_c^2}\right)^n = \frac{(-1)^{(n-1)}}{(n-1)!}\delta^{(n-1)}((p_b-q-k)^2-m_c^2), \quad (27)$$

where the superscript of the delta function denotes the $(n-1)$th derivative with respect to its argument itself. When working with terms involving the derivatives of the delta function, it is important to handle them with care. The first step is to use integration by parts to remove the derivatives from the delta function and transfer them onto other functions by multiplying it. However, it is crucial to use the theta function carefully during this process because it determines the minimum value of the neutrino energy, denoted by E_{ν}. See Appendix B for details of kinematics.

Subsequently, we proceed by combining the terms $\mathcal{M}^{(1)}_{\mu\nu}$, $\mathcal{L}^{\mu\nu(1)}$, and I_1 to compute the imaginary part of the amplitude. Notably, our analysis reveals that no new operators are produced beyond those already present in the decay rate of the $B\to X_c\ell\bar{\nu}_{\ell}$ process. All the pertinent operators, up to dimension five, are listed in Eq. (14).

It is evident that only I_1 contributes to the imaginary part of the matrix element. Hence, we have presented an explicit expression of the matrix element in accordance with the representation outlined in Eq. (25). Each square bracket contains terms with expansion in Π up to second order. The imaginary parts of the coefficients of these square brackets contribute to the delta function and its derivatives. We designate the forward scattering matrix element as

$$\mathcal{J}_1(n;\alpha) = \langle B(v)|\mathrm{Im}\{I_1\mathcal{M}^{(1)}_{\mu\nu}\mathcal{L}^{\mu\nu(1)}\}|B(v)\rangle(n\alpha), \quad (28)$$

where $n=0$, 1, 2 denotes expansion powers of Π and $\alpha=a$, c, d represents square brackets of Eq. (25) in order. For example in $\mathcal{J}(0;a)$, "0" denotes the expansion in Π to $\mathcal{O}(\Pi^0)$ and "a" reveals that elements of the first square bracket of Eq. (25) are chosen. We now explicitly list all the expressions for each of the terms of I_1 without the delta function or its derivatives:

$\mathcal{O}(\Pi^0)$:

$$\mathcal{J}_1(0;a) = \frac{-1}{m_b(p_c.k)^2}16(p_b.p_\ell)((q^2+m_b^2-2(p_b.q))(p_\nu.(p_b+k-q))-2((p_b-q).k)\\(q-p_b).p_\nu+4z^2m_b^2(3p_b.p_\nu+k.p_\nu-3q.p_\nu)). \tag{29}$$

$\mathcal{O}(\Pi)$:

$$\mathcal{J}_1(1;a) = \frac{-1}{3m_b^3(p_c.k)^3}64(\lambda_1+3\lambda_2)(2m_b^2(p_\ell.q)+(p_b.p_\ell)(3m_b^2-5(p_b.q)))((q^2+m_b^2-2(p_b.q))\\(p_\nu.(p_b+k-q))-2((p_b-q).k)(p_\nu.q-p_b.p_\nu)+z^2m_b^2(2(3p_b+k-3q).p_n(2m_b^2(q.p_l)\\+(p_b.p_l)(3m_b^2-5(q.p_b)))+3(k.p_c)(2m_b^2(p_l.p_n)-5(p_b.p_l)(p_b.p_n)))). \tag{30}$$

$\mathcal{O}(\Pi^2)$:

$$\mathcal{J}_1(2;a) = \frac{1}{3m_b^3(p_c.k)^4}32\lambda_1((p_b.p_\ell)(-2(p_c.k)^2(m_b^2(p_\nu.k-5(p_\nu.q))+(p_b.p_\nu)(2(p_b.(q+k))+3m_b^2))\\+((q^2+m_b^2-2(p_b.q))(p_\nu.(p_b+k-q))-2(p_b-q).k((p_b-q).p_\nu))((p_b.q)^2+m_b^2(3(p_c.k)-q^2))\\+4(p_c.k)(m_b^2((p_\ell.p_\nu)(2p_b.(q+k)-4(q.k)-3q^2)+(p_b.p_\nu)(p_b.q+2(q.k+q^2))+2q^2(p_\nu.k))\\+(p_b.q)(2(p_\nu.q)(p_b.(q+k))-2(p_b.q)(p_\nu.k)+(p_b.p_\nu)(q^2+2(q.k)-4p_b.(q+k)))\\-m_b^4(p_\nu.q)))+4z^2m_b^2(-p_b.p_l(3p_b+k-3q).p_n((q.p_b)^2+m_b^2(3k.p_c-q^2))-6(k.p_c)\\(m_b^2((p_b.p_l)(\bar{q}.p_n)+(p_bp_n)(\bar{q}.p_l))-2(p_b.p_l)(p_b.p_n)(q.p_b)))). \tag{31}$$

The sum, $\mathcal{J}_1(0;a)+\mathcal{J}_1(1;a)+\mathcal{J}_1(2;a)$, is multiplied with $\delta(k^2)\theta(k^0)\delta(((p_b-q-k)^2-m_c^2))\theta((p_b-q-k)^0)$. Similarly, the second square brackets of Eq. (25) with hadronic and leptonic tensors [Eq. (23)] produce the terms expanded in powers of Π. Since this set of terms comes multiplied with $\frac{1}{((p_b-q-k)^2-m_c^2)^2}$, i.e., the square of the propagator, the sum of these terms has an overall factor of $\delta(k^2)\theta(k^0)\delta'((p_b-q-k)^2-m_c^2)\theta((p_b-q-k)^0)$.

$\mathcal{O}(\Pi)$:

$$\mathcal{J}_1(1;c) = \frac{-1}{3m_b^3(p_c.k)^2}128(\lambda_1+3\lambda_2)(((m_b^2+q^2-2p_b.q)((p_b-q+k).p_\nu)-2((p_b-q).k)((p_b-q).p_\nu))\\((p_b.p_\ell)(5p_b.(q+k)-3m_b^2)-2m_b^2(p_\ell.(q+k)))+z^2m_b^2(3p_b.p_\nu+k.p_\nu-3(q.p_\nu))(2m_b^2\\(k.p_l+q.p_l)+(p_b.p_l)(3m_b^2-5(k.p_b+q.p_b))))). \tag{32}$$

$\mathcal{O}(\Pi^2)$:

$$\mathcal{J}_1(2;c) = \frac{-1}{3m_b^3(p_c.k)^3}128(\lambda_1(p_b.p_\ell)(8((p_c.k+p_b.q)(p_\nu.q)-(2(p_c.k)+p_b.q)(p_b.p_\nu))(p_b.k)^2\\+2((4(p_\nu.k)(p_c.k)+(7p_c.k-4k.q-4q^2)(p_\nu.q)+(4(q.k)-3(p_c.k)+4q^2)(p_b.p_\nu))m_b^2\\-2(p_b.q)((p_\nu.k)(2(p_c.k)-q^2+2(p_b.q)-m_b^2)+(p_\nu.q)(2k.q-4(p_c.k)+q^2-4(p_b.q)+m_b^2))\\+2(2(q.k)(p_c.k+p_b.q)+(p_c.k)(q^2-8(p_b.q)+m_b^2)+(p_b.q)(q^2-4(p_b.q)+m_b^2))(p_b.p_\nu))(p_b.k)\\+(8(p_b-p_\nu).q(k.q)^2+2(2(3q^2-2(q.p_b)+m_b^2)(p_\nu.q-p_b.p_\nu)+p_c.k(11(p_b.p_\nu)-15(q.p_\nu)))(k.q)\\+4q^2(q^2-2(p_b.q)-m_b^2)(p_b-q).p_\nu+p_c.k((2(p_b.q)-m_b^2)(7q-3p_b).p_\nu+q^2(11p_b-15q).p_\nu))m_b^2\\+4(p_b.q)(p_b.qp_\nu.q(2(p_c-q).k-q^2+2p_b.q-m_b^2)+(2(q.k)(p_c.k+p_b.q)+p_c.k(q^2-4p_b.q+m_b^2)\\+p_b.q(q^2-2p_b.q+m_b^2))(p_b.p_\nu))+(p_\nu.k)(4(q^2-2p_b.q+m_b^2)((p_b.q)^2-(q^2+k.q)m_b^2)\\+(p_c.k)((7q^2+2p_b.q-m_b^2)m_b^2-8(p_b.q)^2))+4z^2m_b^2(p_b.p_l)(m_b^2(3(p_b.p_\nu)(4q^2-3(k.p_c))+4(k.q)\\(3p_b.p_\nu+k.p_n-3(q.p_\nu))+(k.p_\nu)(3k.p_c+4q^2)+3(\bar{q}.p_\nu)(5(k.p_c)-4q^2))-2(k.p_b+q.p_b)\\(3(p_b.p_\nu)(2q.p_b+k.p_c)+2(q.p_b)(k.p_\nu-3(q.p_\nu)))))). \tag{33}$$

In a similar way, we then consider the imaginary part of the third square bracket of Eq. (25), and combine with Eq. (23). Explicitly, the amplitude in the expanded in powers of Π is

$\mathcal{O}(\Pi^2)$:

$$\mathcal{J}_1(2;d) = \frac{1}{3m_b^3(p_c.k)^2} 256\lambda_1(p_b.p_\ell)(((-2p_b.q + m_b^2 + q^2)(p_b + k - q).p_\nu - 2(p_b - q).k(q - p_b).p_\nu) \\ (((q+k).p_b)^2 - m_b^2(2q.k + q^2)) - 4z^2 m_b^2((k.p_b + q.p_b)^2 - m_b^2(2k.q + q^2))(3p_b + k - 3q).p_\nu). \quad (34)$$

Here, "d" refers to the elements of the third square bracket of Eq. (25). Moreover, the $\mathcal{J}_1(2;d)$ has a multiplicative factor of $\delta(k^2)\theta(k^0)\delta''(((p_b - q - k)^2 - m_c^2))\theta((p_b - q - k)^0)$.

Next, combining all the amplitudes, the total forward matrix element for Fig. 2(a) is given by

$$\langle B|\mathrm{Im}\{I_1\mathcal{M}_{\mu\nu}^{(1)}\mathcal{L}_{\mu\nu}^{(1)}|B\}\rangle = \delta(k^2)\theta(k^0)[(\mathcal{J}_1(0;a) + \mathcal{J}_1(1;a) + \mathcal{J}_1(2;a))\delta((p_b - q - k)^2 - m_c^2) \\ + (\mathcal{J}_1(1;c) + \mathcal{J}_1(2;c))\delta'((p_b - q - k)^2 - m_c^2) + (\mathcal{J}_1(2;d))\delta''((p_b - q - k)^2 - m_c^2)] \\ \theta((p_b - q - k)^0). \quad (35)$$

Integration by parts is then used to simplify such expressions:

$$\delta'(x)\theta(x)f(x) = -\delta(x)\delta(x)f(x) - \delta(x)\theta(x)f'(x), \text{ and} \quad (36)$$

$$\delta''(x)\theta(x)f(x) = \delta(x)\delta'(x)f(x) + 2\delta(x)\delta(x)f'(x) + \delta(x)\theta(x)f''(x). \quad (37)$$

These relations will be used to carry out integrals over phase space. In a similar way, the forward matrix elements for the remaining eight Feynman diagrams, as shown in Fig. 2, are calculated. Relevant expressions for $\mathcal{M}_{\mu\nu}^{(m)}$, $\mathcal{L}_{\mu\nu}^{(m)}$, and I_m $m = 2, \ldots 9$ for all the Feynman diagrams are provided in Appendix A.

IV. DIFFERENTIAL RATE FOR $B \to X_u \ell\bar{\nu}\gamma$

In the limit $m_u \to 0$ the above method can be carried to $B \to X_u$ decay. The matrix element for Fig. 3(a) replacing the c quark to u quark is given by

$$\mathcal{M}_{\mu\nu}^{(1)}(u) = 2(-ig^{\alpha\beta})\bar{b}\gamma^\nu(1-\gamma^5)i(\not{p}_b + \not{\Pi} - \not{q})(-ieQ_u)\gamma^\alpha i(\not{p}_b + \not{\Pi} - \not{k} - \not{q})(-ieQ_u)\gamma^\beta i(\not{p}_b + \not{\Pi} - \not{q})\gamma^\mu(1-\gamma^5)b, \\ \mathcal{L}_{\mu\nu}^{(1)} = (\bar{\ell}\gamma^\mu(1-\gamma^5)\nu_\ell)(\bar{\nu}_\ell\gamma^\nu(1-\gamma^5)\ell), \quad (38)$$

and

$$I_1(u) = \frac{1}{(k^2 + i\epsilon)((p_b + \Pi - q)^2 + i\epsilon)((p_b + \Pi - q - k)^2 + i\epsilon)((p_b + \Pi - q)^2 + i\epsilon)}. \quad (39)$$

$I_1(u)$ has a similar expansion as Eq. (25) replacing m_c with m_u. The differential decay rate for $B \to X_u \ell\nu\gamma$ is then calculated following the procedure outlined for $B \to X_c$, but now in the limit $z \to 0$. In the above processes, one difference is the expansion in power of z and another crucial difference is in kinematics.[4]

V. RESULTS

As a general rule, the four-body phase space comprises five distinct variables. In the context of inclusive decays, we have an additional variable, namely the invariant mass squared of the final state meson (p_X^2), which we trade for $q'^2 (= (p_\ell + p_\nu + k)^2)$. The other independent variables are the lepton energy $y(= \frac{2p_B.p_\ell}{m_B^2})$, the energy of the hard photon $x(= \frac{2p_B.k}{m_B^2})$, the neutrino energy, and three angles. These are defined in the rest frame of the B meson. A detailed description

[4]In Appendix B, we have retained m_X in corresponding equations which can be put to zero for the u quark case.

TABLE I. Numerical inputs used for the decay rate calculation.

Parameter	Numerical value	References
$m_{B(b)}$	5.28(4.18) GeV	[24]
m_c	1.27 GeV	[24]
m_μ	0.10 GeV	[24]
G_F	1.166×10^{-5} GeV^{-2}	[24]
α_{em}	1/137	[24]
λ_1	-0.3 GeV^2	[25]
λ_2	0.117 GeV^2	[25]
$\lvert V_{ub}\rvert$	3.82×10^{-3}	[24]
$\lvert V_{cb}\rvert$	41×10^{-3}	[24]

of the kinematics is furnished in Appendix B. All the input parameters required for the evaluation of the differential decay rate are listed in Table I.

We begin by integrating over all variables except y to determine the spectrum of the charged lepton for different values of x. Figures 4 and 5(a) illustrate the differential decay rate as a function of the lepton energy (y) for various photon energy values (x) for $B \to X_c\ell\bar{\nu}_\ell\gamma$ and $B \to X_u\ell\bar{\nu}_\ell\gamma$ respectively. We observe that as the photon becomes increasingly soft, the leptonic energy end point shifts accordingly, which is consistent with the kinematics of the process.

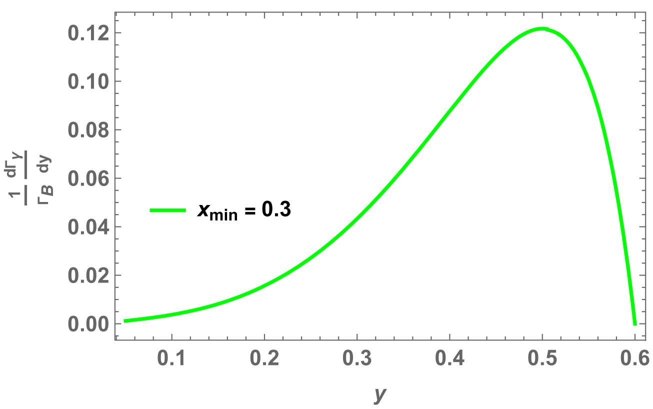


FIG. 4. Differential decay width of $B \to X_c\mu\bar{\nu}_\mu\gamma$ for particular normalized photon energy ($x_{\rm min} = 0.3$).

To provide a complete picture of the distribution for the differential decay rate for the $u-$ quark mode, we present the distribution for $x_{\rm min} = 0.3$, which corresponds to $k_{\rm min} \sim 0.8$ GeV in Fig. 5(b). The plot shows that the distribution ends at the kinematical boundary, which is, as expected, larger than that for $x_{\rm min} = 0.5$, and more toward the nonradiative, $B \to X_u\ell\bar{\nu}_\ell$, case. Further, it is observed that apart from the difference in CKM elements between the X_c and X_u modes, the decay rate for the radiative $B \to X_c$ mode receives a correction of approximately 10% due to mass effects. This finding aligns with the difference of about 12% in the two inclusive rates as per the particle data group values [24] after correcting for the difference in the CKM factors.

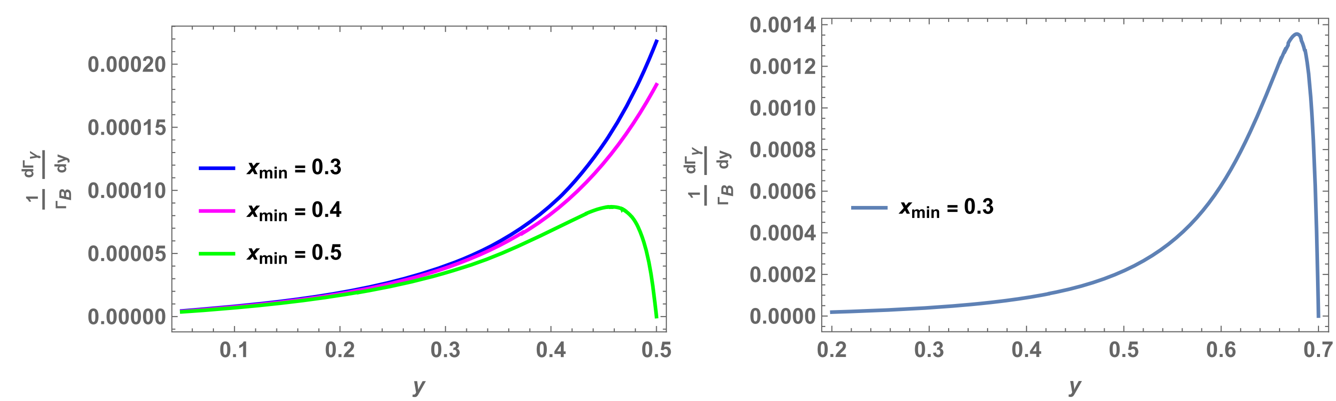
Further, as an example of a possible additional observable, we define the photon (differential) forward-backward asymmetry, $A_{\rm FB}(y)$, with respect to the recoiling final state hadron as

$$A_{\rm FB}(y) = \frac{\int_0^1 dt \frac{d^2\Gamma_\gamma}{dydt} - \int_{-1}^0 dt \frac{d^2\Gamma_\gamma}{dydt}}{\int_{-1}^1 dt \frac{d^2\Gamma_\gamma}{dydt}}, \tag{40}$$

where $t = \cos\theta_{X\gamma}$ is the angle between the outgoing photon and recoiling hadron (X) in the rest frame of the B meson. The forward-backward asymmetry for $B \to X_u\ell\bar{\nu}_\ell\gamma$ is shown in Fig. 6 for λ_1 and λ_2 from Table 1.

Finally, Fig. 7 depicts the differential decay rate as a function of the normalized photon energy (x), which shows that with lowering of the energy of the photon, the decay rate behaves as for the nonradiative mode.

It is worth noting that infrared divergences can be effectively circumvented by assigning sufficient mass to



(a) (b)

FIG. 5. (a) Differential decay width of $B \to X_u\mu\bar{\nu}_\mu\gamma$ for different values of normalized photon energy (x), and (b) differential decay width of $B \to X_u\mu\bar{\nu}_\mu\gamma$ for particular normalized photon energy ($x_{\rm min} = 0.3$).

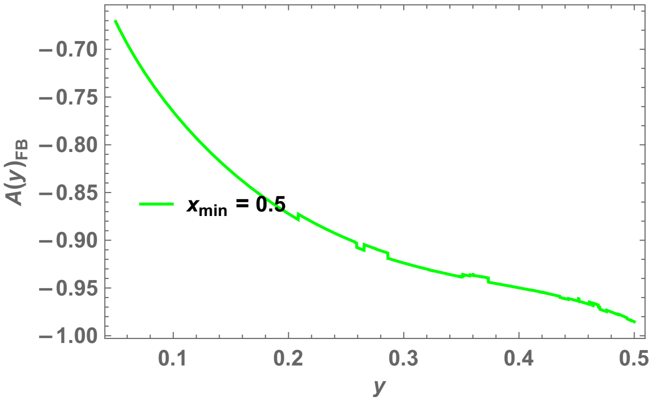


FIG. 6. Forward-backward asymmetry (A_{FB}) with lepton energy (y).

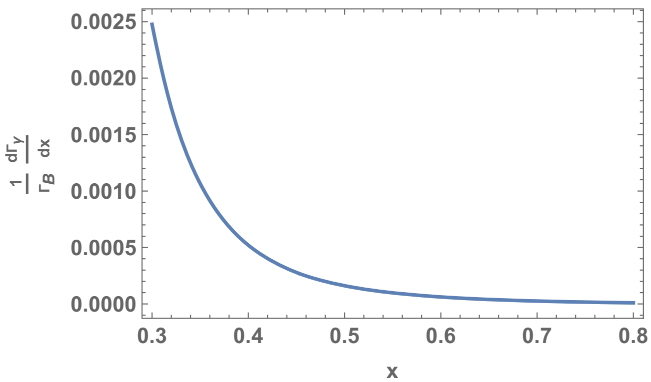


FIG. 7. Differential decay width of $B \to X_u \mu \bar{\nu}_\mu \gamma$ with photon energies (x).

the photon, i.e., the photon is made sufficiently hard. To avoid any contribution from mass singularities, we choose to work with muons in the final state. Furthermore, by choosing a lower cut for the polar angle, we can eliminate any collinear singularities that might arise.

When the photon is hard enough, the total decay width (Γ_γ) for the radiative mode is expected to be suppressed by $\mathcal{O}(\alpha)$ compared to the nonradiative one. In order to check if this expectation is met, we evaluate the ratio of the radiative decay width (Γ_γ) to the nonradiative decay width (Γ) and put $m_q \to 0$. We find that this ratio is approximately 0.01 for hard photons with energy around 1 GeV ($x_{\rm min} = 0.5$), thus confirming the expectation.

A. Nonperturbative parameters

After determining the decay width, we now propose a simple yet efficient method for calculating the nonperturbative parameters λ_1 and λ_2. We observe that using the ratio of the decay widths, instead of the widths themselves, is more suitable since the latter may contain uncertainties due to the CKM element V_{ub}. Moreover, such ratios yield ratios of simple functions of λ_1 and λ_2. By knowing the experimentally measured values of R_1 and R_2, we can simultaneously solve these two linear equations to obtain λ_1 and λ_2. The two ratios that we propose to employ are as in Eq. (3). Both the numerator and denominator have the form $A + B\lambda_1 + C\lambda_2$. Thus, each of the ratios has the form

$$\frac{A + B\lambda_1 + C\lambda_2}{A' + B'\lambda_1 + C'\lambda_2} = R_1 \quad \text{(say)}, \tag{41}$$

and the same for R_2. These are still two linear equations in λ_1 and λ_2. To further illustrate the practicality of our proposed method, we perform a sample calculation for nonperturbative parameters λ_1 and λ_2. At this point, there are no data available to directly determine λ_1 and λ_2. To proceed further in order to demonstrate the case of obtaining λ_1 and λ_2 once the suggested ratios are experimentally available, we obtain a value for R_2 using the known values of λ_1 and λ_2, and for R_1 we assume that the decay rate for $B \to X_c \mu \bar{\nu} \gamma$ is α_{em} times the decay rate for $B \to X_c \mu \bar{\nu}$. With these values, we can then numerically calculate the nonperturbative parameters λ_1 and λ_2. Our results yield $\lambda_1 = -0.29\ \mathrm{GeV}^2$ and $\lambda_2 = 0.13\ \mathrm{GeV}^2$, which are consistent with previously values reported in the literature [22,25,26]. This motivates the need for an experimental measurement of the decay width of $B \to X_c \ell \bar{\nu}_\ell \gamma$. Such a determination will aid in the precise determination of λ_1 and λ_2. As mentioned earlier, we have focused on $\mathcal{O}(\frac{1}{m_b})$ terms in HQE and therefore are sensitive only to λ_1 and λ_2. At higher orders, the expressions develop a dependence on more such nonperturbative parameters. Measurements of the $B \to X_c \ell \bar{\nu}_\ell \gamma$ rate and ratio R_1 will in fact be very helpful in a simultaneous and easy determination of these parameters when combined with $B \to X_c \ell \bar{\nu}_\ell$ data and ratio R_2.

VI. SUMMARY

To summarize, we have proposed a method to calculate the nonperturbative parameters λ_1 and λ_2 in the context of inclusive B meson decays. We first calculated the decay width for the inclusive modes $B \to X_{c(u)} \ell \bar{\nu}_\ell$ and $B \to X_{c(u)} \ell \bar{\nu}_\ell \gamma$ directly using the Cutkosky method and the heavy quark expansion applied to the amplitude, retaining terms to order $(\frac{\Lambda_{\rm QCD}}{m_b})$ in HQE. Since the radiative mode, $B \to X_{c(u)} \ell \bar{\nu}_\ell \gamma$, involves a hard photon, the most general tensorial structure will involve four indices as opposed to two for $B \to X_{c(u)} \ell \bar{\nu}_\ell$. In view of this and related complications to pick out the analytic properties in order to evaluate the decay rate, we decided to compute the relevant amplitude directly (or brute force) by using the Cutkosky method. The differential rate contains two nonperturbative parameters λ_1 and λ_2. Further, the differential and total decay rates were then obtained by integrating the four-body phase space variables. The differential rate and forward-backward asymmetry are plotted with respect to lepton energy for different $x_{\rm min}$ values that define the hardness of the photon. It was found that the decay rate for the radiative mode ($B \to X_{c(u)}\ell\bar{\nu}_\ell\gamma$) is $\mathcal{O}(\alpha)$ times the nonradiative mode ($B \to X_{c(u)}\ell\bar{\nu}_\ell$) when the photon is sufficiently hard.

Next, two ratios, R_1 and R_2, are formed using the differential rate for radiative (for photon energy $\geq$ few times $\Lambda_{\rm QCD} \sim 500$ MeV) and nonradiative modes respectively, in different lepton energy ranges. These ratios are independent of the CKM element and provide two linear equations in λ_1 and λ_2, allowing an unambiguous determination of these parameters. A simple example is also shown in Sec. VA to demonstrate the calculation of these parameters, λ_1 and λ_2, which are found to be consistent with existing values. It should be mentioned that, at present, the radiative mode has not been measured. The results of this exercise encourage us to have precise measurements of the radiative mode, more so since, at higher orders in HQE, there are more nonperturbative parameters that enter the amplitude. The radiative inclusive semileptonic mode will be very useful in aiding a clean determination of these.

The decay rate for free quark decay in $B \to X_{c(u)}\ell\bar{\nu}_\ell \propto 2y^2(2y-3)$, where y is the normalized lepton energy in the rest frame of B meson. This indicates the difference in partonic and hadronic end points, with the former being $\frac{m_b}{2}$ and the latter being at $\frac{m_B}{2}$, resulting in an end point region of order $\bar{\Lambda} = m_B - m_b$. Now, more importantly for the $B \to X_u$ mode, in order to correctly include this region, it is required to sum over the infinite terms present in the heavy quark expansion. However, this expansion in $\frac{\bar{\Lambda}}{m_b}$ brings higher-order derivatives of the delta function at each successive order, resulting in a failure of the OPE and QCD perturbation theory in this region. Therefore in order to incorporate the end-point behavior, the distribution function of the heavy quark, or the shape function, needs to be introduced in the decay rate of the $B \to X_u\ell\bar{\nu}_\ell$ and $B \to X_s\gamma$ mode [10,12].

Now, in the case of the radiative decay ($B \to X_u\ell\bar{\nu}_\ell\gamma$) mode, it is noted that the presence of a hard photon in the final state causes the end point to shift in comparison to the nonradiative decay. The partonic and hadronic end points are at $\frac{m_b}{2} - x_{\rm min}$ and $\frac{m_B}{2} - x_{\rm min}$ respectively. However, the challenge due to the difference in partonic and hadronic end points remains similar to the $B \to X_u\ell\bar{\nu}_\ell$ mode. Hence, the shape function for this process is required. Further, one can try to use a simple form of the shape function, $\propto (1-y-x)^a e^{(1+a)(x+y)}$, similar to the one suggested for the nonradiative decay mode [27,28]. However, this may be very naive at this point, and hence, it is required to compute the shape function for this process and verify its universality before any definitive conclusions can be drawn. While the issue of the form of the shape function remains unsettled at this point and is left for a future study, the main idea that the use of the radiative inclusive decay rate can aid in a quick determination of λ_1 and λ_2 stays unaffected.

In conclusion, our proposed method provides a simple yet efficient way to calculate nonperturbative parameters λ_1 and λ_2 in inclusive B decays once the decay rate for radiative $B \to X_{c(u)}\ell\bar{\nu}_\ell\gamma$ is known experimentally.

APPENDIX A: CONTRIBUTIONS FROM DIFFERENT DIAGRAMS TO THE INCLUSIVE MATRIX ELEMENT

The explicit structures of $\mathcal{M}^{(m)}_{\mu\nu}$, $\mathcal{L}^{(m)}_{\mu\nu}$, and I_m calculated to $\mathcal{O}(z^3)(z = m_c/m_b)$ are listed below.[5] All the relevant expressions for Fig. 2(a) are listed in Sec. III. We now present explicit expressions for the diagrams in Figs. 2(b)–2(i).

(1) Figure 2(b):

$$\begin{aligned}\mathcal{M}^{(2)}_{\mu\nu} &= 2(-ig^{\alpha\beta})\bar{b}(-ieQ_b)\gamma_\alpha i(\not{p}_b + \not{\Pi} - \not{k} + m_b)\gamma^\nu(1-\gamma^5)i(\not{p}_b + \not{\Pi} - \not{k} - \not{q} + m_c)\gamma^\mu(1-\gamma^5)b,\\ \mathcal{L}^{(2)}_{\mu\nu} &= (\bar{\ell}(-ieQ_\ell)\gamma_\beta i(\not{p}_l + \not{k} + m_l)\gamma^\mu(1-\gamma^5)\nu_\ell)(\bar{\nu}_\ell\gamma^\nu(1-\gamma^5)\ell),\end{aligned} \tag{A1}$$

$$\begin{aligned}I_2 = {}&\frac{1}{k^2((p_b-q-k)^2-m_c^2)}\left[\frac{-1}{(p_b.k)(p_\ell.k)} - \frac{(p_b-k).\Pi}{(p_\ell.k)(p_b.k)^2} - \frac{\Pi^2}{2(p_\ell.k)(p_b.k)^2} - \frac{((p_b-k).\Pi)^2}{2(p_\ell.k)(p_b.k)^3}\right]\\ &-\frac{1}{k^2((p_b-q-k)^2-m_c^2)^2}\left[\frac{2p_c.\Pi}{(p_\ell.k)(p_b.k)} + \frac{2(p_c.\Pi)(p_b-k).\Pi}{(p_\ell.k)(p_b.k)^2} + \frac{\Pi^2}{(p_\ell.k)(p_b.k)}\right]\\ &-\frac{1}{k^2((p_b-q-k)^2-m_c^2)^3}\left[\frac{2(p_c.\Pi)^2}{(p_\ell.k)(p_b.k)}\right].\end{aligned} \tag{A2}$$

[5]Mathematical expressions for amplitude are very long, so we have only shown the leading term in z here. However, we have included terms up to $\mathcal{O}(z^3)$ in our numerical computations.

$\mathcal{O}(\Pi^0)$:

$$\mathcal{J}_2(0;a) = \frac{-1}{m_b(p_b.k)(p_\ell.k)} 16(-2(p_b.p_\nu)(2(p_b.p_\ell)^2 + (p_\ell.k)^2 + p_\ell.k(p_b.q + p_\ell.q - 3p_b.p_\ell) - (p_b.k)(p_\ell.k) - p_b.p_\ell(q.k - 2p_b.k + 2p_\ell.q)) + m_b^2(p_\ell.k((p_b + q).p_\nu) + p_\ell.p_\nu(p_b.k - q.k - 2p_\ell.k) - ((q - p_b).p_\ell)(p_\nu.k + 2p_\ell.p_\nu)) - 2((p_b.p_\ell)(p_\nu.k) - (p_b.k)(p_\ell.p_\nu))((q + k - p_b).p_\ell)), \tag{A3}$$

$\mathcal{O}(\Pi)$:

$$\mathcal{J}_2(1;a) = \frac{-1}{3m_b^3(p_b.k)^2(p_\ell.k)} 32(\lambda_1 + 3\lambda_2)(2m_b^2(p_\nu.k(p_b.p_\ell(p_\ell.q - 2q.k - 3p_\ell.k) - (p_b.p_\ell)^2 + 2p_\ell.k(p_b.q + 2p_\ell.k + 2p_\ell.q)) + p_b.p_\nu(6(p_b.p_\ell)^2 - p_b.p_\ell(7p_\ell.k + 3q.k + 6p_\ell.q) + p_\ell.k(3p_b.q + p_\ell.(q + k)))) + (p_b.k)^2(10(p_\ell.p_\nu(q + k - p_b).p_\ell) + p_b.p_\nu(p_\ell.q - 2p_b.p_\ell) + m_b^2 p_\ell.p_\nu) + p_b.k(-10p_b.p_\nu(2(p_b.p_\ell)^2 - p_b.p_\ell(3p_\ell.k + q.k + 2p_\ell.q) + p_\ell.k(p_b.q + (q + k).p_\ell)) + m_b^2(p_\nu.k(9p_b - 5q).p_\ell + p_\ell.k(9p_b - 12p_\ell + q) - p_\ell.p_\nu(12(q - p_b).p_\ell + q.k) - 6p_b.p_\nu(q - 2p_b).p_\ell) - 10(p_b.p_\ell)(p_\nu.k)((q + k - p_b).p_\ell) - 3m_b^4(p_\ell.p_\nu)) + m_b^4(3(p_b - q).p_\ell((k + 2p_\ell).p_\nu) + 3(q.k)(p_\ell.p_\nu) - 3(p_\ell.k)(p_\nu.(p_b - 2p_\ell + q)) + 4p_\nu.k)), \tag{A4}$$

$$\mathcal{J}_2(1;c) = \frac{1}{3m_b^3(p_b.k)(p_\ell.k)} 128(\lambda_1 + 3\lambda_2)(m_b^4(p_\nu.k((q - 3p_b).p_\ell) - p_\ell.k((4k + 5q + 3p_b - 6p_\ell).p_\nu) + p_\ell.p_\nu(5k.q + 6p_\ell.q - 6p_b.p_\ell)) + (2((4k + 2q + p_b).p_\nu)(p_\ell.k)^2 + (2p_\nu.k(2q.p_b + 6q.p_\ell - 3p_b.p_\ell) + 4p_\nu.q((q - 2p_b).p_\ell) + 2q^2(p_b.p_\nu) - 2((q + 7p_b).p_\ell)p_b.p_\nu + p_b.q((5q + 11p_b - 6p_\ell).p_\nu) - 4(q.k)(p_\ell.p_\nu))p_\ell.k + 4(p_\nu.q)(p_b.p_\ell)((p_b - q).p_\ell) + p_\nu.k(4(q.p_\ell)^2 - 2(p_b.p_\ell)^2 - p_b.q((q - 5p_b).p_\ell) - 2p_b.p_\ell(2k.q + q^2 + p_\ell.q)) - 2p_b.p_\nu(2q.p_\ell((k + p_\ell).q) + p_b.p_\ell((3k + 4p_\ell).q) - 6(p_b.p_\ell)^2) + p_\ell.p_\nu(6p_b.q((p_b - q).p_\ell) + k.q(4p_b.p_\ell - 4q.p_\ell - 5q.p_b)))m_b^2 + 10q.p_b(-(p_\nu.k)(p_b.p_\ell) ((k + q - p_b).p_\ell) - p_b.p_\nu(2(p_b.p_\ell)^2 - p_b.p_\ell(k.q + 3k.p_\ell + 2q.p_\ell) + p_\ell.k(k.p_\ell + q.p_b + q.p_\ell))) + (p_b.k)^2(m_b^2 p_\ell.p_\nu + 10((q - 2p_b).p_\ell)(p_b.p_\nu) + ((k + q - p_b).p_\ell)(p_\ell.p_\nu)) + p_b.k(-3(p_\ell.p_\nu)m_b^4 + (p_\nu.k((9p_b - q).p_\ell) + p_\ell.k((q + p_b).p_\nu) + p_\ell.p_\nu(2q^2 + q.p_b - 12k.p_\ell - k.q) - 2((2q + p_b + 6p_\ell).p_\nu) (q.p_\ell) + 4p_b.p_\ell((q + 3p_b + 3p_\ell).p_\nu))m_b^2 + 10(-(p_\nu.k)(p_b.p_\ell)((k + q - p_b).p_\ell) - p_b.p_\nu((p_\ell.k)^2 + p_\ell.k(q.p_b + q.p_\ell - 3p_b.p_\ell) + 2(p_b.p_\ell)^2 - (q.p_b)(q.p_\ell) - ((k - 2p_b + 2p_\ell).q)) + ((k + q - p_b).p_\ell) (p_b.q)(p_\ell.p_\nu)))). \tag{A5}$$

$\mathcal{O}(\Pi^2)$:

$$\mathcal{J}_2(2;a) = \frac{1}{3m_b^3(p_b.k)^2(p_\ell.k)} 16\lambda_1((3(p_\ell.k)((q + p_b).p_\nu) - 3p_\ell.p_\nu((3q + 10p_\ell).k) - 3p_\ell.(q - p_b) ((k + 2p_\ell).p_\nu))m_b^4 - 2(p_\nu.k((7k + 5q - 5p_b).p_\ell)(p_b.p_\ell) + p_b.p_\nu(6(p_b.p\ell)^2 - 3p_b.p_\ell (k.q + 5k.p_\ell + 2q.p_\ell) + p_\ell.k(7p_\ell.k + 3p_b.q + 5p_\ell.q)))m_b^2 + (p_b.k)^2(2p_\ell.q((p_b - p_\ell).p_\nu) - 2p_b.p_\ell((2p_b + p_\ell).p_\nu) - p_\ell.p_\nu(3m_b^2 + 2p_\ell.k)) + p_b.k(3(p_\ell.p_\nu)m_b^4 - 4(p_b.p_\nu)m_\ell^2 m_b^2 + 6((q - 2p_b).p_\ell)(p_b.p_\nu)m_b^2 - (k.q - 8q.p_\ell + 8p_b.p_\ell)(p_\ell.p_\nu)m_b^2 + p_b.p_\nu(2(p_\ell.k)^2 - 4(p_b.p_\ell)^2 + 2(q.k)(p_b.p_\ell) + 4(p_\ell.q)(p_b.p_\ell)) + p_\nu.k(2(pb.p_\ell)((q + p_b).p_\ell) - ((q - 5p_b).p_\ell)m_b^2) + p_\ell.k(((q + 5p_b + 12p_\ell).p_\nu)m_b^2 + 2(p_\nu.k)(p_b.p_\ell) + 2(p_\ell.q - p_b.q + 3p_b.p_\ell)(p_b.p_\nu)))), \tag{A6}$$

$$\mathcal{J}_2(2;c) = \frac{-1}{3m_b^3(p_b.k)^2(p_\ell.k)}128\lambda_1(m_b^4(-k.(2q+3p_b)(k.p_\nu((q-p_b).p_\ell)-k.p_\ell((q+p_b).p_\nu))-p_\ell.p_\nu(2(k.q)^2 + q.k(5p_b.k+4(k+q-p_b).p_\ell)+p_b.k((10k+10q-6p_b).p_\ell-3p_b.k)))+2((p_\ell.p_\nu)(p_b.k)^3 + (p_b.k)^2((p_b.p\nu)(p_\ell.(k+5q))+p_\ell.p_\nu(7k.p_\ell-q^2+q.p_b+7q.p_\ell)+p_b.p_\ell((k+2q-6p_b-3p_\ell).p_\nu)) + (-7(p_b.p_\nu)(p_\ell.k)^2+((p_b.p_\nu)(q^2-4q.p_b-9q.p_\ell)+p_b.p_\ell((15p_b-2q-7k).p_\nu)+2(p_\ell.p_\nu) ((k+p_b).q))p_\ell.k+2(q.p_\nu)(p_b.p_\ell)(p_b.p_\ell-k.q-q.p_\ell)+(p_\nu.k)(p_b.p_\ell)(q^2-q.p_b-7q.p_\ell+5p_b.p_\ell) -2(q.p_\ell)^2(p_b.p_\nu)-(p_b.p_\ell)(p_b.p_\nu)(6p_b.p_\ell-3k.q-12q.p_\ell)+2(p_\ell.p_\nu)((q.p_b)(q.p_\ell)+k.q(q.p_b+q.p_\ell -p_b.p_\ell)))p_b.k+2k.q(-(p_\nu.k)((k+q-p_b).p_\ell)(p_b.p_\ell)-p_b.p_\nu(2(p_b.p_\ell)^2-p_b.p_\ell(k.q+3p_\ell.k+2q.p_\ell) +p_\ell.k(k.p_\ell+q.(p_b+p_\ell)))))m_b^2-4k.p_b((k+q).p_b)(-p_b.p_\nu(p_\ell.k)^2-p_\ell.k((p_\nu.k)(p_b.p_\ell)+(q.p_\ell) (p_b.p_\nu)-(p_b.k)(p_\ell.p_\nu))+p_b.p_\ell(2p_b.p_\nu((k+p_\ell).p_b)-(k.p_\nu)(q.p_\ell))+(k.p_b)(q.p_\ell)(p_\ell.p_\nu))), \tag{A7}$$

$$\mathcal{J}_2(2;d) = \frac{-1}{3m_b^3(p_b.k)(p_\ell.k)}256\lambda_1(((q+k).p_b)^2 - m_b^2(2k.q+q^2))(-2p_b.p_\nu(2(p_b.p_\ell)^2+(p_\ell.k)^2+p_\ell.k (q.(p_\ell+p_b))-3p_b.p_\ell)-(p_b.k)(q.p_\ell)-p_b.p_\ell(k.q+2q.p_\ell-2k.p_b))+m_b^2(k.p_\ell((p_b+q).p_\nu) + (p_\ell.p_\nu)((p_b-2p_\ell-q).k)-((q-p_b).p_\ell)((k+2p_\ell).p_\nu))-2(p_b.p_\ell)(k.p_\nu)(-p_b+k+q).p_\ell + 2(k.p_b)(p_\ell.p_\nu)((k+q-p_b).p_\ell)). \tag{A8}$$

(2) Figure 2(c):

$$\mathcal{M}_{\mu\nu}^{(3)} = 2(-ig^{\alpha\beta})\bar{b}\gamma^\nu(1-\gamma^5)i(\not{p}_b+\not{\Pi}-\not{q}-\not{k}+m_c)\gamma^\mu(1-\gamma^5)i(\not{p}_b+\not{\Pi}-\not{k}+m_b)(-ieQ_b)\gamma_\alpha b,$$
$$\mathcal{L}_{\mu\nu}^{(3)} = (\bar{\ell}\gamma^\mu(1-\gamma^5)\nu_\ell)(\bar{\nu}_\ell\gamma^\nu(1-\gamma^5)(-ieQ_l)\gamma_\alpha i(\not{p}_l+\not{k}+m_l)\ell), \tag{A9}$$

$$I_3 = I_2. \tag{A10}$$

$\mathcal{O}(\Pi^0)$:

$$\mathcal{J}_3(0;a) = \frac{1}{m_b(p_b.k)(p_\ell.k)}16(-(p_b.p_\nu)(p_\ell.k)^2+p_\ell.k(m_b^2((p_b-2p_\ell+q).p_\nu)-2p_b.p_\nu(-2p_b.p_\ell+q.(p_b+p_\ell))) -m_b^2((q.p_\ell-p_b.p_\ell)(k.p_\nu+2p_\ell.p_\nu)+(k.q)(p_\ell.p_\nu))+k.p_b(p_\ell.p_\nu(2q.p_\ell+m_b^2)+2p_b.p_\nu((q-2p_b).p_\ell) +m_\ell^2k.p_\nu+k.p_\ell((p_\ell-2k).p_\nu))+2p_b.p_\ell(p_b.p_\nu(k.q+2q.p_\ell-2p_b.p_\ell)-(k.p_\nu)(q.p_\ell))). \tag{A11}$$

$\mathcal{O}(\Pi)$:

$$\mathcal{J}_3(1;a) = \frac{-1}{3m_b^3(p_b.k)^2(p_\ell.k)}32(\lambda_1+3\lambda_2)((p_b.k)^2(5p_\ell.k(2k.p_\nu-p_\ell.p_\nu)-5m_\ell^2(k.p_\nu)-10p_b.p_\nu((q-2p_b).p_\ell) -P_\ell.p_\nu(10q.p_\ell+m_b^2))+p_b.k(5(p_b.p_\nu)(p_\ell.k)^2+m_b^2(p_\ell.p_\nu(k.q+12q.p_\ell-6p_b.p_\ell)+6p_b.p_\nu(q.p_\ell -2p_b.p_\ell))+k.p_\ell(m_b^2((q-9p_\ell+6k-9p_b).p_\nu)+15(p_b.p_\ell)(k.p_\nu)+10p_b.p_\nu(q.(p_b+p_\ell)-2p_b.p_\ell)) +k.p_\nu(5q.p_\ell(2p_b.p_\ell+m_b^2)+3m_b^2(m_\ell^2-3p_b.p_\ell))-10(p_b.p_\ell)(p_b.p_\nu)(-2p_b.p_\ell+k.q+2q.p_\ell)+3m_b^4 (p_\ell.p_\nu))+m_b^2(-p_\nu.k(4(p_b.p_\ell)^2+p_b.p_\ell(5k.p_\ell-4k.q+2q.p_\ell)+4k.p_\ell(q.p_b+2k.p_\ell+2q.p_\ell)) +p_b.p_\nu((k.p_\ell)^2-2k.p_\ell(3q.p_b+q.p_\ell-4p_b.p_\ell)+6p_b.p_\ell(k.q+2q.p_\ell-2p_b.p_\ell))+m_b^2(k.p_\ell ((3p_b-2p_\ell+q).p_\nu+4k.p_\nu)-3(q.p_\ell-p_b.p_\ell)(k.p_\nu+2p_\ell.p_\nu)-3(k.q)(p_\ell.p_\nu)))), \tag{A12}$$

$$\mathcal{J}_3(1;c)=\frac{1}{3m_b^3(p_b.k)(p_\ell.k)}128(\lambda_1+3\lambda_2)(5(p_b.k)^2(k.p_\ell((2k+4q-4p_b-p_\ell).p_\nu)-m_\ell^2((k+2q-2p_b).p_\nu)$$
$$+2p_b.p_\ell((p_\ell-2(k+q-p_b)).p_\nu))+(5(p_b.p_\nu)(p_\ell.k)^2+(3m_b^2((p_\ell+4p_b-4q).p_\nu)+10p_b.p_\ell$$
$$((2q-3p_b).p_\nu)+5q.p_b((4q-4p_b-p_\ell).p_\nu)+k.p_\nu(-6m_b^2+10q.p_b+15p_b.p_\ell))p_\ell.k+(2(q-p_b).p_\nu$$
$$(6m_b^2-5q.p_b)+p_\nu.k(9m_b^2-5q.p_b))m_\ell^2+2p_b.p_\ell(3m_b^2((2q-2p_b-p_\ell).p_\nu)-10(q.p_b+p_b.p_\ell)$$
$$(q.p_\nu-p_b.p_\nu)+5(q.p_b)(p_\ell.p_\nu)+k.p_\nu(6m_b^2-10q.p_b-15p_b.p_\ell)))p_b.k+2m_b^2(k.q((k+2q-2p_b).p_\nu)$$
$$+3((k+q-p_b).p_\nu)(q.p_b-m_b^2))m_\ell^2+2p_b.p_\ell(((2q.p_\ell+3p_b.p_\ell)(3k.p_\nu+2q.p_\nu-2p_b.p_\nu)+k.q$$
$$((4(k+q-p_b)-2p_\ell).p_\nu))m_b^2-5(q.p_b)(p_b.p_\ell)(3k.p_\nu+2q.p_\nu-2p_b.p_\nu))+p_\ell.k((k.p_\nu(7p_b.p_\ell-6q.p_\ell)$$
$$-8(q.p_\ell)(q.p_\nu)+2(4q.p_\ell+5p_b.p_\ell)p_b.p_\nu+2k.q(4p_b.p_\nu+p_\ell.p_\nu-4q.p_\ell-2k.p_\nu))m_b^2+5(q.p_b)(p_b.p_\ell)$$
$$((3k+4q-6p_b).p_\nu))+(p_\ell.k)^2(5p_b.p_\nu(m_b^2+q.p_b)-2(4k.p_\nu+5q.p_\nu)m_b^2)). \qquad \text{(A13)}$$

$\mathcal{O}(\Pi^2)$:

$$\mathcal{J}_3(2;a)=\frac{1}{3m_b^3(p_b.k)^2(p_\ell.k)}16\lambda_1((p_b.k)^2(-m_\ell^2(p_\nu.k)+2((q-2p_b).p_\ell)(p_b.p_\nu)+k.p_\ell((2k-p_\ell).p_\nu)$$
$$-p_\ell.p_\nu(9m_b^2+2q.p_\ell+4p_b.p_\ell))+(3(p_\ell.p_\nu)m_b^4+(p_b.p_\nu)m_b^2(2m_\ell^2+6q.p_\ell-12p_b.p_\ell)-(k.q-8q.p_\ell$$
$$+2p_b.p_\ell)(p_\ell.p_\nu)m_b^2+((k.p_\ell)^2-4(p_b.p_\ell)^2+2(k.q)(p_b.p_\ell)+4(q.p_\ell)(p_b.p_\ell))(p_b.p_\nu)+p_\ell.k$$
$$(((q+2p_b+9p_\ell).p_\nu)m_b^2+2(q.p_\ell+4p_b.p_\ell-q.p_b)p_b.p_\nu+3(p_\nu.k)(p_b.p_\ell-2m_b^2))+p_\nu.k(3m_b^2m_\ell^2$$
$$+q.p_\ell(2p_b.p_\ell-m_b^2)+4p_b.p_\ell(2m_b^2+p_b.p_\ell)))p_b.k+m_b^2(-11(p_b.p_\nu)(p_\ell.k)^2$$
$$+(m_b^2((3q+3p_b-10p_\ell).p_\nu)-17(k.p_\nu)(p_b.p_\ell)-2p_b.p_\nu(3q.p_b+5q.p_\ell-12p_b.p_\ell))p_\ell.k$$
$$+3(k.q+2q.p_\ell-2p_b.p_\ell)(2(p_b.p_\ell)(p_b.p_\nu)$$
$$-(p_\ell.p_\nu)m_b^2)+k.p_\nu(p_b.p_\ell(3m_b^2+4p_b.p_\ell)-q.p_\ell(3m_b^2+10p_b.p_\ell)))), \qquad \text{(A14)}$$

$$\mathcal{J}_3(2;c)=\frac{-1}{3m_b^3(p_b.k)^2(p_\ell.k)}128(2((2q.p_\nu+k.p_\nu)m_l^2+k.p_\ell(-2k.p_\nu+p_\ell.p_\nu-4q.p_\nu)+p_b.p_\ell$$
$$(4p_b.p_\nu-2p_\ell.p_\nu))(k.p_b)^3+(-2(p_b.p_\nu)(k.p_\ell)^2+k.p_\ell((28q.p_\nu-20(p_b.p_\nu)-3(p_\ell.p_\nu))m_b^2$$
$$+(22m_b^2-4q.p_b-6p_b.p_\ell)k.p_\nu+4p_b.p_\ell(p_b.p_\nu-2q.p_\nu)+2q.p_b(p_\ell.p_\nu-4q.p_\nu))+m_l^2((-11k.p_\nu$$
$$+6p_b.p_\nu-14q.p_\nu)m_b^2+2q.p_b(2q.p_\nu+k.p_\nu))+4p_b.p_\ell((p_b.p_\ell)(2p_b.p_\nu+k.p_\nu)+q.p_b(2p_b.p_\nu-p_\ell.p_\nu))$$
$$+2m_b^2(2(2q.p_\ell-5p_b.p_\ell)k.p_\nu+4q.p_\ell(q.p_\nu-p_b.p_\nu)+p_b.p_\ell(-10q.p_\nu+6p_b.p_\nu+3p_\ell.p_\nu)))$$
$$(k.p_b)^2+2(k.q)m_b^2(p_b.p_\nu(k.p_\ell)^2+2m_b^2m_l^2(-p_b.p_\nu+.kp_\nu+.qp_\nu)-2(p_b.p_\ell)^2(3k.p_\nu+2q.p_\nu$$
$$-2p_b.p_\nu)+(k.p_\ell)(p_b.p_\ell)(3k.p_\nu+4q.p_\nu-6p_b.p_\nu))+k.p_b((3m_b^2-2q.p_b)(p_b.p_\nu)(k.p_\ell)^2$$
$$-2m_b^2((3p_b.p_\nu-5(k.p_\nu+q.p_\nu))m_b^2+2q.p_b(k.p_\nu+q.p_\nu)+k.q(2q.p_\nu+k.p_\nu-2p_b.p_\nu))m_l^2$$
$$+2p_b.p_\ell((2p_b.p_\nu(3p_b.p_\ell-2q.p_\ell)+(4q.p_\ell-13p_b.p_\ell)k.p_\nu+2(2q.p_\ell-5p_b.p_\ell)q.p_\nu+2(k.q)$$
$$(p_\ell.p_\nu-2(-p_b.p_\nu+.kp_\nu+.qp_\nu)))m_b^2+2(q.p_b)(p_b.p_\ell)(2p_b.p_\nu+k.p_\nu))+(k.p_\ell)$$
$$((p_b.p_\ell(25k.p_\nu+28q.p_\nu-26p_b.p_\nu)+(k.q)(4k.p_\nu+8q.p_\nu-8(p_b.p_\nu)-2(p_\ell.p_\nu)))m_b^2$$
$$-2(q.p_b)(p_b.p_\ell)(3k.p_\nu+4q.p_\nu-2p_b.p_\nu)))), \qquad \text{(A15)}$$

$$\mathcal{J}_3(2;d) = \frac{1}{3m_b^3(p_b.k)(p_\ell.k)}256\lambda_1((k.p_b + q.p_b)^2 - m_b^2(2k.q + q^2))(-2(p_b.p_\ell)^2(-2p_b.p_\nu + 3k.p_\nu + 2q.p_\nu) + (p_b.p_\nu)(k.p_\ell)^2 + 2m_b^2m_l^2(-p_b.p_\nu + k.p_\nu + q.p_\nu) + k.p_b(m_l^2(2p_b.p_\nu - k.p_\nu - 2q.p_\nu) + (k.p_\ell)(-4p_b.p_\nu + 2k.p_\nu - p_\ell.p_\nu + 4q.p_\nu) + 2(p_b.p_\ell)(p_\ell.p_\nu - 2(-p_b.p_\nu + k.p_\nu + q.p_\nu))) + (p_b.p_\ell)(k.p_\ell)(-6p_b.p_\nu + 3k.p_\nu + 4q.p_\nu)). \tag{A16}$$

(3) Figure 2(d):

$$\mathcal{M}_{\mu\nu}^{(4)} = \bar{b}\gamma^\nu(1-\gamma^5)i(\not{p}_b + \not{\Pi} - \not{q} - \not{k} + m_b)\gamma^\mu(1-\gamma^5)b,$$
$$\mathcal{L}_{\mu\nu}^{(4)} = (-ig^{\alpha\beta})(\overline{\ell}(-ieQ_l)\gamma_\alpha i(\not{p}_l + \not{k} + m_l)\gamma^\mu(1-\gamma^5)\nu_\ell)(\bar{\nu}_\ell\gamma^\nu(1-\gamma^5)i(\not{p}_l + \not{k} + m_l)(-ieQ_l)\gamma_\beta\ell), \tag{A17}$$

$$I_4 = \frac{1}{k^2((p_b - q - k)^2 - m_c^2)}\left[\frac{1}{(p_\ell.k)^2}\right] - \frac{1}{k^2((p_b - q - k)^2 - m_c^2)^2}\left[\frac{2p_c.\Pi}{(p_\ell.k)^2} + \frac{\Pi^2}{(p_\ell.k)^2}\right] + \frac{1}{k^2((p_b - q - k)^2 - m_c^2)^3}\left[\frac{2(p_c.\Pi)^2}{(p_\ell.k)^2}\right]. \tag{A18}$$

$\mathcal{O}(\Pi^0)$:

$$\mathcal{J}_4(0;a) = \frac{1}{m_b(p_\ell.k)^2}64p_b.p_\nu(k.p_\ell(k.q - k.p_b) - m_\ell^2(k.p_\ell + k.q + q.p_\ell - k.p_b - p_b.p_\ell)). \tag{A19}$$

$\mathcal{O}(\Pi)$:

$$\mathcal{J}_4(1;a) = \frac{-1}{3m_b^3(p_\ell.k)^2}64(\lambda_1 + 3\lambda_2)(m_\ell^2(2m_b^2(k.p_\nu + p_\ell.p_\nu) - 5p_b.p_\nu(k.p_b + p_b.p_\ell)) + k.p_\ell(5(k.p_b)(p_b.p_\nu) - 2m_b^2k.p_\nu)), \tag{A20}$$

$$\mathcal{J}_4(1;c) = \frac{1}{3m_b^3(p_\ell.k)^2}512(\lambda_1 + 3\lambda_2)(m_\ell^2(k.p_\ell + k.q + q.p_\ell - k.p_b - p_b.p_\ell) + k.p_\ell(k.p_b - k.q))(2m_b^2(k.p_\nu + q.p_\nu) + p_b.p_\nu(3m_b^2 - 5(k.p_b + q.p_b))). \tag{A21}$$

$\mathcal{O}(\Pi^2)$:

$$\mathcal{J}_4(2;a) = 0, \tag{A22}$$

$$\mathcal{J}_4(2;c) = \frac{-1}{3m_b^3(p_\ell.k)^2}256\lambda_1(4p_b.p_\nu(m_\ell^2 - k.p_\ell)(k.p_b)^2 + 2k.p_b(m_\ell^2(m_b^2(k.p_\nu + q.p_\nu) + p_b.p_\nu(2(p_b.p_\ell + q.p_b) + 3m_b^2)) - (p_b.p_\nu)(k.p_\ell)(2q.p_b + 3m_b^2) - m_b^2m_\ell^2(k.p_\nu + q.p_\nu)) + m_\ell^2(m_b^2(p_b.p_\ell(6p_b.p_\nu + k.p_\nu + q.p_\nu) - p_b.p_\nu(11(k.p_\ell + q.p_\ell) + 12k.q)) + 4(p_b.p_\ell)(p_b.p_\nu)(q.p_b)) + m_b^2(10(k.q)(p_b.p_\nu)(k.p_\ell) + m_\ell^2((p_b.p_\nu)(k.p_\ell + 2k.q + q.p_\ell) - (p_b.p_\ell)(k.p_\nu + q.p_\nu)))), \tag{A23}$$

$$\mathcal{J}_4(2;d) = \frac{1}{3m_b^3(p_\ell.k)^2}1024\lambda_1p_b.p_{nu}((k.p_b + q.p_b)^2 - m_b^2(2k.q + q^2))(m_\ell^2(-k.p_b - p_b.p_\ell + k.p_\ell + k.q + q.p_\ell) + k.p_\ell(k.p_b - k.q)). \tag{A24}$$

(4) Figure 2(e):

$$\mathcal{M}^{(5)}_{\mu\nu} = (-ig^{\alpha\beta})\bar{b}(-ieQ_b)\gamma^\alpha i(\not{p}_b + \not{\Pi} - \not{k} + m_b)\gamma^\nu(1-\gamma^5)i(\not{p}_b + \not{\Pi} - \not{q} - \not{k} + m_c)\gamma^\mu(1-\gamma^5)i(\not{p}_b + \not{\Pi} - \not{k} + m_b)(-ieQ_b)\gamma^\beta b,$$

$$\mathcal{L}^{(5)}_{\mu\nu} = (\bar{\ell}\gamma^\mu(1-\gamma^5)\nu_\ell)(\bar{\nu}_\ell\gamma^\nu(1-\gamma^5)\gamma_\alpha\ell), \tag{A25}$$

$$I_5 = \frac{1}{k^2((p_b-q-k)^2-m_c^2)}\left[\frac{1}{(p_b.k)^2} + \frac{2(p_b-k).\Pi}{(p_b.k)^3} + \frac{\Pi^2}{(p_b.k)^3} + \frac{((p_b-k).\Pi)^2}{(p_b.k)^4}\right] + \frac{1}{k^2((p_b-q-k)^2-m_c^2)^2}\left[\frac{-2p_c.\Pi}{(p_b.k)^2} - \frac{4(p_c.\Pi)(p_b-k).\Pi}{(p_b.k)^3} - \frac{\Pi^2}{(p_b.k)^2}\right] + \frac{1}{k^2((p_b-q-k)^2-m_c^2)^3}\frac{2(p_c.\Pi)^2}{(p_b.k)^2}. \tag{A26}$$

$\mathcal{O}(\Pi^0)$:

$$\mathcal{J}_5(0;a) = \frac{1}{m_b(p_b.k)^2}32(m_b^2(k.p_\nu(p_b.p_\ell - 2q.p_\ell) + k.p_\ell(p_b.p_\nu - 2k.p_\nu) - p_b.p_\ell(q.p_\nu) + p_b.p_\nu(q.p_\ell)) + 2k.p_b(k.p_\nu)((k+q-p_b).p_\ell)). \tag{A27}$$

$\mathcal{O}(\Pi)$:

$$\mathcal{J}_5(1;a) = \frac{1}{3m_b^3(p_b.k)^3}128(\lambda_1+3\lambda_2)(10k.p_\nu(k.p_b)^2((k+q-p_b).p_\ell) + m_b^4(k.p_\nu(8q.p_\ell - 5p_b.p_\ell) + k.p_\ell((2q+10k-5p_b).p_\nu) + 3p_b.p_\ell(q.p_\nu) - 3p_b.p_\nu(q.p_\ell)) + m_b^2 p_b.k(k.p_n(11p_b.p_\ell - 16q.p_\ell) + k.p_\ell(p_b.p_\nu - 16k.p_\nu) - 5p_b.p_\ell(q.p_\nu) + p_b.p_\nu(4p_b.p_\ell + q.p_\ell))), \tag{A28}$$

$$\mathcal{J}_5(1;c) = \frac{1}{3m_b^3(p_b.k)^2}256(\lambda_1+3\lambda_2)(m_b^2(4p_\ell.p_\nu(k.p_b)^2 + k.p_b(k.p_\nu(11p_b.p_\ell - 16q.p_\ell) + k.p_\ell(p_b.p_\nu - 4(4k.p_\nu + q.p_\nu)) - q.p_\nu(p_b.p_\ell + 4q.p_\ell) + p_b.p_\nu(4p_b.p_\ell + q.p_\ell)) - 4k.q(k.p_\nu - p_b.p_\nu)((k+q-p_b).p_\ell) + q.p_b(k.p_\nu(p_b.p_\ell - 6q.p_\ell) + k.p_\ell(p_b.p_\nu - 6k.p_\nu) - 5p_b.p_\ell(q.p_\nu) + p_b.p_\nu(4p_b.p_\ell + q.p_\ell))) + m_b^4(5k.p_\nu(2q.p_\ell - p_b.p_\ell) + k.p_\ell((10k+44q-5p_b).p_\nu) + q.p_\nu(p_b.p_\ell + 4q.p_\ell) - 5p_b.p\nu(q.p_\ell)) + 10k.p_b(k.p_\nu)(k.p_b + q.p_b)(k.p_\ell + q.p_\ell - p_b.p_\ell)). \tag{A29}$$

$\mathcal{O}(\Pi^2)$:

$$\mathcal{J}_5(2;a) = \frac{1}{3m_b^3(p_b.k)^3}64\lambda_1(2k.p_\nu(k.p_b)^2((p_b+k+q).p_\ell) + m_b^4(k.p_\nu(10q.p_\ell - 7p_b.p_\ell) + 7k.p_\ell(2k.p_\nu - p_b.p_\nu) - 2k.p_b(p_\ell.p_\nu) + 3p_b.p_\ell(q.p_\nu) - 3(p_b.p_\nu)(q.p_\ell)) + m_b^2 k.p_b(k.p_\nu(7p_b.p_\ell - 12q.p_\ell) - 8k.p_\ell(p_b.p_\nu + 2k.p_\nu) + 2k.p_b(p_\ell.p_\nu) - p_b.p_\ell(q.p_\nu - 13p_b.p_\nu) - 6p_b.p_\nu(q.p_\ell))), \tag{A30}$$

$$\mathcal{J}_5(2;c) = \frac{-1}{3m_b^3(p_b.k)^3}256\lambda_1(4(k.p_b + q.p_b)(k.p_b)^2(2k.p_\nu(k.p_\ell + q.p_\ell - p_b.p_\ell) - p_b.p_\nu(k.p_\ell)) + m_b^4(4k.q(p_b.p_\ell(q.p_\nu) - p_b.p_\nu(k.p_\ell + q.p_\ell) - p_b.p_\ell(k.p_\nu) + 2q.p_\ell(k.p_\nu)) + k.p_b(k.p_\nu(p_b.p_\ell + 2q.p_\ell) + k.p_\ell(p_b.p_\nu - 2k.p_\nu - 8q.p_\nu) + q.p_\nu(7p_b.p_\ell - 4q.p_\ell) + p_b.p_\nu(q.p_\ell))) + 2m_b^2 p_b.k(2q.p_b(p_b.p_\nu((q+2k-2p_b).p_\ell) - 2k.p_\nu((q+k-p_b).p_\ell)) + 2k.q(k.p_\nu(p_b.p_\ell - 2q.p_\ell) + p_b.p_\ell(p_b.p_\nu - q.p_\nu) - 2k.p_\ell(k.p_\nu)) + k.p_b(k.p_\nu(7p_b.p_\ell - 5q.p_\ell) + k.p_\ell(2p_b.p_\nu - 3k.p_\nu + 4q.p_\nu) - 4p_b.p_\ell(p_b.p_\nu) + 2q.p_\ell(q.p_\nu)))), \tag{A31}$$

$$\mathcal{J}_5(2;d) = \frac{-1}{3m_b^3(p_\ell.k)^2}512\lambda_1((pb.k + p_b.q)^2 - m_b^2(2k.q + q^2))(m_b^2(p_\nu.k(p_b.p_\ell - 2q.p_\ell) + k.p_\ell(p_b.p_\nu - 2k.p_\nu) - p_b.p_\ell(q.p_\nu) + (p_b.p_\nu)(q.p\ell)) + 2k.p_b(k.p_\nu)(k.p_\ell + q.p_\ell - p_b.p_\ell)). \tag{A32}$$

(5) Figure 2(f):

$$\mathcal{M}_{\mu\nu}^{(6)} = (-ig^{\alpha\beta})\bar{b}\gamma^\nu(1-\gamma^5)i(\not{p}_b + \not{\Pi} - \not{q} - \not{k} + m_c)(-ieQ_u)\gamma^\alpha i(\not{p}_b + \not{\Pi} - \not{q} + m_c)\gamma^\mu(1-\gamma^5)b,$$
$$\mathcal{L}_{\mu\nu}^{(6)} = (\bar{\ell}\gamma^\mu(1-\gamma^5)\nu_\ell)(\bar{\nu}_\ell\gamma^\nu(1-\gamma^5)i(\not{p}_l + \not{k} + m_l)(-ieQ_l)\gamma_\beta\ell), \tag{A33}$$

$$I_6 = I_7 = \frac{1}{k^2((p_b-q-k)^2 - m_c^2)}\left[\frac{1}{(p_c.k)(p_\ell.k)} - \frac{(p_b-q).\Pi}{(p_\ell.k)(p_c.k)^2} - \frac{\Pi^2}{2(p_\ell.k)(p_c.k)^2} + \frac{((p_b-q).\Pi)^2}{2(p_\ell.k)(p_c.k)^3}\right] + \frac{1}{k^2((p_b-q-k)^2 - m_c^2)^2}\left[\frac{-2p_c.\Pi}{(p_\ell.k)(p_c.k)} + \frac{2(p_c.\Pi)(p_b-q).\Pi}{(p_\ell.k)(p_c.k)^2} - \frac{\Pi^2}{(p_\ell.k)(p_c.k)}\right] + \frac{1}{k^2((p_b-q-k)^2 - m_c^2)^3}\frac{2(p_c.\Pi)^2}{(p_\ell.k)(p_c.k)}. \tag{A34}$$

$\mathcal{O}(\Pi^0)$:

$$\mathcal{J}_6(0;a) = \frac{1}{m_b(p_\ell.k)(p_c.k)}32p_b.p_\nu(2(q.p_\ell - p_b.p_\ell)(k.p_\ell + k.q + q.p_\ell - k.p_b - p_b.p_\ell) - m_\ell^2(k.q + m_b^2 + q^2 - k.p_b - 2q.p_b)). \tag{A35}$$

$\mathcal{O}(\Pi)$:

$$\mathcal{J}_6(1;a) = \frac{1}{3m_b^3(p_\ell.k)(k.p_c)^2}64(\lambda_1 + 3\lambda_2)(2m_b^2q.p_\nu + p_b.p_\nu(3m_b^2 - 5q.p_b))(2(q.p_\ell - p_b.p_\ell)(k.p_\ell + k.q + q.p_\ell - p_b.p_\ell - k.p_b) - m_\ell^2(k.q + q^2 + m_b^2 - k.p_b - 2q.p_b)), \tag{A36}$$

$$\mathcal{J}_6(1;c) = \frac{1}{3m_b(p_\ell.k)(p_c.k)}256(\lambda_1 + 3\lambda_2)(m_\ell^2(k.q + q^2 + m_b^2 - k.p_b - 2q.p_b) - 2(q.p_\ell - p_b.p_\ell)(k.p_\ell + k.q + q.p_\ell - k.p_b - p_b.p_\ell))(p_b.p_\nu(5(k.q + q.p_b) - 3m_b^2) - 2m_b^2(k.p_\nu + q.p_\nu)). \tag{A37}$$

$\mathcal{O}(\Pi^2)$:

$$\mathcal{J}_6(2;a) = \frac{1}{3m_b^3(p_\ell.k)(p_c.k)^3}32\lambda_1 p_b.p_\nu(2k.p_c(m_\ell^2(m_b^2(k.q + 2q^2) - 2(q.p_b)^2) + 2(k.p_\ell + 2q.p_\ell - 2p_b.p_\ell)((p_b.p_\ell)(q.p_b) - m_b^2(q.p_\ell)) + 2k.q(p_b.p_\ell(q.p_b + m_b^2) - 2m_b^2q.p_\ell) + k.p_b(2m_b^2q.p_\ell + q.p_b(2q.p_\ell - 4p_b.p_\ell - m_\ell^2))) - (2(q.p_\ell - p_b.p_\ell)(k.p_\ell + k.q + q.p_\ell - k.p_b - p_b.p_\ell) - m_\ell^2(k.q + q^2 + m_b^2 - k.p_b - 2q.p_b))((q.p_b)^2 + m_b^2(3k.p_c - q^2)) - 2(k.p_c)^2(m_b^2(m_\ell^2 - 2k.p_\ell) + 2p_b.p_\ell(k.p_b + p_b.p_\ell))), \tag{A38}$$

$$\mathcal{J}_6(2;c) = \frac{-1}{3m_b^3(p_\ell.k)(p_c.k)^2}128\lambda_1(m_b^2(k.p_c)(q.p_\nu)(2(p_b.p_\ell)^2 - m_b^2p_\ell.k + k.p_\ell(q.p_b) - 2(k.q)(p_b.p_\ell) - 3(k.p_\ell)(p_b.p_\ell) - 4(q.p_\ell)(p_b.p_\ell)) + m_b^2(q.p_\ell)^2(p_b.p_\nu)(8k.q - 24k.p_c + 8q^2) + m_b^2(p_b.p_\ell)^2(p_b.p_\nu)(8k.q - 12k.p_c + 8q^2) + m_b^2p_b.p_\nu((p_c.k)(p_\ell.k)(q.p_b - 6p_\ell.k - 8k.q - q^2) +)), \tag{A39}$$

$$\mathcal{J}_6(2;d) = -\frac{1}{3(k.p_\ell)(k.p_c)^2}128\lambda_1((q.p_\nu)(k.p_c)(-(k.p_\ell)m_b^4 - 4(q.p_\ell)(p_b.p_\ell)m_b^2 - 2(k.q)(p_b.p_\ell)m_b^2 \\ -3(k.p_\ell)(p_b.p_\ell)m_b^2) + (k.p_c)(p_b.p_\nu)(-24(q.p_\ell)^2m_b^2 - 12(p_b.p_\ell)^2m_b^2) + q^2(p_b.p_\nu)m_b^2(8(q.p_\ell)^2 \\ +8(p_b.p_\ell)^2 - (k.p_\ell)(k.p_c)) + 8(k.q)(p_b.p_\nu)m_b^2((q.p_\ell)^2 + (p_b.p_\ell)^2 + (k.q+q^2)q.p_\ell) + (q.p_\ell)(p_b.p_\nu) \\ m_b^2(8(q^2+k.q)(k.p_\ell) - 6(4k.q+5k.p_\ell)k.p_c) - 8(p_b.p_\nu)m_b^2(p_b.p_\ell(k.q+k.p_\ell+2q.p_\ell)(q^2+k.q) \\ -(6k.p_\ell+8k.q)(k.p_\ell)(k.p_c)) + (k.p_c)m_b^2(q.p_\nu(2(p_b.p_\ell)^2 + (q.p_b)(k.p_\ell)) + (q.p_b)(k.p_\ell)(p_b.p_\nu) \\ +(19k.q+26k.p_\ell+38q.p_\ell+2q^2+3k.q)(p_b.p_\ell)(p_\ell.p_\nu)) + (4(((q^2+k.q+p_b.q)m_b^2 + q.p_b(q^2+k.q \\ -3q.p_b))m_l^2 - 2(q.p_\ell - p_b.p_\ell)((q^2+k.q)m_b^2 + q.p_b(k.q - p_b.q + p_\ell.q - p_b.p_\ell + k.p_\ell)))(p_b.p_\nu) \\ +(-(p_\ell.p_\nu)(q.p_b)^2 - m_b^2((k.p_\ell+2q.p_\ell)(q.p_\nu) + 6(2(p_b.p_\ell)+m_l^2)(p_b.p_\nu)) + ((p_b.p_\ell)^2 \\ -(2q.p_\ell + k.p_\ell - 2p_b.p_\ell)m_b^2)(k.p_\nu) + 2(p_b.p_\ell + m_b^2)(p_b.p_\ell)(q.p_\nu) + (p_b.p_\nu)(-16(p_b.p_\ell)^2 + 8m_b^2(k.p_\ell) \\ +19m_b^2(q.p_\ell) + 7(p_b.p_\ell)(k.q+k.p_\ell+2q.p_\ell)) + (q^2+2k.q)m_b^2(p_\ell.p_\nu) + (q.p_b)(-9(p_b.p_\nu)m_l^2 \\ +(7q.p_\ell + .kp_\ell)(p_b.p_\nu) + (p_b.p_\ell)(2q.p_\nu - 16(p_b.p_\nu) - 3(p_\ell.p_\nu)))))(k.p_c))(k.p_b) - 8((q.p_b)^2(q.p_\ell)^2 \\ +(q.p_b + 2k.p_c)(q.p_b)(p_b.p_\ell)^2)(p_b.p_\nu) + (-4(k.q)^2m_b^2 - 4(q^2+m_b^2-2(q.p_b))(q^2m_b^2 - (q.p_b)^2) \\ +(-4m_b^4 + (-8q^2 + 8.p_bq + 15k.p_c)m_b^2 + 4(q.p_b)^2)(k.q) + 6(m_b^4 + 2(q^2 - q.p_b)m_b^2 - (q.p_b)^2) \\ (k.p_c))m_l^2(p_b.p_\nu) - 8(q.p_b)^2(q.p_\ell)(p_b.p_\nu)(k.q+k.p_\ell) + 8(q.p_b)^2(p_b.p_\ell)(p_b.p_\nu)(k.q+k.p_\ell+2q.p_\ell) \\ -(k.p_c)(p_\ell.p_\nu)(2(q.p_b)^2(p_b.p_\ell) + (k.p_b)^3) + (k.p_b)^2(4(q.p_b)(-m_l^2 + 2q.p_\ell - 2(p_b.p_\ell))(p_b.p_\nu) \\ +(p_b.p_\ell)(k.p_\nu)(k.p_c) + (-3(p_b.p_\nu)m_l^2 + (7q.p_\ell + k.p_\ell)(p_b.p_\nu) - 2(q.p_b)(p_\ell.p_\nu) + (p_b.p_\ell) \\ (2q.p_\nu - p_\ell.p_\nu - 16p_b.p_\nu))k.p_c) + (-(k.p_\ell)m_b^4 + p_b.p_\ell(q^2 - 2k.p_\ell + 2p_b.p_\ell - 3q.p_\ell)m_b^2 \\ +((p_b.p_\ell)^2 + m_b^2k.p_\ell)q.p_b - (q.p_b)^2(p_b.p_\ell))(k.p_\nu)(k.p_c) + (2(p_b.p_\ell)(q.p_\nu) + 7(k.q+k.p_\ell+2q.p_\ell) \\ p_b.p_\nu)(q.p_b)(p_b.p_\ell)(k.p_c)). \tag{A40}$$

(6) Figure 2(g):

$$\mathcal{M}_{\mu\nu}^{(7)} = (-ig^{\alpha\beta})\bar{b}\gamma^\nu(1-\gamma^5)i(\not{p}_b + \not{\Pi} - \not{q} + m_c)(-ieQ_u)\gamma^\alpha i(\not{p}_b + \not{\Pi} - \not{q} - \not{k} + m_c)\gamma^\mu(1-\gamma^5)b, \tag{A41}$$

$$\mathcal{L}_{\mu\nu}^{(7)} = (\bar{\ell}(-ieQ_l)\gamma_\beta i(\not{p}_l + \not{k} + m_l)\gamma^\mu(1-\gamma^5)\nu_\ell)(\bar{\nu}_\ell\gamma^\nu(1-\gamma^5)\ell), \\ I_7 = I_6. \tag{A42}$$

$$\mathcal{O}(\Pi^0):\ \mathcal{J}_7(0;a) = \mathcal{J}_6(0;a), \tag{A43}$$

$$\mathcal{O}(\Pi):\ \mathcal{J}_7(1;a) = \mathcal{J}_6(1;a), \qquad \mathcal{J}_7(1;c) = \mathcal{J}_6(1;c), \tag{A44}$$

$$\mathcal{O}(\Pi^2): \mathcal{J}_7(2;a) = \mathcal{J}_6(2;a), \qquad \mathcal{J}_7(2;c) = \mathcal{J}_6(2;c), \qquad \mathcal{J}_7(2;d) = \mathcal{J}_6(2;d). \tag{A45}$$

(7) Figure 2(h):

$$\mathcal{M}_{\mu\nu}^{(8)} = (-ig^{\alpha\beta})\bar{b}\gamma^\nu(1-\gamma^5)i(\not{p}_b + \not{\Pi} - \not{q} + m_c)(-ieQ_u)\gamma^\alpha i(\not{p}_b + \not{\Pi} - \not{q} - \not{k} + m_c)\gamma^\mu(1-\gamma^5)i(\not{p}_b \\ + \not{\Pi} - \not{k} + m_b)(-ieQ_b)\gamma^\alpha b, \\ \mathcal{L}_{\mu\nu}^{(8)} = (\bar{\ell}\gamma^\mu(1-\gamma^5)\nu_\ell)(\bar{\nu}_\ell\gamma^\nu(1-\gamma^5)\ell), \tag{A46}$$

$$I_8 = \frac{1}{k^2((p_b-q-k)^2-m_c^2)}\left[\frac{-1}{(p_c.k)(p_b.k)}+\frac{(p_b-q).\Pi}{(p_b.k)(p_c.k)^2}-\frac{(p_b-k).\Pi}{(p_b.k)^2(p_c.k)}+\frac{\Pi^2}{2(p_b.k)(p_c.k)^2}\right.$$
$$\left.-\frac{\Pi^2}{2(p_b.k)^2(p_c.k)}-\frac{((p_b-q).\Pi)^2}{2(p_b.k)(p_c.k)^3}-\frac{((p_b-k).\Pi)^2}{2(p_b.k)^3(p_c.k)}+\frac{((p_b-q).\Pi)((p_b-k).\Pi)}{(p_c.k)^2(p_b.k)^2}\right]$$
$$+\frac{1}{k^2((p_b-q-k)^2-m_c^2)^2}\left[\frac{2p_c.\Pi}{(p_b.k)(p_c.k)}-\frac{2(p_c.\Pi)(p_b-q).\Pi}{(p_b.k)(p_c.k)^2}+\frac{2(p_c.\Pi)(p_b-k).\Pi}{(p_b.k)^2(p_c.k)}+\frac{\Pi^2}{(p_b.k)(p_c.k)}\right]$$
$$+\frac{1}{k^2((p_b-q-k)^2-m_c^2)^3}\frac{-2(p_c.\Pi)^2}{(p_b.k)(p_c.k)}. \tag{A47}$$

$\mathcal{O}(\Pi^0)$:

$$\mathcal{J}_8(0;a) = \frac{1}{m_b(k.p_b)(k.p_c)}16(-2(p_\ell.p_\nu)(k.p_b)^2+m_b^2(-(p_\ell.p_\nu)(-2(k.p_b)-2(q.p_b)+k.q+q^2)$$
$$-2(p_b.p_\ell)(q.p_\nu)+2(q.p_\ell)(p_b.p_\nu+q.p_\nu))+(k.p_\nu)((q.p_\ell)(2(q.p_b)-m_b^2)-(p_b.p_\ell)(-2k.p_b$$
$$+2k.q+q^2))+(k.p_\ell)(2(k.p_\nu)(q.p_b-m_b^2)-2(k.p_b)(q.p_\nu)+(p_b.p_\nu)(2(k.p_b)-q^2)+m_b^2(q.p_\nu))$$
$$-2(k.p_b)(p_b.p_\nu)(q.p_\ell)-2(k.p_b)(q.p_\ell)(q.p_\nu)+2(k.p_b)(p_b.p_\ell)(q.p_\nu)+2(k.q)(p_b.p_\nu)(q.p_\ell)$$
$$+q^2(k.p_b)(p_\ell.p_\nu)+2(k.q)(k.p_b)(p_\ell.p_\nu)-2(k.p_b)(q.p_b)(p_\ell.p_\nu)+m_b^4(-(p_\ell.p_\nu))-4(p_b.p_\nu)(q.p_b)$$
$$(q.p_\ell)+2q^2(p_b.p_\ell)(p_b.p_\nu)). \tag{A48}$$

$\mathcal{O}(\Pi)$:

$$\mathcal{J}_8(1;a)=\frac{1}{3m_b^3(k.p_b)^2(k.p_c)^2}16(\lambda_1+3\lambda_2)(2p_\ell.p_\nu(k.p_b)^3(5q.p_b-3m_b^2)+(p_b.k)^2(6m_b^4(p_\ell.p_\nu)+m_b^2(-2q.p_\nu$$
$$(3k.p_\ell+7q.p_\ell-5p_b.p_\ell)+k.p_\nu(4q.p_\ell+6p_b.p_\ell)+6p_b.p_\nu(k.p_\ell-q.p_\ell)+(2k.q+8k.p_c+5q^2-16q.p_b)$$
$$p_\ell.p_\nu)+10(-(k.p_c-q.p_b)(k.p_\ell+q.p_\ell)q.p_\nu-q.p_b(q.p_\ell-k.p_\ell)(p_b.p_\nu))+5(2(q.p_b)^2-(2k.q+q^2)q.p_b$$
$$+k.p_c(2k.q+q^2))p_\ell.p_\nu)+p_b.k(-3(p_\ell.p_\nu)m_b^6+m_b^4(-8(q.p_\nu)(p_b.p_\ell)+2q.p_\ell(5q.p_\nu+4p_b.p_\nu)$$
$$+p_\ell.p_\nu(k.q-6k.p_c-3q^2+9q.p_b))+m_b^2(2q.p_\nu(5(k.p_c-q.p_b)q.p_\ell+p_b.p_\ell(2k.q-4k.p_c+q^2+3q.p_b))$$
$$+2p_b.p_\nu(q.p_\ell(5k.q+2k.p_c-q^2-13q.p_b)+(5q^2-2k.q)p_b.p_\ell)+p_\ell.p_\nu(4(k.q)^2+k.q(2q^2-15k.p_c$$
$$-3q.p_b)+3q.p_b(q^2-2q.p_b)+k.p_c(8q.p_b-5q^2)))+10p_b.p_\nu(q.p_\ell(2(q.p_b)^2+k.q(k.p_c-q.p_b))-q^2$$
$$(q.p_b)(p_b.p_\ell))+k.p_\nu(-7(q.p_\ell)m_b^4-m_b^2(q.p_\ell(4k.q+k.p_c-2q^2-15q.p_b)+p_b.p_\ell(6k.q+2k.p_c+9q^2$$
$$-4q.p_b))+5(k.p_c-q.p_b)(2(q.p_b)(q.p_\ell)-(2k.q+q^2)p_b.p_\ell))+k.p_\ell(3m_b^4(q.p_\nu-2k.p_\nu)-(4k.p_\nu$$
$$(2k.p_c-q^2-3q.p_b)+q.p_\nu(4k.q-15k.p_c+2q^2+q.p_b)+p_b.p_\nu(5(2k.p_c+q^2)-4k.q))m_b^2+5(k.p_c$$
$$-q.p_b)(2k.p_\nu(q.p_b)-q^2(p_b.p_\nu)))))+(k.p_c)m_b^2(3(p_b.p_\ell)m_b^4-m_b^2(k.p_\nu(q.p_\ell-2p_b.p_\ell)-6(q.p_\nu)(p_b.p_\ell)$$
$$+6q.p_\ell(q.p_\nu+p_b.p_\nu)-3(q^2-2q.p_b)p_\ell.p_\nu)-(2(q.p_b)(q.p_\ell)-q^2(p_b.p_\ell))(k.p_\nu-6p_b.p_\nu)+k.q(3m_b^2$$
$$(p_\ell.p_\nu)+k.p_\nu(14p_b.p_\ell-8q.p_\ell)-2p_b.p_\nu(q.p_\ell+2p_b.p_\ell))+k.p_\ell(-3m_b^2(q.p_\nu)+p_b.p_\nu(-2m_b^2+q^2$$
$$+4q.p_b)+2k.p_\nu(5m_b^2+2q^2-7q.p_b)))), \tag{A49}$$

$$\begin{aligned}\mathcal{J}_8(1;c)=&\frac{1}{3(k.p_b)(k.p_c)m_b}128(-3(p_\ell.p_\nu)m_b^6+((p_\ell.p_\nu)(-3q^2+k.q+9p_b.q)+(3q.p_\ell+2p_b.p_\ell)(k.p_\nu)\\&+2(5q.p_\ell+4p_b.p_\ell)(q.p_\nu)-8(q.p_\ell)(p_b.p_\nu)-(k.p_\ell)(10k.p_\nu+3q.p_\nu+2p_b.p_\nu))m_b^4+(-2(k.p_\nu)\\&(q.p_\ell)q^2-3(k.p_\nu)(p_b.p_\ell)q^2-2(q.p_\nu)(p_b.p_\ell)q^2+2(q.p_\ell)(p_b.p_\nu)q^2+10(p_b.p_\ell)(p_b.p_\nu)q^2-(k.p_\nu)\\&(q.p_b)(q.p_\ell)-10(q.p_b)(q.p_\nu)(q.p_\ell)-4(k.p_\nu)(q.p_b)(p_b.p_\ell)-26(q.p_b)(q.p_\nu)(p_b.p_\ell)-4(k.q)(q.p_\ell)\\&(k.p_\nu)+4(k.q)(p_b.p_\ell)(k.p_\nu)+6(k.q)(p_b.p_\ell)(q.p_\nu)+4(k.q)(q.p_\ell)(p_b.p_\nu)+6(q.p_b)(q.p_\ell)(p_b.p_\nu)\\&+(-6(q.p_b)^2+2(k.q)(q^2+2k.q)+3(q^2-k.q)(q.p_b))(p_\ell.p_\nu)+(k.p_\ell)((q.p_b)(20k.p_\nu+11q.p_\nu\\&+4p_b.p_\nu)+(q^2+2k.q)(2(q.p_\nu)-7(p_b.p_\nu))))m_b^2+(9(p_\ell.p_\nu)m_b^4+((p_\ell.p_\nu)(8q^2+7k.q-22(q.p_b))\\&-4(5q.p_\ell+4p_b.p_\ell)(q.p_\nu)+16(q.p_\ell)(p_b.p_\nu))m_b^2+(-11(q.p_\ell)m_b^2+(5q^2+6m_b^2)(p_b.p_\ell)+10(q.p_b)\\&(q.p_\ell-p_b.p_\ell))(k.p_\nu)+10((q.p_b)(q.p_\ell)-(k.q-3(q.p_b))(p_b.p_\ell))(q.p_\nu)-10((p_b.p_\ell)q^2+(q.p_b)\\&(q.p_\ell))(p_b.p_\nu)-5(q^2+2k.q-2(q.p_b))(q.p_b)(p_\ell.p_\nu)+(k.p_\ell)((10k.p_\nu+14p_b.p_\nu+q.p_\nu)m_b^2+5\\&(q^2+2k.q)(p_b.p_\nu)-10(q.p_b)(.kp_\nu+.qp_\nu+.p_bp_\nu)))(k.p_b)+10(k.p_b)^3(p_\ell.p_\nu)+(k.p_b)^2(-16\\&(p_\ell.p_\nu)m_b^2-5(q^2+2k.q-4(q.p_b))(p_\ell.p_\nu)+10(-(p_b.p_\nu)(k.p_\ell+q.p_\ell)+(q.p_\ell-p_b.p_\ell)(k.p_\nu)\\&+(q.p_\ell+p_b.p_\ell)(q.p_\nu)))+5(q.p_b)((p_b.p_\ell)(q^2(k.p_\nu-2(p_b.p_\nu))-2(k.q-2(q.p_b))(q.p_\nu))+(k.p_\ell)\\&((q^2+2k.q)(p_b.p_\nu)-2(q.p_b)(k.p_\nu+q.p_\nu)))))(\lambda_1+3\lambda_2).\end{aligned} \tag{A50}$$

$\mathcal{O}(\Pi^2)$:

$$\begin{aligned}\mathcal{J}_8(2;a)=&\frac{1}{3(k.p_b)^2(k.p_c)^3m_b^3}16(2(k.p_c)^2(k.p_b)(2(3q.p_\ell+k.p_\ell)(p_b.p_\nu)+(-3m_b^2+4k.p_b+2q.p_b)(p_\ell.p_\nu)\\&-2(p_b.p_\ell)(3k.p_\nu+q.p_\nu))m_b^2+((k.p_c)(2(k.q)-3(k.p_c))m_b^2+((3(k.p_c)-q^2)m_b^2+(k.p_c-q.p_b)^2)\\&(k.p_b))((p_\ell.p_\nu)m_b^4+((p_\ell.p_\nu)(q^2+k.q-2(k.p_b)-2(q.p_b))+2(p_b.p_\ell)(q.p_\nu)-2(q.p_\ell)\\&(q.p_\nu+p_b.p_\nu))m_b^2+(-(q.p_\nu)m_b^2+2(m_b^2-q.p_b)(k.p_\nu)+2(k.p_b)(q.p_\nu)+(q^2-2(k.p_b))(p_b.p_\nu))\\&(k.p_\ell)-2(k.p_b)(q.p_\nu)(p_b.p_\ell)+((p_b.p_\ell)(q^2+2k.q-2(k.p_b))+(m_b^2-2(q.p_b))(q.p_\ell))(k.p_\nu)\\&+2(k.p_b)(q.p_\ell)(q.p_\nu)-2(k.q)(q.p_\ell)(p_b.p_\nu)+2(k.p_b)(q.p_\ell)(p_b.p_\nu)+4(q.p_b)(q.p_\ell)(p_b.p_\nu)\\&-2q^2(p_b.p_\ell)(p_b.p_\nu)+2(((k.p_b)))^2(p_\ell.p_\nu)-(k.p_b)q^2(p_\ell.p_\nu)-2(k.q)(k.p_b)(p_\ell.p_\nu)+2(k.p_b)\\&(q.p_b)(p_\ell.p_\nu))-2(k.p_c)(((k.p_\ell)(k.p_c)(2k.p_\nu-p_b.p_\nu-2q.p_\nu)+(k.p_b)(-2(p_\ell.p_\nu)q^2\\&-(q.p_\nu)(p_b.p_\ell)-2(q.p_\ell)(k.p_\nu)+(q.p_\ell)(2q.p_\nu+p_b.p_\nu))+((p_b.p_\ell)(k.p_\nu)+2(k.q)(p_\ell.p_\nu))(k.p_c))m_b^4\\&+(2((q^2+k.q)(p_\ell.p_\nu)-(q.p_\nu)(-p_b.p_\ell+k.p_\ell+q.p_\ell))(k.p_b)^2+(k.p_b)(-(p_b.p_\ell)(q.p_\nu)q^2+(q.p_\ell)\\&(p_b.p_\nu)q^2+2(p_b.p_\ell)(p_b.p_\nu)q^2+2((q.p_b)(q.p_\ell+p_b.p_\ell)-(q^2+k.q+2k.p_c)(p_b.p_\ell))(k.p_\nu)\\&+2(q.p_b)(q.p_\ell)(q.p_\nu)-2(q.p_b)(p_b.p_\ell)(q.p_\nu)+2(k.q)(q.p_\ell)(p_b.p_\nu)-2(q.p_b)(q.p_\ell)(p_b.p_\nu)-2(k.q)\\&(p_b.p_\ell)(p_b.p_\nu)-2((q.p_b)(k.p_c-q.p_b)+(k.q)(k.p_c))(p_\ell.p_\nu)+2(p_b.p_\ell)(q.p_\nu)(k.p_c)+2(k.p_\ell)\\&(k.p_\nu+q.p_\nu)(k.p_c))+((p_b.p_\nu)((k.p_\ell)(q^2+2p_b.q)-2(k.q)(q.p_\ell+p_b.p_\ell))+((q^2+2k.q)(p_b.p_\ell)\\&-2(q.p_b)(k.p_\ell+q.p_\ell))(k.p_\nu))(k.p_c))m_b^2+2(k.p_b)(k.p_c-q.p_b)((p_\ell.p_\nu)(((k.p_b)))^2-((k.p_\ell-q.p_\ell)\\&(p_b.p_\nu)-(q.p_b)(p_\ell.p_\nu)+(p_b.p_\ell)(.kp_\nu+.qp_\nu))(k.p_b)+(2(q.p_b)(q.p_\ell)-q^2(p_b.p_\ell))(p_b.p_\nu))))\lambda_1,\end{aligned} \tag{A51}$$

$$\begin{aligned}\mathcal{J}_8(2;c) = &-\frac{1}{3(k.p_b)^2(k.p_c)^2}128(4(q.p_b)(p_\ell.p_\nu)(k.p_b)^4+2(-2(q^2+k.q+p_b.q)(p_\ell.p_\nu)m_b^2-2(k.p_\nu)((q.p_b)\\
&(p_b.p_\ell-q.p_\ell)+(q.p_\ell)(k.p_c))+(q.p_b)(-(p_\ell.p_\nu)(q^2+2k.q-4(q.p_b))+2(q.p_\ell+p_b.p_\ell)(q.p_\nu)\\
&-2(k.p_\ell+q.p_\ell)(p_b.p_\nu))+((p_\ell.p_\nu)(q^2+2k.q+5m_b^2)-2(q.p_\ell)(q.p_\nu))(k.p_c))(k.p_b)^3+(2(2(q^2\\
&+k.q-2(k.p_c))+q.p_b)(p_\ell.p_\nu)m_b^4+((p_\ell.p_\nu)(4(k.q)^2-8(q.p_b)^2-2(-3q^2+p_b.q+6k.p_c)(k.q)\\
&-2(q^2-5(k.p_c))(q.p_b)+q^2(2q^2-9(k.p_c)))+2((q.p_\ell)(p_b.p_\nu)(2k.q+2(q^2+p_b.q)-5(k.p_c))\\
&+(p_b.p_\ell)((k.p_\nu)(2q^2+2k.q-3(k.p_c))-(2k.q+2(q^2+p_b.q)-5(k.p_c))(q.p_\nu))+(q.p_\ell)(-(q.p_\nu)\\
&(2k.q+2(q^2+p_b.q)-7(k.p_c))+(2(q^2+k.q-4(k.p_c))+q.p_b)(-k.p_\nu))))m_b^2+2((2(p_b.p_\nu)(q^2\\
&+k.q)+(q.p_b)(q.p_\nu))m_b^2+2(q.p_b)(m_b^2-q.p_b+k.p_c)(k.p_\nu)+(q.p_b)((q^2+2k.q)(p_b.p_\nu)-2(q.p_b)\\
&(q.p_\nu+p_b.p_\nu))+(-(q.p_\nu)m_b^2-(p_b.p_\nu)(q^2+2k.q+7m_b^2)+2(q.p_b)(q.p_\nu))(k.p_c))(k.p_\ell)\\
&+2(((p_b.p_\ell)((q^2-2(q.p_b))(q.p_b)-(k.p_c)q^2)+2(q.p_b)(q.p_\ell)(q.p_b-k.p_c))(k.p_\nu)+2((p_b.p_\ell)\\
&(3(q.p_b)^2+(k.q)(k.p_c-q.p_b))+(q.p_b)(q.p_\ell)(q.p_b-k.p_c))(q.p_\nu)-2(q.p_b)((p_b.p_\ell)q^2+(q.p_b)\\
&(q.p_\ell))(p_b.p_\nu)+(2(q.p_b)^2-(q^2+2k.q)(q.p_b)+(q^2+2k.q)(k.p_c))(q.p_b)(p_\ell.p_\nu)))(k.p_b)^2\\
&+((-2q^2-2k.q+3k.p_c)(p_\ell.p_\nu)m_b^6+(-2(p_\ell.p_\nu)(k.q)^2+((2q^2-7(k.p_c))(q.p_\ell)-2(k.p_c)(p_b.p_\ell))\\
&(k.p_\nu)+2(2(p_b.p_\ell)(q^2-2(k.p_c))+(q.p_\ell)(2q^2-5(k.p_c)))(q.p_\nu)+4(2(k.p_c)-q^2)(q.p_\ell)(p_b.p_\nu)\\
&+(2(q.p_b)^2+q^2(7k.p_c-2q^2)+4(q.p_b)(q^2-2k.p_c))(p_\ell.p_\nu)+(k.q)((p_\ell.p_\nu)(-4q^2+4p_b.q+3k.p_c)\\
&-4(p_b.p_\nu)(q.p_\ell)+2(q.p_\ell)(k.p_\nu)+4(q.p_\ell+p_b.p_\ell)(q.p_\nu)))m_b^4+(4((q.p_\nu)(p_b.p_\ell)-(k.p_c)(p_\ell.p_\nu))\\
&(k.q)^2+2(-2(p_\ell.p_\nu)(q.p_b)^3+((p_\ell.p_\nu)q^2-2(q.p_\ell+p_b.p_\ell)(q.p_\nu)+2(q.p_\ell)(p_b.p_\nu))(q.p_b)^2+q^2\\
&(((q.p_\ell)(k.p_c)+(p_b.p_\ell)(2q^2-5(k.p_c)))(p_b.p_\nu)-(k.p_c)(q.p_\nu)(p_b.p_\ell))+(q.p_b)(2(5(k.p_c)-2q^2)\\
&(q.p_\nu)(p_b.p_\ell)-(k.p_c)q^2(p_\ell.p_\nu)))+((p_b.p_\ell)((k.p_c)(5q^2+4.p_bq)-2q^4)+2(q.p_b)(k.p_c-q.p_b)\\
&(q.p_\ell))(k.p_\nu)+2(k.q)((p_\ell.p_\nu)(((q.p_b)))^2+(2(k.p_c)(q.p_\ell)-(q^2+2k.p_c)(p_b.p_\ell))(k.p_\nu)\\
&+2(p_b.p_\ell)((p_b.p_\nu)q^2+(q^2-2(q.p_b))(q.p_\nu))+((p_\ell.p_\nu)(q.p_b-q^2)-5(p_b.p_\ell)(q.p_\nu)+2(q.p_\ell)(q.p_\nu\\
&-p_b.p_\nu))(k.p_c)))m_b^2+(2(q.p_b-m_b^2)(-2(((q.p_b)))^2+2(q^2+k.q)m_b^2+(2(q.p_b)-5m_b^2)(k.p_c))\\
&(k.p_\nu)-2((q.p_b)^2-(q^2+k.q)m_b^2)((q.p_\nu)(2(q.p_b)-m_b^2)-(q^2+2k.q)(p_b.p_\nu))+((7q.p_\nu+2p_b.p_\nu)\\
&m_b^4+(9q^2+10k.q)(p_b.p_\nu)m_b^2+2(-8(q.p_\nu)m_b^2-(p_b.p_\nu)(q^2+2k.q+2m_b^2))(q.p_b)+4(q.p_b)^2(q.p_\nu))\\
&(k.p_c))(k.p_\ell)+2(q.p_b)(p_b.p_\ell)(-2(q.p_b)(p_b.p_\nu)q^2+(k.p_\nu)(q.p_b-k.p_c)q^2+2(2(q.p_b)^2+(k.q)\\
&(k.p_c-q.p_b))(q.p_\nu)))(k.p_b)+2(k.q)m_b^2((p_\ell.p_\nu)m_b^4+((p_\ell.p_\nu)(q^2+k.q-2(q.p_b))-2(q.p_\nu)(q.p_\ell\\
&+p_b.p_\ell)+2(q.p_\ell)(p_b.p_\nu)+(q.p_\ell)(-k.p_\nu))m_b^2+(k.p_\ell)((2k.p_\nu+q.p_\nu)m_b^2+(q^2+2k.q)(p_b.p_\nu)\\
&-2(q.p_b)(k.p_\nu+q.p_\nu))+(p_b.p_\ell)(q^2(k.p_\nu-2(p_b.p_\nu))-2(k.q-2(q.p_b))(q.p_\nu)))(k.p_c))\lambda_1,\end{aligned} \tag{A52}$$

$$\begin{aligned}\mathcal{J}_8(2;d) = &\frac{1}{3m_b^3(k.p_b)(k.p_c)}256\lambda_1((k.p_b+q.p_b)^2-m_b^2(2k.q+q^2))(2(p_\ell.p_\nu)(k.p_b)^2+m_b^2((p_\ell.p_\nu)\\
&(-2(q.p_b)+k.q+q^2)-2(q.p_\nu)(p_b.p_\ell+q.p_\ell)+2(p_b.p_\nu)(q.p_\ell)-(k.p_\nu)(q.p_\ell))+(k.p_\ell)\\
&(m_b^2(2k.p_\nu+q.p_\nu)+(p_b.p_\nu)(-2(k.p_b)+2k.q+q^2)-2(q.p_b)(k.p_\nu+q.p_\nu))+(k.p_b)\\
&(-(p_\ell.p_\nu)(-2(q.p_b)+2k.q+q^2+2m_b^2)+2(k.p_\nu)(q.p_\ell-p_b.p_\ell)-2(p_b.p_\nu)(q.p_\ell)+2(q.p_\nu)\\
&(p_b.p_\ell+q.p_\ell))+(p_b.p_\ell)(q^2(k.p_\nu-2(p_b.p_\nu))-2(q.p_\nu)(k.q-2(q.p_b)))+m_b^4(p_\ell.p_\nu)).\end{aligned} \tag{A53}$$

(8) Figure 2(i):

$$\mathcal{M}^{(9)}_{\mu\nu} = (-ig^{\alpha\beta})\bar{b}(-ieQ_b)\gamma^\alpha i(\not{p}+\not{\Pi}-\not{k}+m_b)\gamma^\nu(1-\gamma^5)i(\not{p}+\not{\Pi}-\not{q}-\not{k}+m_c)(-ieQ_u)\gamma^\beta i(\not{p} + \not{\Pi}-\not{q}+m_c)\gamma^\mu(1-\gamma^5)b, \tag{A54}$$

$$\mathcal{L}^{(9)}_{\mu\nu} = (\bar{\ell}\gamma^\mu(1-\gamma^5)\nu_\ell)(\bar{\nu}_\ell\gamma^\nu(1-\gamma^5)\ell), \quad I_9 = I_8. \tag{A55}$$

$\mathcal{O}(\Pi^0)$:

$$\mathcal{J}_9(0;a) = \frac{1}{m_b(k.p_b)(k.p_c)}16(-2(p_\ell.p_\nu)(k.p_b)^2+m_b^2((p_\ell.p_\nu)(2k.p_b+2q.p_b-k.q-q^2)-2(p_b.p_\ell)(q.p_\nu) +2(q.p_\ell)(p_b.p_\nu+q.p_\nu))+(k.p_\nu)((q.p_\ell)(2(q.p_b)-m_b^2)-(p_b.p_\ell)(-2k.p_b+2k.q+q^2))+(k.p_\ell) (2(k.p_\nu)(q.p_b-m_b^2)-2(k.p_b)(q.p_\nu)+(p_b.p_\nu)(2(k.p_b)-q^2)+m_b^2(q.p_\nu))-2(k.p_b)(p._bp_\nu)(q.p_\ell) -2(k.p_b)(q.p_\ell)(q.p_\nu)+2(k.p_b)(p_b.p_\ell)(q.p_\nu)+2(k.q)(p_b.p_\nu)(q.p_\ell)+q^2(k.p_b)(p_\ell.p_\nu)+2(k.q)(k.p_b) (p_\ell.p_\nu)-2(k.p_b)(q.p_b)(p_\ell.p_\nu)-m_b^4p_\ell.p_\nu-4(p_b.p_\nu)(q.p_b)(q.p_\ell)+2q^2(p_b.p_\ell)(p_b.p_\nu)). \tag{A56}$$

$\mathcal{O}(\Pi)$:

$$\mathcal{J}_9(1;a) = \frac{1}{3(k.p_b)^2(k.p_c)^2m_b^3}16(2(5(q.p_b)-3m_b^2)(p_\ell.p_\nu)(k.p_b)^3+(6(p_\ell.p_\nu)m_b^4+((p_\ell.p_\nu)(5q^2+2k.q -16(q.p_b)+8k.p_c)+(4q.p_\ell+6p_b.p_\ell)(k.p_\nu)-2(3k.p_\ell+7q.p_\ell-5(p_b.p_\ell))(q.p_\nu)+6(k.p_\ell-q.p_\ell) (p_b.p_\nu))m_b^2+5(2(q.p_b)^2-(q^2+2k.q)(q.p_b)+(q^2+2k.q)(k.p_c))(p_\ell.p_\nu)+10(-(q.p_b)(k.p_\nu +q.p_\nu)(p_b.p_\ell)-(k.p_c-q.p_b)(k.p_\ell+q.p_\ell)(q.p_\nu)+(q.p_b)(q.p_\ell-k.p_\ell)(p_b.p_\nu)))(k.p_b)^2+(k.p_b) (-3(p_\ell.p_\nu)m_b^6+((p_\ell.p_\nu)(-3q^2+.kq+9.p_bq-6(k.p_c))-8(p_b.p_\ell)(q.p_\nu)+2(q.p_\ell)(5q.p_\nu +4p_b.p_\nu))m_b^4+((p_\ell.p_\nu)(4(k.q)^2+3(q.p_b)(q^2-2(q.p_b))+(k.q)(2q^2-3(q.p_b)-18k.p_c) +(8(q.p_b)-5q^2)(k.p_c))+2(5(q.p_\ell)(k.p_c-q.p_b)+(q^2+2k.q+3.p_bq-4(k.p_c))(p_b.p_\ell))(q.p_\nu) +2((p_b.p_\ell)(5q^2-2(k.q))+(-q^2+5k.q-13(q.p_b)+2k.p_c)(q.p_\ell))(p_b.p_\nu))m_b^2+(-7(q.p_\ell)m_b^4 -((p_b.p_\ell)(9q^2+6k.q-4(q.p_b)+2k.p_c)+(-2q^2+4k.q-15(q.p_b)+4k.p_c)(q.p_\ell))m_b^2+5(k.p_c -q.p_b)(2(q.p_b)(q.p_\ell)-(q^2+2k.q)(p_b.p_\ell)))(k.p_\nu)+10((2(q.p_b)^2+(k.q)(k.p_c-q.p_b))(q.p_\ell) -q^2(q.p_b)(p_b.p_\ell))(p_b.p_\nu)+(k.p_\ell)(3(q.p_\nu-2(k.p_\nu))m_b^4-((p_b.p_\nu)(5(q^2+2k.p_c)-4(k.q))+4(-q^2 -3(q.p_b)+2k.p_c)(k.p_\nu)+(2q^2+4k.q+.p_bq-18(k.p_c))(q.p_\nu))m_b^2+5(k.p_c-q.p_b)(2(k.p_\nu)(q.p_b) -q^2(p_b.p_\nu))))+m_b^2(3(p_\ell.p_\nu)m_b^4-(-3(p_\ell.p_\nu)(q^2-2(q.p_b))-6(q.p_\nu)(p_b.p_\ell)+(q.p_\ell-2(p_b.p_\ell)) (k.p_\nu)+6(q.p_\ell)(.qp_\nu+.p_bp_\nu))m_b^2+(-3(q.p_\nu)m_b^2+2(2q^2+5m_b^2-7(q.p_b))(k.p_\nu)+(q^2+4.p_bq -2m_b^2)(p_b.p_\nu))(k.p_\ell)-(2(q.p_b)(q.p_\ell)-q^2(p_b.p_\ell))(k.p_\nu-6(p_b.p_\nu))+(k.q)(3(p_\ell.p_\nu)m_b^2+(14 (p_b.p_\ell)-8(q.p_\ell))(k.p_\nu)-2(2p_b.p_\ell+.qp_\ell)(p_b.p_\nu)))(k.p_c))(\lambda_1+3\lambda_2), \tag{A57}$$

$$\mathcal{J}_9(1;c) = -\frac{1}{3(k.p_b)(k.p_c)m_b^3}128(-3(p_\ell.p_\nu)m_b^6+((p_\ell.p_\nu)(-3q^2+k.q+9p_b.q)-(k.p_\nu)(3q.p_\ell+2p_b.p_\ell) +2(5(q.p_\ell)-4(p_b.p_\ell))(q.p_\nu)+8(q.p_\ell)(p_b.p_\nu)+(k.p_\ell)(-10k.p_\nu+3q.p_\nu+2p_b.p_\nu))m_b^4+(-2 (p_b.p_\nu)(q.p_\ell)q^2-7(k.p_\nu)(p_b.p_\ell)q^2+2(q.p_\ell)(k.p_\nu)q^2+2(p_b.p_\ell)(q.p_\nu)q^2+10(p_b.p_\ell)(p_b.p_\nu)q^2$$

$$- 10(q.p_b)(q.p_\nu)(q.p_\ell) - 26(q.p_b)(p_b.p_\nu)(q.p_\ell) - 14(k.q)(k.p_\nu)(p_b.p_\ell) + 4(k.q)(q.p_\ell)(k.p_\nu) + 11(q.p_b) (q.p_\ell)(k.p_\nu) + 4(q.p_b)(p_b.p_\ell)(k.p_\nu) + 4(k.q)(p_b.p_\ell)(q.p_\nu) + 6(q.p_b)(p_b.p_\ell)(q.p_\nu) + 6(k.q)(q.p_\ell) (p_b.p_\nu) + (-6(q.p_b)^2 + 2(k.q)(q^2 + 2k.q) + 3(q^2 - k.q)(q.p_b))(p_\ell.p_\nu) + (k.p_\ell)(20(q.p_b)(k.p_\nu) - (2q^2 + 4k.q + .p_b q)(q.p_\nu) + (-3q^2 + 4k.q - 4(q.p_b))(p_b.p_\nu)))m_b^2 + (9(p_\ell.p_\nu)m_b^4 + ((p_\ell.p_\nu)(8q^2 + 7k.q - 22(q.p_b)) + 16(p_b.p_\ell)(q.p_\nu) - 4(q.p_\ell)(5q.p_\nu + 4p_b.p_\nu))m_b^2 + ((q.p_\ell)m_b^2 + (5q^2 + 10k.q + 14m_b^2)(p_b.p_\ell) - 10(q.p_b)(.qp_\ell + .p_b p_\ell))(k.p_\nu) + 10(q.p_b)(q.p_\ell - p_b.p_\ell)(q.p_\nu) - 10((p_b.p_\ell)q^2 + (k.q - 3(q.p_b))(q.p_\ell))(p_b.p_\nu) - 5(q^2 + 2k.q - 2(q.p_b))(q.p_b)(p_\ell.p_\nu) + (k.p_\ell)(5(p_b.p_\nu)q^2 - 10(q.p_b)(-q.p_\nu + k.p_\nu + p_b.p_\nu) + m_b^2(10k.p_\nu + 6p_b.p_\nu - 11q.p_\nu)))(k.p_b) + (k.p_b)^2(-(p_\ell.p_\nu)(16m_b^2 + 5(q^2 + 2k.q - 4(q.p_b))) - 10(p_b.p_\ell)(k.p_\nu + q.p_\nu) + 10(q.p_\ell)(q.p_\nu + p_b.p_\nu) + 10(k.p_\ell)(q.p_\nu - p_b.p_\nu)) + 10(k.p_b)^3 (p_\ell.p_\nu) + 5(q.p_b)((p_b.p_\nu)(q^2(k.p_\ell - 2p_b.p_\ell) - 2(k.q - 2(q.p_b))(q.p_\ell)) + ((q^2 + 2k.q)(p_b.p_\ell) - 2(q.p_b)(k.p_\ell + q.p_\ell))(k.p_\nu)))(\lambda_1 + 3\lambda_2). \tag{A58}$$

$\mathcal{O}(\Pi^2)$:

$$\mathcal{J}_9(2;a) = -\frac{1}{3(k.p_b)^2(k.p_c)^3 m_b^3} 16(2(-(q.p_b)^2 + (k.p_c)^2 + q^2 m_b^2 - 3m_b^2(k.p_c))(p_\ell.p_\nu)(k.p_b)^3 + (2(3k.p_c - q^2)(p_\ell.p_\nu)m_b^4 + ((p_\ell.p_\nu)(2(q.p_b)^2 + q^2(9k.p_c - q^2) + (k.q)(6k.p_c - 2q^2) + 2(q.p_b)(q^2 - 5k.p_c)) + 2((p_b.p_\ell)(5(k.p_c) - q^2) + (q.p_\ell)(q^2 - 7k.p_c))(q.p_\nu) + 2(q.p_\ell)(p_b.p_\nu)(q^2 - 3(k.p_c)))m_b^2 + (-(2 p_b.p_\nu + q.p_\nu)(((k.p_c)))^2 - 2(q.p_\nu - p_b.p_\nu)((((q.p_b)))^2 - q^2 m_b^2) + 2((3(p_b.p_\nu) - 5(q.p_\nu))m_b^2 + 2(q.p_b)(q.p_\nu))(k.p_c))(k.p_\ell) - 2(k.p_c)^2(q.p_\nu)(p_b.p_\ell) + (-(2p_b.p_\ell + q.p_\ell)(k.p_c)^2 + 2((q.p_b)^2 - q^2 m_b^2)(p_b.p_\ell) + 6m_b^2(p_b.p_\ell)(k.p_c))(k.p_\nu) - 2(q.p_b)^2(q.p_\ell)(q.p_\nu) - 2(k.p_c)^2(q.p_\ell)(q.p_\nu) + 2(q.p_b)^2 (p_b.p_\ell)(q.p_\nu) - 2(q.p_b)^2(q.p_\ell)(p_b.p_\nu) + 2(k.p_c)^2(q.p_\ell)(p_b.p_\nu) - 2(q.p_b)^3(p_\ell.p_\nu) + (q.p_b)^2 q^2(p_\ell.p_\nu) + (k.p_c)^2 q^2(p_\ell.p_\nu) + 2(q.p_b)^2(k.q)(p_\ell.p_\nu) + (k.p_c)^2(k.q)(p_\ell.p_\nu) + 2(k.p_c)^2(q.p_b)(p_\ell.p_\nu) - 2(k.p_c) q^2(q.p_b)(p_\ell.p_\nu) - 4(k.q)(k.p_c)(q.p_b)(p_\ell.p_\nu) + 4(q.p_b)(q.p_\ell)(q.p_\nu)(k.p_c))(k.p_b)^2 + (k.p_b)((p_\ell.p_\nu) (q^2 - 3(k.p_c))m_b^6 + (2(q.p_\ell)(p_b.p_\nu)(4(k.p_c) - q^2) + 2((q.p_\ell)(5(k.p_c) - q^2) + (p_b.p_\ell)(q^2 - 4k.p_c)) (q.p_\nu) + (q^4 - 2(q.p_b)q^2 - (q.p_b)^2 - (k.p_c)^2 + (k.q)(q^2 + .kp_c) + (8(q.p_b) - 7q^2)(k.p_c))(p_\ell.p_\nu)) m_b^4 + ((p_\ell.p_\nu)(2(q.p_b)^3 - (q^2 + k.q)(q.p_b)^2 + (2(k.q - 2(k.p_c))q^2 + (k.q)(4(k.q) - 13(k.p_c))) (k.p_c) + 2(q^2 - 2(k.q))(q.p_b)(k.p_c)) + 2((p_b.p_\ell)((k.p_c)(2(k.q) - q^2) - (q.p_b)^2) + ((q.p_b)^2 + 4(k.p_c)^2 - 2(k.q)(k.p_c))(q.p_\ell))(q.p_\nu) + 2(((q.p_b)^2 + (k.q)(5(k.p_c) - q^2) + (k.p_c)(q^2 - 2(k.p_c)) + 2(q.p_b)(q^2 - 5(k.p_c)))(q.p_\ell) - ((q^2 - 5k.p_c)q^2 + 2(k.q)(k.p_c))(p_b.p_\ell))(p_b.p_\nu))m_b^2 + (-(-13 (q.p_\nu)m_b^2 + (q.p_b)(q.p_\nu) + (q^2 + 10m_b^2)(p_b.p_\nu))(k.p_c)^2 + (q^2 m_b^2 - (q.p_b)^2)(q^2(p_b.p_\nu) - (q.p_\nu)m_b^2) + 2((q^2 - 3k.p_c)m_b^4 + (-(q.p_b)^2 + (k.p_c)^2 + (5k.p_c - q^2)(q.p_b))m_b^2 + (q.p_b)(k.p_c - q.p_b)^2)(k.p_\nu) + (3(q.p_\nu)m_b^4 + 4(k.q)(p_b.p_\nu - q.p_\nu)m_b^2 + q^2(2(q.p_b) - 5m_b^2)(p_b.p_\nu))(k.p_c))(k.p_\ell) + (((q.p_\ell)m_b^2 - (p_b.p_\ell)(q^2 + 2k.q + 2m_b^2) + 3(q.p_b)(q.p_\ell))(k.p_c)^2 + ((q.p_b)^2 - q^2 m_b^2)((q.p_\ell)(2(q.p_b) - m_b^2) - (q^2 + 2k.q)(p_b.p_\ell)) + (-7(q.p_\ell)m_b^4 - 3(3q^2 + 2k.q)(p_b.p_\ell)m_b^2 + 2(7(q.p_\ell)m_b^2 + (q^2 + 2k.q + 2m_b^2)(p_b.p_\ell)) (q.p_b) - 4(q.p_b)^2(q.p_\ell))(k.p_c))(k.p_\nu) - 4(q.p_b)^3(q.p_\ell)(p_b.p_\nu) + 2(q.p_b)^2(k.q)(q.p_\ell)(p_b.p_\nu) + 2(k.p_c)^2(k.q)(q.p_\ell)(p_b.p_\nu) + 4(k.p_c)^2(q.p_b)(q.p_\ell)(p_b.p_\nu) - 4(k.q)(k.p_c)(q.p_b)(q.p_\ell)(p_b.p_\nu) + 2 (q.p_b)^2 q^2(p_b.p_\ell)(p_b.p_\nu) - 2(k.p_c)^2 q^2(p_b.p_\ell)(p_b.p_\nu) + (k.p_c)^2(k.q)(q.p_b)(p_\ell.p_\nu)) + m_b^2(-2((p_\ell.p_\nu)$$

$$m_b^2 + 2(p_b.p_\ell)(k.p_\nu) - 2(q.p_\ell)(p_b.p_\nu))(k.q)^2 + (k.q)(-2(p_\ell.p_\nu)m_b^4 + ((p_\ell.p_\nu)(-2q^2 + 4.p_b q + 7k.p_c) - 2(q.p_\ell)(k.p_\nu) + 4(q.p_\ell - p_b.p_\ell)(q.p_\nu) + 4(q.p_\ell)(p_b.p_\nu) + 2(k.p_\ell)(q.p_\nu - 2(k.p_\nu)))m_b^2 + 2((p_b.p_\ell)(7(k.p_c) - q^2) + 2(q.p_b)(.kp_\ell + .qp_\ell))(k.p_\nu) - 2((k.p_\ell)q^2 + (4q.p_b + 5k.p_c)(q.p_\ell) + 2(k.p_c - q^2)(p_b.p_\ell))(p_b.p_\nu)) + (3(p_\ell.p_\nu)m_b^4 + (3(p_\ell.p_\nu)(q^2 - 2(q.p_b)) + (3q.p_\ell + 2p_b.p_\ell)(k.p_\nu) + 6(p_b.p_\ell)(q.p_\nu) - 6(q.p_\ell)(.qp_\nu + .p_b p_\nu))m_b^2 + (-7(q.p_\nu)m_b^2 + 2(5m_b^2 - 7(q.p_b))(k.p_\nu) + (5q^2 + 4.p_b q - 2m_b^2)(p_b.p_\nu))(k.p_\ell) - (2(q.p_b)(q.p_\ell) - q^2(p_b.p_\ell))(5(k.p_\nu) - 6(p_b.p_\nu)))(k.p_c))(k.p_c))\lambda_1, \tag{A59}$$

$$\mathcal{J}_9(2;c) = -\frac{1}{3(k.p_b)^2(k.p_c)^2}128(4(q.p_b)(p_\ell.p_\nu)(k.p_b)^4 + 2(-2(q^2 + k.q + p_b.q)(p_\ell.p_\nu)m_b^2 + (q.p_b)(-(p_\ell.p_\nu)(q^2 + 2k.q - 4(q.p_b)) - 2(p_b.p_\ell)(k.p_\nu + q.p_\nu) + 2(q.p_\ell)(q.p_\nu + p_b.p_\nu)) - 2(k.p_\ell)((q.p_b)(p_b.p_\nu - q.p_\nu) + (q.p_\nu)(k.p_c)) + ((p_\ell.p_\nu)(q^2 + 2k.q + 3m_b^2) - 2(q.p_\ell)(q.p_\nu))(k.p_c))(k.p_b)^3 + (2(2(q^2 + k.q - 2(k.p_c)) + q.p_b)(p_\ell.p_\nu)m_b^4 + ((p_\ell.p_\nu)(4(k.q)^2 - 8(q.p_b)^2 - 2(-3q^2 + p_b.q + 7k.p_c)(k.q) - 2(q^2 - 3(k.p_c))(q.p_b) + q^2(2q^2 - 11k.p_c)) + 2((p_b.p_\ell)(2q^2 + 2k.q - 9(k.p_c)) + (q.p_b)(q.p_\ell))(k.p_\nu) + 2((2k.q + 2(q^2 + .p_b q) - 3(k.p_c))(p_b.p_\ell) - (2k.q + 2(q^2 + p_b.q) - 9(k.p_c))(q.p_\ell))(q.p_\nu) - 2(2k.q + 2(q^2 + .p_b q) - 3(k.p_c))(q.p_\ell)(p_b.p_\nu))m_b^2 + 2(-((2(q^2 + k.q) + q.p_b)(q.p_\nu) - 2(q^2 + k.q)(p_b.p_\nu))m_b^2 - (k.p_c)(-9(q.p_\nu)m_b^2 + 2(q.p_b)(q.p_\nu) + (q^2 + m_b^2)(p_b.p_\nu)) + 2(q.p_b)(m_b^2 - q.p_b + .kp_c)(k.p_\nu) + (q.p_b)(2(q.p_b)(q.p_\nu) + (q^2 - 2(q.p_b))(p_b.p_\nu)))(k.p_\ell) + 2(-2(q.p_b)(q.p_\nu)((q.p_b)(p_b.p_\ell - q.p_\ell) + (q.p_\ell)(k.p_c)) + ((p_b.p_\ell)(-2(q.p_b)^2 - (k.p_c)(q^2 + 2k.q) + (q^2 + 2k.q)(q.p_b)) + 2(q.p_b)(k.p_c - q.p_b)(q.p_\ell))(k.p_\nu) + 2((3(q.p_b)^2 + (k.q)(k.p_c - q.p_b))(q.p_\ell) - q^2(q.p_b)(p_b.p_\ell))(p_b.p_\nu) + (2(q.p_b)^2 - (q^2 + 2k.q)(q.p_b) + (q^2 + 2k.q)(k.p_c))(q.p_b)(p_\ell.p_\nu)))(k.p_b)^2 + ((-2q^2 - 2k.q + 3k.p_c)(p_\ell.p_\nu)m_b^6 + (-2(p_\ell.p_\nu)(k.q)^2 + ((k.p_c)(7q.p_\ell + 2p_b.p_\ell) - 2q^2(q.p_\ell))(k.p_\nu) + 2(2(p_b.p_\ell)(2(k.p_c) - q^2) + (q.p_\ell)(2q^2 - 5k.p_c))(q.p_\nu) + (2(q.p_b)^2 + q^2(7(k.p_c) - 2q^2) + 4(q.p_b)(q^2 - 2(k.p_c)))(p_\ell.p_\nu) + (k.q)((p_\ell.p_\nu)(-4q^2 + 4.p_b q + 3k.p_c) - 4(q.p_\nu)(p_b.p_\ell) - 2(q.p_\ell)(k.p_\nu) + 4(q.p_\ell)(.qp_\nu + .p_b p_\nu)) + 4(q.p_\ell)(p_b.p_\nu)(q^2 - 2(k.p_c)))m_b^4 + (-4((p_b.p_\ell)(k.p_\nu) - (q.p_\ell)(p_b.p_\nu) + (p_\ell.p_\nu)(k.p_c))(k.q)^2 + ((p_b.p_\ell)((k.p_c)(11q^2 - 4(q.p_b)) - 2q^4) + 2(q.p_b)(2q^2 + p_b.q - 9k.p_c)(q.p_\ell))(k.p_\nu) + 2(-2(p_\ell.p_\nu)(q.p_b)^3 + ((p_\ell.p_\nu)q^2 + 2(p_b.p_\ell)(q.p_\nu) - 2(q.p_\ell)(q.p_\nu + p_b.p_\nu))(q.p_b)^2 + q^2((k.p_c)(q.p_\nu)(p_b.p_\ell) - ((p_b.p_\ell)(5(k.p_c) - 2q^2) + (q.p_\ell)(k.p_c))(p_b.p_\nu)) + (q.p_b)(2(5(k.p_c) - 2q^2)(q.p_\ell)(p_b.p_\nu) - (k.p_c)q^2(p_\ell.p_\nu))) + 2(k.q)((p_\ell.p_\nu)(q.p_b)^2 + ((p_b.p_\ell)(7(k.p_c) - 3q^2) + 2(q.p_b)(q.p_\ell))(k.p_\nu) + 2((p_b.p_\ell)q^2 + (q^2 - 2(q.p_b))(q.p_\ell))(p_b.p_\nu) + (-(p_\ell.p_\nu)(q^2 - 2(q.p_b)) + 2(q.p_\ell - p_b.p_\ell)(q.p_\nu) - 7(q.p_\ell)(p_b.p_\nu))(k.p_c)))m_b^2 + (-2((q^2 + .kq)m_b^2 - (q.p_b)^2)(q^2(p_b.p_\nu) - (q.p_\nu)m_b^2) + 2((k.p_c)(5m_b^4 - 9(q.p_b)m_b^2 + 2(q.p_b)^2) - 2(q.p_b - m_b^2)((q.p_b)^2 - (q^2 + k.q)m_b^2))(k.p_\nu) + (-(7q.p_\nu + 2p_b.p_\nu)m_b^4 + (4(k.q)(q.p_\nu) + (7q^2 - 4k.q + 4.p_b q)(p_b.p_\nu))m_b^2 - 2q^2(q.p_b)(p_b.p_\nu))(k.p_c))(k.p_\ell) + 2(q.p_b)(2(p_b.p_\nu)((2(q.p_b)^2 + (k.q)(k.p_c - q.p_b))(q.p_\ell) - q^2(q.p_b)(p_b.p_\ell)) + (k.p_c - q.p_b)(2(q.p_b)(q.p_\ell) - (q^2 + 2k.q)(p_b.p_\ell))(k.p_\nu)))(k.p_b) + 2(k.q)m_b^2((p_\ell.p_\nu)m_b^4 + ((p_\ell.p_\nu)(q^2 + k.q - 2(q.p_b)) + 2(p_b.p_\ell)(q.p_\nu) - 2(q.p_\ell)(q.p_\nu + p_b.p_\nu))m_b^2 + ((p_b.p_\nu)q^2 + 2(m_b^2 - q.p_b)(k.p_\nu) + m_b^2(-(q.p_\nu)))(k.p_\ell) + ((p_b.p_\ell)(q^2 + 2k.q) + (m_b^2 - 2(q.p_b))(q.p_\ell))(k.p_\nu) - 2((p_b.p_\ell)q^2 + (k.q - 2(q.p_b))(q.p_\ell))(p_b.p_\nu))(k.p_c))\lambda_1, \tag{A60}$$

$$\mathcal{J}_9(2;d)=\frac{1}{3m_b^3(k.p_b)(k.p_c)}256\lambda_1((k.p_b+q.p_b)^2-m_b^2(2k.q+q^2))(2(p_\ell.p_\nu)(k.p_b)^2+m_b^2((p_\ell.p_\nu)(-2(k.p_b)-2(q.p_b)+k.q+q^2)+2(p_b.p_\ell)(q.p_\nu)-2(q.p_\ell)(p_b.p_\nu+q.p_\nu))+(k.p_\nu)((p_b.p_\ell)(-2(k.p_b)+2k.q+q^2)+(q.p_\ell)(m_b^2-2q.p_b))+(k.p_\ell)(2(k.p_\nu)(m_b^2-q.p_b)+2(k.p_b)(q.p_\nu)+(p_b.p_\nu)(q^2-2(k.p_b))+m_b^2(-q.p_\nu))-2(k.q)(p_b.p_\nu)(q.p_\ell)+2(k.p_b)(q.p_\ell)(q.p_\nu)-2(k.p_b)(p_b.p_\ell)(q.p_\nu)+2(k.p_b)(p_b.p_\nu)(q.p_\ell)-q^2(k.p_b)(p_\ell.p_\nu)-2(k.q)(k.p_b)(p_\ell.p_\nu)+2(k.p_b)(q.p_b)(p_\ell.p_\nu)+m_b^4(p_\ell.p_\nu)+4(p_b.p_\nu)(q.p_b)(q.p_\ell)-2q^2(p_b.p_\ell)(p_b.p_\nu)). \tag{A61}$$

APPENDIX B: KINEMATICS

In this section, we describe the kinematics involved in the decay. However four-body decay generally consists of five independent kinematical variables. The inclusive four-body decay consists of six independent variables where one extra variable is due to invariant mass squared for decayed hadron (p_X^2). Here, we have traded p_X^2 with $q'^2(=(p_\ell+p_\nu+k)^2)$. Further we define two Lorentz invariant variables as

$$y=\frac{2p_B.p_\ell}{m_B^2} \quad \text{and} \quad x=\frac{2p_B.k}{m_B^2}. \tag{B1}$$

The variables y and x are the normalized lepton and photon energy, respectively in the B meson rest frame. The other three variables are neutrino energy (E_ν), and two angles: (a) $\theta_{X\gamma}$ is the angle between the recoiling hadron and hard photon and (b) $\theta_{X\ell}$ is the angle between the final state recoiling hadron (X) and charged lepton. The general form of triple differential decay,

$$\frac{d^3\Gamma}{dq'^2dE_\ell dE_\nu}=\int\frac{d^4p_\ell}{(2\pi)^4}2\pi\delta(p_\ell^2-m_\ell^2)\theta(p_\ell^0)\int\frac{d^4p_\nu}{(2\pi)^4}2\pi\delta(p_\nu^2)\theta(p_\nu^0)\delta(E_\ell-p_\ell^0)\delta(E_\nu-p_\nu^0)\delta(q'^2-(q+k)^2)\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2((p_b+\Pi-q-k)^2-m_c^2)}|\mathcal{M}|^2(2\pi)^4\delta^4(p_b-q'-p_X). \tag{B2}$$

The Cutcosky method implies that

$$\int\frac{d^4k}{(2\pi)^4}\to\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4p_X}{(2\pi)^4}(2\pi)^4\delta^4(p_b-q-p_X-k), \tag{B3}$$

and the propagator is replaced with delta functions. For example, in Fig. 2(a), the propagators are

$$\frac{1}{k^2}\to-2\pi i\delta(k^2)\theta(k^0),\ \text{and} \tag{B4}$$

$$\frac{1}{((p_b+\Pi-q-k)^2-m_c^2)}\to-2\pi i\delta(((p_b+\Pi-q-k)^2-m_c^2))\theta((p_b+\Pi-q-k)^0). \tag{B5}$$

Incorporating the Cutkosky method, the differential decay width is

$$\begin{aligned}\frac{d^3\Gamma}{dE_\ell dq'^2 dE_\nu} &= \int \frac{d^3 p_\ell}{(2\pi)^3}\delta(p_\ell^2)\theta(p_\ell^0)\int \frac{d^3 p_\nu}{(2\pi)^3}\delta(p_\nu^2)\theta(p_\nu^0)\delta(q'^2-(q+k)^2)\int \frac{d^4 k}{(2\pi)^4}\frac{d^4 p_X}{(2\pi)^4}(-2\pi i)\delta(k^2)\theta(k^0)\\ &\quad (-2\pi i)\delta((p_b+\Pi-q-k)^2-m_c^2)(2\pi)^4\delta^4(p_b-q'-p_X)(2\pi)^4|\mathcal{M}|^2\delta^4(p_B-q-p_X-k),\\ &= -\frac{1}{8\pi^2}E_\ell E_\nu\int d(\cos\theta_{\ell\nu})\delta(q'^2-(q+k)^2)\int\frac{d^3k}{2E_\gamma}\delta((p_B+\Pi-q-k)^2-m_c^2)|\mathcal{M}|^2\\ &\quad \delta^4(p_B-q-p_X-k),\\ &= -\frac{1}{8\pi^2}\int\frac{d^3k}{2E_\gamma}\delta((p_B+\Pi-q-k)^2-m_c^2)|\mathcal{M}|^2\delta^4(p_B-q-p_X-k),\\ &= -\frac{1}{16\pi^2}\int|k|dE_\gamma\int d\Omega_k\delta((p_B+\Pi-q-k)^2-m_c^2)|\mathcal{M}|^2\delta^4(p_B-q-p_X-k),\end{aligned}\tag{B6}$$

where $\theta_{\ell\nu}$ is the angle between the lepton and neutrino. Equation (B6) can be translated in terms of variables x and y. Hence, one can write Eq. (B6) as

$$\frac{d^2\Gamma}{dydx} = -\frac{m_B^2}{64\pi^2}\int dq'^2\int d\Omega_k|k|\int dE_\nu\delta((p_B+\Pi-q-k)^2-m_c^2)|\mathcal{M}|^2\delta^4(p_B-q-p_X-k).\tag{B7}$$

Further, the differential rate with respect to lepton energy is

$$\frac{d\Gamma}{dy} = \int_{x_{\min}}^{x_{\max}} dx\frac{d^2\Gamma}{dydx},\tag{B8}$$

where $m_\gamma^2 \le q'^2 \le \frac{ym_B^2(1-y-z^2)}{1-y}$, $m_\gamma \le x \le 1-y-z^2$, and $0\le y\le 1-z^2$. The variable $z = m_{u/c}/m_b$ is defined in Sec. IV.

Next, the delta function with Π can be expanded in the power of Π. Explicitly, it is given by

$$\begin{aligned}\delta((p_b+\Pi-q-k)^2) &= \delta((p_b-q-k)^2)+2\Pi\cdot(p_b-q-k)\delta'((p_b-q-k)^2)+\Pi^2(\delta'((p_b-q-k)^2)\\ &\quad +2(p_b-q-k)^2\delta''((p_b-q-k)^2))+\cdots\end{aligned}\tag{B9}$$

In the rest frame of the B meson, the momentas involved are given by

$$p_B = (m_B,\mathbf{0}),\qquad q' = (q^0,|\mathbf{q}'|),\qquad p_X = (E_X,-|\mathbf{q}'|),\tag{B10}$$

where

$$E_X = \frac{m_B^2-q'^2+m_X^2}{2m_B},\qquad q^0 = \frac{m_B^2+q'^2-m_X^2}{2m_B},\quad \text{and}\quad |\mathbf{q}'| = \frac{\lambda^{1/2}(m_B^2,q'^2,m_X^2)}{2m_B}.\tag{B11}$$

Another important point to note is that the integration domain in E_ν has a boundary from below:

$$E_\nu \ge \frac{q'^2-2q.k-m_\ell^2}{4E_\ell}.\tag{B12}$$

Therefore, it should be ensured that E_ν does not cross the boundary. This is enforced by introducing an appropriate theta function in the integral. This plays an important role in the integration of delta functions and their derivatives present in the differential rate in Eq. (B6).

[1] N. Isgur, D. Scora, B. Grinstein, and M. B. Wise, Semileptonic B and D decays in the quark model, Phys. Rev. D **39**, 799 (1989).

[2] S. Nussinov and W. Wetzel, Comparison of exclusive decay rates for $b \to u$ and $b \to c$ transitions, Phys. Rev. D **36**, 130 (1987).

[3] V. A. Khoze, M. A. Shifman, N. G. Uraltsev, and M. B. Voloshin, On inclusive hadronic widths of beautiful particles, Sov. J. Nucl. Phys. **46**, 112 (1987).

[4] B. Blok, L. Koyrakh, M. A. Shifman, and A. I. Vainshtein, Differential distributions in semileptonic decays of the heavy flavors in QCD, Phys. Rev. D **49**, 3356 (1994); **50**, 3572(E) (1994).

[5] M. A. Shifman and M. B. Voloshin, On production of d and D* mesons in B meson decays, Sov. J. Nucl. Phys. **47**, 511 (1988).

[6] K. G. Wilson, Non-Lagrangian models of current algebra, Phys. Rev. **179**, 1499 (1969).

[7] K. Symanzik, Small distance behavior analysis and Wilson expansion, Commun. Math. Phys. **23**, 49 (1971).

[8] E. Eichten and B. R. Hill, An effective field theory for the calculation of matrix elements involving heavy quarks, Phys. Lett. B **234**, 511 (1990).

[9] H. Georgi, An effective field theory for heavy quarks at low-energies, Phys. Lett. B **240**, 447 (1990).

[10] M. Neubert, Heavy quark symmetry, Phys. Rep. **245**, 259 (1994).

[11] A. V. Manohar and M. B. Wise, *Heavy Quark Physics* (Cambridge University Press, 2000), Vol. 10, 10.1017/CBO9780511529351.

[12] I. I. Y. Bigi, M. A. Shifman, N. G. Uraltsev, and A. I. Vainshtein, On the motion of heavy quarks inside hadrons: Universal distributions and inclusive decays, Int. J. Mod. Phys. A **09**, 2467 (1994).

[13] N. Uraltsev, Topics in the heavy quark expansion, in *At The Frontier of Particle Physics* (World Scientific Publishing, Singapore, 2000), pp. 1577–1670, 10.1142/9789812810458_0034.

[14] E. Bagan, P. Ball, V. M. Braun, and H. G. Dosch, QCD sum rules in the effective heavy quark theory, Phys. Lett. B **278**, 457 (1992).

[15] M. Neubert, Symmetry breaking corrections to meson decay constants in the heavy quark effective theory, Phys. Rev. D **46**, 1076 (1992).

[16] P. Ball and V. M. Braun, Next-to-leading order corrections to meson masses in the heavy quark effective theory, Phys. Rev. D **49**, 2472 (1994).

[17] I. I. Y. Bigi, M. A. Shifman, N. G. Uraltsev, and A. I. Vainshtein, Sum rules for heavy flavor transitions in the SV limit, Phys. Rev. D **52**, 196 (1995).

[18] Z. Ligeti and Y. Nir, Phenomenological constraints on $\bar{\Lambda}$ and λ_1, Phys. Rev. D **49**, R4331 (1994).

[19] A. Kapustin and Z. Ligeti, Moments of the photon spectrum in the inclusive $B \to X_s\gamma$ decay, Phys. Lett. B **355**, 318 (1995).

[20] A. F. Falk, M. E. Luke, and M. J. Savage, Phenomenology of the $\frac{1}{m_Q}$ expansion in inclusive B and D meson decays, Phys. Rev. D **53**, 6316 (1996).

[21] M. Gremm, A. Kapustin, Z. Ligeti, and M. B. Wise, Implications of the $B \to X\ell\bar{\nu}_\ell$ lepton spectrum for heavy quark theory, Phys. Rev. Lett. **77**, 20 (1996).

[22] M. Gremm and A. Kapustin, Order $1/m_b^3$ corrections to $B \to X_c\ell\bar{\nu}$ decay and their implication for the measurement of $\bar{\Lambda}$ and λ_1, Phys. Rev. D **55**, 6924 (1997).

[23] S. Aoki *et al.*, Heavy quark expansion parameters from lattice NRQCD, Phys. Rev. D **69**, 094512 (2004).

[24] P. A. Zyla *et al.*, Review of particle physics, Prog. Theor. Exp. Phys. **2020**, 083C01 (2020).

[25] G. Finauri and P. Gambino, The q^2 moments in inclusive semileptonic B decays, J. High Energy Phys. 02 (2024) 206.

[26] J. Abdallah *et al.*, Determination of heavy quark non-perturbative parameters from spectral moments in semi-leptonic B decays, Eur. Phys. J. C **45**, 35 (2006).

[27] A. L. Kagan and M. Neubert, QCD anatomy of $B \to X_s\gamma$ decays, Eur. Phys. J. C **7**, 5 (1999).

[28] F. De Fazio and M. Neubert, $B \to X_u l\bar{\nu}l$ decay distributions to order α_s, J. High Energy Phys. 06 (1999) 017.