Type II string theory on $AdS_3 \times S^3 \times T^4$ and symmetric orbifolds

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We discuss in detail the $1+1$-dimensional superconformal field theory dual to type II string theory on $AdS_3 \times S^3 \times T^4$, emphasizing the string theoretic aspects of this duality. For one unit of Neveu-Schwarz (NS-NS) 5-brane flux ($Q_5 = 1$), this string theory has been suggested to be dual to a grand-canonical ensemble of T^{4N}/S_N free symmetric orbifold conformal field theories (CFTs). We show how the string genus expansion emerges to all orders for the free orbifold grand-canonical correlation functions. We also discuss how the strong coupling limit of the NS-NS string theory arises (even at large N) in the free orbifold description, and argue why this limit does not have a weakly coupled RR description. The dual conformal field theory (CFT) includes (for all values of Q_5) an extra T^4 factor that is decoupled from perturbative string theory. We discuss the exactly marginal deformations that relate the different values of Q_5, including the precise $J\bar{J}$ deformations mixing this extra T^4 with the symmetric orbifold.

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I. INTRODUCTION AND SUMMARY

The duality between type II string theory on $AdS_3 \times S^3 \times T^4$ (which can be viewed as the near-horizon limit of Q_1 D1-branes and Q_5 D5-branes wrapped on T^4) and a specific $1+1$-dimensional $\mathcal{N} = (4,4)$ superconformal field theory (SCFT) (the "D1-D5 SCFT") is one of the original examples of the AdS/CFT correspondence [1,2]. It shares many features with other examples of the AdS/CFT correspondence; for $Q_1 \gg Q_5 \gg 1$ the theory has one limit of its continuous parameters where it is described by string theory on a weakly coupled and weakly curved background, and another limit where it is a free field theory [a free symmetric orbifold $(T^4)^{Q_1 Q_5}/S_{Q_1 Q_5}$]. However, it also has several advantages compared to higher-dimensional examples, most notably the fact that when the background has purely Neveu-Schwarz (NS-NS) charges the string world sheet theory is under very good control, either in the Ramond-Neveu-Schwarz (RNS) formalism [3–10] (for $Q_5 > 1$) or in the hybrid formalism [11–14].

Many features of this theory, such as its moduli space [15], chiral ring [16,17], protected [18–24] and unprotected [25–36] correlators, and its perturbative string spectrum [14,37] were studied and understood over the years. However, some of its features, in particular features related to string perturbation theory, are more subtle. In this paper we review what is known about this theory, highlighting three specific confusing issues and how they are resolved.

We begin in Sec. II by reviewing this theory from various different points of view—the supergravity approximation, the CFT, and the string world sheet. In particular, we describe the parameter space of the theory and the relations between theories with different (relatively prime) values of Q_1 and Q_5 (and the same $N = Q_1 Q_5$).

In Sec. III we review one of the strange features of this theory, the fact that perturbative string theory in the background with only NS-NS fluxes on AdS_3 and on S^3 is not dual to a specific SCFT but rather to a grand-canonical ensemble of SCFTs (with different values of Q_1). This fact was suspected based on the world sheet in the RNS formalism for $Q_5 > 1$ [7,38]. For $Q_5 = 1$, where the theory is dual to a free symmetric orbifold, the relation between string theory and the SCFT can be analyzed in great detail. We describe (following [39–41]; see also [20,29,42]) how correlation functions of free symmetric orbifolds $\mathrm{Sym}_N(\mathcal{M})$ have a $1/N$ expansion. However, this expansion does not map directly to a string theory, while its grand-canonical version does, and we write in detail the relation between string theory and CFT correlation functions. The main result is Eq. (3.12), which shows how appropriately defined grand-canonical correlation functions exhibit an exact string genus expansion (generalizing the relation between the partition functions, discussed in [43–45]). We comment on the generalization of this relation to $Q_5 > 1$.

In Sec. IV we discuss the fact that even though the $Q_5 = 1$ background [with vanishing Ramond-Ramond (RR)

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scalars] maps to a free orbifold theory for any value of its parameters, with the $1/N$ expansion of the orbifold mapping to perturbative string theory in the NS-NS description of this background, there is a region in parameter space where the NS-NS string becomes strongly coupled and an S-dual string seems to become weakly coupled. We show that in this region of parameter space the $1/N$ expansion of the free orbifold breaks down, and discuss the fact that even though naively the string coupling in the dual RR description can become arbitrarily weakly coupled, this description is never really weakly coupled (similar to the behavior of type IIB string theory on $\mathrm{AdS}_5 \times S^5$ with a small integer flux N, which does not become weakly coupled even when $g_s \to 0$).

Finally, in Sec. V, we describe a mysterious feature of the CFT, which is that it includes an extra $\hat{T}^4$ factor, which is completely decoupled when the rest of the CFT is a free symmetric orbifold (for general values of the parameters, this factor only couples to the rest of the SCFT through $J\bar{J}$ deformations). Usually, string theory does not allow for decoupled sectors (since all states couple to gravity), but in this case, the decoupled sector is a topological Chern-Simons (CS) theory on AdS_3, and we discuss to what extent and how it couples to the rest of string theory for general values of the parameters.

Our analysis suggests various interesting future directions. When N is not prime, the CFT has a singular limit corresponding to every $N = Q_1 Q_5$ factorization with $Q_5 > 1$, in which its spectrum becomes continuous (described by "long strings" on AdS_3 [46]). The CFT also has another singular limit where it is described by the orbifold $(T^4)^N/S_N$ with a vanishing theta angle at the $\mathbb{Z}_2$ singularity, and it would be nice to know if this limit has some controllable string theory description. Our results in Sec. V suggest various properties of string theory on AdS_3 and its D-branes, and it would be nice to confirm these properties directly. It would also be interesting to understand in more detail the precise grand-canonical ensemble that is dual to string theory on the NS-NS background with $Q_5 > 1$.

The discussion of the free orbifold limit can be generalized to other purely NS-NS type IIB backgrounds with 1 unit of 3-form flux on an S^3, such as $\mathrm{AdS}_3 \times S^3 \times K3$ or $\mathrm{AdS}_3 \times S^3 \times S^3 \times S^1$ [47]. However, in those cases there is no direct relation to other values of the flux, so it is not clear what can be said about them. It would be interesting to understand what can be said about other backgrounds, like orbifolds of AdS_3 and/or S^3 [48–51], type IIA string theory on $\mathrm{AdS}_3 \times S^3 \times K3$ [52] and heterotic string theories (including nonsupersymmetric ones [53,54]) on AdS_3.

Last but not least, it would be nice if the detailed understanding of the duality between free orbifolds and string theory could be generalized to the case of free gauge theories with continuous gauge groups.

II. A REVIEW OF TYPE II STRING THEORY ON $\mathrm{AdS}_3 \times S^3 \times T^4$ AND ITS DUAL CFT

One way to obtain type II string theory on $\mathrm{AdS}_3 \times S^3 \times T^4$ is by considering the near-horizon limit [1] of Bogomol'nyi-Prasad-Sommerfield (BPS) strings preserving 2d $\mathcal{N} = (4,4)$ supersymmetry in type II string theory on T^4 (coming from fundamental strings, wrapped NS5-branes or 8 types of wrapped D-branes). The type IIA and type IIB cases are related by T duality so there is no need to discuss them separately, for convenience we will use the type IIB language throughout this paper. Type II string theory on T^4 has an $SO(5,5,\mathbb{Z})$ U-duality group, under which the ten string charges transform as a vector. By a U-duality transformation one can go to a frame where we have only D-string charge Q_1 and wrapped D5-brane charge Q_5, and we will focus on this case in our discussion. By S duality these configurations are identical to configurations carrying Q_1 units of fundamental string charge and Q_5 units of wrapped NS5-brane charge. In this section, we review what is known about these configurations from three different points of view—supergravity, the string world sheet, and the dual CFT. We will then review how U duality relates different AdS_3 backgrounds.

A. Supergravity solutions

The near-horizon limit of Q_1 D1-branes and Q_5 D5-branes wrapped on a T^4 is $\mathrm{AdS}_3 \times S^3 \times T^4$. String theory on T^4 has 25 massless scalar fields, including the metric and B-field on the T^4 (16 scalars), the dilaton (1 scalar), and 8 RR scalars. When both Q_1 and Q_5 are nonzero (as we will assume throughout this paper) 5 of these scalars (4 of the RR scalars and the overall volume of the T^4) are fixed in the near-horizon limit, while the other 20 scalars remain as moduli.

We will begin by focusing on the case where the RR scalars vanish, while the string coupling takes some arbitrary value g_{10}, and the torus has some fixed shape and B field. In this case the near-horizon limit is $\mathrm{AdS}_3 \times S^3 \times T^4$, with RR 3-form flux on AdS_3 and on S^3, where the radius R of AdS_3 and S^3 and the volume of T^4 are given by (up to numerical constants that will not be important for our discussion):

$$\frac{R^2}{\alpha'} = g_{10} Q_5 = g_6 \sqrt{Q_1 Q_5}, \qquad \frac{\mathrm{vol}(T^4)}{\alpha'^2} = \frac{Q_1}{Q_5}, \qquad g_{10}^2 = g_6^2 \frac{Q_1}{Q_5}. \tag{2.1}$$

Here we defined the six-dimensional string coupling g_6 in the standard way, and the expressions are derived from the supergravity solution for these branes, so they are reliable when R and $\mathrm{vol}(T^4)$ are much larger than the string scale. Naively, perturbative string theory in this RR background is reliable when $g_6, g_{10} \ll 1$, while supergravity is valid when $Q_1 \gg Q_5$, $g_{10} Q_5 \gg 1$. The 6- and 10-dimensional Newton constants (in AdS units) in the background described above are

$$\frac{G_N^{(6)}}{R^4} = \frac{1}{Q_1 Q_5}, \qquad \frac{G_N^{(10)}}{R^8} = \frac{1}{g_6^2 Q_1 Q_5^3}. \tag{2.2}$$

Performing S duality, we find a purely NS-NS background (that also arises as the near-horizon limit of Q_1 fundamental strings and Q_5 NS5-branes). In this background, there is NS-NS 3-form flux on AdS_3 and on S^3 instead of RR 3-form flux. While the ten-dimensional string coupling in this new frame, $g'_{10} = 1/g_{10}$, is still a free parameter, the six-dimensional string coupling is now fixed, while the volume of the torus in string units in this frame is a free parameter which we will denote by v_4, related to the RR frame by $v_4 = g_6^{-2}$. The new background is given by

$$\frac{R^2}{\alpha'} = Q_5, \qquad \frac{\mathrm{vol}(T^4)}{\alpha'^2} = v_4, \qquad g_6'^2 = \frac{Q_5}{Q_1}, \qquad g_{10}'^2 = v_4 \frac{Q_5}{Q_1}. \tag{2.3}$$

In this background all 16 scalars parametrizing the metric and B field on the T^4 are massless moduli, as well as 4 RR scalars (which we are still setting to zero). Perturbative string theory is now valid whenever $g'_6, g'_{10} \ll 1$, while supergravity is valid for $Q_5, v_4 \gg 1$. The NS-NS description is invariant under a T-duality transformation inverting the four cycles of the torus, so we can choose without loss of generality $v_4 \geq 1$ (implying that for fixed Q_1, Q_5, the ten-dimensional string coupling cannot be arbitrarily small). Newton's constants remain the same, and in terms of v_4 they are given by (2.2)

$$\frac{G_N^{(6)}}{R^4} = \frac{1}{Q_1 Q_5}, \qquad \frac{G_N^{(10)}}{R^8} = \frac{v_4}{Q_1 Q_5^3}. \tag{2.4}$$

B. The dual conformal field theory

The general arguments of the AdS/CFT correspondence [1] imply that string theory on the AdS_3 background discussed above should be dual to the 2d $\mathcal{N} = (4,4)$ SCFT that arises at low energies on a bound state of Q_1 D strings and Q_5 D5-branes wrapped on T^4. In this section we will discuss this theory when all the RR scalars vanish. One way to describe this theory is as a sigma model on the moduli space of Q_1 instantons of a $U(Q_5)$ theory on T^4 [T duality implies that another way to describe the same low-energy theory is as a sigma model on the moduli space of Q_5 instantons of a $U(Q_1)$ gauge theory on a dual T^4]. The dimension of this moduli space is $4Q_1Q_5 + 4$, where the first factor comes from the moduli space of Q_1 instantons of $SU(Q_5)$ on T^4, and the second factor from the Wilson lines of the overall $U(1)$ in $U(Q_5)$ [which do not affect the $SU(Q_5)$ gauge fields, except through global constraints]. The corresponding conformal field theory thus has central charge $c = 6Q_1Q_5 + 6$, consistent at large Q_1, Q_5 with the central charge following from the supergravity background of the previous section (2.2), (2.4). We will denote $N \equiv Q_1 Q_5$.

In general, this sigma model is a complicated SCFT, and no simple theory that flows to it at low energies is known. In the limit that the volume of T^4 becomes infinite, it can be identified (for $Q_5 > 1$) as the conformal field theory describing the Higgs branch of a $U(Q_1)$ $\mathcal{N} = (4,4)$ supersymmetric QCD theory with one hypermultiplet in the adjoint representation and Q_5 hypermultiplets in the fundamental representation (this theory decouples at low energies from the theory of the Coulomb branch [55,56], which corresponds to separating the D strings from the D5-branes).

The sigma model on the instanton moduli space is singular for $Q_5 > 1$ when instantons go to zero size; for a single $SU(2)$ instanton on $\mathbb{R}^4$ becoming small, this singularity is locally an $\mathbb{R}^4/\mathbb{Z}_2$ singularity with a vanishing theta angle. At these singularities, the sigma model develops semi-infinite throats (which before taking the low-energy limit connect the Higgs branch to the Coulomb branch [57]), with a continuous spectrum of dimensions above a gap of order Q_5.

The moduli space simplifies significantly for $Q_5 = 1$; in this case all instantons are pointlike, so the moduli space of instantons becomes $(T^4)^{Q_1}/S_{Q_1}$. Including the Wilson lines, and noting that in this case $N = Q_1$, the CFT is expected to be a sigma model on

$$(T^4)^N / S_N \times \hat{T}^4 \tag{2.5}$$

[where the second $\hat{T}^4$, arising from the $U(1)$ Wilson lines, is inversely related to the first T^4]. This moduli space has orbifold singularities when instantons come together on the T^4 (these are not related to the small-instanton singularities discussed above). *A priori* the value of the theta angle at these singularities is not clear, but we will review below the arguments that it has the value $\theta = \pi$ corresponding to a free orbifold (rather than the value $\theta = 0$ that appeared above).[1]

[1]Naively this statement contradicts statements made in the previous paragraphs, because we stated that the $(Q_1, Q_5) = (N, 1)$ theory is the same as the $(Q_1, Q_5) = (1, N)$ theory (possibly at different values of the moduli), and naively the latter theory of a single instanton in $U(N)$ is expected to have a continuous spectrum because of the small-instanton singularity. However, it turns out that there are actually no smooth $SU(N)$ instantons on T^4 [58], so the naive expectation that the local small-instanton singularity looks the same on T^4 and on $\mathbb{R}^4$ does not hold in the $Q_1 = 1$ case. Our arguments imply that the sigma model on the one-instanton moduli space of $U(N)$ on T^4 should be given by the free orbifold (2.5), which one would naively obtain from the position of a zero-size instanton and from the $U(N)$ Wilson lines in this theory, and it would be interesting to confirm this directly.

Returning to the case of general Q_5, as in any $\mathcal{N} = (4,4)$ SCFT, the super-Virasoro algebra contains $SU(2)_L \times SU(2)_R$ affine Lie algebras of level $Q_1 Q_5 + 1$. In addition, the SCFT described above has 8 left-moving and 8 right-moving $U(1)$ affine Lie algebras. $4+4$ of these symmetries have level Q_1, and are related to the compact scalars describing the center of mass position of the Q_1 instantons (which lives on T^4). The other $4+4$ affine Lie algebras have level Q_5, and are related to the $U(1) \subset U(Q_5)$ Wilson lines (which live on the dual $\hat{T}^4$). These symmetries are manifest in the $Q_5 = 1$ case (2.5). In the dual supergravity description of the previous subsection, if we consider the NS-NS background, the first $8U(1)$'s arise from the NS-NS sector (as the $g_{\mu i}$ and $B_{\mu i}$ components of the metric and B field, where i labels the four coordinates of T^4), and the other $8U(1)$'s arise from the RR sector (by taking suitable components of the RR 2-form and 4-form potentials). The level of the affine Lie algebras in the CFT is related to the Chern-Simons level of the corresponding $U(1)$ gauge fields on AdS_3, and the levels mentioned above agree with type II supergravity on $AdS_3 \times S^3 \times T^4$.

C. The string world sheet

There are various methods for studying string theory in the RR $AdS_3 \times S^3 \times T^4$ background—the Green-Schwarz string, the pure spinor string and a hybrid formalism. However, as in other RR backgrounds, quantization of the string is not yet well-understood in any of these approaches.

The situation is much better in the NS-NS background. This background can be described in the standard RNS formalism [with $\mathcal{N} = (1,1)$ supergravity on the world sheet], in which the world sheet action is a sum of supersymmetric Wess-Zumino-Witten (WZW) models of $SL(2)$ (describing AdS_3 with NS-NS flux) and $SU(2)$ (describing S^3 with NS-NS flux), both at level Q_5, and a supersymmetric sigma model on T^4 (together these give a critical type II superstring). The supersymmetric $SU(2)$ WZW model can be written as a direct sum of a bosonic $SU(2)$ model at level $(Q_5 - 2)$ and three free fermions, so this description makes sense for $Q_5 \geq 2$, and it was investigated in detail in [5,7].

One unusual property of string theory in these backgrounds is that it has a continuous spectrum. From the point of view of the world sheet this arises because $SL(2)$ has representations with continuous "spins," while in spacetime it is related to long strings [46] winding around the angular direction of AdS_3, that can go all the way to the boundary at a finite cost in energy (so their radial momentum is a continuous parameter). This continuous spectrum can be related to the continuous spectrum arising at small-instanton singularities in the SCFT, discussed in the previous subsection.

The NS-NS background may also be described in the hybrid formalism for the world sheet, where the $AdS_3 \times S^3$ part corresponds [11] to a super-WZW model on the supergroup $PSU(1,1|2)$ at level Q_5 (for a recent review see [59]). This description makes sense for any integer value of Q_5. For $Q_5 > 1$ it agrees with the RNS formalism discussed above, while for $Q_5 = 1$ it behaves very differently and, in particular, it does not have a continuous spectrum. It was shown in [60] that the spectrum [13,14] and correlation functions [42,61–63] of string theory in this background with $Q_5 = 1$ precisely reproduce those of the symmetric orbifold $(T^4)^{Q_1}/S_{Q_1}$, and we will discuss the precise form of this matching (and its consistency with our discussion in the previous subsection) in more detail below. Together with the discussion of the previous subsection, and with the fact that this background (like the free orbifold) maps to itself under T duality, this strongly suggests that string theory on the $(N,1)$ NS-NS AdS_3 background with vanishing RR scalars is dual to the CFT (2.5) with $\theta = \pi$, the free orbifold value.

D. Relations between different backgrounds

Naively, the backgrounds with different values of (Q_1, Q_5) are distinct, and each of them corresponds to a different SCFT. But in fact, as discussed in detail in [15],[2] all the backgrounds with the same $N \equiv Q_1 Q_5$ and with mutually prime (Q_1, Q_5) are related by $SO(5,5,\mathbb{Z})$ U-duality transformations of type II string theory on T^4. Those $SO(5,5,\mathbb{Z})$ transformations that preserve (Q_1, Q_5) remain as duality symmetries of string theory on $AdS_3 \times S^3 \times T^4$, and form a subgroup of $SO(4,5,\mathbb{Z})$ [among these, we already mentioned the $SO(4,4,\mathbb{Z})$ T-duality in the NS-NS description]. On the other hand, the remaining transformations relate the theories with different values of (Q_1, Q_5); we already described the S-duality transformation that maps NS-NS charges to RR charges, and in this subsection we focus on transformations between backgrounds with purely NS-NS charges (a more general discussion of these dualities will be given in Sec. V). In our descriptions in the previous subsections, the theories with different Q_1 and Q_5 looked very different, because we focused on the codimension 4 subspace of their moduli space where the RR scalars vanish. However, the duality transformations do not leave this subspace invariant, but rather they relate configurations with vanishing RR scalars for one value of (Q_1, Q_5) to configurations with nonzero RR scalars for other values. When the string coupling in one description is small, it is large in all other descriptions, so there is no direct relation between

[2]In [15] it was assumed that the free orbifold (2.5) sits at $Q_5 = 1$ at a nonzero value of the RR scalar χ, since at the time it was believed that the $\chi = 0$ theory has a continuous spectrum (as it does for higher values of Q_5). However, up to moving the subspace associated with the free orbifold to sit at $\chi = 0$, the analysis of the dualities relating different backgrounds in [15] is still valid, and we review it in this subsection and in Sec. V.

the perturbative string expansions for different values of (Q_1, Q_5).

Turning on the RR scalars corresponds to an exactly marginal deformation of the CFT, so this implies that all the theories with different mutually prime (Q_1, Q_5) but the same $N = Q_1 Q_5$ sit on the same moduli space of exactly marginal deformations. In particular, they can all be described as exactly marginal deformations of the free sigma model on (2.5). This sigma model actually has an 84-dimensional space of exactly marginal deformations. Sixteen of these are the metric and B field on the T^4 in $(T^4)^N/S_N$ (which map to the metric and B field of the T^4 in the NS-NS description). Four additional ones are blowup modes of the $\mathbb{Z}_2$ singularity of this orbifold. The other 64 exactly marginal deformations take the form $J_i \bar{J}_j$, where i (j) go over the 8 left-moving (right-moving) $U(1)$ currents of the SCFT. 16 of these deformations are the metric and B field on the $\hat{T}^4$ in (2.5), 16 of them can be thought of as changing the metric and B field on the “center of mass” T^4 in the orbifold, and the remaining 32 mix the $U(1)$ symmetries of the orbifold and those of $\hat{T}^4$. As discussed in [15], turning on the 4 RR scalars on AdS_3 corresponds in the CFT to a linear combination of turning on the blowup modes of the orbifold, and specific $J\bar{J}$ deformations mixing the T^4 and $\hat{T}^4$ symmetries. For every mutually prime pair (Q_1, Q_5) there is a codimension 4 subspace of the 20-dimensional space of non-$J\bar{J}$ deformations that maps to the (Q_1, Q_5) background with vanishing RR scalars, and on that subspace the specific $J\bar{J}$ deformations that are related to string theory on AdS_3 deform the currents such that there are $4+4$ $U(1)$ currents of level Q_1, and $4+4$ additional currents (orthogonal to them) of level Q_5. In Fig. 1 we show a slice of these subspaces for $N = 30$, where we turn on only the string coupling and the 10-dimensional RR scalar field χ (this determines also the value of the RR 4-form on the T^4); we draw this slice in the language of the RR background with $Q_1 = 30$ and $Q_5 = 1$ (with a fixed T^4 shape), and describe the positions of the subspaces mentioned above in this parametrization (a similar figure for $Q_1 = 6$ appears in [15]). This background is mapped to itself under a $\Gamma_0(N)$ subgroup of an $SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$ subgroup of the U-duality group, which acts on $\tau = \chi + \frac{i}{g_{10}}$ in the same way as the $\Gamma_0(N)$ subgroup of $SL(2,\mathbb{Z})$, and we draw a fundamental domain containing all inequivalent theories (in this slice).[3] In Sec. V we will describe in detail where all the subspaces mentioned above [of (Q_1, Q_5) charges with vanishing RR fields] sit inside this slice; they are denoted by thick black lines in the figure. Note that the subspaces corresponding to charges (Q_1, Q_5) and (Q_5, Q_1) are identical, since the two descriptions are related (for fixed T^4 shape) by a combination of S duality, T duality on all 4 cycles, S duality and another T duality on all 4 cycles.

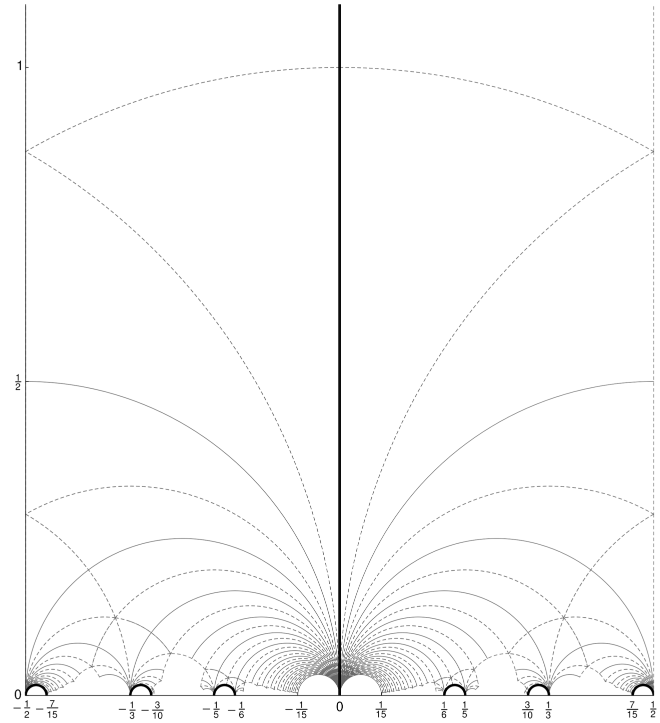

FIG. 1. The subspace of the moduli space parametrized by $\tau_{\mathrm{RR}} = (\chi + i/g_{10})$ in the RR description of the $Q_1 = 30$, $Q_5 = 1$ background, for fixed T^4 moduli. A fundamental domain under the U-duality subgroup $\Gamma_0(N)$ [that keeps both the charges $(Q_1, Q_5) = (N, 1)$ and the T^4 normalized metric fixed] is bounded by the grey lines, and divided into different $SL(2,\mathbb{Z})$ fundamental domains. The $(Q_1, Q_5) = (N, 1)$ NS-NS background with vanishing RR scalars sits on the imaginary axis at $\chi = 0$, and is weakly coupled close to $\tau = 0$. Its S dual, the $(N, 1)$ RR background, is weakly coupled close to $\tau = i\infty$. The two limits are continuously connected to each other as a function of g_{10}, drawn in a vertical thick black line, and this line maps to the free orbifold (2.5). For every other (Q_1, Q_5) background with $Q_1 Q_5 = N$, there is a line describing its singular ($\chi' = 0$) locus, connecting the weakly coupled (Q_1, Q_5) NS-NS and RR backgrounds. These are the rest of the thick black lines in the figure. Up to the sign of χ, the (15,2) line connects the NS-NS theory at $\tau = 1/2$ to the RR theory at $\tau = 7/15$, the (10,3) line connects the NS-NS theory at $\tau = 1/3$ to the RR theory at $\tau = 3/10$, and the (6,5) line connects the NS-NS theory at $\tau = 1/5$ to the RR theory at $\tau = 1/6$.

The other 64 exactly marginal deformations, that are not related to scalar fields in AdS_3, are all related to changes in the boundary conditions for the $U(1)^8 \times U(1)^8$ CS gauge fields on AdS_3 (these can be written as 8 copies of a $\frac{k}{4\pi} \int A \wedge dB$ theory, where k is equal to either Q_5 or Q_1, depending on the gauge field). For a given choice of complex coordinates in the CFT, they can be thought of as choosing which 8 bulk gauge fields have a boundary condition in which their z component is fixed at the

[3]The number of fundamental domains of $SL(2,\mathbb{Z})$ inside a fundamental domain of $\Gamma_0(N)$ is $N \prod_{p|N, p \text{ prime}} (1 + p^{-1})$ [64]. For $N = 30$, which we draw in the figure, it is 72.

boundary (so that they correspond to antiholomorphic currents), while the other 8 gauge fields have their $\bar{z}$ component fixed at the boundary.

Note that for $Q_5 > 1$ with vanishing RR scalars the CFT contains a long-string sector with a continuous spectrum, which is far from evident in our description of the CFT as a deformation of the symmetric orbifold. The subsector of the CFT describing the long strings is believed to be described as an S_{Q_1} symmetric orbifold [46,65–67], and it would be nice to understand how this is related to the picture described above.[4]

There is another subspace of the moduli space of the deformed free orbifold which is singular, which is the subspace where the metric is given by (2.5) but the theta angle on the vanishing 2-cycle at the $\mathbb{Z}_2$ orbifold singularity of (2.5) is taken to $\theta = 0$ (instead of its value $\theta = \pi$ at the free orbifold point [69]). This subspace is not the same as the singular subspaces that are weakly coupled NS-NS-background strings; we believe that in the RR background of Fig. 1 it sits at $\chi = \pm 1/2$, on the boundary of the fundamental domain.

E. Questions

The picture described in this section raises 3 questions:

(1) String theory in NS-NS AdS_3 backgrounds is believed [7,8,38,45,70] to describe not a background with fixed Q_1, but rather a grand-canonical ensemble of theories with different Q_1 (and the same Q_5). How is this consistent with the picture described above, and how is it consistent with the detailed matching of the correlation functions of the $Q_5 = 1$ string to the free orbifold (2.5)?

(2) For $Q_5 = 1$ we mentioned that the NS-NS perturbative expansion matches the $1/N$ expansion of the free orbifold (2.5). However, the orbifold is free for any value of the moduli of the torus, while (2.3) implies that the $(N, 1)$ NS-NS string theory becomes strongly coupled once $v_4 > N$. Moreover, for $v_4 \gg N$ it seems that the same theory should have a different weakly coupled description, as string theory on the RR background. How is this consistent?

(3) The SCFT (2.5) (dual to string theory with $Q_5 = 1$) has a decoupled $\hat{T}^4$ sector, which did not show up in the mapping of this string theory to the free orbifold. Moreover, this sector is still essentially decoupled from the rest of the SCFT even after we deform it by $J\bar{J}$ deformations to go to other values of (Q_1, Q_5) (one can think of these deformations as just modifying the energies of states with given charges). How is this decoupled sector realized in string theory?

The answers to these questions will be discussed in the next three sections of the paper, respectively. The three sections are almost completely independent of each other, so readers who are just interested in one question can jump directly to the relevant section.

III. STRING THEORY AND THE GRAND-CANONICAL ENSEMBLE

The near-horizon limit suggests that (the nonperturbative completion of) string theory on $AdS_3 \times S^3 \times T^4$ with fixed fluxes Q_1 and Q_5 should be equivalent to the conformal field theories discussed above, at any point in their moduli space. In principle, one can consider instead of the quantum gravity theory (and its dual CFT) with some fixed flux Q, the grand-canonical ensemble defined as the sum over the theories with flux Q, weighted by $e^{-\mu Q}$. In the bulk this can be thought of as choosing a different boundary condition for the 2-form field potential on AdS_3, for which the integral over AdS_3 of its field strength is equal to Q.

Somewhat surprisingly, it turns out that perturbative string theory on the NS-NS background with NS5-brane flux Q_5 computes such a grand-canonical ensemble with respect to Q_1, rather than being dual to a specific conformal field theory.[5] This was first understood in the RNS formalism for $Q_5 \geq 2$ [7,8,38], and it was then claimed to be the case also in the hybrid formalism for $Q_5 = 1$ [45,70]. In this section, we explain in detail how the grand-canonical ensemble for $Q_5 = 1$ is consistent with the relation between free orbifold correlation functions and perturbative string theory.

A. A brief review of string theory in the NS-NS background

In this subsection we review the relevant information about string theory on $AdS_3 \times S^3 \times T^4$ with a purely NS-NS background.

As reviewed in Sec. II A, when we take the near-horizon limit of Q_1 fundamental strings and Q_5 NS5-branes wrapped on T^4, we obtain this AdS_3 background, with a fixed value for the six-dimensional string coupling (related to the vacuum expectation value of the dilaton field) given by $g_6'^2 = Q_5/Q_1$. So from this point of view, we would expect string theory on the NS-NS background to have a fixed value of Q_1 and not to have any continuous NS-NS parameters beyond the moduli of the T^4.

However, if we directly study string theory on the NS-NS AdS_3 background, either in the RNS formalism (for $Q_5 \geq 2$) [5,7,8,38] or in the hybrid formalism [45,70], we find a rather different picture. The world sheet action depends explicitly on Q_5, but the parameter Q_1 does not appear in it. On the world sheet one can have as usual a

[4]It was suggested in [68] that perhaps even the full CFT has a description as a deformed symmetric orbifold of this type.

[5]Note that perturbative string theory in the RR background is believed to correspond to fixed fluxes; in particular string theory in this background has a T-duality symmetry exchanging Q_1 and Q_5.

continuous parameter g_s related to the coefficient of $\int d^2\sigma \sqrt{\det(h)} R[h]$ where h is the world sheet metric, which weighs connected string diagrams arising from genus g surfaces by g_s^{2g-2}. Unlike in most other string theories, here this parameter is not related to the expectation value of any field in space-time, but it is still present on the world sheet.

In addition, when computing from the world sheet the central charge appearing in the space-time operator product expansion (OPE) of two CFT energy-momentum tensors or NS-NS $U(1)$ currents, one finds that it is given by $c = 6Q_5 I$ and by $k = I$, respectively, where I is a specific operator on the world sheet (naively it depends on a position on the boundary of AdS space, but in fact this dependence is trivial) [5]. If the string theory corresponded to the CFT suggested by the near-horizon limit, we would expect to have Q_1 appearing in these equations instead of I, but instead the operator I appears, whose correlation functions are not equivalent to replacing it by a constant. The correlation functions of this operator are not arbitrary, but are fixed by the world sheet theory. The fact that I is not a constant prevents the interpretation of string theory as dual to a specific CFT with a given central charge, and from having the expected space-time factorization properties.

The interpretation of this property was suggested in [38,45,70]. Since I is a vertex operator on the world sheet, it is natural to turn on a coupling μ for it, like for any other vertex operator. Then, naively one obtains a string theory labeled by two continuous parameters, g_s and μ. However, the properties of correlation functions of I imply a specific dependence of correlation functions on μ [8,38], and this dependence is such that its effect can be swallowed into a rescaling of the string coupling g_s and a μ-dependent normalization of the world sheet vertex operators. So, in this normalization of the vertex operators, string theory on AdS_3 depends just on a single continuous parameter g_s. It was then suggested that this string theory should be identified with a grand-canonical ensemble of the (Q_1, Q_5) CFTs reviewed in Sec. II, with the same Q_5 but with different values of Q_1, weighted by p^{Q_1}, with some specific relation between the continuous parameters p and g_s. It was argued in [38] that one can obtain the CFT with fixed Q_1 by a Legendre transform of the string theory. In this section, following the analysis of $Q_5 = 1$ partition functions in [45], we describe the precise relation between the grand-canonical ensemble of CFTs and the dual string theory. For $Q_5 = 1$ we argue that the precise way to obtain fixed-Q_1 CFT correlation functions from string theory is more complicated than a Legendre transform (though it agrees with it at leading order), and we conjecture that this should be true also for $Q_5 > 1$.

B. The genus expansion for symmetric orbifolds

Before introducing the grand-canonical ensemble, it is illuminating to understand why the fixed N orbifold theory (2.5) does not behave as a string theory. Specifically, it does not have a proper genus expansion. Even though we focus on the $(T^4)^N/S_N$ orbifold in this paper, our analysis of the free orbifold in this section is relevant for any symmetric orbifold.

We begin by briefly reviewing the construction of the genus expansion for the (fixed N) $(T^4)^N/S_N$ symmetric orbifold, following [39,40] (see also [18,20,29,42]). The Hilbert space of the symmetric orbifold at low energies and large N can be understood as a Fock space of single-cycle-twist operators, divided into different w sectors. The basic operator for a given such sector $1 < w \leq N$ is the w-cycle twist operator, defined as

$$\sigma_w(z) = \frac{1}{w(N-w)!} \sum_{h \in S_N} \sigma_{h^{-1}(1,\ldots,w)h}(z), \qquad (3.1)$$

where $(1, \ldots, w)$ is the cyclic permutation of the first w elements, and $\sigma_g(z)$ is the twist operator with holonomy $g \in S_N$ permuting the N copies as one goes around it. The sum over S_N ensures the gauge invariance of the operator. Our choice of normalization may appear nonstandard, but as we will see below, this is the correct normalization for comparison with string theory vertex operators. It is mathematically natural, as the denominator is the stabilizer size of $(1, \ldots, w)$ in S_N (by S_N conjugations), so that each different permutation is counted in (3.1) with weight 1. In this section we will consider only correlation functions of σ_w's, but the extension to more general single-cycle operators (with additional CFT excitations in addition to the twist operator) and to untwisted-sector operators (which have $w = 1$) is straightforward.

The insertion of $\sigma_w(0)$ can be understood as cutting a hole around $z = 0$, with a cyclic gluing of w copies of T^4 around the boundary circle [different ones for each term in (3.1)], and a trivial gluing for the rest of the $(N - w)$ copies. A correlation function of $\sigma_w(z)$'s will get contributions from consistent gluings between the insertions. Each such consistent gluing between the N copies gives a nontrivial covering of the original CFT manifold. The contribution of each gluing to the correlation function is proportional to the partition function of the T^4 CFT on the covering space. For this reason, the covering space is believed [40] (and later also shown [42,61–63,71]) to be identified with the string world sheet.

We begin with the two-point function of two σ_w's on the plane (which can be conformally mapped to the sphere), given by [40]

$$\langle \sigma_w(z) \sigma_{w'}(0) \rangle = \frac{N!}{w(N-w)!} \frac{\delta_{w,w'}}{|z|^{2\Delta_w}}, \qquad (3.2)$$

with $\Delta_w = \frac{c}{24}(w - 1/w)$ (where in our case $c = 6$ is the central charge of the T^4 SCFT). In this case all consistent gluings are related by a gauge transformation, and the topology of the gluing is of a sphere with two insertions of degree-w branch points (more precisely, this is the topology of the nontrivial part of the correlation function which involves w copies of the T^4, ignoring all other copies). We thus understand (3.2) as a single world sheet diagram contribution with genus zero. The combinatorial factor of the diagram can be understood as follows. In the normalization we choose, each gluing is summed over once, and we only need to count how many different permutations are gauge equivalent to $(1,\ldots,w)$. This is given by $N!$ divided by the size of the stabilizer of $(1,\ldots,w)$, which is $w(N-w)!$.

In a general diagram contributing to a correlation function on the plane, there will be N_c copies of T^4 which are involved in nontrivial permutations, while the other $(N-N_c)$ copies may be viewed as "vacuum diagrams". We will call a diagram "connected" if all the N_c copies are permuted nontrivially (rather than having some of them only permuted among themselves). By the Riemann-Hurwitz formula, in connected diagrams N_c is related to the genus g of the covering space by

$$N_c = 1 - g + \frac{1}{2}\sum_{j=1}^{n}(w_j - 1). \tag{3.3}$$

The contribution from such connected diagrams goes as

$$\langle \sigma_{w_1}(z_1)\ldots\sigma_{w_s}(z_s)\rangle_{\text{conn}} = \sum_{g=0}^{g_{\max}} \frac{N!}{N_c(N-N_c)!}\cdot(\ldots), \tag{3.4}$$

where $(\ldots)$ stands for the contribution of diagrams with the appropriate value of N_c, given in terms of the path integral over the appropriate branched covering, divided by the appropriate vacuum path-integral [39]. For a given N_c, the combinatorial factor in (3.4) is simply the orbit size of the cyclic permutation $(1,\ldots,N_c)$ in S_N; in a specific diagram, after choosing the N_c participating sheets and their cyclic ordering, there is just one permutation from (3.1) which contributes for each operator. Here we are interested in the large N limit in which the w_i are kept fixed; for finite values of N there is an extra restriction that $N_c \le N$.

In the literature [39,40] one usually defines a normalized version $\tilde{\sigma}_w$ of the twisted operators (3.1), in which their two-point function (3.2) is normalized to one. Then one can ask about the leading large N behavior of a given connected diagram to (3.4). The result from genus g diagrams is

$$\langle \tilde{\sigma}_{w_1}\cdots\tilde{\sigma}_{w_n}\rangle_{\text{conn},g} \propto \frac{N!}{N_c(N-N_c)!}\prod_{j=1}^{n}\sqrt{\frac{w_j(N-w_j)!}{N!}} \sim N^{N_c-\frac{1}{2}\sum_{j=1}^{n} w_j} = N^{1-g-n/2}. \tag{3.5}$$

This result looks very appealing, after comparing it to the general structure expected in string theory,

$$\langle O_1\cdots O_n\rangle_{\text{conn}} = \sum_{g=0}^{\infty} g_s^{2g-2+n}\cdot \sum_{\text{Diagrams with genus } g}(\ldots.). \tag{3.6}$$

A naive comparison gives $g_s^2 = 1/N$, with an identification between string theory diagrams and the symmetric orbifold's sum over branched coverings.

However, this identification actually works only at leading order in $1/N$. We will give two different reasons to suspect the identification. The first reason, which was noticed already in [20], is that the right-hand side of (3.5) describes only the leading $1/N$ behavior from a given genus, and receives $1/N$ corrections from expanding the product in the middle. Thus, at subleading order in $1/N$, a given correlator will receive contributions from higher-genus diagrams, but also from the subleading corrections to the planar diagrams, and we find

$$\langle O_1\cdots O_n\rangle_{\text{conn}} = g_s^{n-2}\cdot(\text{genus}\,0) + g_s^{n}\cdot(\text{genus}\,1 + \#\cdot\text{genus}\,0) + \cdots. \tag{3.7}$$

This is not what we expect in (3.6). Note that because of the structure of (3.5), it is not possible to fix this by redefining g_s and/or the operator normalizations at higher orders in $1/N$.

The second problem with identifying $g_s^2 = 1/N$ arises for disconnected diagrams. For simplicity, consider the four-point function of two w_1 and two w_2 twist operators. Diagrammatically the four-point function has a leading disconnected piece, with w_1 sheets permuted among themselves and w_2 other sheets permuted among themselves. Following the string theory picture, one would expect this contribution to be equal to the product of the two two-point functions. However, for a given N, the combinatorial factor associated with this disconnected piece also accounts for the fact that the w_1 and w_2 cycles should not overlap. The exact ratio between the disconnected piece and the product of the connected diagrams is thus

$$\frac{\langle \sigma_{w_1}(z_1)\sigma_{w_1}(z_2)\sigma_{w_2}(z_3)\sigma_{w_2}(z_4)\rangle_{\text{disc}}}{\langle \sigma_{w_1}(z_1)\sigma_{w_1}(z_2)\rangle\langle \sigma_{w_2}(z_3)\sigma_{w_2}(z_4)\rangle} = \frac{(N-w_1)!(N-w_2)!}{N!(N-w_1-w_2)!} = 1 + O(1/N), \tag{3.8}$$

and is not exactly equal to 1. As a result, at fixed N, correlation functions do not obey the expected "cluster decomposition" of string theory beyond the leading order in $1/N$.[6]

[6]We stress that the CFT still maintains standard field-theory cluster decomposition at fixed N. It is only the string theory interpretation that breaks. It was noticed in [38] that the string theory dual for $Q_5 > 1$ does not obey field-theory cluster decomposition, due to the Legendre transform; we will comment further on $Q_5 > 1$ below.

C. The grand-canonical ensemble

To solve this issue we would like to work not with fixed N but rather in the grand-canonical ensemble with a chemical potential for N. As we reviewed at the beginning of this section, this is generally expected for string theory on AdS_3 with NS-NS backgrounds. It was suggested in [38] that the string theory at $Q_5 > 1$ is related to a fixed CFT by taking a Legendre transform with respect to the coefficient of the operator I. A slightly different suggestion was made in [45] for $Q_5 = 1$, that string theory and the dual CFT are related by a Laplace transform, so that the string theory corresponds to a grand-canonical ensemble of the (Q_1, Q_5) theories with the same NS-NS moduli, with a chemical potential for Q_1. In this subsection we analyze this suggestion in detail for $Q_5 = 1$, generalizing the analysis of the partition function in [45] to general correlation functions. In this case we just have a grand-canonical ensemble of (2.5) with respect to N; since this does not affect the $\hat{T}^4$ factor, we will ignore it here and return to it in Sec. V.

Denoting by $Z_N[J]$ the partition function of the $\mathrm{Sym}_N(T^4)$ CFT on some manifold with sources J for various operators, the grand-canonical partition function is defined as[7]

$$Z_p[J] = \sum_{N=0}^{\infty} p^N Z_N[J], \tag{3.9}$$

with $Z_0[J] = 1$. Sources for untwisted-sector operators exist for any N, while sources for twisted operators start appearing at some $N_{\min}$ that depends on the twisted sector. Note that we are summing here $Z_N[J]$ rather than $Z_N[J]/Z_N[0]$ [which would be the generating function for correlation functions in the $\mathrm{Sym}_N(T^4)$ CFT], since already the vacuum partition function (discussed in [45]) is nontrivial when the CFT lives on higher-genus manifolds.

Simplifying to the case where we turn on sources just for the leading twist operators in each twisted sector of the orbifold, the $\mathrm{Sym}_N(T^4)$ generating function is given schematically by[8]

$$Z_N[J] = \frac{1}{N!}\int D^N X e^{-S[X]} \exp\left(\sum_{w=1}^{\infty}\int d^2 z J_w(z)\sigma_w(z)\right). \tag{3.10}$$

The expansion of $Z_N[J]$ generally includes disconnected coverings. Some of the connected components include operator insertions and others, the vacuum diagrams, do not. The normalization (3.1) exactly allows us to compute the combinatorial factor from each component on equal footing. Let us denote by $N_1, \ldots, N_k$ and $n_1, \ldots, n_k$ the number of copies of T^4 of the different connected components (which can differ in the number of copies and/or in the operator insertions) and their multiplicity, respectively, so that $\sum_{i=1}^{k} N_i n_i = N$. The generalization of (3.4) is given by

$$Z_N[J] = \sum_{n=0}^{\infty} \frac{\left(\sum_{w=1}^{\infty}\int d^2 z J_w(z)\right)^n}{n!} \times \sum_{\substack{\text{Disconnected diagrams}\\ \sum_{i=1}^{k} n_i N_i = N}} \prod_{i=1}^{k} \frac{1}{n_i! N_i^{n_i}} \left(Z^{(i)}_{\text{covering}}\right)^{n_i}, \tag{3.11}$$

with $Z^{(i)}_{\text{covering}}$ the partition function coming from the ith connected component [such that the product of the partition functions contains also insertions of n $\sigma_{w_j}(z_j)$ operators].[9] We stress again that some of these components include no operator insertions and correspond to vacuum diagrams. For example, in (3.4) there is a single $n_1 = 1, N_1 = N_c$ connected diagram with all the insertions, and $n_2 = N - N_c$ vacuum diagrams with $N_2 = 1$ each. For given operator insertions, not all the possible N_i's and n_i's are included in the sum, as only some will correspond to a possible diagram. For example, only $N_i = 1$ diagrams are possible when all $w_j = 1$. Of course, for some values of $\{w_j\}$ there are no covering spaces at all if the corresponding permutations cannot be multiplied to give the identity permutation (and in particular there are no diagrams if any $w_j > N$).

Importantly, the only dependence on N in (3.11) is in the definition of the sum over N_i, n_i. Considering now the grand-canonical ensemble (3.9), the sum over the diagrams is no longer constrained by N. Notice that as (3.1) is cyclic, each insertion will appear in a single connected component.[10] As a result, the now unconstrained sums over n and n_i nicely exponentiate into connected components

[7]As we will discuss below, p is related to the string coupling. In general it is not clear if the sum (3.9) converges, and correspondingly we do not expect string perturbation theory to converge but just to be an asymptotic series. Our statement is that the asymptotic series on the two sides of the duality are the same. In the special case of the free orbifold CFT on a sphere we will see that the sum does converge (at least for any finite number of operator insertions).

[8]We include here the $w = 1$ case even though $\sigma_1(z)$ is trivial, because the same formulas will be relevant also for sources for any other operators from the same sectors.

[9]We are not writing down explicitly the dependence of the CFT partition function on the metric (some such dependence is required by the conformal anomaly), and, if it is on a manifold of genus $G \geq 1$, on the spin structure. The covering spaces inherit the metric and spin structure in an obvious way from those of the original manifold.

[10]The fact that only for single-cycle operators the grand-canonical partition function exponentiates further supports their identification as single-string states.

$$Z_p[J] = \exp\left\{\sum_{n=0}^{\infty}\frac{(\sum_{w=1}^{\infty}\int d^2zJ_w(z))^n}{n!}\sum_{\substack{\text{Connected}\\ \text{diagrams}}}\frac{p^{N_c}}{N_c}Z^{\{w_j\}}_{\text{covering}}\right\}, \tag{3.12}$$

with N_c the number of T^4's in the connected covering, and $Z^{\{w_j\}}_{\text{covering}}$ the corresponding partition function in the presence of the n $\sigma_{w_j}(z_j)$ operators. In this expression, the vacuum diagrams appear in the $n = 0$ term in the exponent. The vacuum partition function is known to exponentiate nicely in the grand-canonical ensemble [43,44]. Using the normalization (3.1), (3.12) is a generalization of that statement which incorporates operator insertions.[11]

Up to now our discussion was valid for the CFT on any manifold, but let us now consider the case of the CFT on the sphere (higher-genus surfaces will be discussed in the next subsection). In this case there is a single vacuum diagram in Z_N, which gives $Z_N[0] = Z_1[0]^N/N!$. Thus, the grand-canonical partition function for the sphere can also be written as

$$Z_p^{\text{sphere}}[J] = \exp\left\{\sum_{n=1}^{\infty}\frac{\left(\sum_{w_n=1}^{\infty}\int d^2zJ_w(z)\right)^n}{n!}\times\sum_{\substack{\text{Connected}\\ \text{non-vacuum}}}\frac{p^{N_c}}{N_c}Z^{\{w_j\}}_{\text{covering}} + p\cdot Z_1[0]\right\}. \tag{3.13}$$

Unlike the fixed N ensemble discussed above, the exponential form of (3.12) has a nice interpretation as a perturbative string theory. What is the corresponding string coupling? Redefining our operators by

$$\sigma'_w = p^{-w/2}\sigma_w, \tag{3.14}$$

and using (3.3), the grand-canonical generating function for σ'_w takes the form

$$Z_p[J'] = \exp\left\{\sum_{n=0}^{\infty}\frac{\left(\sum_{w=1}^{\infty}\int d^2zJ'_w(z)\right)^n}{n!}\times\sum_{\substack{\text{Connected}\\ \text{diagrams}}}\frac{p^{1-g-n/2}}{N_c}Z^{\{w_j\}}_{\text{covering}}\right\}. \tag{3.15}$$

Comparing with (3.6) gives exactly the string theory genus expansion with

$$g_s^{-2} \propto p. \tag{3.16}$$

[11]It would be interesting to study extensions of (3.12) to incorporate defects [72], and other permutation groups [73,74].

The problems discussed in the previous subsection disappear when going to the grand-canonical ensemble.

Note that *a priori* one may expect string theory to compute a grand-canonical generating function for the correlation functions, where the coefficient of p^N in the expansion of an n-point correlation function would be the correlation function in the Nth CFT. Instead, we find that string theory computes correlation functions that are derivatives of $Z_p[J]$ divided by $Z_p[0]$, and these are not directly related to correlators for specific values of N (which are derivatives of $Z_N[J]$ divided by $Z_N[0]$). To obtain correlators with a specific N one has to separately inverse-Laplace transform the correlation functions and the vacuum diagrams of the string theory. In particular, it seems that one should view the grand-canonical ensemble not as a sum over different independent theories, but more like summing over some extra "particle number' charge in a given theory, that is present also in the vacuum diagrams.[12]

To make the matching to string theory more precise, we can define an operator "I_{CFT}" that would couple to $\log(p)$, and that should be related to the operator I discussed above on the world sheet [5,8,45]. For the free orbifold ($Q_5 = 1$) this can be identified with "N" inside the sum over N in (3.9). The vacuum expectation value of I_{CFT} for the CFT on the sphere (from vacuum diagrams), related to string theory on Euclidean AdS_3, is given by

$$\langle I_{\text{CFT}}\rangle = \partial_{\log(p)}\log(Z_p[0]) = pZ_1[0]. \tag{3.17}$$

This agrees with the expectation that for large p (corresponding to weak string coupling) the sum over N should have a saddle point with $N \propto p$, with a smaller and smaller variation as p increases. Note that because of the conformal anomaly, $Z_1[0]$ depends on the radius of the sphere that the CFT lives on as $Z_1[0] \propto R^2$. So the relation between p and the typical value of N depends on this radius (see the related discussion in [75]). However, the relation between the string coupling and this typical value does not depend on the radius. This can be seen by rewriting $p^{N_c}Z^{\{w_j\}}_{\text{covering}}$ in (3.13) as $(pZ_1[0])^{N_c}(Z^{\{w_j\}}_{\text{covering}}/Z_1[0]^{N_c})$, where the factor in the second parenthesis is independent of the radius, suggesting that a more precise version of (3.16) is $g_s^{-2} \propto pZ_1[0]$.

To find the expectation value of I_{CFT} for a given connected diagram, we can compute the derivative of the generating function $\log(Z_p[J])$ by $\log(p)$. Directly from the form of $\log(Z_p[J])$ in (3.12) we get

[12]This suggests in particular that the string theory dual to the CFT on a disconnected surface should also correspond to a product of the corresponding Z_p's, rather than the naive product of the corresponding Z_N's, as discussed in [45].

$$\left\langle I_{\rm CFT}\prod_{j=1}^{n}\sigma_w(z_j)\right\rangle_{{\rm conn},g} = N_c\cdot\left\langle\prod_{j=1}^{n}\sigma_w(z_j)\right\rangle_{{\rm conn},g}. \quad (3.18)$$

This agrees with the arguments of [45] for correlation functions of I in the corresponding string theory.

In [38] it was noted that the string theory disconnected 4-point function (for $Q_5 > 1$) does not satisfy the naively expected field-theoretic cluster decomposition, due to the appearance of the central element I in the appropriate channel. Similarly, for $Q_5 = 1$ where we can compute the correlation functions from Z_p, what would naively be identified as a connected correlation function in the dual CFT is given by

$$\begin{aligned}&\langle\sigma_{w_1}(z_1)\sigma_{w_1}(z_2)\sigma_{w_2}(z_3)\sigma_{w_2}(z_4)\rangle_p\\&\quad-\langle\sigma_{w_1}(z_1)\sigma_{w_1}(z_2)\rangle_p\langle\sigma_{w_2}(z_3)\sigma_{w_2}(z_4)\rangle_p\\&=\frac{\partial^2_{J_{w_1}}\partial^2_{J_{w_1}}Z_p[0]}{Z_p[0]}-\frac{\partial^2_{J_{w_1}}Z_p[0]}{Z_p[0]}\frac{\partial^2_{J_{w_1}}Z_p[0]}{Z_p[0]}\\&=(1-\exp(-p\cdot Z_1[0]))\cdot\langle\sigma_{w_1}(z_1)\sigma_{w_1}(z_2)\rangle\\&\quad\times\langle\sigma_{w_2}(z_3)\sigma_{w_2}(z_4)\rangle+(\text{Connected}).\end{aligned} \quad (3.19)$$

The first term on the right-hand side does not vanish when the distance between the pairs of operators becomes large, as expected for a standard CFT. Expanding in orders of p, the sum can be interpreted as the OPE contribution of $(I-\langle I\rangle)^k$ operators.

D. Higher-genus partition functions

Equation (3.12) holds also for the CFT on higher-genus Riemann surfaces (which is expected to be dual to string theory on orbifolds of AdS_3). The main difference is that already without any operator insertions, we now have a sum over holonomies in S_N around the various cycles of the Riemann surface (subject to a global consistency condition), and a similar sum over holonomies appears in the correlation functions of vertex operators. So, in (3.12) the sum over coverings should also allow different S_N holonomies (already for the $n = 0$ term). The definition of the N_c “participating” sheets now involves both the sheets that are permuted by vertex operators, and the ones that are permuted by nontrivial holonomies (it is still the degree of the covering). The splitting into “connected” contributions (which are linked by some permutation, either from a vertex operator or a holonomy) is the same. Because of the sum over the holonomies, there is no longer a simple relation between $Z_N[0]$ and $Z_1[0]$.

Surprisingly, Z_p has peculiar properties for higher genus. Let us denote the genus of the CFT manifold by G, and assume $G > 1$. To repeat the calculation of the string coupling, we can use the generalization of (3.3) to general genus G

$$(1-G)N_c = (1-g) + \frac{1}{2}\sum_{j=1}^{n}(w_j - 1). \quad (3.20)$$

Unlike the $G = 0$ case where we had contributions from $g = 0, \ldots, g_{\max}$, for $G > 1$ the world sheet genus g can range from $g_{\min} = G + \sum_j (w_j - 1)/2$ to values of order N.

Repeating the same redefinition (3.14) of the operators, the generating function for σ'_w now gives the string coupling identification

$$p \propto g_s^{2G-2}, \quad (3.21)$$

as also found by [45]. Note that while for $G = 0$ small coupling corresponded to large p (and therefore large N), for $G > 1$ the string coupling appears to become large for large p. To understand the reason behind this, we repeat the calculation of $\langle I_{\rm CFT}\rangle$ for general G,

$$\langle I_{\rm CFT}\rangle_G = \sum_{N_c=1} p^{N_c} Z^{(G)}_{{\rm Vac},N_c} = \sum_g g_s^{2g-2} Z^{(G)}_{{\rm Vac},g}, \quad (3.22)$$

with $Z^{(G)}_{\rm Vac}$ the corresponding vacuum diagram. For the sphere $G = 0$, we only had the sphere $g = 0$ diagram, which gave the expected leading behavior $\langle I_{\rm CFT}\rangle \sim g_s^{-2}$. For $G > 1$ one would expect a similar relation to arise from the sphere diagram. However because in this theory the world sheet is localized to be a covering of the CFT manifold, the minimal world sheet genus is $g = G$, so we find (for small g_s) the peculiar relation

$$\langle I_{\rm CFT}\rangle_G \sim p \sim g_s^{2G-2}. \quad (3.23)$$

This relation explains why the large p (or N) limit does not have the same relation to the string coupling for different G’s. We emphasize that in any correlation function, (3.20) forbids tree-level $g = 0$ diagrams for $G \geq 1$. This is a unique property of the $Q_5 = 1$ string theory, due to the localization of the world sheet to the boundary, and we do not expect it (and the related properties discussed in this subsection) to persist for $Q_5 > 1$.

An even more peculiar case is the CFT on the torus $G = 1$ (related to the string theory on thermal AdS, which is an orbifold of AdS_3). In this case (3.20) degenerates, so that the world sheet genus is completely determined by the operator insertions (with $g = 1$ for vacuum diagrams), independently of N_c. For that reason we can set $p = 1$, and “mimic” a genus expansion by simply appropriately defining a normalized σ'_w.

E. Final comments

It is natural to guess that a similar relation between string theory and the CFT applies also to $Q_5 > 1$. The natural guess is that while there is an exact CFT for every Q_1 and Q_5, the NS-NS string theory backgrounds for $Q_5 > 1$ are given by a Laplace transform over Q_1

$$Z_{p,Q_5}[J] = \sum_{Q_1=0}^{\infty} p^{Q_5 Q_1} Z_{Q_1,Q_5}[J], \tag{3.24}$$

such that only Z_{p,Q_5} has a well-behaved string perturbation theory. This suggestion is a refinement of the Legendre transform suggested in [38] (which gives a good approximation for large Q_1 or small g_s). Note that when Q_1 is relatively prime to Q_5 the CFTs appearing here are also on the moduli space of the $N = Q_1 Q_5$ free orbifold as discussed above, while for other values of Q_1 they are not; it is not obvious if all values of Q_1 should appear in (3.24) or just the relatively prime ones.

To be precise, for (3.24) to make sense it is essential to properly normalize the sources in the CFT as a function of Q_1 [as we did in (3.1) above]. Furthermore, we saw that to get the expected powers of g_s, it was necessary to also normalize the sources by a power of p (3.14). This is important if we would like to perform an inverse Laplace transform to fixed Q_1. For $Q_5 > 1$, the operator I (related to Q_1) was found to satisfy [8,38]

$$\left\langle I \prod_{j=1}^{n} \Phi_{h_j}(z_j) \right\rangle_{\mathrm{conn},g} = \frac{1}{Q_5}\left(1-g+\sum_{j=1}^{n}(h_j-1)\right) \cdot \left\langle \prod_{j=1}^{n} \Phi_{h_j}(z_j) \right\rangle_{\mathrm{conn},g}, \tag{3.25}$$

where h_j is the CFT dimension of the operator. Comparing with (3.18), this suggests that only in the normalization of Φ_h where each string diagram comes with a power of $p^{1-g+\sum_{j=1}^{n}(h_j-1)}$ does an inverse Laplace transform give the fixed Q_1-generating function. One can further wonder if something can be said about the appropriate fixed-Q_1 normalization. For large Q_1, using the relation $Q_1 \propto p$, the two-point function will scale as $\langle \Phi_h(z_1)\Phi_h(z_2)\rangle_{Q_1} \sim Q_1^{2h-1}$. To find the exact normalization, we need to compute the inverse Laplace transform of the full string theory calculation.

For $Q_5 = 1$, since we have an equality in (3.13) for finite values of p, it seems that the perturbative string theory here captures the full partition function of the free symmetric orbifold, with no nonperturbative contributions. Naively, for higher-genus manifolds, one would expect (based on weakly curved examples) to have different bulk backgrounds that give saddle points leading to nonperturbative contributions (in g_s) to the same CFT partition function. But, as noted in [45], this is not the case here, suggesting that the perturbative expansion around any bulk background (with the appropriate boundary) is complete by itself, giving a strong version of background independence (and in [45] this was shown explicitly for some orbifolds of AdS_3, see also [71]). This is obviously related to the localization of the string world sheets on the boundary. Presumably, all of these properties will no longer be present once we deform away from the free orbifold theory (and, in particular, for $Q_5 > 1$).

IV. BREAKDOWN OF STRING PERTURBATION THEORY

A. The large-volume limit and S duality

As reviewed in Sec. II, the symmetric orbifold (2.5) is believed to be dual (up to the Laplace transformation reviewed in the previous section) to the NS-NS background with $(Q_1, Q_5) = (N, 1)$ and vanishing RR scalars, where the moduli of the T^4 in the symmetric orbifold are identified with the moduli of T^4 in string theory [we will discuss the other $\hat{T}^4$ in (2.5) in the next section]. The six-dimensional string coupling in this background is $g_6^2 = 1/N$ (2.3), so this description is weakly coupled when $N \gg 1$, and the $1/N$ expansion of the free orbifold correlation functions [39,41,42] was argued to exactly match with the NS-NS perturbative expansion. However, while for some observables the perturbative expansion is governed by g_6, for other observables it is governed by g_{10}, and, in particular, the latter parameter is expected to control string perturbation theory in the large-volume limit. We denote the volume of the torus in string units in this frame, which is also the dual symmetric orbifold metric, by V. Since $g_{10}^2 = V/N$ (2.3), this implies that string perturbation theory should break down when V becomes as large as N, even though the orbifold remains free for any value of V. Moreover, for $g_{10}^2 \gg 1$ the S-dual RR background becomes weakly coupled, so we may expect to obtain a different perturbative expansion for the free orbifold in this large-volume regime, that would reproduce this different string perturbation theory. How is this consistent with the free orbifold?

The point is that even though the orbifold (2.5) is free for any V and N, its (nontrivial) $1/N$ expansion, reviewed in the previous section, can sometimes break down. In particular, it was shown in [42] that this expansion can be mapped to the NS-NS perturbative expansion. In this expansion, for large $V \gg 1$, higher-genus diagrams are proportional to V^g, because this factor would arise (for $V \gg 1$) from the momentum states running in the loops. For instance, the torus partition function of a scalar of radius $R \gg 1$ is proportional to R. The general structure of the $1/N$ perturbation theory that arises from the free orbifold (on a sphere) for $V \gg 1$ is thus schematically

$$\langle O_{w_1}(z_1)\ldots O_{w_n}(z_n)\rangle = N^{1-\frac{n}{2}}\left((g=0)+\frac{V}{N}\cdot(g=1)+\cdots + \left(\frac{V}{N}\right)^{g_{\max}}\cdot(g=g_{\max})\right), \tag{4.1}$$

where $g_{\max}$ is the maximal genus arising in that correlation function. In the language of the free orbifold, even though it lives on a topologically trivial space, there are nontrivial loops going around branch points where twist operators are inserted, along which the different T^4 factors are permuted, and there are nontrivial zero modes of the T^4 scalars along these loops, such that the integrals over them give the factors of V in (4.1). Thus, even though the description (2.5) is always free, its $1/N$ expansion indeed breaks down (in any correlation function that contains $g > 0$ contributions) once V becomes as large as N, consistent with the bulk string theory.

As mentioned above, S duality in string theory seems to imply that for $V \gg N$ we should find a new perturbative description of the free orbifold, that would match with the RR string theory perturbative expansion, that has $g_{10}^2 = N/V$. Indeed, our analysis in Sec. II A tells us that this string theory has a $(Q_1, Q_5) = (N, 1)$ RR description with parameters

$$g_6^2 = \frac{1}{V}, \quad g_{10}^2 = \frac{N}{V}, \quad \frac{R^2}{\alpha'} = \sqrt{\frac{N}{V}}, \quad \frac{\mathrm{Vol}(T^4)}{\alpha'^2} = N, \qquad (4.2)$$

such that the theory with $V \gg N \gg 1$ seems to be weakly coupled.

While individual correlation functions like (4.1) can be rewritten as expansions in N/V, it is easy to check that there is no sensible expansion in this parameter of the full theory. The resolution of this apparent paradox is that (4.2), which follows from supergravity, is not valid in this regime, since it gives an anti–de Sitter (AdS) radius that is much smaller than the string scale. When this happens, we expect all the stringy modes to be at the AdS energy scale, rather than the string scale. The physics (at least for the gravitational sector) is expected to be governed by Newton's constants in AdS units, which are given by the S-duality invariant values

$$G_N^{(6)}/R^4 = 1/N, \qquad G_N^{(10)}/R^8 = V/N. \qquad (4.3)$$

The fact that $G_N^{(10)}$ is large in AdS units implies that perturbation theory is not valid, despite the naive expectations from (4.2). The bottom line is, as discussed in [23], that for $Q_5 = 1$ the RR picture is always strongly coupled, but the NS-NS picture is weakly coupled for $V \ll N$. For $V \gg N$ both pictures are strongly coupled. It is only for $Q_5 \gg 1$ that there is a region where the RR description is truly weakly coupled, and both the string couplings and Newton's constants in AdS units are small.

B. Breakdown of perturbation theory at high energies

There is another regime where we expect string perturbation theory to break down, which is the regime of high energies (scaling as some power of the inverse string scale and the inverse string coupling). In this subsection, we study the n-point function of single-cycle operators, all with dimensions scaling as $E \gg 1$, and we will see for which energies the $1/N$ expansion breaks down. Note that in a highly curved background it is not clear from the spacetime point of view at which energies this should happen.

If we look at an operator of dimension $L_0 = h$ in the wth twisted sector of the orbifold, its energy (the conformal dimension of the corresponding twisted sector operator) is given by

$$L_0 = \frac{h}{w} + \frac{w - w^{-1}}{4}, \qquad (4.4)$$

where the second term comes from the dimension of the twist field (in the NS-NS sector of the CFT). For a given operator with some $h \gg 1$, this implies that the lowest-energy state it gives will come from the sector with $w \simeq \sqrt{h}$ and will have $E = L_0 + \bar{L}_0 \simeq \sqrt{h}$. Conversely, this implies that typical operators with energy E will come from sectors with $w \simeq E$.

We start by estimating the number of diagrams with genus g that contribute to the n-point correlation function, following [76]. Each genus g diagram satisfies

$$n - n_E + n_F = 2 - 2g, \qquad (4.5)$$

with n_E, n_F the number of edges and faces of the skeleton graph. For high enough E, we can approximate the graphs being triangular $3n_F = 2n_E$, which gives

$$n_E = 3(2g - 2 + n). \qquad (4.6)$$

The diagram is completely determined by the number of edges between each pair of vertices i and j, which we denote by n_{ij}. In a graph with n_E edges, n_{ij} has n_E (4.6) nontrivial elements, and it has to satisfy $\sum_j n_{ij} = w_i$, giving n constraints. So overall there are $3(2g-2) + 2n$ independent variables n_{ij}, each of order $\sim w \sim E$. In this way we can approximate the number of diagrams by

$$\#(\text{coverings with genus } g) \simeq E^{3(2g-2)+2n}. \qquad (4.7)$$

It is not clear how the contribution of individual diagrams scales in the large energy/cycle length limit. If we assume that the contribution of each diagram separately does not scale with the energy, then the $1/N$ expansion breaks down when the number of diagrams grows faster than N^g, which happens [using (4.7)] when $E \sim N^{1/6}$. If individual diagrams grow with the energy in a way that depends on the genus, the expansion could break down at lower scales, such as the naive ten-dimensional Planck scale coming from (2.4), which scales as $N^{1/8}$. In any case, we find that perturbation theory breaks down at high energies going as a negative power of the string coupling, as expected.

V. THE DECOUPLED SECTOR

A. The decoupled sector of the $Q_5 = 1$ CFT

In Sec. II B we reviewed the argument that the SCFT dual of the $Q_5 = 1$, $Q_1 = N$ NS-NS background with vanishing RR scalars is given by a product of the symmetric orbifold over T^4 and an extra $\hat{T}^4$ (2.5). The symmetric orbifold $\mathrm{Sym}_N(T^4)$ has 4 holomorphic and 4 antiholomorphic $U(1)$ currents associated with the "center-of-mass" momentum and winding of the T^4, whose currents are given by the sums of the corresponding $U(1)$ currents over the N copies of the T^4, such that they have level N. The $\hat{T}^4$ CFT has its own 8 (winding and momentum) currents, all at level 1.

Invoking the coset construction, it is possible to rewrite this theory as the semidirect product

$$(T^4)^N/S_N \times \hat{T}^4 \equiv \left((T^4)^N/S_N\right)/U(1)^8_N \rtimes \left(U(1)^8_N \times \hat{U}(1)^8_1\right). \tag{5.1}$$

We will name the first term on the right-hand side the "coset CFT," and the second the "Sugawara CFT." The operators of the full theory are spanned by products of operator pairs, one from the coset and one in the Sugawara $U(1)^8_N \times \hat{U}(1)^8_1$ theory; the product with $U(1)^8_N$ is semidirect, meaning that not all possible products of operators appear in the theory, while the product with $\hat{U}(1)^8_1$ is direct, such that the theory contains any operator in this sector multiplied by any operator in the rest of the CFT.

The left-moving and right-moving energies of the full theory can each be written as a sum

$$E^{L/R} = E^{L/R}_{\mathrm{coset}} + E^{L/R}_{\mathrm{Sugawara}}, \tag{5.2}$$

with E_{Sugawara} given by the Sugawara stress tensor of the 8 left-moving or right-moving currents. Due to unitarity, the Sugawara energy of a state gives a lower bound for the left- and right-moving energies

$$E^{L/R} \geq E^{L/R}_{\mathrm{Sugawara}}. \tag{5.3}$$

We denote the metric and B field of the T^4 in the orbifold by $E = G + B$, and $\hat{E} = \hat{G} + \hat{B}$ for the decoupled $\hat{T}^4$. We also denote the integer winding and momentum charges of the orbifold by w^i, p_i, and the $\hat{T}^4$ charges by $\hat{w}^i$, $\hat{p}_i$ $(i = 1, \ldots, 4)$. In this notation, the left/right Sugawara energy is given by

$$E^{L/R}_{\mathrm{Sugawara}} = \frac{1}{4}\begin{bmatrix} w & p & \hat{w} & \hat{p}\end{bmatrix}\begin{bmatrix} \frac{1}{N}(G + B^T G^{-1} B) & \frac{1}{N}(\pm 1 + B G^{-1}) & 0 & 0 \\ \frac{1}{N}(\pm 1 - G^{-1} B) & \frac{1}{N} G^{-1} & 0 & 0 \\ 0 & 0 & \hat{G} + \hat{B}^T \hat{G}^{-1}\hat{B} & \pm 1 + \hat{B}\hat{G}^{-1} \\ 0 & 0 & \pm 1 - \hat{G}^{-1}\hat{B} & \hat{G}^{-1}\end{bmatrix}\begin{bmatrix} w \\ p \\ \hat{w} \\ \hat{p}\end{bmatrix}. \tag{5.4}$$

We now turn to the string theory background with NS-NS fluxes.[13] The charges w^i and p_i correspond to fundamental string winding and momentum on the T^4, respectively. The extra $\hat{T}^4$ accounts for the charges of RR D-branes wrapping on the T^4. In type IIB string theory, we have 4 charges from D1-branes winding on one of the T^4 circles w^i_{D1}, and $\binom{4}{3} = 4$ from D3-branes winding on 3 circles of the T^4, $w_i^{\mathrm{D3}} = \epsilon_{ijkl} w^{jkl}_{\mathrm{D3}}$. We identify the $\hat{T}^4$ charges with the bulk RR charges by

$$\hat{w}^i = w^i_{\mathrm{D1}}, \qquad \hat{p}_i = w_i^{\mathrm{D3}}. \tag{5.5}$$

What is the meaning of the Sugawara energy bound from the string theory side? This question was answered in [15]. The authors used the description of the $\mathrm{AdS}_3 \times S^3 \times T^4$ background as a near-horizon limit of a flat space brane configuration. The flat-space BPS formula holds even in the decoupling limit, and when subtracting the energies of the strings and 5-branes making up the background, it gives an energy bound[14] on particlelike excitations in AdS_3.

[13]In this section we ignore the fact that perturbative string theory in this background actually involves a grand-canonical ensemble of CFTs, as discussed in Sec. III; namely, we consider the theory with fixed Q_1 and Q_5, even though its perturbative string description is more complicated. The U-duality group that we discuss in this section acts naturally on the backgrounds with fixed Q_1 and Q_5, rather than on the grand-canonical ensemble that appears in the perturbative NS-NS-background string.

[14]This is obvious in the RR sector of the CFT, in which the fermions are periodic around the circle and supersymmetry is preserved. Here we discuss the NS-NS sector of the CFT which is dual to string theory on AdS_3 (where the spatial circle of the CFT is contractible), where the fermions are antiperiodic, so that the configuration does not directly arise as a near-horizon limit. However, $\mathcal{N} = (4,4)$ supersymmetry implies that this sector is related by spectral flow to the RR sector, so the same BPS bounds hold.

We will discuss the energy formula for a general background in the next subsection. For the $(Q_1, Q_5) = (N, 1)$ NS-NS background with T^4 metric G and vanishing B field and RR scalars, the resulting BPS formula is [15]

$$\begin{aligned} E^{L/R}_{\rm BPS} &= \frac{1}{4N^{1/2}}(g_{10}w^2 + g_{10}^{-1}w_{D1}^2 + (g_{10}/V)p^2 \\ &\quad + (V/g_{10})(w^{D3})^2) \pm \frac{1}{2N}p\cdot w \pm \frac{1}{2}w_{D1}\cdot w^{D3} \\ &= \frac{1}{4N}(V^{1/2}w^2 + V^{-1/2}p^2 \pm 2w\cdot p) \\ &\quad + \frac{1}{4}(V^{-1/2}\hat{w}^2 + V^{1/2}\hat{p}^2 \pm 2\hat{w}\cdot\hat{p}). \end{aligned} \tag{5.6}$$

All the (implicit) T^4 metric contractions in this equation are written in terms of the unit-volume metric $(G_0)_{ij} = V^{-1/2}G_{ij}$, where V is the T^4 volume $V = \sqrt{\det(G)}$. In the second line we used (5.5) and the relation (2.3)

$$V = Ng_{10}^2. \tag{5.7}$$

To find the relation to the CFT parameters, we compare the bulk BPS bound (5.6) to the CFT Sugawara bound (5.4). The orbifold metric is immediately identified with the bulk T^4 metric, while the $\hat{T}^4$ metric has the same shape but an inverse volume, $\hat{G} = \frac{1}{V}G$. The fact that we got an inverse volume can be traced to the fact that the D1-brane tension is proportional to $g_{10}^{-1} \sim V^{-1/2}$ (or, alternatively, to the fact that this extra $\hat{T}^4$ comes from Wilson lines which live on a dual torus). Choosing a different $\hat{G}$ in the CFT corresponds in the bulk to different boundary conditions for the $(U(1)\times U(1))^4$ RR gauge fields (analogous to a double-trace deformation).

We can generalize (5.6) to nonzero bulk B fields, by performing the transformation [15]

$$p_i \mapsto p_i + B_{ij}w^j, \quad \hat{w}^i \mapsto \hat{w}^i - \frac{1}{2}\epsilon^{iljk}B_{jk}\hat{p}_l. \tag{5.8}$$

The first transformation gives a B field (identical to the one in the bulk) to the T^4 in the orbifold CFT. The second (coming from the $C_2 \wedge B_2$ coupling in the D3-brane action) amounts to adding $B'^{ij} = -\frac{1}{2}\epsilon^{ijkl}B_{kl}$ in the T-dual frame of the $\hat{T}^4$ sigma model. Altogether, the CFT moduli are related to the string theory moduli (for vanishing RR scalars) by

$$E_{ij} = G_{ij} + B_{ij}, \quad (\hat{E})^{-1,ij} = V\cdot G^{-1,ij} - \frac{1}{2}\epsilon^{ijkl}B_{kl}. \tag{5.9}$$

What is the world sheet interpretation of the decoupled sector? The momentum and winding currents of the orbifold have a clear world sheet dual, coming from the metric and B field of the bulk T^4, whose $U(1)^8$ currents can be used to construct the corresponding currents of the CFT (for $Q_5 > 1$ this was discussed in [7,12], and for $Q_5 = 1$ in [14]). Perturbative string states are charged under these currents. On the other hand, the charged states under the $\hat{U}(1)^8_1$ RR currents can all be constructed as "boundary modes" of the bulk Chern-Simons theory (for general level k this is only true for states whose charges are integer multiples of k, while here $k = 1$). In the bulk they are pure-gauge modes of the RR gauge potentials. One would naively expect the $\hat{T}^4$ currents to have their own world sheet vertex operator, describing the RR potential in the bulk,[15] and to have D-branes (though no perturbative string states) that carry their charge. However, the structure of the dual CFT contradicts this expectation. Notice that each single-string state has its single-string descendants under the NS-NS $U(1)^8_N$ current algebra. In the CFT these are descendants in the core CFT whose symmetric orbifold we are considering. More generally, the existence of a world sheet vertex for a CFT current means there is a $U(1)^8$ current algebra module already in the single-string partition function $Z_{1\text{-string}} = \log(Z)$. However, the partition function of the $\hat{T}^4$ CFT is simply an overall factor in the full partition function. In terms of the string theory, for every multistring state, each of its single strings has its own NS-NS $U(1)^8_N$ descendants, but only one multistring RR $\hat{U}(1)^8_1$, or $\hat{T}^4$, descendant. This suggests that the $\hat{T}^4$ $\hat{U}(1)^8_1$ currents have no local string theory vertex.[16]

An alternative perspective on this is that a $(U(1)\times U(1))^4_1$ Chern-Simons theory on AdS_3 is dual by itself to a $\hat{T}^4$ CFT, with the moduli of the $\hat{T}^4$ determined by the boundary conditions on the CS fields (see, for instance, [77,78]). So this sector of the theory, which is modular invariant by itself, decouples completely from the rest of the string theory; it has no interactions with any string states (and does not even depend on the topology of the interior of the AdS space, just on the boundary conditions). In particular, even though naively one may expect the string theory with $Q_5 = 1$ to contain particlelike D-branes wrapped on cycles of the T^4 which are charged under these $\hat{U}(1)$'s, such D-branes have not yet been found, and this analysis implies that they should not exist as nontrivial boundary states.

This behavior is special to level $k = 1$, and does not apply to the decoupled $U(1)^8_N$ sector; we will discuss in what sense it is decoupled later, after generalizing to other values of Q_5.

B. String dualities and exact CFT deformations

We now extend our discussion to more general string theory backgrounds with different charges, following [15].

[15]One reason why this is not obvious is that in flat space, vertex operators for RR potentials, as opposed to field strengths, exist in some pictures for the world sheet ghosts but not in others.

[16]It is possible that world sheet operators for the total charge of these currents can be defined.

We write the string and 5-brane integer fluxes as a charge matrix

$$Q = \begin{bmatrix} f_1 & d_1 \\ -d_5 & n_5 \end{bmatrix}, \tag{5.10}$$

which labels the (integer) number of F1, D1, D5, and NS5-branes in the original flat-space construction (we assume for simplicity there are no wrapped D3-branes). Before taking the near-horizon limit, the backgrounds are also labeled by $\tau = \chi + i g_{10}^{-1}$ and by $\tilde{\tau} = A_4 + i v_4 g_{10}^{-1}$. Here g_{10} is the ten-dimensional string coupling, $v_4 = \sqrt{\det(G)}$ the volume of the T^4, and χ and A_4 are the RR scalar C_0 and the holonomy of the RR 4-form C_4 on the T^4, respectively.[17] It is useful to write $\tau, \tilde{\tau}$ also as matrices

$$\mathcal{T} = g_s \begin{bmatrix} 1 & -\chi \\ -\chi & g_s^{-2} + \chi^2 \end{bmatrix},$$
$$\tilde{\mathcal{T}} = \frac{g_s}{v_4} \begin{bmatrix} 1 & -A_4 \\ -A_4 & v_4^2/g_s^2 + A_4^2 \end{bmatrix}. \tag{5.11}$$

When taking the near-horizon limit of a brane configuration with a charge matrix Q, the attractor mechanism gives a relation between τ and $\tilde{\tau}$ of the form

$$\det(Q) \cdot 1 = \tilde{\mathcal{T}} Q \mathcal{T} Q^T. \tag{5.12}$$

This minimizes the tension formula $\mathrm{tr}(\tilde{\mathcal{T}} Q \mathcal{T} Q^T)$. For example, in the case of the d_1, d_5 background the equation is $\tilde{\tau} = (d_1/d_5)\tau$.

As in the $Q_5 = 1$ case, we now consider particle excitations in this background. These are labeled by their T^4 winding and momentum $w^i_{\mathrm{F1}}, \tilde{p}_i$, and by the RR winding charges w^i_{D1} and w_i^{D3}. It is useful to organize them in pairs based on their natural T^4 indices

$$q^i = \begin{bmatrix} w^i_{F1} \\ w^i_{D1} \end{bmatrix}, \quad q_i = \begin{bmatrix} \tilde{p}_i \\ w_i^{D3} \end{bmatrix}. \tag{5.13}$$

For a general string theory background, the energy bound found by [15] is given by

$$E^{L/R}_{\mathrm{BPS}} = \frac{1}{4} \begin{bmatrix} q^i & q_j \end{bmatrix} \begin{bmatrix} \frac{\mathcal{T}}{\det^{1/2}(Q)} & \pm Q^{-1} \\ \pm (Q^{-1})^T & \frac{\tilde{\mathcal{T}}}{\det^{1/2}(Q)} \end{bmatrix} \begin{bmatrix} q^k \\ q_l \end{bmatrix}, \tag{5.14}$$

where any T^4 index contraction is written in terms of the unit metric $v_4^{-1/2} G$ (the v_4 dependence appears through $\mathcal{T}, \tilde{\mathcal{T}}$).

The U-duality group of the system is $SO(5,5;\mathbb{Z})$. In this section we are interested only in the $SL(2,\mathbb{Z})_L \times SL(2,\mathbb{Z})_R$ U-duality subgroup which fixes the general form (5.10) of the charges, the vanishing of B_2 and C_2, and the unit-volume metric on the T^4. $SL(2,\mathbb{Z})_R$ is generated by the S-duality of type IIB string theory and by the transformation $\chi \mapsto \chi + 1$. $SL(2,\mathbb{Z})_L$ is generated by $T_{1234} S T_{1234}$ (where T_{1234} means a T-duality transformation on the four cycles of the T^4) and by the transformation $A_4 \mapsto A_4 + 1$. For $g_L, g_R \in SL(2,\mathbb{Z})$, the background parameters are transformed by

$$Q' = g_L Q g_R^T, \quad \mathcal{T}' = g_R^{-1,T} \mathcal{T} g_R^{-1}, \quad \tilde{\mathcal{T}}' = g_L^{-1,T} \tilde{\mathcal{T}} g_L^{-1}. \tag{5.15}$$

The particle charges are transformed by

$$q^{i\prime} = g_R \cdot q^i, \qquad q_j' = g_L \cdot q_j. \tag{5.16}$$

We note that the energy formula (5.14) [as well as the relation (5.12)] is duality invariant under the transformation of the background parameters and the charge lattice.

We would like to describe the structure of the conformal manifold for (5.1). Different values of $\tau, \tilde{\tau}$ correspond to different deformations of (5.1), but because τ and $\tilde{\tau}$ are related through (5.12), each deformation can be labeled by τ alone. The $(N,1)$ NS-NS background is given by

$$Q = \begin{bmatrix} N & 0 \\ 0 & 1 \end{bmatrix}, \tag{5.17}$$

and we consider values of the charges (5.10) that can be mapped to these charges by a U-duality transformation. We can then describe the deformations by τ in this background, and two deformations are identical if and only if there is a U-duality transformation mapping one τ to the other, while keeping the canonical background (5.17) fixed. In other words, the conformal manifold is given by the fundamental domain of the U-duality subgroup which stabilizes the canonical background (5.17).[18] As explained in [15], this subgroup is isomorphic to the congruence modular subgroup $\Gamma_0(N)$, or the set of pairs

$$g_L = \begin{bmatrix} \alpha & \beta N \\ \gamma & \delta \end{bmatrix}, \quad g_R = \begin{bmatrix} -\delta & -\gamma N \\ -\beta & -\alpha \end{bmatrix}, \tag{5.18}$$

with $\alpha\delta - \beta\gamma N = 1$. Figure 1 depicts the fundamental domain of $\Gamma_0(N)$ for $N = 30$, in terms of the S-dual parameter $\tau_{RR} = -1/\tau$. The grey dashed lines separate the domain into (halves) of $SL(2,\mathbb{Z})$ fundamental domains. The number of $SL(2,\mathbb{Z})$ fundamental domains in the fundamental domain of $\Gamma_0(N)$ is given by [15,64]

[17] We assume here vanishing B_2 and C_2 fields on the T^4, although the generalization is straightforward [15].

[18] Of course, we could just as well describe the geometry of the conformal manifold in terms of deformations of any other string theory background U dual to (5.17).

$$(SL(2,\mathbb{Z}):\Gamma_0(N)) = N \prod_{\substack{p|N \\ p \text{ is prime}}} (1+p^{-1}). \qquad (5.19)$$

The vertical black line corresponds to the $(N,1)$ background with vanishing RR scalars $\chi = A_4 = 0$. The point $\tau_{\rm RR} = 0$ corresponds to the $g_{10} = 0$ limit of the NS-NS background. The point $\tau_{\rm RR} = i\infty$ is the strongly coupled limit, or, by S-duality, the $g_{10} = 0$ limit of the $(N,1)$ RR background.

Every NS-NS (and RR) (Q_1, Q_5) background with Q_1,Q_5 mutually prime is related to the canonical $(N,1)$ background (5.17) with $N = Q_1 \cdot Q_5$ by a U-duality transformation, and can be described as a CFT deformation of (5.1). We are specifically interested in the singular locus of these backgrounds (with vanishing RR scalars $\chi = A_4 = 0$), given by

$$\mathcal{Q} = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_5 \end{bmatrix}, \quad \mathcal{T} = \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix},$$
$$\tilde{\mathcal{T}} = \begin{bmatrix} t^{-1}Q_5/Q_1 & 0 \\ 0 & tQ_1/Q_5 \end{bmatrix}, \qquad (5.20)$$

with t the ten-dimensional string coupling in the NS-NS (Q_1, Q_5) background. These backgrounds are related to the canonical background (5.17) by the duality transformation

$$g_L = \begin{bmatrix} aQ_5 & bQ_1 \\ 1 & 1 \end{bmatrix}, \quad g_R = \begin{bmatrix} Q_5 & -Q_1 \\ -b & a \end{bmatrix}, \qquad (5.21)$$

with a, b integers satisfying $aQ_5 - bQ_1 = 1$. Under the transformation (5.15), the dual canonical parameters are

$$g_{10} = a^2 t + b^2 t^{-1}, \quad \chi = -\frac{Q_1 at + Q_5 bt^{-1}}{a^2 t + b^2 t^{-1}}. \qquad (5.22)$$

Written in terms of $\tau_{\rm RR} = -(\chi + i/g_{10})^{-1}$, these lines are drawn as thick black arcs in Fig. 1. Each arc is drawn twice, related by a sign $\chi \mapsto -\chi$, but the two arcs are identified by the action of $\Gamma_0(N)$. The limit $t = 0$ is the weakly coupled limit of the NS-NS (Q_1, Q_5) background, which is also dual by a $T_{1234}ST_{1234}$ duality transformation to the weakly coupled RR (Q_5, Q_1) background.[19] Both are mapped to the point $\tau_{\rm RR} = \pm b/Q_5$ in the figure. The limit $t = \infty$ in (5.20) corresponds by S duality to the weakly coupled RR (Q_1, Q_5) background, and by another $T_{1234}ST_{1234}$, to the weakly coupled NS-NS (Q_5, Q_1) background. Both are mapped to $\tau_{\rm RR} = \pm a/Q_1$ in the figure. Generally, the arcs of the fundamental domain's boundary are identified in a complicated way. We conjecture that the CP-invariant line $\chi = \pm 1/2$ in the figure (in the RR background) corresponds to the orbifold (2.5) at the singular point $\theta = 0$ for the 2-cycle at the $\mathbb{Z}_2$ fixed point; note that this line has $A_4 = \pm N/2$ both in the RR and in the NS-NS descriptions.[20]

For a given t, the string theory background (5.20) is a deformation of the CFT (5.1); this deformation involves both a blowup of the $\mathbb{Z}_2$ singularity of the orbifold, which affects only the coset CFT, and a $J\bar{J}$ deformation, which affects only the Sugawara CFT. In the dual $(N,1)$ background, it is described by τ and $\tilde{\tau}$ given by (5.22). Consider a general deformation labeled by $\tau, \tilde{\tau}$. For the canonical background (5.12) the two are related by

$$A_4 = -v_4\chi, \qquad v_4 = \frac{V}{1+\chi^2 \frac{V}{N}}, \qquad (5.23)$$

with $V = Ng_{10}^2$. Labeling a general deformation by V, χ, the BPS energy bound (5.14) gives in the canonical frame (assuming $B = 0$)

$$E^{L/R}_{\rm BPS} = \frac{1}{4}[w \;\; p \;\; \hat{w} \;\; \hat{p}] \begin{bmatrix} \frac{V^{1/2}}{N} & \pm\frac{1}{N} & -\chi\frac{V^{1/2}}{N} & 0 \\ \pm\frac{1}{N} & \frac{1+\frac{V}{N}\chi^2}{NV^{1/2}} & 0 & \chi\frac{V^{1/2}}{N} \\ -\chi\frac{V^{1/2}}{N} & 0 & \frac{1+\frac{V}{N}\chi^2}{V^{1/2}} & \pm 1 \\ 0 & \chi\frac{V^{1/2}}{N} & \pm 1 & V^{1/2} \end{bmatrix} \begin{bmatrix} w \\ p \\ \hat{w} \\ \hat{p} \end{bmatrix}$$
$$= \frac{1}{4N}\left(V^{1/2}(w-\chi\hat{w})^2 + V^{-1/2}p^2 \pm 2w\cdot p\right)$$
$$+ \frac{1}{4}\left(V^{-1/2}\hat{w}^2 + V^{1/2}\left(\hat{p}+\frac{\chi}{N}p\right)^2 \pm 2\hat{w}\cdot\hat{p}\right). \qquad (5.24)$$

This generalizes (5.6) to the case of nonzero χ. We see that in the BPS formula the χ deformation is equivalent to the shift

$$w^i \mapsto w^i - \chi\hat{w}^i, \qquad \hat{p}_j \mapsto \hat{p}_j + \frac{\chi}{N}p_j. \qquad (5.25)$$

We would like to repeat the exercise of the previous section. Interpreting the BPS bound (5.24) as the Sugawara energy, we can find the Sugawara CFT deformation for every V, χ. However, unlike (5.6), (5.24) is no longer decoupled between the charges of T^4 and $\hat{T}^4$. It is therefore not enough to simply deform the G, $\hat{G}$ metrics. As explained in [15], the Sugawara CFT needs to be further deformed by a mixed $J\bar{J}$ deformation to match with (5.24).

Let us denote the $U(1)^8_N$ orbifold currents by J^i, $\bar{J}^i$, and the $U(1)^8_1$ $\hat{T}^4$ currents by $\hat{J}^i$ and $\hat{\bar{J}}^i$ ($i = 1,\ldots,4$). All the

[19]By weak coupling here we mean a small ten-dimensional string coupling; as discussed above, this does not necessarily imply that the six-dimensional string coupling or Newton's constant are small, so the theory may not really be weakly coupled.

[20]The line $\chi = \pm N/2$ in the $(N,1)$ NS-NS background, which is related to the $A_4 = \pm N/2$ line by T duality in the NS-NS frame, and that appears in the figure as the two (identified) boundary arcs emanating from $\tau_{\rm RR} = 0$, is also singular.

currents are normalized such that the corresponding charges are integers (see the Appendix). A general $J\bar{J}$ marginal deformation is of the form

$$\Delta S = 2\pi \int d^2z \left(M_{ij}^{11} J^i \bar{J}^j + M_{ij}^{12} J^i \hat{\bar{J}}^j + M_{ij}^{21} \hat{J}^i \bar{J}^j + M_{ij}^{22} \hat{J}^i \hat{\bar{J}}^j \right), \tag{5.26}$$

for some 8×8 matrix $M_{ij}^{\alpha\beta}$ ($\alpha, \beta = 1, 2, i, j = 1, \ldots, 4$). The M^{22} matrix simply deforms the $\hat{T}^4$ moduli, $\hat{E} \mapsto \hat{E} + M^{22}$. M^{11}, M^{12} and M^{21} do not have such an interpretation, but together with M^{22} they label an $SO(8,8)/(O(8) \times O(8))$ conformal manifold. In the appropriate sense (which takes care of the different levels), $M_{ij}^{\alpha\beta}$ deforms the metric and the B field on the $T^4 \times \hat{T}^4$ manifold.

For every V and χ, it is possible to find the exact $M_{ij}^{\alpha\beta}$ deformation such that the new E_{Sugawara} will exactly agree with the string theory BPS bound (5.24). Starting from the moduli (5.9) with the appropriate V, the answer is

$$\begin{aligned} M_{ij}^{11} &= 0, \\ M_{ij}^{12} &= M_{ij}^{21} = -\frac{\chi}{N} G_{ij}, \\ M_{ij}^{22} &= \frac{V}{N} \chi^2 \hat{G}_{ij} = \frac{\chi^2}{N} G_{ij}, \end{aligned} \tag{5.27}$$

where in the last equation we used $\hat{G} = \frac{1}{V} G$. Specifically, plugging in (5.22) will give us the exact Sugawara CFT deformation which corresponds to the singular (Q_1, Q_5) string theory backgrounds.

Let us emphasize that the deformation labeled by χ on the string theory side deforms both the coset CFT and the Sugawara CFT in (5.1). The coset deformation is known to correspond at leading order to one of the blowup modes of the $\mathbb{Z}_2$ orbifold singularity [79]. Following the deformation is a highly complicated task, even in conformal perturbation theory, and it is beyond the scope of this paper (see [80–88] and references therein for recent progress on this). In this section we identified the exact deformation only of the Sugawara part of the CFT, not the coset CFT. This was possible because the $J\bar{J}$ deformation is integrable from the CFT perspective, and can be matched exactly to the string theory using string dualities.

C. The decoupled sector of the $Q_5 > 1$ CFT

In Sec. V A we asked for the world sheet interpretation of the decoupled $\hat{T}^4$ currents. For the backgrounds (5.20) at small $g_{10} = t$ there is a different perturbative world sheet description with $Q_5 > 1$ [5,7,8,37]. We would like to ask more generally, what is the $Q_5 > 1$ world sheet interpretation of the deformed Sugawara CFT (5.27)?

First, it is useful to rewrite the Sugawara energy (5.24) in terms of the (Q_1, Q_5) charges, and not the original CFT charges. The two charge lattices are related through (5.16), (5.21) by

$$\begin{aligned} \begin{bmatrix} w_{F1}^i \\ w_{D1}^i \end{bmatrix} &= \begin{bmatrix} a & Q_1 \\ b & Q_5 \end{bmatrix} \begin{bmatrix} w^i \\ \hat{w}^i \end{bmatrix}, \\ \begin{bmatrix} \tilde{p}_j \\ w_j^{D3} \end{bmatrix} &= \begin{bmatrix} 1 & -bQ_1 \\ -1 & aQ_5 \end{bmatrix} \begin{bmatrix} p_j \\ \hat{p}_j \end{bmatrix}. \end{aligned} \tag{5.28}$$

In terms of the (Q_1, Q_5) string theory charges, the Sugawara energy [also given directly from (5.14)] takes the simple form

$$\begin{aligned} E_{\text{Sugawara}}^{L/R} = &\frac{1}{4Q_1} \left(v_4^{1/2} w_{\text{F1}}^2 + v_4^{-1/2} \tilde{p}^2 \pm 2 w_{\text{F1}} \cdot \tilde{p} \right) \\ &+ \frac{1}{4Q_5} \left(v_4^{1/2} (w^{\text{D3}})^2 + v_4^{-1/2} w_{\text{D1}}^2 \pm 2 w_{\text{D1}} \cdot w^{\text{D3}} \right), \end{aligned} \tag{5.29}$$

where $v_4 = t^2 Q_1 / Q_5$ is the world sheet T^4 volume (2.3). Equation (5.29) should be understood as the generalization of (5.6) for $Q_5 > 1$. It agrees with the fact that, as reviewed in Sec. II, the bulk theory includes a $U(1)_{Q_1}^8 \times U(1)_{Q_5}^8$ CS theory in AdS_3, from the $4 + 4$ NS-NS and $4 + 4$ RR bulk gauge fields, respectively. The $J\bar{J}$ deformation at these points in the moduli space has precisely the effect of forming orthogonal linear combinations of the original currents of (5.1) that have these new levels. Each set of currents involves linear combinations of the currents from the orbifold and from $\hat{T}^4$.

Just as in the $Q_5 = 1$ case, the NS-NS currents at level Q_1 have a string world sheet vertex operator, found explicitly in [7]. As the authors comment there, the RR currents do not seem to appear in the string spectrum, consistent with the fact discussed above that single-string states should have descendants under 8 of the $U(1)$ symmetries but not under the other 8.[21] This also agrees with the string 1-loop calculation done recently for $Q_5 > 1$ [89], in which the $U(1)_{Q_5}^8$ RR currents were absent.

After discussing the currents of the decoupled Sugawara sector, we would next like to understand the bulk interpretation of its charged states. Consider first the undeformed theory (5.1) for $Q_5 = 1$. A general charged state will satisfy the Sugawara/BPS bound, but will not saturate it. The operators that saturate the bound are exactly the operators that are given entirely by the Sugawara CFT, with

[21]Note that the higher Kaluza-Klein modes of the RR potential which have a nonzero RR field strength do have a world sheet vertex operator [7], but here we consider the pure-gauge mode of the RR potential.

only the identity in the coset CFT. These operators can have any $\hat{w}$, $\hat{p}$ integer charges (any operator from $\hat{T}^4$), but the charges w, p must be integer multiples of N. We term this charge sublattice the BPS lattice.[22] We can immediately generalize this statement to the (Q_1, Q_5) backgrounds. Under the χ, V deformation, the energy of the coset changes in a nontrivial way. However, for the BPS lattice, the coset operator is simply the identity. Therefore, also at finite χ these operators remain BPS, as their energy is given exactly by the (deformed) Sugawara energy (5.29). In order to understand what these operators are from the perspective of the (Q_1, Q_5) background, we rewrite (5.28) as

$$\begin{bmatrix} \frac{w^i_{F1}}{Q_1} \\ \frac{w^i_{D1}}{Q_5} \end{bmatrix} = \begin{bmatrix} aQ_5 & 1 \\ bQ_1 & 1 \end{bmatrix} \begin{bmatrix} \frac{w^i}{N} \\ \hat{w}^i \end{bmatrix},$$

$$\begin{bmatrix} \frac{\tilde{p}_j}{Q_1} \\ \frac{w^{D3}_j}{Q_5} \end{bmatrix} = \begin{bmatrix} Q_5 & -b \\ -Q_1 & a \end{bmatrix} \begin{bmatrix} \frac{p_j}{N} \\ \hat{p}_j \end{bmatrix}. \tag{5.30}$$

The matrices appearing here also have determinant 1. Therefore, in terms of the (Q_1, Q_5) NS-NS theory, the BPS sub-lattice $w, p = 0 \bmod N$ is exactly

$$w^i_{\mathrm{F1}} = \tilde{p}_j = 0 \mod Q_1,$$
$$w^i_{\mathrm{D1}} = w^{\mathrm{D3}}_j = 0 \mod Q_5. \tag{5.31}$$

Namely, the charges are all integer multiples of the corresponding Chern-Simons levels. States carrying other charges, including now also singly wrapped D-branes, are not BPS.

We now discuss the bulk interpretation of this BPS lattice. Already in the undeformed case, the interpretation of the $\hat{T}^4$ charged states seems to have a paradoxical nature. On the one hand, they carry RR charge ($\hat{p}$, $\hat{w}$), with the interpretation of D-branes. On the other hand, these states exactly decouple from the entire closed string spectrum (and for example do not emit gravitons). This seems to contradict the basic property of D-branes as world sheet boundary conditions. For $Q_5 > 1$ we found a generalization of that question. The BPS states do not just saturate the bound. As they live entirely in the Sugawara CFT, they also decouple exactly (in correlation functions) from operators in the coset CFT. These modes appear to be completely topological, and yet they carry (Q_5-quantized) RR charges like D-branes.[23] Similarly, we find states carrying the same charge as Q_1 winding fundamental strings, that decouple from all operators in the coset CFT.

The solution to the paradox is that these states are pure bulk gauge modes, not D-branes, that nevertheless carry RR (and/or NS-NS) charges. One way to think of these modes is as boundary modes of the low-energy Chern-Simons theories, as in our discussion of the $Q_5 = 1$ case above. Alternatively, the existence of such charged modes in string theory in the presence of nonzero Kalb-Ramond H_3 flux was described in [91]. Consider the $AdS_3 \times S^3 \times T^4$ type IIA background with $H_3 = Q_5\omega_3$, with ω_3 the unit volume-form on the S^3 (the AdS_3 component of the flux is not important for our analysis).[24] Denote the pure-gauge mode of the C_3 field on S^3 by ψ,

$$C_3 = \psi\, \omega_3 + c_3, \tag{5.32}$$

where c_3 includes the rest of the directions. In type IIA, the gauge transformation of $\delta C_1 = d\Lambda$ is followed by $\delta C_3 = \Lambda H_3$ [92], with 0-form Λ. In our case,

$$\delta\psi = Q_5 \Lambda. \tag{5.33}$$

Because ψ is the C_3 holonomy over S^3, $\psi \sim \psi + 2\pi$. The holonomy $\exp(i\psi)$ is thus an operator with charge Q_5 under the RR gauge field C_1. In terms of type IIB charges (upon T dualizing one of the T^4 directions), those are pure C_4 (wrapping $S^3 \times T^1$) gauge modes with $w_{\mathrm{D1}} = Q_5$ RR charge. Applying T duality on the four directions of the torus also reveals pure $\tilde{C}_6$ modes (wrapping $S^3 \times T^3$) with Q_5-quantized w^{D3} charge. Applying $ST_{1234}S$ gives the topological modes with Q_1-quantized momentum and winding modes. In the language of [91], the full string theory includes a $U(1)^{16}$ gauge theory with $\mathbb{Z}^{16}$ charges, but the coset has only $\mathbb{Z}^8_{Q_1} \oplus \mathbb{Z}^8_{Q_5} \equiv \mathbb{Z}^8_{Q_1 Q_5}$ charges.

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[22]Note that the states saturating the Sugawara/BPS bound are BPS states before taking the near-horizon limit, where the supersymmetry algebra has central charges corresponding to the momenta and winding numbers on the T^4. These states are not BPS states of the theory on AdS_3, where there are different BPS states, carrying charges under the $SU(2)_L \times SU(2)_R$ symmetry; we do not discuss these states here. The bound on the energy of the states we discuss in AdS_3 comes from the form of the CFT energy-momentum tensor discussed above, rather than from the supersymmetry algebra.

[23]For $Q_5 > 1$ there are also operators with $O(1)$ RR charge that are not BPS, and we expect them to take the form of D-branes in the bulk. For $Q_5 = 1$ all the operators with RR charge are BPS, and in this theory we expect no D-brane particle excitations (see also [72,90]).

[24]We use the conventions of [91] which are slightly different than [92], but more natural for these purposes.

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APPENDIX: TOROIDAL SIGMA MODELS AND $J\bar{J}$ DEFORMATIONS

In this Appendix we write down basic facts about 1 + 1-dimensional sigma models on tori and their $J\bar{J}$ deformations.

1. Compact bosons

We will follow Polchinski's notations [93] with

$$z=\sigma^1+i\sigma^2, \quad \bar{z}=\sigma^1-i\sigma^2,$$
$$\partial=\frac{1}{2}(\partial_1-i\partial_2), \quad \bar{\partial}=\frac{1}{2}(\partial_1+i\partial_2), \tag{A1}$$

and $d^2z=2d^2\sigma$. We take d compact scalar fields X^i ($i=1,\ldots,d$) with

$$X^i\sim X^i+2\pi. \tag{A2}$$

In these conventions, we take the (Euclidean) action

$$S=\frac{1}{4\pi}\int d^2\sigma(G_{ij}g^{\mu\nu}+iB_{ij}\epsilon^{\mu\nu})\partial_\mu X^i\partial_\nu X^j$$
$$=\frac{1}{2\pi}\int d^2z(G_{ij}\partial X^i\bar{\partial}X^j+B_{ij}\partial X^i\bar{\partial}X^j), \tag{A3}$$

with some constant metric and B field $E_{ij}=G_{ij}+B_{ij}$. The momentum and winding Noether currents are

$$J_i^\mu=\frac{1}{2\pi}(G_{ij}g^{\mu\nu}+iB_{ij}\epsilon^{\mu\nu})\partial_\nu X^j,$$
$$\tilde{J}^i_\mu=\epsilon_{\mu\nu}G^{ij}J^\nu_j=\frac{1}{2\pi}(\delta^i_j\epsilon_{\mu\nu}-iG^{ik}B_{kj}g_{\mu\nu})\partial^\nu X^j. \tag{A4}$$

We can write these in a holomorphic/antiholomorphic basis

$$J_i=\frac{1}{2}(J_{i,z}-i\tilde{J}_{i,z})=\frac{1}{2\pi}(G_{ij}-B_{ij})\partial X^j,$$
$$\bar{J}_i=\frac{1}{2}(J_{i,\bar{z}}+i\tilde{J}_{i,\bar{z}})=\frac{1}{2\pi}(G_{ij}+B_{ij})\bar{\partial}X^j, \tag{A5}$$

which satisfy $\bar{\partial}J_i=\partial\bar{J}_i=0$.

To consider the Hilbert space on a circle, we use the transformation $z=\exp(-iw)$, with

$$w=\theta+i\tau\sim w+2\pi. \tag{A6}$$

To consider the Lorentzian version we further take $\tau=it$. The different particle sectors are labeled by the winding $w^j\in\mathbb{Z}$

$$X^j(w+2\pi)=X^j(w)+2\pi w^j, \tag{A7}$$

and the center of mass momentum $p_i\in\mathbb{Z}$

$$p_i=\int d\theta J_i^t=G_{ij}v^j+B_{ij}w^j, \tag{A8}$$

with the target space velocity v^i, with the quantization

$$v_i=p_i-B_{ij}w^j. \tag{A9}$$

The Hilbert space is labeled by the integers p_i, w^j together with the usual harmonic oscillators α_n. The Hamiltonian (ignoring the Casimir energy) is given by

$$H=\frac{1}{4}(v_R^2+v_L^2)+N+\bar{N}, \tag{A10}$$

with

$$v_{i,L/R}=v_i\pm G_{ij}w^j. \tag{A11}$$

The Sugawara energy is given solely by the first term in (A10), which gives a lower bound on $L_0,\bar{L}_0$,

$$E^{L/R}_{\text{Sugawara}}=\frac{1}{4}v^2_{L/R}=\frac{1}{4}v^2+\frac{1}{4}w^2\pm\frac{1}{2}v\cdot w$$
$$=\frac{1}{4}\begin{bmatrix}w^i & p_j\end{bmatrix}\begin{bmatrix}G+B^TG^{-1}B & \pm1+BG^{-1}\\ \pm1-G^{-1}B & G^{-1}\end{bmatrix}\begin{bmatrix}w^i\\ p_j\end{bmatrix}, \tag{A12}$$

where we omitted the G contractions in the first line. The last formula is explicitly invariant under T duality, which acts as

$$p_i\leftrightarrow w^i, \quad E\leftrightarrow E^{-1}. \tag{A13}$$

Back to the Euclidean plane $z,\bar{z}$. For a given G, B we would like to know the metric on the currents. The two-point function

$$\langle\partial X^i(z)\partial X^j(0)\rangle=\frac{G^{ij}}{2z^2},$$
$$\langle\bar{\partial}X^i(\bar{z})\bar{\partial}X^j(0)\rangle=\frac{G^{ij}}{2\bar{z}^2}, \tag{A14}$$

gives the current two-point function for the chiral components of the momentum current (A5)

$$\langle J_i(z)J_j(0)\rangle = \frac{1}{8\pi^2}\frac{G_{ij}+B_{ik}G^{kl}B_{lj}}{z^2},$$
$$\langle \bar{J}_i(\bar{z})\bar{J}_j(0)\rangle = \frac{1}{8\pi^2}\frac{G_{ij}+B_{ik}G^{kl}B_{lj}}{\bar{z}^2}. \qquad \text{(A15)}$$

This exactly the $G+B^TG^{-1}B$ term that we have in (A12). The winding Noether current is given by $\hat{J}^{\mu,i} = \frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\mu X^i$. Its holomorphic and antiholomorphic components are[25]

$$J^i = \frac{i}{2\pi}\partial X^i, \qquad \bar{J}^j = -\frac{i}{2\pi}\bar{\partial}X^j, \qquad \text{(A16)}$$

which are also given by raising the target space index of (A5) using E. The two-point function is

$$\langle J^i(z)J^j(0)\rangle = -\frac{1}{8\pi^2}\frac{G^{ij}}{z^2},$$
$$\langle \bar{J}^i(\bar{z})\bar{J}^j(0)\rangle = -\frac{1}{8\pi^2}\frac{G^{ij}}{\bar{z}^2}, \qquad \text{(A17)}$$

with the mixed terms

$$\langle J_i(z)J^j(0)\rangle = \frac{i}{8\pi^2}\frac{\delta_i^j - B_{ik}G^{kj}}{z^2},$$
$$\langle \bar{J}_i(\bar{z})\bar{J}^j(0)\rangle = -\frac{i}{8\pi^2}\frac{\delta_i^j + B_{ik}G^{kj}}{\bar{z}^2}. \qquad \text{(A18)}$$

We can directly see that the matrix (A12) is also the matrix of the current-current two-point functions, as required in the Sugawara construction.

The $J\bar{J}$ deformation [94] (see also the recent [95]) is given by the general deformation

$$\partial_\lambda S = 2\pi\int d^2z M^{ij}J_i\bar{J}_j = \frac{1}{2\pi}\int d^2z E_{ki}M^{ij}E_{jl}\partial X^k\bar{\partial}X^l, \qquad \text{(A19)}$$

for some matrix M^{ij} and currents $J_i, \bar{J}_j$ depending on λ. In other words, the sigma-model moduli exactly changes with

$$\partial_\lambda E_{ij} = E_{ik}M^{kl}E_{lj}. \qquad \text{(A20)}$$

These deformations cover the entire space of E_{ij} deformations. They can be labeled by the $O(d,d)/(O(d)\times O(d))$ moduli.

2. Going to the next level

We will now specialize to the (bosonic version of) theory (2.5). Namely, we have $4N$ fields $X^{I,i}$ ($I=1,\ldots,N$ and $i=1,\ldots,4$) which will constitute the T^{4N}/S_N orbifold, and 4 more fields $\hat{X}^i$. We assume the moduli on the $X^{I,i}$ are independent of I and consist of "single-trace" moduli. The action is given by

$$S = \frac{1}{2\pi}\int d^2z(E_{ij}\partial X^{I,i}\bar{\partial}X^{I,j} + \hat{E}_{ij}\partial\hat{X}^i\bar{\partial}\hat{X}^j). \qquad \text{(A21)}$$

The single-trace momentum currents are

$$J_i = \frac{1}{2\pi}\sum_{I=1}^N E_{ji}\partial X^{I,j}, \qquad \bar{J}_i = \frac{1}{2\pi}\sum_{I=1}^N E_{ij}\bar{\partial}X^{I,j},$$
$$\hat{J}_i = \frac{1}{2\pi}\hat{E}_{ji}\partial\hat{X}^j, \qquad \hat{\bar{J}}_i = \frac{1}{2\pi}\hat{E}_{ij}\bar{\partial}\hat{X}^j. \qquad \text{(A22)}$$

In the normalization we use, the w-cycle twisted sectors (including the untwisted sector) have integer charges Q_i, Q^j. We denote the integer charges of these currents by w^i, p_j, $\hat{w}^k$ and $\hat{p}_l$, respectively.

Setting $E=G+B$, $\hat{E}=\hat{G}+\hat{B}$, the two-point function for the currents is

$$\langle J_i(z)J_j(0)\rangle = \frac{N}{8\pi^2}\frac{G_{ij}+B_{ik}G^{kl}B_{lj}}{z^2},$$
$$\langle \hat{J}_i(z)\hat{J}_j(0)\rangle = \frac{1}{8\pi^2}\frac{\hat{G}_{ij}+\hat{B}_{ik}\hat{G}^{kl}\hat{B}_{lj}}{z^2}. \qquad \text{(A23)}$$

Inverting the matrix, the Sugawara energy bound is given by

$$E^{L/R}_{\text{Sugawara}} = \frac{1}{4}\begin{bmatrix} w & p & \hat{w} & \hat{p}\end{bmatrix}\begin{bmatrix} \frac{G+B^TG^{-1}B}{N} & \frac{\pm 1+BG^{-1}}{N} & 0 & 0 \\ \frac{\pm 1-G^{-1}B}{N} & \frac{1}{N}G^{-1} & 0 & 0 \\ 0 & 0 & \hat{G}+\hat{B}^T\hat{G}^{-1}\hat{B} & \pm 1+\hat{B}\hat{G}^{-1} \\ 0 & 0 & \pm 1-\hat{G}^{-1}\hat{B} & \hat{G}^{-1}\end{bmatrix}\begin{bmatrix} w \\ p \\ \hat{w} \\ \hat{p}\end{bmatrix}. \qquad \text{(A24)}$$

[25]These i's are an outcome of the holomorphic coordinates in Euclidean space with $\epsilon_{z,\bar{z}} = \frac{i}{2}$. After continuing to the circle $w,\bar{w} = \theta \mp t$ we have $\epsilon_{w,\bar{w}} = \frac{1}{2}$, which gives the familiar formulas.

Notice that this is exactly two blocks of (A12), only with the p, w block multiplied by $1/N$ (as the two-point functions of J are proportional to N).

A general $J\bar{J}$ deformation in this theory can be written as

$$\partial_\lambda S = 2\pi \int d^2z (M^{11}_{ij} J^i \bar{J}^j + M^{12}_{ij} J^i \hat{\bar{J}}^j + M^{21}_{ij} \hat{J}^i \bar{J}^j + M^{22}_{ij} \hat{J}^i \hat{\bar{J}}^j), \tag{A25}$$

for some 8×8 matrix $M^{\alpha\beta}_{ij}$ $(\alpha,\ \beta = 1,\ 2,\ i, j = 1, \ldots, 4)$. In (A25) we raise indices of $J, \bar{J}$ by E, and those of $\hat{J}, \hat{\bar{J}}$ by $\hat{E}$. The M^{22} matrix simply deforms the $\hat{T}^4$ moduli just like (A20),

$$\partial_\lambda \hat{E}_{ij} = M^{22}_{ij}. \tag{A26}$$

M^{11}, M^{12} and M^{21} do not have that interpretation anymore, but together with M^{22} they label an $SO(8,8)/(O(8)\times O(8))$ manifold.

To make contact with the χ deformation of (5.24), we set $B = \hat{B} = 0$ and consider the $J\bar{J}$ deformation

$$M^{11}_{ij} = 0,$$
$$M^{12}_{ij} = M^{21}_{ij} = -\frac{\chi}{N} G_{ij},$$
$$M^{22}_{ij} = \frac{V}{N}\chi^2 \hat{G}_{ij} = \frac{\chi^2}{N} G_{ij}, \tag{A27}$$

where we took $\hat{G} = \frac{1}{V} G$ as in Sec, V. In terms of the (ungauged) $4N+4$ fields, we deformed the $(4N+4)\times(4N+4)$ metric to

$$G_{4N+4} = \left[\begin{array}{c|c} \delta_{IJ} & -\chi\frac{1}{N} \\ \hline -\chi\frac{1}{N} & V^{-1} + \frac{\chi^2}{N} \end{array}\right] \otimes G, \tag{A28}$$

with the inverse

$$G^{-1}_{4N+4} = \left[\begin{array}{c|c} \delta_{IJ} + \frac{V\chi^2}{N^2} & \chi\frac{V}{N} \\ \hline \chi\frac{V}{N} & V \end{array}\right] \otimes G^{-1}. \tag{A29}$$

Projecting to the single-trace currents (A22) gives the Sugawara energy

$$E^{L/R}_{\text{Sugawara}} = \frac{1}{4}\begin{bmatrix} w & \hat{w} & p & \hat{p} \end{bmatrix} \begin{bmatrix} \frac{1}{N}G & -\chi\frac{1}{N}G & \pm\frac{1}{N} & 0 \\ -\chi\frac{1}{N}G & \frac{1+\frac{V}{N}\chi^2}{V}G & 0 & \pm 1 \\ \pm\frac{1}{N} & 0 & \frac{1+\frac{V}{N}\chi^2}{N}G^{-1} & \chi\frac{V}{N}G^{-1} \\ 0 & \pm 1 & \chi\frac{V}{N}G^{-1} & VG^{-1} \end{bmatrix} \begin{bmatrix} w \\ \hat{w} \\ p \\ \hat{p} \end{bmatrix}. \tag{A30}$$

Rewriting in terms of contractions with the unit metric $V^{-1/2}G$ (and changing the order of the vector components) gives exactly (5.24).

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