Fermi model of a quantum black hole

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(Received 25 April 2024; accepted 21 June 2024; published 1 August 2024)

We propose a quantum model of the Schwarzschild black hole as a quantum mechanics of a system of fermionic degrees of freedom. The system has a constant density of states and a Fermi energy that is inversely proportional to the size of the system. Assuming the equivalence principle, we show that the degeneracy pressure of the Fermi degrees of freedom is able to withstand the collapse of gravity if the radius of the system is given precisely by the horizon radius of the Schwarzschild black hole. In our model, the fermionic degrees of freedom at each energy level can be entangled in certain different ways, giving rise to a multitude of degenerate ground states of the system. The counting of these microstates reproduces precisely the Bekenstein-Hawking entropy. This simple Fermi model is universal and works also for the Reissner-Nordström charged black hole as well as black holes with a cosmological constant. From the properties of the Fermi variables, we propose that quantum gravity is characterized by a principle of *maximal capacity of states* where there can be no more than V/l_P^3 quantum states in any volume V. It implies a loss of spatial locality below the Planck length and suggests that any singularity predicted by general relativity is resolved and replaced by a quantum space in quantum gravity. In our model, a black hole spacetime is equipped with a uniform distribution of energy levels. This is another reason why black holes can be considered a simple harmonic oscillator of quantum gravity.

DOI: 10.1103/PhysRevD.110.046001

I. INTRODUCTION

The nature of quantum spacetime and quantum gravity is one of the most important questions for fundamental physics. Black holes arise as solution of the classical general relativity. However, the quantum property of black holes is puzzling and it is believed that only in a full theory of quantum gravity can they be properly understood. In this paper we propose a model of the interior of a black hole as a bottom-up approach to learn about the construction of quantum gravity. In our model, the following puzzling properties of black holes are addressed and resolved.

(i) *Bekenstein-Hawking entropy*: The laws of black hole mechanics [1], for example, the first law

$$dU = \frac{\kappa}{8\pi G} dA + \cdots \tag{1.1}$$

was originally derived from the classical theory of Einstein gravity. That these mechanical relations are, in fact, thermodynamic relations was made evident with the discovery of the quantum Hawking radiation [2], which makes the black hole appear like a thermal object to the outside world with the Hawking temperature

$$T_H = \frac{1}{8\pi GM}. \tag{1.2}$$

As a result, the perturbation relation (1.1) becomes the first law of thermodynamics if the black hole carries the Bekenstein-Hawking entropy [3,4]

$$S_{\rm BH} = \frac{A}{4G}. \tag{1.3}$$

The thermodynamic nature of black hole mechanics has led Jacobson to the very interesting proposal [5] that Einstein's theory of gravity is a thermodynamic expression of spacetime.

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The possession of the entropy (1.3) by a quantum black hole is puzzling. Instead of being extensive as in standard thermodynamic systems, the area dependence of the Bekenstein-Hawking entropy (1.3) has suggested holography [6,7] as a fundamental principle for quantum gravity and has inspired the formulation of the AdS/CFT correspondence [8]. In effect, any quantum gravitational theory of black holes should have the gravitational degrees of freedom inside the black hole properly identified such that an account of them would give rise to the area-dependent entropy (1.3) of black holes. Therefore, the reproduction of (1.3) becomes an important testing stone for a consistent model of quantum black holes. See, for example, [9] for progress in microstates counting in supersymmetric brane black holes and [10] for the fuzzy ball proposal.

(ii) *Horizon radius*: Classically, the black hole is a vacuum solution of spacetime. Outside the horizon, black holes behave very much like an ordinary gravitational massive object. For the Schwarzschild black hole, the horizon radius depends on the mass M in a specific way:

$$R = 2GM \coloneqq R_S, \tag{1.4}$$

where R_S is the Schwarzschild radius. This is strikingly different from the typical relation $R \sim M^{-\beta}, \beta > 0$ (e.g., $\beta = 1/3$ for a nonrelativistic neutron star) for stellar objects made of ordinary matter without any internal energy source, e.g., fusion. That the radius is inversely correlated with the mass is a simple consequence of the fact that such matter becomes more compactly packed as the force of gravity increases. On the other hand, the unusual behavior (1.4) that a black hole gets bigger as it get more massive means that the degrees of freedom of the black hole cannot be of an ordinary variety. For a charged black hole, the horizon radius satisfies the relation

$$R^2 - RR_S + R_Q^2 = 0, \tag{1.5}$$

where $R_Q \coloneqq \sqrt{GQ^2}$ is the charge radius for the Reissner-Nordström black hole. Likewise, the horizon radius R of a black hole with a cosmological constant Λ satisfies the relation

$$R - R_S - \frac{\Lambda}{3} R^3 = 0. \tag{1.6}$$

As a result, the reproduction of relations (1.4)–(1.6) for the horizon radius in the respective cases is another important testing stone for a consistent quantum model of black holes.

(iii) *Nonsingular interior*: Most troubling is that as long as certain energy conditions of matter holds, the prediction of the black hole singularity in general relativity cannot be avoided [11]. Although it is generally expected that quantum gravity will play a crucial role in resolving the singularity of classical spacetime and provide a complete quantum description of black holes, there are different ways it could happen. It is possible that the Einstein equation is modified so that the curvature singularity is avoided. A more dramatic possibility is that the classical spacetime of the black hole is replaced by a completely novel version of quantum existence where the singularity theorem does not apply. This possibility is interesting not just because it offers a valuable window to look into the novel properties of quantum spacetime but also because, in a quantum theory of spacetime, the black hole interior would automatically be equipped with a Hilbert space of states, which potentially could explain the origin of the black hole entropy (1.3).

(iv) *Unitarity*: Assuming quantum mechanics holds, black holes should be described by a unitary quantum system. However, apparently a black hole form in a pure state can evolve into a mixed state as it ends its life through the evaporation of thermal Hawking radiation [12]. This information problem contradicts the unitarity of quantum mechanics. Unitarity also requires the entanglement entropy of the Hawking radiation to follow a Page curve [13,14] instead of a Hawking curve. The recovery of quantum information and the reproduction of the Page curve behavior of the Hawking radiation entropy are necessary requirements for a consistent quantum model of black holes.

It is clear from the above discussion that the crux of all these problems is a lack of understanding of the quantum properties of the interior of black hole. In principle, a consistent quantum model of black holes without singularity is needed in order to allow for a proper identification of the Hilbert space of states of the black hole, and in turn this is needed to express the quantum states of the black hole interior spacetime and which in turn is needed to explain the Bekenstein-Hawking entropy (1.3). And again, only with the availability of the black hole Hilbert space does one have the necessary states to possibly describe the unitarity evolution of the black hole without the risk of missing anything.

So far the most commonly adopted approach to the problem of black holes is a top-down one where one starts with a candidate theory of quantum gravity and studies the properties of the black hole as a solution of it. For example, brane models of black holes have been considered in nonperturbative string theory, with the Berkenstein-Hawking entropy reproduced successfully from a microstates counting [9,10]. Recently progress has been achieved for black holes constructed in the AdS/CFT correspondence [8] where the Page curve behavior of the entanglement entropy of the Hawking radiation is remarkably obtained [15–19]. Nevertheless, despite this success, it is still highly desirable to have a direct formulation of the quantum gravity itself and be able to describe the fundamental degrees of freedom of quantized spacetime and the set of microstates directly without resorting to supersymmetry or duality.

In a recent paper [20], we have initiated an investigation of quantum black holes along these lines. There, we proposed that the interior of black holes is filled with a thin shell of entangled Bell states located just underneath the horizon. We argued that the configuration can be stabilized by a new kind of degeneracy pressure that originated from the noncommutative spacetime. We showed that quantum tunneling [21], which occurs near the horizon, results in the Page curve behavior of the entanglement entropy of the Hawking radiation, just as islands in the AdS/CFT [15–19]. Moreover, it was shown that the entanglement information initially stored in the black hole can all be returned to the environment at the end of life of the black hole. That the origin of the Page curve and the return of information can be elucidated in terms of explicit spacetime quantum mechanical processes is interesting. However, some of the basic properties of the model in [20] were assumed and not derived. For example, how does the degeneracy pressure arise? What kind of quanta are these? To provide a more complete model from which the results of [20] can be derived and firmly justified is a main motivation of this work.

Instead of a top-down approach where the fundamental theory is assumed, we will take a bottom-up approach to use the black hole properties as "empirical input" to help us to learn about the fundamental theory of quantum gravity. The only assumption we make is that quantum gravity can be formulated in terms of a quantum mechanics of fermionic and bosonic degrees of freedom. Generically, the Lagrangian takes the form

$$L = i\psi^{+}\dot{\psi} + \psi^{+}h(X)\psi - V(X), \tag{1.7}$$

where $h(X)$ denotes some Yukawa coupling and $V(X)$ the self-interaction. We emphasis that these degrees of freedom are not particles, nor are they described by an energy-momentum tensor, both notions of which would only make sense when there is a spacetime. Interestingly, Maldacena has recently proposed quantum mechanics of oscillators and Majorana fermions to describe a black hole [22]. It is interesting that the fermionic nature of black holes is also recognized in their considerations.

Here is the plan of the paper. In Sec. II A, we perform a general analysis for a quantum Fermi system described by a density of energy eigenstates. We show that it admits a degenerate pressure in general and there is no need to assume the degrees of freedom are particles. In Sec. II B, we apply it to the system of a neutron and obtain the usual mass radius relation of a neutron star. In Sec. II C, we consider a quantum mechanical system of fermionic degrees of freedom as a model of the Schwarzschild black hole. Outside the system, we assume that the low energy effective description emerges where the usual spacetime and general relativity become valid and the system is seen by the outside observer as an object with a gravitational mass M and some horizon radius R. We show that the desired properties of the quantum black hole can be precisely reproduced by the microscopic Fermi system. In fact, by assuming the equivalence principle, we show that the degeneracy pressure of the Fermi degrees of freedom is just right to withstand the pressure of gravitational collapse when the size of the Fermi system is precisely given by the Schwarzschild horizon radius (1.4). We also show that the Fermi system admits a large number of degenerate ground states, each of which is given by a richly entangled configuration of the bosonic and fermionic degrees of freedom, and the counting of the ground state degeneracy reproduces precisely the Bekenstein-Hawking entropy (1.3). This gives evidence that our Fermi model has indeed captured the microstates of the black hole. The Page curve behavior of the Hawking entropy is also reproduced in our model [23]. In Secs. II D and II E, we extend the analysis to the more general cases of a black hole with charge or with a cosmological constant [24] and show that all the desired properties of the black hole are reproduced. That the properties of various black holes could be accounted for universally provides support to the validity of our description. Our model suggests that holography is a consequence of the fermionic nature of the quantum spacetime. Our analysis also suggests that quantum gravity is characterized by a principle of *maximal capacity of states* where there can be no more than V/l_p^3 of quantum states in any volume V; see (2.33). This is satisfied by our model of black holes. The loss of locality below the Planck length suggests that the usual singularity theorem is avoided and our model is consistent. Further discussions are presented in Sec. III.

II. A FERMIONIC MODEL FOR THE QUANTUM BLACK HOLE

Black holes in general relativity are described by a metric with a singularity. The existence of a singularity means that the classical model of black holes as a vacuum spacetime is not trustable. The formation of a singularity may be avoided if the gravitational collapse is somehow counterbalanced by some internal pressure. Classically this is impossible due to the singularity theorem [11]. In the following we consider a replacement of the interior spacetime of black holes by a quantum mechanical system of fermionic degrees of freedom. To an outside observer, the system occupies a volume of space and possesses an internal energy. We will show that the volumetric response of the internal energy creates a pressure that is just right to precisely balance out the pressure from the gravitational spacetime outside. We emphasis that we are not considering some field theoretic exotic matter or vacuum system of the black hole; see, e.g., [25–27]. Instead we propose that the classical interior spacetime of black holes is replaced by a quantum space and the fermionic degrees of freedom describe the quantum fluctuations over it.

A. Degenerate Fermi system

To start with, let us determine the degeneracy pressure of a system of Fermi degrees of freedom. Consider a system of fermionic degrees of freedom contained in a spatial volume V. An important specification of the system is the density of energy eigenstates $g(E)$ where $g(E)dE$ is the number of energy eigenstates contained within the energy interval $(E, E+dE)$. Let us consider $g(E)$ of the general form

$$g(E) = c_0 V E^{\alpha}, \tag{2.1}$$

where c_0 and α are constants. This covers the case of nonrelativistic particles with $\alpha = 1/2$, $c_0 = 2^{1/2} m^{3/2}/\pi^2$ and the relativistic case with $\alpha = 2$, $c_0 = 1/\pi^2$, where the spin factor $g_s = 2s+1 = 2$ has been included. This gives the total number of states $N_S = \int_0^{\mu} dE g(E)$

$$N_S = \frac{c_0 V}{\alpha+1}\mu^{\alpha+1} \tag{2.2}$$

and the total internal energy $U = \int_0^{\mu} dE g(E) E$

$$U = \frac{c_0 V}{\alpha+2}\mu^{\alpha+2}, \tag{2.3}$$

where μ is the Fermi energy.

As a result, the response of the energy to change in volume gives rise to an "internal energy pressure" $P_U \coloneqq -(\partial U/\partial V)_N$

$$P_U = -\frac{\alpha+1}{\alpha+2} N_S \left(\frac{\partial \mu}{\partial V}\right)_N. \tag{2.4}$$

In this system, the total number of fermions is given by $N = 2N_S$ since each energy state is occupied by two fermions with opposite spins. Therefore, the degeneracy pressure is determined by the volume dependence of the Fermi level μ. Note that from (2.3), μ is approximately equal to the average energy of the states. Note also that the form of P_U depends crucially on the volume dependence of the Fermi energy. We will see later that the Fermi energy (2.18) in our black hole model has an entirely different behavior from that of a system of ordinary fermionic particles; see, e.g., (2.5). This crucial difference allows our system of Fermi degrees of freedom to model a black hole.

B. Matter degeneracy pressure: Neutron star

As a warm up, let us apply this model to consider the example of a nonrelativistic neutron star ($\alpha = 1/2$). It follows from (2.2) that

$$\mu = \frac{1}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3} \tag{2.5}$$

and hence

$$P_U \sim \frac{N^{5/3}\hbar^2}{mR^5}, \tag{2.6}$$

where $\sim$ means equality up to a proportional constant of order 1. Here, the total number of particles is given by

$$N = M/m \tag{2.7}$$

where m is the mass of the neutron. In the case of a neutron star, the internal energy and the pressure P_U arises from the kinetic motion of the neutrons. Now for a neutron star of mass M and radius R, the gravitational energy is $E_g = -\gamma GM^2/R$ where γ is a constant which depends on the distribution of the masses. This gives the gravitational pressure $P_g = -\frac{\partial E_g}{\partial(V_0 - V)} = \partial E_g/\partial V$, where $V = 4\pi R^3/3$ is the volume of the star and V_0 is the fixed volume of an infrared box where the star sits in. We obtain the gravitational pressure

$$P_g = \frac{\gamma}{4\pi}\frac{GM^2}{R^4} \tag{2.8}$$

acting on the star. The star settles at a radius when equilibrium $P_g = P_U$ is attained. This gives immediately the mass-radius relation

$$R \sim M^{-1/3}. \tag{2.9}$$

The requirement that R is greater than the Schwarzschild radius gives the mass limit $M \lesssim 2M_{\odot}$ for the neutron star.

One may wonder if one could achieve the black hole mass-radius relation (1.4) with some choice of α. The answer is no. In fact, as long as N is given by (2.7) for a fixed mass m of the fermions, the scaling relation is

$$R \sim M^{-\alpha/(2-\alpha)} \tag{2.10}$$

and it clearly shows that no choice of α can reproduce the black hole mass-radius relation. This is just a simple way to see that a Fermi system of particles where (2.7) holds can never model a black hole. In addition to this, the usual Fermi gas model as described above also seems not to be able to account for the Bekenstein-Hawking entropy. In fact, in the standard analysis, the degenerate Fermi gas is in a ground state obtained by filling up all the energy levels up to the Fermi energy in accordance with the Pauli exclusion principle. This gives a unique ground state and hence a vanishing microstate entropy

$$S = \log \mathcal{G} = 0, \tag{2.11}$$

where $\mathcal{G} = 1$ is the ground state degeneracy. In the next subsection, we will see that both problems can be resolved together by considering a system of Fermi quanta whose Fermi energy has a simple dependence on the size of the system.

C. Schwarzschild black hole as a system of Fermi quanta

The fact that a black hole has an entropy means that a black hole is not a classical vacuum as described in general relativity but a nontrivial quantum system of microstates. Obviously, these microstates cannot arise from the ordinary elementary particles of the standard model since, according to the singularity theorem, the ordinary matter energy-momentum cannot withstand gravity collapse. Instead, the universal nature of black holes suggests that these quanta should have a generic identity independent of the matter that has been collapsed to form the black hole. It then seems natural to consider the black hole interior a region filled with elementary quanta of the quantized spacetime. As a Fermi system is naturally equipped with a degeneracy pressure, which has the potential to counter the collapse and keep the system from further collapsing under the force of gravity, let us, therefore, consider a system of fermionic degrees of freedom in a spherical volume of radius R as a model of the interior spacetime of black hole. As we will show now, by taking R to be the horizon radius of the black hole and with a specification of its density of eigenstates, this simple Fermi system reproduces all the desired properties of the quantum black hole.

Outside the black hole, Einstein gravity and the Schwarzschild metric are good descriptions of the long range physics. From the outside observer point of view, the Schwarzschild black hole is described by a vacuum solution of the Einstein equation with a horizon radius $R = 2GM$ where M is the mass of the black hole. In general, given an asymptotically flat spacetime with a timelike Killing vector t^α, the energy of the solution is given by the Komar mass

$$M = -\frac{1}{8\pi G}\int_{S_\infty} \nabla^\alpha t^\beta dS_{\alpha\beta}, \tag{2.12}$$

where t^α has been normalized such that $t^2 = -1$ asymptotically. The energy M, however, contains both the gravitational energy and other energy in the spacetime, e.g., rotational energy or Coulombic energy. This motivates the adoption of a surface integral over the horizon [28]

$$E_g \coloneqq -\frac{1}{8\pi G}\int_{H} \nabla^\alpha t^\beta dS_{\alpha\beta} \tag{2.13}$$

as the gravitational energy of the black hole. E_g may be expressed in terms of the surface gravity κ of the horizon as

$$E_g \coloneqq \frac{\kappa A}{4\pi G}. \tag{2.14}$$

For the Schwarzschild black hole, we have

$$E_g = M. \tag{2.15}$$

This give rises to the gravitational pressure

$$P_g = \frac{\partial E_g}{\partial V} = \frac{1}{8\pi G R^2} \tag{2.16}$$

which acts on the Fermi system from outside. Here we have used $M = R/2G$ for the Schwarzschild black hole. The Fermi system is stable if the *gravitational pressure* P_g *from outside* can be balanced by some *internal pressure* P_U *from* the black hole

$$P_g = P_U. \tag{2.17}$$

To determine P_U, one needs to know the density of energy eigenstates $g(E)$ and the energy μ of the Fermi system. In the context of the matrix model, the fermionic energy eigenstates arise from a quantization of the quantum mechanical Hamiltonian in some background of the bosonic matrices which corresponds to the black hole. In any case, it is natural that the energy eigenvalues are related to the inverse size of the system. Therefore let us propose a Fermi energy of the form

$$\mu = \frac{a}{R}, \tag{2.18}$$

where a is some numerical constant to be fixed. Substituting this into (2.4), we obtain the pressure

$$P_U = \frac{U}{4\pi R^3} \tag{2.19}$$

due to the internal energy of the Fermi system. Note that it is independent of a. Note also that $P_U = \frac{U}{3V}$ implies the Fermi system is Weyl invariant, i.e., $\rho_U = 3P_U$. Now if we assume the equivalence principle holds, then the internal energy U of the Fermi system is equal to the gravitational mass M of the black hole

$$U = M. \tag{2.20}$$

From this we arrive immediately at (2.17) and so our Fermi model for the interior of the Schwarzschild black hole is stable against the squash of gravity. This result is interesting and deserves a couple of remarks.

(1) In general relativity, the black hole is a vacuum solution with vanishing energy density and vanishing pressure. With a vanishing pressure, the squash by gravity is unavoidable. In our model, the black hole is populated with a set of fermionic quantum states and a nonvanishing degeneracy pressure.
(2) The presence of a positive pressure is crucial to the balance of the system against gravitational collapse. A deeper understanding of the theory of quantum gravity is needed to understand the nature of the quantum spacetime of the black hole.
(3) In our discussion above, we have assumed that the mass M is sufficiently large so that the spacetime outside the black hole can be described semiclassically in terms of a metric. For smaller mass where the semiclassical approximation fails, the exterior region of the black hole should also be described quantum mechanically in terms of the Fermi quanta.
(4) It is interesting to note that the Newton constant was not present in the original expression (2.19) but appears only in the gravity description after the equivalence principle is invoked. It suggests that gravity is, in fact, an effective theory that emerges from an underlying fundamental theory of quantum spacetime. The coupling constant for this fundamental interaction of spacetime would be a dimensionless pure number.

Next we look at the entropy. We note from (2.20) that $U \propto R$. This gives $\alpha = 0$ in (2.3) and hence

$$U = \frac{1}{2} N_S \mu \tag{2.21}$$

Or, on using (1.4) and (2.20), we obtain the total number of energy eigenstates

$$N_S = \frac{1}{\pi a} \cdot \frac{\pi R^2}{G\hbar}, \tag{2.22}$$

where we have restored $\hbar$ in order to emphasis the quantum nature of (2.22). We note that, interestingly, (2.22) has an area dependence and suggests that the Bekenstein-Hawking entropy may find an explanation in our model. In fact if somehow there is a power like ground state degeneracy $\mathcal{G} = (\text{something})^{N_S}$, then the Bekenstein-Hawking entropy would follow immediately as a coarse grain entropy.

It looks, however, a little puzzling in our present picture since in the standard treatment of the degenerate Fermi system, the ground state is solely dedicated by energetics and is obtained by filling up all of these N_S states by a pair of fermions with opposite spins $|\pm\rangle$. This results in an unique ground state wave function and zero entropy. However, this is not the only thing that can happen. The fermion degrees of freedom may also be described additionally by some bosonic wave functions. This is, for example, the case in an atom where the electronic states are given by a spin wave function together with an orbital wave function. Therefore let us consider the simple setup where there are two bosonic states χ_1, χ_2 corresponding to each energy level. Then, after taking into account of Fermi statistics, there are four two-body wave functions

$$\begin{gathered}\chi_1(1)\chi_1(2) \times |1,2\rangle_-, \qquad \chi_2(1)\chi_2(2) \times |1,2\rangle_-, \\ (\chi_1(1)\chi_2(2) + \chi_1(2)\chi_2(1)) \times |1,2\rangle_-, \\ (\chi_1(1)\chi_2(2) - \chi_1(2)\chi_2(1)) \times |1,2\rangle_+ \end{gathered} \tag{2.23}$$

where

$$|1,2\rangle_\pm \coloneqq \frac{1}{\sqrt{2}}(|+\rangle_1|-\rangle_2 \pm |-\rangle_1|+\rangle_2) \tag{2.24}$$

are the spin wave functions. Let us denote the set (2.23) of wave functions by $\psi_{E,n_E}, n_E = 1, \ldots, 4$. Then the ground state wave function is given by

$$\Psi_{\{n_E\}} = \prod_{0<E<\mu} \psi_{E,n_E}, \tag{2.25}$$

for a given specification $\{n_E\}$ of symmetry of the ground state. See an example in Fig. 1. We remark that in the spinor space, the fermion pairs at the same energy level are entangled in Bell states. This explains the origin of the assumption made in [20]. We remark that since each energy level has a degeneracy of 4 as in (2.23), the number of energy levels is given by

$$N_L = N_S/4, \tag{2.26}$$

and since each state in (2.23) describes 2 fermionic degrees of freedom, the number N of fermionic degrees

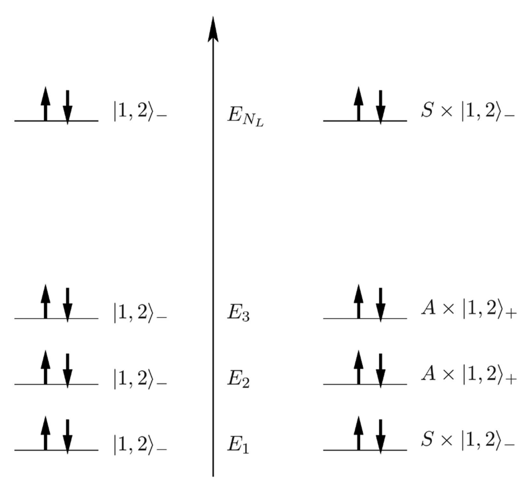

FIG. 1. Left: typical nondegenerate ground state of Fermi system. Right: an entangled ground state of spacetime of black hole.

of freedom contained in each of the $\mathcal{G}$ ground states is given by [29]

$$N = 2N_L \tag{2.27}$$

and

$$N = \frac{A}{4G} \times \frac{1}{2\pi a}. \tag{2.28}$$

This is in sharp contrast with the expression (2.7) for ordinary particles. Now as there is a choice of 4 states ψ_{E,n_E} for each energy level E, the ground state (2.25) has a degeneracy of

$$\mathcal{G} = 4^{N_L} = 2^{N_S/2}. \tag{2.29}$$

The coarse graining of these entangled microstates then give rises to the entropy

$$S = \log \mathcal{G} = N_S/2, \tag{2.30}$$

where a base 2 logarithm is used. This reproduces the Bekenstein-Hawking entropy if $a = \frac{1}{2\pi}$. As a consistency check, (2.22) can also be reproduced from (2.2) if $c_0 = 3\pi/G$. Summarizing, our model is defined by the Fermi level

$$\mu = \frac{1}{2\pi R} \tag{2.31}$$

and a constant density of energy eigenstates

$$g(E) = \frac{3\pi V}{G}. \tag{2.32}$$

We emphasis that the density of energy eigenstates $g(E)$ is not to be confused with the density of microstates $\omega(E)$, which appears directly in the counting of the coarse grain entropy. In the above, we have derived that a constant $g(E)$ give rises to the needed number of microstates to account for the Bekenstein-Hawking entropy. In the context of the matrix model, e.g., Banks-Fischler-Shenker-Susskind (BFSS), it is interesting to find out what background of the bosonic matrices would give rises to a fermionic quantum mechanics with a constant density of eigenstates.

In general, any classical spacetime should admit a fermionic description at the fundamental quantized level. It is an interesting question to understand how a quantum spacetime other than a quantum black hole is characterized by a density of states $g(E)$. It is often assumed that black hole is the closest packed object in nature. This would imply an upper bound on $g(E)$ for an arbitrary quantized spacetime. Since the maximal allowed energy is the Planck scale [30] $\Lambda_P = 1/(3\pi l_P)$, we obtain an upper bound on the number of energy eigenstates $N_S = \int dE g(E)$:

$$N_S \le \frac{V}{l_P^3} \tag{2.33}$$

which is a statement on the *maximal capacity of states* in any volume V. Here we have used $G = l_P^2$ where l_P is the Planck length. Because of the general character of this statement, we propose this to be a fundamental principle of quantum gravity. A possible origin of the upper bound (2.33) of states is that there is a lower limit on the locality in space

$$\Delta V \gtrsim l_P^3. \tag{2.34}$$

Because of (2.34), quantum gravity is manifestly nonlocal below the Planck length scale. This provides a way out of the usual singularity theorem and suggests that our model is in fact consistent without singularity. It will be interesting to understand deeper the meaning and origin of (2.34). It may be related to some kind of uncertainty relation of the quantized spacetime.

We note that the above construction of the ground state wave function can be easily generalized to the case when there is a bosonic degeneracy of g for each energy level E. Denote the single particle wave functions by χ_a, $a = 1, \ldots, g$; then after taking Fermi statistics into account, the two-body wave functions are given by

$$S_{ab}(1,2) \times |1,2\rangle_-, \qquad A_{ab}(1,2) \times |1,2\rangle_+, \tag{2.35}$$

where S_{ab} (respectively A_{ab}) are the symmetric (respectively antisymmetric) two-body wave functions formed by χ_a, χ_b. There are in total g^2 of them. The ground state wave function of the system is given by (2.25) and there is a degeneracy of $\mathcal{G} = (g^2)^{N_L}$ where the number of energy levels is now $N_L = N_S/g^2$. This gives rise to the entropy $\sim 2N_S/g^2$ and is equal to (1.3) for $g = 2$.

We note also that our construction of entangled ground state wave functions can be generalized easily to ground state wave functions with more complicated entanglement, e.g., multibody. In principle, it could mean that black holes can exist in other quantum states with different entropy content. But it may also mean that these more general entangled states are actually forbidden in quantum gravity. Formulating the selection rule and understanding how and why it works would give useful hints on the properties of quantum gravity.

We remark that in our model, the area dependence of the Bekenstein-Hawking entropy is explained by a set of fermionic degrees of freedom, which are gravitational in origin and live in the interior of the black hole. On the other hand, in the picture of [7], the explanation is found in terms of a two-state spin system associated with each site of a two-dimensional lattice of quanta living on the horizon area. The two descriptions are supposed to be related by holography. In our model, the Bekenstein-Hawking entropy originates in the entanglement of the fermionic degrees of freedom. This is nothing but another illustration of the celebrated observation of Ryu and Takayanagi [31] that entanglement is at the heart of holography. Our model suggests that holography is a consequence of the fermionic nature of the quantum spacetime.

We remark that the area law receives corrections in higher derivative modified gravity. In principle, such corrections can be accounted for in our model by using a modified Fermi level and a modified density of states. However, deriving these results will require the knowledge of a quantum theory of gravity. Please see [32,33] for a recent proposal of a theory of quantum gravity as a quantum mechanics of quantum spaces. It is quite amusing that most of the features of the phenomenological Fermi model appear there in a consistent setup.

Finally, let us comment on the first law of black hole thermodynamics. Using (2.21), (2.26), and (2.27) in our model, we find $U = N\mu$ for the energy of the Fermi system, where the number of fermions $N = A/4G$ and $\mu = 1/(2\pi R)$. This gives immediately [34] the pressure

$$P_U = -\left(\frac{\partial U}{\partial V}\right)_{N,S} = \frac{1}{8\pi G R^2} \tag{2.36}$$

and the chemical potential

$$\mu_C = \left(\frac{\partial U}{\partial N}\right)_{V,S} = \mu. \tag{2.37}$$

As a consistency check, due to the fact that $S = N$ in our model, we get a vanishing temperature

$$T = \left(\frac{\partial U}{\partial S}\right)_{V,N} = 0, \tag{2.38}$$

which is in fact what we have considered for a completely degenerate system. It is easy to see that the following relation holds:

$$dU = -P_U dV + \mu_C dN. \tag{2.39}$$

At this point, it might be a bit puzzling as to why (2.39) is different from the usual statement of black hole thermodynamics

$$dU = T_H dS, \tag{2.40}$$

which involves only the entropy term, and has neither a pressure term nor a chemical potential term as in (2.39). The puzzle is resolved once we realize that the form (2.40) of the first law was deduced from the physics outside the black hole, e.g., from the properties of the Hawking radiation or from the Brown-York stress tensor defined outside the black hole [35]. In particular, T_H is the temperature of the Hawking radiation and (2.40) is the first law of the black hole as observed by an asymptotic observer based on the degrees of freedom *outside* the black hole. In comparison, (2.39) is the first law for the *interior* of the black hole. The existence of a nonvanishing pressure (2.36) and chemical potential (2.37) follows from the properties of the fermionic degrees of freedom of the black hole. Note that while classically not even light can escape a black hole, it is possible to access the interior of a black hole through quantum mechanical effects such as tunneling or entanglement. For example, one can create an entangled pair of states and allows one of them to fall into the black hole. With careful handling of the state, the entanglement can be preserved. This would allow the outside world to probe the interior of the black hole through an entanglement measurement.

D. Reissner-Nordström charged black hole

Next, let us consider a charged black hole with mass M and charge Q. In general relativity, it is described by the Reissner-Nordström metric with the horizon radius R satisfying the quadratic relation

$$R^2 - RR_S + R_Q^2 = 0, \tag{2.41}$$

where $R_S = 2GM$ and $R_Q = \sqrt{GQ^2}$. We now show that the properties of the charged black hole can also be reproduced by the Fermi system. From our analysis in the last subsection, we have learned that the system of Fermi quanta has a constant density of states

$$g(E) = \frac{3\pi V}{G} \tag{2.42}$$

and a Fermi energy

$$\mu = \frac{1}{2\pi R}. \tag{2.43}$$

It is natural to assume that these properties are fundamental quantum properties of black hole spacetime and holds also here. As a result, we obtain the total number of states

$$N_S = \frac{2\pi R^2}{G}, \tag{2.44}$$

the internal energy of the Fermi sea

$$U = \frac{1}{2} N_S \mu = \frac{R}{2G} \tag{2.45}$$

and the internal energy pressure $P_U = -(\partial U/\partial V)_N$,

$$P_U = \frac{1}{8\pi G R^2}. \tag{2.46}$$

We note that (2.44)–(2.46) are obtained from the properties (2.42) and (2.43) and are independent of the charge.

Next we have the gravitational pressure. Using (2.14), the gravitational energy of the charged black hole is given by

$$E_g = \frac{R}{G} - M. \tag{2.47}$$

In addition, there is also a Coulombic energy U_Q due to the presence of an electric field. We can compute $U_Q = \int_R^\infty u\, 4\pi r^2 dr$ from the electric energy density $u = E^2/(8\pi) = Q^2/(8\pi r^4)$ outside the black hole, which gives

$$E_Q = \frac{Q^2}{2R}. \tag{2.48}$$

Invoking the equivalence principle, the total energy $E_g + E_Q$ of the black hole solution should be equal to the internal energy U of the Fermi system. This gives precisely the mass-charge-radius relation (2.41), which in turn implies that

$$E_g = \frac{R}{2G} - \frac{Q^2}{2R}. \tag{2.49}$$

This leads to the gravitational pressure

$$P_g = \frac{1}{8\pi G R^2} + \frac{Q^2}{8\pi R^4} \tag{2.50}$$

acting from outside on the horizon of the charged black hole. Added to it is the electromagnetic pressure $P_Q = \partial E_Q / \partial V$

$$P_Q = -\frac{Q^2}{8\pi R^4}. \tag{2.51}$$

As a result we find a balance of pressure $P_U = P_g + P_Q$ for the Fermi system and our model for the black hole is stable against the gravitational squash.

We remark that the radius relation (2.41) was originally derived from the condition $1/g_{rr}|_{r=R} = 0$ at the horizon of the Reissner-Nordström metric

$$ds^2 = -\left(1 - \frac{R_S}{r} + \frac{R_Q^2}{r^2}\right) dt^2 + \left(1 - \frac{R_S}{r} + \frac{R_Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{2.52}$$

As such, there is no simple understanding of (2.41) apart from the fact that it comes from the Einstein equation. It is quite remarkable that in our model of the black hole, the radius relation (2.41) admits a direct physical interpretation in terms of the equivalence principle, which in turn guarantees the stability of the black hole model. We also note that the counting for the Bekenstein-Hawking entropy works the same way as discussed above.

E. Black hole with a cosmological constant

In this subsection, we show that our model also works for black holes with a cosmological constant. Let us start first with the de Sitter space [36] with a positive cosmological constant $\Lambda > 0$. In the static coordinates, the metric,

$$ds^2 = -\left(1 - \frac{\Lambda r^2}{3}\right) dt^2 + \left(1 - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{2.53}$$

is a solution to the Einstein equation with a vacuum background energy density $\rho_\Lambda = \Lambda/(8\pi G)$ and a background pressure

$$P_\Lambda = -\frac{\Lambda}{8\pi G}. \tag{2.54}$$

The de Sitter black hole is a solution in this background with the metric

$$ds^2 = -\left(1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}\right) dt^2 + \left(1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{2.55}$$

For $M < M_c$ less than a certain critical mass M_c, the condition

$$1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3} = 0 \tag{2.56}$$

has two positive roots, giving the location of the black hole horizon and the cosmological horizon, respectively. At $M = M_c$, the two positive roots meet, meaning that there is a maximum size black hole which can fit the de Sitter space before the black hole horizon hits the cosmological horizon.

Let us consider the case $M < M_c$ of interest. Let R be the black hole horizon radius and L be the cosmological horizon radius. The black hole interior region $0 < r < R$ is described by a system of quanta with the Fermi energy and the density of states

$$\mu = \frac{1}{2\pi R}, \qquad g(E) = \frac{3\pi}{G} V_R, \tag{2.57}$$

where $V_R := 4\pi R^3/3$. This gives the internal energy

$$U = \frac{R}{2G} \tag{2.58}$$

for the Fermi system and the degeneracy pressure

$$P_U = \frac{1}{8\pi G R^2}. \tag{2.59}$$

In addition, the system has an energy

$$E_\Lambda = \frac{\Lambda}{6G} R^3 \tag{2.60}$$

and the pressure (2.54) due to the presence of a cosmological constant. As for the gravitational energy, we note that the definition (2.14) requires an asymptotic flat background as a reference level for the energy. Now since there is no asymptotically flat region in the de Sitter black hole spacetime and also because there is not a timelike Killing vector in the global de Sitter spacetime, the definition of mass in de Sitter spacetime is nontrivial. One can employ a convenient definition of the black hole mass by subtracting away the background contribution from the pure de Sitter space ($M = 0$):

$$E_g \coloneqq \frac{\kappa A}{4\pi G} - \left(\frac{\kappa A}{4\pi G}\right)_{M=0}. \tag{2.61}$$

This gives for the de Sitter (dS) black hole

$$E_g = M. \tag{2.62}$$

Actually one can use any other definition as long as the relation (2.62) is reproduced. The principle of equivalence then states that equality $U = E_g + E_\Lambda$ between the energy of the Fermi system and the total gravitational energy of the black hole solution. This gives immediately the relation (2.55) satisfied by the horizon radius of a dS black hole. At the same time, we have from (2.62) the gravitational pressure

$$P_g = \frac{1}{8\pi G R^2} - \frac{\Lambda}{8\pi G} \tag{2.63}$$

in the outside region of the black hole. Note that the background pressure of de Sitter space is automatically reproduced here. All in all, the pressure is balanced:

$$P_U = P_g + (-P_\Lambda), \tag{2.64}$$

where the term $-P_\Lambda$ is the pressure acting on the system externally due to the background energy density. Our model for the black hole is stable. We remark that as the Fermi degrees of freedom are supposed to be elementary quanta of the quantized space, it should be possible to build a Fermi model that describes the cosmological horizon as well. This is an interesting problem to consider for the quantum mechanics of quantum space [32,33].

Finally, let us consider the case of the anti–de Sitter (AdS) black hole. Classically, the metric is given by

$$ds^2 = -\left(1 - \frac{2GM}{r} + \frac{|\Lambda| r^2}{3}\right) dt^2 + \left(1 - \frac{2GM}{r} + \frac{|\Lambda| r^2}{3}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{2.65}$$

In this case, the black hole horizon radius satisfies the relation

$$1 - \frac{2GM}{R} + \frac{|\Lambda| R^2}{3} = 0 \tag{2.66}$$

and the black hole is situated in the background AdS space with a boundary at infinity. We model the black hole interior with a Fermi system of the same properties (2.57)–(2.59). The gravitational energy for the AdS black hole is well defined and we have

$$E_g = M. \tag{2.67}$$

Equating the energy of the Fermi system $\frac{R}{2G}$ with the total gravitational energy $M - \frac{|\Lambda|}{6G} R^3$ of the black hole spacetime, we obtain the mass-horizon radius relation for the AdS black hole. In the same way, we have the gravitational pressure

$$P_g = \frac{1}{8\pi G R^2} + \frac{|\Lambda|}{8\pi G} \tag{2.68}$$

in the outside region of the black hole. Pressure is again balanced and the Fermi model of the black hole interior is stable against collapse. The Bekenstein-Hawking entropy works out the same way as before.

III. CONCLUSION AND DISCUSSIONS

In this paper, we have proposed a quantum mechanical model of the black hole as a system of fermionic degrees of freedom residing within the horizon. Quite amazingly, with only a few simple assumptions, our bottom-up approach can resolve immediately all the mentioned puzzling properties of black holes, all at once in the same model. In our model, there is a pressure of the black hole, which balances out the gravitational collapse, and there is a chemical potential that measures the amount of energy carried away by the Hawking radiation quanta. The temperature is zero, yet a Bekenstein-Hawking entropy arises due to the presence of entanglement of states. We also find a new expression of the black hole mechanics in terms of the black hole interior degrees of freedom. We show that the fermionic nature of the black hole ground state, together with its entanglement structure, is crucial in obtaining a nonsingular interior and providing the needed microstates counting.

In general, we propose that quantum gravity can be described in terms of fermionic quantum mechanics. In this paper, we took a bottom-up approach to build the quantum mechanics by invoking only what is necessary [37]. According to this picture, ordinary spacetime and gravity should arise as some effective description and interaction of the Fermi variables, with the Newton constant emerges in the gravitational description. Accordingly, a general spacetime would have an entangled wave function associated with it. It is interesting to ask whether and how the entanglement asset contained in spacetime is observable in general. It is also interesting to understand how the Einstein equation may emerge as effective dynamics of the underlying fermionic degrees of freedom, perhaps due to the entropic property of these variables.

We comment that the Page curve behavior of the entanglement entropy of Hawking radiation may be analyzed in our model. In [20], the set of Bell states was assumed to be located in a thin layer underneath the horizon, and it was shown that tunneling results in the Page curve. In our present model, the entangled states are distributed all over the "interior." Nevertheless, the same analysis can be carried out, and one can show that the tunneling process also resulted in the Page curve [23].

We remark that despite the fact that our model is stable against gravitational collapse in terms of pressure, it is necessary to understand more precisely how the prediction of the singularity in general relativity is avoided in our model. This may be related to our proposal that the interior of black holes is a quantum spacetime with a nonvanishing lower bound (2.34) on the spatial nonlocality. In fact, the relation (2.34) suggests that spacetime at the Planck scale is characterized by some form of noncommutative geometry. A better understanding of this fundamental issue is crucial but is beyond the scope of the current work.

As matter collapses and forms the black hole, the energy and information of the collapsed matter are supposed to be transformed into the energy and entanglement of the quantized spacetime stored and encoded by the fermionic variables. Recently, based on observations on the growth rate of astronomical black holes in the Universe, the authors of [38,39] proposed that dark energy need not be a property of the vacuum, but a black hole could be a source of dark energy. It is interesting to study further the properties of this kind of dark energy using our model. It is also interesting to study the response to gravitational waves [40].

We remark that our proposal shared some similarities with the fuzzy ball proposal [10] in that the classical vacuum of the black hole is replaced by a set of quantum states: the fuzzy ball of strings in the fuzzy ball proposal and the set of entangled fermionic states in our proposal. Our proposal is more generic in character. It will be interesting to understand better the differences and possible connections. It will also be interesting to understand how our model may arise from existing proposals of quantum gravity, such as string theory or the matrix model. In particular, a fermionic string theory may provide the right framework in which the quantum geometry of the black hole and the considered fermionic degrees of freedom may be obtained as string quanta from the string quantization [41–45].

Finally, we emphasize that our model of the black hole is just a first step toward the theory of quantum spacetime. Historically, the Bohr atomic model, with its postulate about the existence of quantized orbits, has given valuable hints to the development of quantum mechanics. It is hoped that our simple model of black holes has provided useful clues to the construction of the theory of quantum gravity.

ACKNOWLEDGMENTS

C. S. C. likes to thank Nick Dorey, Dimitris Giataganas, Pei-Ming Ho, Hikaru Kawai, and Himanshu Parihar for helpful discussion. C. S. C. acknowledges support of this work by NCTS and Grant No. 110-2112-M-007-015-MY3 of the National Science and Technology Council of Taiwan. R. X. M. thanks the National Natural Science Foundation of China (No. 12275366) for their support.

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