Double copy of 3D Chern-Simons theory and 6D Kodaira-Spencer gravity

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We apply an algebraic double copy construction of gravity from gauge theory to three-dimensional (3D) Chern-Simons theory. The kinematic algebra $\mathcal{K}$ is the 3D de Rham complex of forms equipped, for a choice of metric, with a graded Lie algebra that is equivalent to the Schouten-Nijenhuis bracket on polyvector fields. The double copied gravity is defined on a subspace of $\mathcal{K} \otimes \bar{\mathcal{K}}$ and yields a topological double field theory for a generalized metric perturbation and two 2-forms. This local and gauge invariant theory is non-Lagrangian but can be rendered Lagrangian by abandoning locality. Upon fixing a gauge this reduces to the double copy of Chern-Simons theory previously proposed by Ben-Shahar and Johansson. Furthermore, using complex coordinates in $\mathbb{C}^3$ this theory is related to six-dimensional (6D) Kodaira-Spencer gravity in that truncating the two 2-forms and one equation yields the Kodaira-Spencer equations on a 3D real slice of $\mathbb{C}^3$. The full 6D Kodaira-Spencer theory can instead be obtained as a consistent truncation of a chiral double copy.

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I. INTRODUCTION

The double copy denotes modern amplitude techniques that relate the scattering amplitudes of gauge theory to those of gravity. The prime example is the double copy of pure Yang-Mills theory which yields at least at tree-level Einstein gravity coupled to a B-field (2-form) and a scalar (dilaton), a theory also known as "$\mathcal{N} = 0$ supergravity" [1,2]. The double copy was originally defined at the level of scattering amplitudes and hence is *a priori* only meaningful for on-shell and gauge-fixed fields, but there are several reasons why one would like to go beyond this. For instance, one would like to double copy classical solutions of Yang-Mills theory to obtain classical gravity solutions, see [3–9], or one would like to have better control over double copy at loop level [10–17].

In recent years there has been significant progress in this direction using the framework of homotopy algebras, based on the general dictionary between (classical) field theories and homotopy Lie or L_∞ algebras [18–24]. These are generalizations of differential graded Lie algebras, i.e., superalgebras equipped with a nilpotent operator, in which the Jacobi identity may only hold "up to homotopy" controlled by higher brackets. L_∞ algebras naturally encode the data of a gauge field theory, including Yang-Mills theory and gravity, the latter written perturbatively as an expansion about a background metric.

Homotopy algebras are a powerful framework for double copy since, among other reasons, they allow one to give meaning to the notion of "stripping off" the color factors associated with the gauge group. Specifically, the L_∞ algebra of Yang-Mills theory factorizes into the tensor product $\mathcal{X}_{\rm YM} = \mathcal{K} \otimes \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of the gauge group and $\mathcal{K}$ the "kinematic" algebra, which is a homotopy generalization of a differential graded commutative associative algebra or C_∞ algebra [20]. Importantly for double copy, the kinematic space $\mathcal{K}$ also carries hidden structures that define what has been termed a $\mathrm{BV}_\infty^\square$ algebra [25–27], a generalization of a Batalin-Vilkovisky algebra [28,29], where $\square$ denotes the wave operator. Upon taking the tensor product of two copies $\mathcal{K}$ and $\bar{\mathcal{K}}$ of the kinematic $\mathrm{BV}_\infty^\square$ algebra, one can construct the L_∞ algebra of $\mathcal{N} = 0$ supergravity, formulated as a double field theory [30–32], on a suitable subspace of $\mathcal{K} \otimes \bar{\mathcal{K}}$. More precisely, so far this has been established to the order corresponding to quartic couplings in an action [27,33,34].

In this paper we apply the general algebraic double copy construction of [27] to the toy model of three-dimensional (3D) Chern-Simons theory. This theory is topological, and hence all of its scattering amplitudes vanish, but it is still a fruitful toy model for color-kinematics duality (a necessary prerequisite for the double copy), which can be studied at the level of off-shell correlation functions [35]. Here the "kinematic Lie algebra" underlying color-kinematics duality has been identified recently and just lives on the familiar de Rham complex of differential forms in 3D. This algebra

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is part of a BV$^\square$ algebra, which is "strict," meaning that no higher maps actually appear. One can thus straightforwardly apply the double copy construction of [27] to 3D Chern-Simons theory and obtain a double copied 3D gravity theory in a formulation that is by construction local and gauge invariant. Here we spell out this 3D gravity theory. To the best of our knowledge this is the first complete first-principle double copy construction of a diffeomorphism invariant gravity theory from a gauge theory. While the double copy of Chern-Simons theory has been explored previously by Ben-Shahar and Johansson, their gravity action is gauge fixed and nonlocal [35], see also [36,37]. The double copied gravity theory constructed in this paper is gauge invariant and local but non-Lagrangian. One can write down a gauge invariant action, however, upon partial gauge fixing and abandoning locality, which then reduces upon further gauge fixing and truncation to the action given in [35].

Concretely, the 3D gravity theory obtained as the double copy of 3D Chern-Simons theory can be written in a form that exhibits a formal six-dimensional (6D) covariance as follows. The gauge field is a 6D 2-form $\Psi = \frac{1}{2}\Psi_{MN}\theta^M\theta^N$, where we view all differential forms as functions of even coordinates $x^M = (x^\mu, \bar{x}^{\bar{\mu}})$ and odd coordinates $\theta^M \simeq \mathbf{d}x^M$. The field equations read

$$\mathbf{d}\Psi + \frac{1}{2}[\Psi, \Psi] = 0, \tag{1.1}$$

which are invariant under gauge transformations with 1-form parameters $\Lambda = \Lambda_M\theta^M$,

$$\delta\Psi = \mathbf{d}\Lambda + [\Psi, \Lambda], \tag{1.2}$$

where the graded symmetric bracket on general 6D forms is given by

$$[F_1, F_2] \coloneqq \eta^{MN}\frac{\partial F_1}{\partial\theta^M}\frac{\partial F_2}{\partial x^N} \pm (1 \leftrightarrow 2), \quad \eta_{MN} = \begin{pmatrix} \delta_{\mu\nu} & 0 \\ 0 & -\delta_{\bar{\mu}\bar{\nu}} \end{pmatrix}, \tag{1.3}$$

with the $O(3,3)$ invariant metric η_{MN} built from two copies of a fiducial 3D metric indicated here by Kronecker δ's. More precisely, this theory is only gauge invariant and hence consistent when the fields are subject to the "strong constraint" making them effectively 3D. The reason is that, while the bracket (1.3) defines a genuine graded Lie algebra in 6D (that written in terms of polyvector fields is known as the Schouten-Nijenhuis bracket), the 6D de Rham differential $\mathbf{d} = \theta^M\partial_M$ does not act via the Leibniz rule on the bracket. The failure of $\mathbf{d}$ to act via the Leibniz rule involves terms of the structural form $\eta^{MN}\partial_M F_1 \partial_N F_2$ and is due to $\Delta \coloneqq \eta^{MN}\partial_M\partial_N = \square - \bar{\square}$ being a second-order differential operator. This failure is cured by imposing the so-called strong constraint $\Delta = 0$ in the sense of double field theory [30–32], i.e., Δ annihilates all fields and all their products. An obvious solution to this constraint is to identify $x = \bar{x}$. (One may also attempt to define a "weakly constrained" version following the recent progress in [38,39], but we will not do so here.)

Decomposing the 2-form Ψ_{MN} into 3D objects one obtains $e_{\mu\bar{\nu}}$, the generalized metric fluctuation of double field theory [32], and two 2-forms. While to the best of our knowledge this (topological) 3D gravity model has not been explored independently, we will point out a curious relation to Kodaira-Spencer gravity, a topological gravity theory in 6D with the complex structure deformation of a Hermitian manifold as fundamental field [40,41]. The relation of the field equations (1.1) to the Kodaira-Spencer equations is as follows: Truncating the two 2-forms, which is a consistent truncation in the technical sense that all solutions of the truncated theory uplift to solutions of the full theory, one is left with *two* equations for $e_{\mu\bar{\nu}}$, reflecting the fact that the theory is non-Lagrangian. Picking one of the two equations only (which of course is no longer a consistent truncation) and rewriting it in terms of complex coordinates of $\mathbb{C}^3$ one obtains the Kodaira-Spencer equations on a 3D real slice of $\mathbb{C}^3$. Once we restrict to one of the equations one may relax the strong constraint as the Kodaira-Spencer theory is consistent in 6D thanks to its invariance under only "holomorphic" 3D diffeomorphisms. We will show that the full 6D Kodaira-Spencer gravity can instead be obtained as a consistent truncation of a certain "chiral" double copy.

The remainder of this paper is organized as follows. In Sec. II we review the kinematic BV$^\square$ algebra of Chern-Simons theory. We then work out the double copied gravity theory and give the double field theory formulation with 6D covariance, which is diffeomorphism invariant and local but non-Lagrangian. In Sec. III we show that a gauge invariant action can be written at the cost of abandoning locality. In Sec. IV we rewrite the theory in terms of complex coordinates of $\mathbb{C}^3$ and describe the relation to Kodaira-Spencer gravity, which is a consistent truncation of a chiral double copy. We close with a summary and outlook in Sec. V.

II. DOUBLE COPY OF CHERN-SIMONS THEORY

A. Kinematic algebra of Chern-Simons theory

We begin by recalling the kinematic algebra of Chern-Simons theory. The kinematic algebra is defined on the de Rham complex of an arbitrary 3-manifold. This chain complex is the graded vector space $\mathcal{K} = \bigoplus_{p=0}^3 K_p$, with K_p the space of p-forms and the differential given by the de Rham differential d satisfying $d^2 = 0$,

$$\begin{array}{ccccccc} K_0 & \xrightarrow{d} & K_1 & \xrightarrow{d} & K_2 & \xrightarrow{d} & K_3 \\ \lambda & & A & & \mathcal{E} & & \mathcal{N} \end{array}. \tag{2.1}$$

Here we also indicated the field theory interpretations in the second line (that will emerge after taking the tensor product with the color Lie algebra $\mathfrak{g}$). Specifically, the gauge parameter λ is a 0-form, the field A is a 1-form, the "equation of motion" is a 2-form, and the "Noether identity" $\mathcal{N}$ is a 3-form.

This chain complex carries a graded commutative and associative algebra structure, given by the familiar wedge product of differential forms. We use the standard normalization

$$\omega_p = \frac{1}{p!}\omega_{\mu_1\cdots\mu_p}(x)\theta^{\mu_1}\cdots\theta^{\mu_p}, \tag{2.2}$$

where we identified the basis 1-forms with odd variables θ^μ of degree $+1$: $dx^\mu \simeq \theta^\mu$. Thus, we can think of differential forms as functions $\omega(x,\theta)$, $\eta(x,\theta)$, etc., of even coordinates x^μ and odd coordinates θ^μ. The wedge product is then encoded in the ordinary pointwise product of functions, which is graded commutative and associative. Specifically, this defines a differential graded commutative associative algebra (DGCA) (the strict case of a C_∞ algebra) with differential m_1 of degree $+1$ and a product m_2 of degree 0, given by

$$m_1 \equiv d \equiv \theta^\mu \partial_\mu, \qquad m_2(\omega,\eta) \equiv \omega\cdot\eta. \tag{2.3}$$

This form makes it clear that $m_1 = d$ is a first-order differential operator and hence obeys the Leibniz rule with respect to m_2.

While the above kinematic DGCA exists for any topological 3-manifold, the full kinematic algebra, denoted $\mathrm{BV}^\square$, requires more structure, as given by a choice of metric. Given such a choice of metric, which for simplicity we take to be flat and Euclidean, we have the Hodge star operation defined by

$$\star\omega_p = \frac{1}{p!(3-p)!}\epsilon_{\mu_1\cdots\mu_{3-p}\nu_1\cdots\nu_p}\omega^{\nu_1\cdots\nu_p}\theta^{\mu_1}\cdots\theta^{\mu_{3-p}}, \tag{2.4}$$

satisfying $\star^2 = 1$, where the metric is used to raise and lower indices. Furthermore, we have the adjoint or divergence operator $d^\dagger$ acting on a p-form as

$$d^\dagger \coloneqq (-1)^{p+1}\star d\star \equiv \frac{\partial^2}{\partial x_\mu \partial\theta^\mu}, \tag{2.5}$$

where again the metric is used to lower the index on x^μ. Note that $d^\dagger$ defines a second differential, satisfying $(d^\dagger)^2 = 0$, that is of opposite degree to d. Moreover, d and $d^\dagger$ satisfy the familiar relation

$$dd^\dagger + d^\dagger d = \square \equiv \partial^\mu\partial_\mu. \tag{2.6}$$

Notice that the second form of $d^\dagger$ in (2.5) makes it clear that it is a second-order operator on the associative product of graded functions (also known as the wedge product of forms).

In line with the general literature we also denote $b \equiv d^\dagger$, so that (2.5) reads

$$b = \partial^\mu \mathcal{D}_\mu, \quad \text{where } \mathcal{D}_\mu \coloneqq \frac{\partial}{\partial\theta^\mu}, \tag{2.7}$$

and (2.6) becomes the graded commutator relation $[m_1, b] = \square$. In addition to the graded commutative associative product m_2 there is a derived (kinematic) Lie bracket, defined by the "failure of b to act as a derivation on m_2." This so-called antibracket can be written as the graded commutator $b_2 \coloneqq [b, m_2]$, for which a brief computation with (2.7) gives

$$b_2(\omega_1,\omega_2) = \mathcal{D}_\mu\omega_1\partial^\mu\omega_2 + (-1)^{\omega_1\omega_2}\mathcal{D}_\mu\omega_2\partial^\mu\omega_1. \tag{2.8}$$

This is a graded Lie bracket, which generalizes the familiar Poisson bracket to the graded setting and is equivalent to the Schouten-Nijenhuis bracket on polyvector fields [42] (see also [26] and Sec. II of [27]). Specifically, given two polyvector fields $\Pi_1 = \frac{1}{p!}\Pi_1^{\mu_1\ldots\mu_p}\partial_{\mu_1}\wedge\ldots\wedge\partial_{\mu_p}$ and $\Pi_2 = \frac{1}{q!}\Pi_2^{\mu_1\ldots\mu_q}\partial_{\mu_1}\wedge\ldots\wedge\partial_{\mu_q}$ the Schouten-Nijenhuis bracket acts as

$$\begin{aligned}[\Pi_1,\Pi_2]_{\mathrm{SN}} = \nabla\cdot(\Pi_1\wedge\Pi_2) - \nabla\cdot\Pi_1\wedge\Pi_2 \\ - (-1)^p\Pi_1\wedge\nabla\cdot\Pi_2,\end{aligned} \tag{2.9}$$

where $\nabla\cdot$ is the covariant divergence of polyvector fields. Since we are working on flat space, the covariant divergence is simply

$$(\nabla\cdot\Pi)^{\mu_1\cdots\mu_{p-1}} = \partial_\nu\Pi^{\nu\mu_1\cdots\mu_{p-1}}. \tag{2.10}$$

One can straightforwardly identify polyvector fields and differential forms, for which the Schouten-Nijenhuis bracket and the bracket b_2 in (2.8) are equivalent.

There is also a Poisson compatibility relation between b_2 and m_2. While b_2 is a graded Lie bracket, (m_1, b_2) fail to define a differential graded Lie algebra (DGLA or a strict L_∞ algebra) due to the box obstruction for the Leibniz relation following from $[m_1, b] = \square$,

$$[m_1, b_2] = [\square, m_2]. \tag{2.11}$$

In contrast, (b, b_2) define a DGLA since b acts via the Leibniz rule on b_2, as one may quickly verify. With these relations, the data (m_1, b, m_2, b_2) together define a $\mathrm{BV}^\square$ algebra [25].

We finally point out that the integration of differential forms equips $\mathcal{K}$ with an inner product. Specifically, integration of top forms coincides with the integration of graded functions,

$$\int \omega_3 = \int d^3\theta d^3x \omega_3(x,\theta), \qquad \int d^3\theta \theta^\mu \theta^\nu \theta^\rho := \epsilon^{\mu\nu\rho}, \qquad (2.12)$$

which provides a degree -3 inner product on $\mathcal{K}$,

$$\langle \omega, \eta \rangle := \int d^3\theta d^3x \omega(x,\theta)\eta(x,\theta) \equiv \int \omega \wedge \eta. \qquad (2.13)$$

The pairing is nonvanishing only between K_1 and K_2 and between K_0 and K_3. This encodes the usual pairings between fields and equations and between gauge parameters and Noether identities. The operator $\star$ realizes the inner product isomorphisms $K_1 \simeq K_2$ and $K_0 \simeq K_3$.

B. The double copy complex

We will now realize an exact double copy of Chern-Simons theory. This is possible since the product m_2 (pointwise product of graded functions) and the bracket b_2 (Schouten-Nijenhuis bracket) are part of a *strict* $\mathrm{BV}^\square$ algebra on $\mathcal{K}$. The double copy is then defined on $\mathcal{K} \otimes \bar{\mathcal{K}}$, which carries a BV^Δ algebra, where $\Delta = \square - \bar{\square}$ [39]. In order to define a consistent field theory on the tensor product $\mathcal{K} \otimes \bar{\mathcal{K}}$, we impose the strong constraint $\Delta = 0$ or $\square = \bar{\square}$ on all elements $\Psi \in \mathcal{K} \otimes \bar{\mathcal{K}}$ and their products, which leads to a (strict) L_∞ algebra underlying a non-Lagrangian gravity theory.

First of all, the double copy space is given by $\mathcal{X} = \mathcal{K} \otimes \bar{\mathcal{K}}$. All elements of $\mathcal{X}$ are (p,q)-forms on the doubled space $\mathbb{R}^3 \times \mathbb{R}^3$, in the sense that

$$\Omega_{p,q} = \frac{1}{p!q!} \Omega_{\mu_1 \cdots \mu_p \bar{\nu}_1 \cdots \bar{\nu}_q}(x,\bar{x}) \theta^{\mu_1} \cdots \theta^{\mu_p} \bar{\theta}^{\bar{\nu}_1} \cdots \bar{\theta}^{\bar{\nu}_q}, \qquad (2.14)$$

with the degree in $\mathcal{X}$ given by total form degree: $|\Omega_{p,q}| = p + q$. Integration of top forms extends naturally to the doubled space,

$$\int \Omega_{3,3} = \int d^3\bar{\theta} d^3\theta d^3x d^3\bar{x} \Omega_{3,3}(x,\bar{x},\theta,\bar{\theta}),$$
$$\int d^3\bar{\theta} d^3\theta \theta^\mu \theta^\nu \theta^\rho \bar{\theta}^{\bar{\mu}} \bar{\theta}^{\bar{\nu}} \bar{\theta}^{\bar{\rho}} := \epsilon^{\mu\nu\rho} \epsilon^{\bar{\mu}\bar{\nu}\bar{\rho}}. \qquad (2.15)$$

The total space $\mathcal{X}$ decomposes as follows:

$$\begin{array}{ccccccccccccc} X_0 & \xrightarrow{\mathbf{d}} & X_1 & \xrightarrow{\mathbf{d}} & X_2 & \xrightarrow{\mathbf{d}} & X_3 & \xrightarrow{\mathbf{d}} & X_4 & \xrightarrow{\mathbf{d}} & X_5 & \xrightarrow{\mathbf{d}} & X_6 \\ & & & & & & \mathcal{E}_{3,0} & & & & & & \\ & & & & C_{2,0} & & & & \mathcal{N}_{3,1} & & & & \\ & & \lambda_{1,0} & & & & \mathcal{E}_{2,1} & & & & \mathcal{R}_{3,2} & & \\ \chi_{0,0} & & & & e_{1,1} & & & & \mathcal{N}_{2,2} & & & & \mathcal{V}_{3,3} \\ & & \bar{\lambda}_{0,1} & & & & \bar{\mathcal{E}}_{1,2} & & & & \bar{\mathcal{R}}_{2,3} & & \\ & & & & \bar{C}_{0,2} & & & & \bar{\mathcal{N}}_{1,3} & & & & \\ & & & & & & \bar{\mathcal{E}}_{0,3} & & & & & & \end{array} \qquad (2.16)$$

where the differential is $\mathbf{d} = d + \bar{d}$. Above we denoted the (p,q)-form degrees by subscripts and aligned vertically the elements according to the twist $p - q$. The integration (2.15) induces a pairing on $\mathcal{X}$ given by

$$\langle \Omega, H \rangle := \int d^3\bar{\theta} d^3\theta d^3x d^3\bar{x} \Omega(x,\bar{x},\theta,\bar{\theta}) H(x,\bar{x},\theta,\bar{\theta}), \qquad (2.17)$$

which, for biforms $\Omega_{p,q}$ and $H_{r,s}$, is nonvanishing only if $p + r = q + s = 3$. We give a field theory interpretation to the chain complex (2.16) by identifying fields as elements of X_2. This choice is due to the fact that the metric fluctuation is expected to arise from the "tensor product of two gauge fields." More precisely, given the Chern-Simons complex (2.1) the spin two fluctuation resides in $K_1 \otimes \bar{K}_1$. Following this interpretation gauge parameters are elements of X_1 and field equations live in X_3, while elements in higher degree correspond to a cascade of Noether and Noether-for-Noether identities. The degree convention differs from the standard L_∞ one (where fields are in degree 0), but for differential forms the form degree is more natural. From the explicit form (2.16) of the complex, it is clear that the field theory described by the L_∞ algebra on $\mathcal{X}$ is non-Lagrangian, at least if one insists in identifying fields in X_2 and field equations in X_3, for then there are more field equations than fields.

Keeping this standard interpretation, we begin by working out the linear theory. The fields living in X_2 can be parametrized as

$$\Psi = e + C + \bar{C} = e_{\mu\bar{\nu}} \theta^\mu \bar{\theta}^{\bar{\nu}} - \frac{1}{2} C_{\mu\nu} \theta^\mu \theta^\nu + \frac{1}{2} \bar{C}_{\bar{\mu}\bar{\nu}} \bar{\theta}^{\bar{\mu}} \bar{\theta}^{\bar{\nu}}, \qquad (2.18)$$

while the gauge parameters living in X_1 can be written as

$$\Lambda = \lambda + \bar{\lambda} = -\lambda_\mu \theta^\mu + \bar{\lambda}_{\bar{\mu}} \bar{\theta}^{\bar{\mu}}. \qquad (2.19)$$

The linear gauge transformations $\delta\Psi = \mathbf{d}\Lambda$ then read

$$\delta e = d\bar{\lambda} + \bar{d}\lambda \longrightarrow \delta e_{\mu\bar{\nu}} = \partial_\mu \bar{\lambda}_{\bar{\nu}} + \bar{\partial}_{\bar{\nu}} \lambda_\mu, \qquad (2.20)$$

$$\delta C = d\lambda \longrightarrow \delta C_{\mu\nu} = \partial_\mu \lambda_\nu - \partial_\nu \lambda_\mu, \qquad (2.21)$$

$$\delta\bar{C} = \bar{d}\,\bar{\lambda} \longrightarrow \delta\bar{C}_{\bar{\mu}\bar{\nu}} = \bar{\partial}_{\bar{\mu}}\bar{\lambda}_{\bar{\nu}} - \bar{\partial}_{\bar{\nu}}\bar{\lambda}_{\bar{\mu}}. \tag{2.22}$$

The above gauge transformations are reducible, with trivial parameters given by $\lambda_\mu^{\text{triv}} = -\partial_\mu\chi$ and $\bar{\lambda}_{\bar{\mu}}^{\text{triv}} = \bar{\partial}_{\bar{\mu}}\chi$. The reducibility parameter χ is the single element of X_0. The gauge invariant field equations $\mathbf{d}\Psi = 0$ split similarly into (3, 0), (2, 1), (1, 2), and (0, 3) components,

$$\begin{aligned} dC = 0 &\longrightarrow \partial_\mu C_{\nu\rho} + \partial_\nu C_{\rho\mu} + \partial_\rho C_{\mu\nu} = 0,\\ de + \bar{d}C = 0 &\longrightarrow \partial_\mu e_{\nu\bar{\rho}} - \partial_\nu e_{\mu\bar{\rho}} - \bar{\partial}_{\bar{\rho}} C_{\mu\nu} = 0,\\ \bar{d}e + d\bar{C} = 0 &\longrightarrow \bar{\partial}_{\bar{\mu}} e_{\rho\bar{\nu}} - \bar{\partial}_{\bar{\nu}} e_{\rho\bar{\mu}} - \partial_\rho \bar{C}_{\bar{\mu}\bar{\nu}} = 0,\\ \bar{d}\,\bar{C} = 0 &\longrightarrow \bar{\partial}_{\bar{\mu}}\bar{C}_{\bar{\nu}\bar{\rho}} + \bar{\partial}_{\bar{\nu}}\bar{C}_{\bar{\rho}\bar{\mu}} + \bar{\partial}_{\bar{\rho}}\bar{C}_{\bar{\mu}\bar{\nu}} = 0. \end{aligned} \tag{2.23}$$

In previous works on the double copy of Yang-Mills theory [27,33], the gravity theory is defined on a subspace of $\mathcal{K}\otimes\bar{\mathcal{K}}$. This subspace, besides imposing the "weak constraint" $\Box = \bar{\Box}$, is defined to be $\ker b^-$, where $b^- \coloneqq b - \bar{b}$, and in the present context is given by

$$b^- = d^\dagger - \bar{d}^\dagger, \quad (b^-)^2 = 0, \quad \mathbf{d}b^- + b^-\mathbf{d} = \Box - \bar{\Box} = 0. \tag{2.24}$$

For now Eq. (2.23) are totally unconstrained. We have neither imposed $b^- = 0$ nor $\Box - \bar{\Box} = 0$ on the elements of $\mathcal{X}$, but let us next explore what consequences these constraints would have. For gauge parameters and fields, the b^- constraint would result in

$$\begin{aligned} \partial\cdot\lambda + \bar{\partial}\cdot\bar{\lambda} &= 0,\\ \partial^\rho e_{\rho\bar{\mu}} - \bar{\partial}^{\bar{\rho}}\bar{C}_{\bar{\rho}\bar{\mu}} &= 0,\\ \bar{\partial}^{\bar{\rho}} e_{\mu\bar{\rho}} - \partial^\rho C_{\rho\mu} &= 0, \end{aligned} \tag{2.25}$$

yielding a constrained gauge symmetry. Upon imposing these constraints the field equations (2.23) imply as integrability condition a gauge-fixed version of the double field theory (DFT) equations in 3D. To see this we take the divergence of (2.23), use (2.25) and the weak constraint $\Box = \bar{\Box}$ to obtain

$$\begin{aligned} \Box e_{\mu\bar{\nu}} - \partial_\mu\partial\cdot e_{\bar{\nu}} - \bar{\partial}_{\bar{\nu}}\bar{\partial}\cdot e_\mu &= 0,\\ \Box C_{\mu\nu} - 2\partial_{[\mu}\partial\cdot C_{\cdot\nu]} &= 0,\\ \Box\bar{C}_{\bar{\mu}\bar{\nu}} - 2\bar{\partial}_{[\bar{\mu}}\bar{\partial}\cdot\bar{C}_{\cdot\bar{\nu}]} &= 0. \end{aligned} \tag{2.26}$$

Notice that $\partial\cdot\bar{\partial}\cdot e = 0$ is implied by taking a divergence of the constraint (2.25). The $e_{\mu\bar{\nu}}$ equation in (2.26) coincides with a gauge-fixed form of the DFT equation in 3D, where the DFT dilaton is set to zero [30], which indeed leaves residual gauge transformations with $\partial\cdot\lambda + \bar{\partial}\cdot\bar{\lambda} = 0$.

Let us remark that the b^- constraint is not needed in order to have a consistent double copy. Imposing $b^- = 0$ is equivalent to a partial gauge fixing and subsequent solution of some components of the unconstrained field equations (2.23). We thus conclude that the first-order equations (2.23) imply the standard equations for a topological graviton and three 2-forms (C, $\bar{C}$, and the B-field). We have also performed a light cone analysis to show that the first-order system has no propagating degrees of freedom.

C. Nonlinear double copy

We now turn to the nonlinear structure on the double copied space $\mathcal{K}\otimes\bar{\mathcal{K}}$. The general double copy prescription defines the differential $\mathbf{d}$ and bracket B_2 by [27]

$$\begin{aligned} \mathbf{d} &= m_1\otimes 1 + 1\otimes\bar{m}_1,\\ B_2 &= b_2\otimes\bar{m}_2 - m_2\otimes\bar{b}_2, \end{aligned} \tag{2.27}$$

where the bar denotes the $\mathrm{BV}^{\Box}$ maps associated with $\bar{\mathcal{K}}$. The space $\mathcal{K}\otimes\bar{\mathcal{K}}$ is also endowed with the b^- operator discussed above and an associative product $M_2 = m_2\otimes\bar{m}_2$. The latter allows one to rewrite the bracket in (2.27) as $B_2 = [b^-, M_2]$, showing that it also has the form of a BV antibracket. Using the $\mathrm{BV}^{\Box}$ relations, including $[m_1, b_2] = [\Box, m_2]$, it is straightforward to verify that $(\mathbf{d}, b^-, M_2, B_2)$ define a BV^{Δ} algebra, where $\Delta = \Box - \bar{\Box}$. Since we need a genuine L_∞ algebra to define a field theory on $\mathcal{K}\otimes\bar{\mathcal{K}}$, we restrict to a subspace by imposing the strong constraint that $\Delta = 0$ or $\Box = \bar{\Box}$, for instance, by identifying x coordinates with $\bar{x}$ coordinates. Imposing the strong constraint, the space $\mathcal{K}\otimes\bar{\mathcal{K}}$ equipped with $(\mathbf{d}, B_2)$ defines a strict L_∞ algebra (DGLA) that describes a gravity theory.

In the following we give a more explicit form of the DGLA given by $(\mathbf{d}, B_2)$. To this end it is convenient to introduce a covariant notation for the doubled space: We define coordinates in $\mathbb{R}^6$ as $x^M \coloneqq (x^\mu, \bar{x}^{\bar{\mu}})$, together with the odd elements $\theta^M = (\theta^\mu, \bar{\theta}^{\bar{\mu}})$. Arbitrary elements of $\mathcal{X}$ are differential forms in $\mathbb{R}^6$, which we do not need to split into 3D components. The differential $\mathbf{d}$ is of course the de Rham differential in 6D,

$$\mathbf{d} = \theta^M\partial_M, \tag{2.28}$$

while the gauge parameter and the field are generic 1- and 2-forms, respectively,

$$\Lambda = \Lambda_M\theta^M, \quad \Psi = \frac{1}{2}\Psi_{MN}\theta^M\theta^N. \tag{2.29}$$

The free theory describes elements of the de Rham cohomology in form degree 2,

$$\mathbf{d}\Psi = 0, \quad \delta\Psi = \mathbf{d}\Lambda. \tag{2.30}$$

In order to describe the nonlinear theory, we shall introduce the $O(3,3)$ metric

$$\eta_{MN} = \begin{pmatrix} \delta_{\mu\nu} & 0 \\ 0 & -\delta_{\bar{\mu}\bar{\nu}} \end{pmatrix}, \tag{2.31}$$

and its inverse η^{MN}. In this section we will raise indices, when convenient, with η^{MN}.

Denoting $\mathcal{D}_M = \frac{\partial}{\partial\theta^M}$, the b^- operator is given by $b^- = \partial^M \mathcal{D}_M$. The two-bracket of general forms F and G (viewed as graded functions of x^M and θ^M) is thus given by the failure of $\partial^M \mathcal{D}_M$ to act via the Leibniz rule on the pointwise product (the 6D wedge product) and is hence the 6D version of the bracket (2.8),

$$B_2(F,G) = \eta^{MN}\big(\mathcal{D}_M F \partial_N G + (-1)^{FG} \mathcal{D}_M G \partial_N F\big). \tag{2.32}$$

This defines the nonlinear field equations in the Maurer-Cartan form,

$$\mathcal{E} \coloneqq \mathbf{d}\Psi + \frac{1}{2} B_2(\Psi,\Psi) = 0. \tag{2.33}$$

This equation is gauge covariant under

$$\delta\Psi = D\Lambda \coloneqq \mathbf{d}\Lambda + B_2(\Psi,\Lambda), \tag{2.34}$$

using the strong constraint in the form $\partial^M A \partial_M B = 0$ for arbitrary A, B,

$$\delta_\Lambda \mathcal{E} = D^2\Lambda = B_2(\Lambda,\mathcal{E}) + 2\partial^M\Psi\partial_M\Lambda = B_2(\Lambda,\mathcal{E}). \tag{2.35}$$

Note that, while (2.32) defines a genuine graded Lie bracket on unconstrained forms in 6D, the differential acts only via the Leibniz rule after imposing the strong constraint.

III. NONLOCAL LAGRANGIAN DOUBLE COPY

The nonlinear theory presented in the previous section is non-Lagrangian, as the number of equations of motion differs from the number of field components. In this section we are first going to impose the b^- constraint (which, as we have discussed, entails a partial gauge fixing and solving field equations). We will then show that it is possible to construct a nonlocal inner product, and thus an action, if one assumes that the $\mathbb{R}^3$ Laplacian $\square$ can be inverted.

A. Imposing the b^- constraint

In the 6D covariant formulation, elements of the chain complex $\mathcal{X}$ are differential forms of arbitrary rank in $\mathbb{R}^6$ subject to the constraint $\square - \bar{\square} = \eta^{MN}\partial_M\partial_N = 0$, written in terms of the $O(3,3)$ invariant metric η_{MN}. We also recall that the b^- operator takes the form

$$b^- = d^\dagger - \bar{d}^\dagger = \eta^{MN}\partial_M \frac{\partial}{\partial\theta^N} \equiv \partial^M \mathcal{D}_M, \tag{3.1}$$

which is the 6D divergence operator constructed from η^{MN}. We now subject all elements of $\mathcal{X}$ to the b^- constraint, i.e., we consider only forms F obeying $\partial^M \mathcal{D}_M F = 0$, meaning that they are transverse with respect to the $O(3,3)$ metric. For the gauge parameter 1-form Λ and the 2-form field Ψ this results in the constraints (2.25) discussed previously. On this subspace the bracket B_2 reduces to

$$B_2(F,G) = b^-(FG), \quad \forall\ F,G \in \ker b^-, \tag{3.2}$$

where FG is the pointwise product of graded functions encoding the 6D wedge product of forms. For fields $\Psi \in \ker b^-$ the Maurer-Cartan equation reduces to

$$\mathcal{E} = \mathbf{d}\Psi + \frac{1}{2} b^-(\Psi^2), \tag{3.3}$$

and obeys $b^-\mathcal{E} = 0$ thanks to (2.24). Similarly, the gauge transformation with constrained parameter satisfying $b^-\Lambda = 0$ takes the form $\delta\Psi = \mathbf{d}\Lambda + b^-(\Psi\Lambda)$.

B. Nonlocal c^- operator and Lagrangian

In certain cases, such as in the local double copy of Yang-Mills theory of [27,33], it is possible to define an operator c^- obeying $(c^-)^2 = 0$ and $b^-c^- + c^-b^- = 1$. The chain complex $\mathcal{X}$ of the double copy then splits into $\ker b^- \oplus \mathrm{im} c^-$, using that b^-c^- and c^-b^- are projectors. One can then define an inner product on $\ker b^- \subset \mathcal{X}$, via a c^- insertion[1] in the natural pairing inherited from $\mathcal{K}\otimes\bar{\mathcal{K}}$ [cf. (2.15)]. Given such an inner product, the known examples suggest that the corresponding double copy ought to be Lagrangian [33,37].

In the present case of Chern-Simons theory there seems to be no local candidate for such a c^- operator. However, if one assumes $\square$ to be invertible, a nonlocal c^- can be defined by $c^- = \frac{m_1}{\square} - \frac{\bar{m}_1}{\bar{\square}}$. Let us then assume that the 3D Laplacian $\square$ can be inverted and set

$$c^- \coloneqq \frac{1}{2}\left(\frac{d}{\square} - \frac{\bar{d}}{\bar{\square}}\right) = \frac{1}{2\square}(d - \bar{d}), \tag{3.4}$$

where d and $\bar{d}$ are the two copies of the 3D de Rham differential, and we used the strong constraint $\square = \bar{\square}$. This operator indeed obeys

$$(c^-)^2 = 0, \quad \mathbf{d}c^- + c^-\mathbf{d} = 0, \quad b^-c^- + c^-b^- = 1, \tag{3.5}$$

which can be easily seen upon using $\mathbf{d} = d + \bar{d}$ and $b^- = d^\dagger - \bar{d}^\dagger$. The graded commutator $[b^-, c^-] = 1$

[1]This is analogous to the c_0^- insertion used to define the inner product in closed string field theory.

ensures that any element in $\ker b^-$ is b^- exact, since for any F obeying $b^-F = 0$ one has

$$F = (b^-c^- + c^-b^-)F = b^-c^-F. \tag{3.6}$$

We now use this to prove that the Maurer-Cartan equation $\mathcal{E} = 0$ is equivalent to the nonlocal equation $c^-\mathcal{E} = 0$. The implication $\mathcal{E} = 0 \Rightarrow c^-\mathcal{E} = 0$ is obvious. In the other direction we apply b^- to $c^-\mathcal{E}$,

$$c^-\mathcal{E} = 0 \Rightarrow 0 = b^-c^-\mathcal{E} = \mathcal{E}, \tag{3.7}$$

where we used that $\mathcal{E} = b^-c^-\mathcal{E}$, since $\mathcal{E} \in \ker b^-$.

The next step is to show that $c^-\mathcal{E} = 0$ is a Lagrangian equation. To this end we recall the pairing (2.17), written here in 6D form,

$$\langle F, G\rangle = \int d^6x d^6\theta F(x,\theta)G(x,\theta),$$
$$\langle F, G\rangle = (-1)^{FG}\langle G, F\rangle, \tag{3.8}$$

which picks the (3, 3)-form component of FG, thanks to (2.15). We claim that the equation $c^-\mathcal{E} = c^-\mathbf{d}\Psi + \frac{1}{2}c^-b^-(\Psi^2) = 0$ results from varying the action[2]

$$S = \frac{1}{2}\langle \Psi, c^-\mathbf{d}\Psi\rangle + \frac{1}{3!}\langle \Psi, \Psi^2\rangle. \tag{3.9}$$

To prove this we need the integration by parts relations

$$\langle \mathbf{d}F, G\rangle = (-1)^{F+1}\langle F, \mathbf{d}G\rangle,$$
$$\langle c^-F, G\rangle = (-1)^{F+1}\langle F, c^-G\rangle,$$
$$\langle b^-F, G\rangle = (-1)^{F}\langle F, b^-G\rangle, \tag{3.10}$$

which are derived by expressing $\mathbf{d}$ and c^- in terms of d and $\bar{d}$, together with $b^- = \partial^M \frac{\partial}{\partial\theta^M}$. We now compute the variation of (3.9),

$$\delta S = \frac{1}{2}\langle \delta\Psi, c^-\mathbf{d}\Psi\rangle + \frac{1}{2}\langle \Psi, c^-\mathbf{d}\delta\Psi\rangle + \frac{1}{2}\langle \delta\Psi, \Psi^2\rangle, \tag{3.11}$$

where for the last term we used that $\langle \Psi, \Psi^2\rangle = \int d^6x d^6\theta \Psi^3$ is manifestly symmetric. The second term can be rewritten using (3.10) and graded symmetry of the pairing as

$$\langle \Psi, c^-\mathbf{d}\delta\Psi\rangle = \langle c^-\mathbf{d}\delta\Psi, \Psi\rangle = \langle \mathbf{d}\delta\Psi, c^-\Psi\rangle$$
$$= -\langle \delta\Psi, \mathbf{d}c^-\Psi\rangle = \langle \delta\Psi, c^-\mathbf{d}\Psi\rangle, \tag{3.12}$$

thus yielding $\delta S = \langle \delta\Psi, c^-\mathbf{d}\Psi + \frac{1}{2}\Psi^2\rangle$. One has to be careful in reading off the equation of motion from the variation, since Ψ is constrained to obey $b^-\Psi = 0$. Using (3.6), one has $\Psi = b^-c^-\Psi$. Taking the variation with the explicit projector on $\ker b^-$ we find

$$\begin{aligned}\delta S &= \left\langle b^-c^-\delta\Psi, c^-\mathbf{d}\Psi + \frac{1}{2}\Psi^2\right\rangle \\ &= -\left\langle c^-\delta\Psi, b^-c^-\mathbf{d}\Psi + \frac{1}{2}b^-(\Psi^2)\right\rangle \\ &= \left\langle \delta\Psi, c^-b^-c^-\mathbf{d}\Psi + \frac{1}{2}c^-b^-(\Psi^2)\right\rangle \\ &= \left\langle \delta\Psi, c^-\mathbf{d}\Psi + \frac{1}{2}c^-b^-(\Psi^2)\right\rangle,\end{aligned} \tag{3.13}$$

where in the last equality we used $b^-c^-\mathbf{d}\Psi = \mathbf{d}b^-c^-\Psi = \mathbf{d}\Psi$. This proves that the action (3.9) yields $c^-\mathcal{E} = 0$ as field equations which are, in turn, equivalent to the local first-order equations $\mathcal{E} = 0$. Using the above formula for the general variation one proves that the action (3.9) is gauge invariant under $\delta\Psi = \mathbf{d}\Lambda + b^-(\Psi\Lambda)$, with $b^-\Lambda = 0$.

We conclude this section by writing explicitly the action (3.9) in terms of the component fields $e_{\mu\bar{\nu}}$, $C_{\mu\nu}$, and $\bar{C}_{\bar{\mu}\bar{\nu}}$. Using the decomposition (2.18) of Ψ and the explicit form (2.15) for the pairing we find

$$\begin{aligned}S = \int d^3x d^3\bar{x} \epsilon^{\mu\nu\rho}\epsilon^{\bar{\mu}\bar{\nu}\bar{\rho}}\Big[&-\frac{1}{2}\partial_\mu e_{\nu\bar{\rho}}\frac{1}{\square}\bar{\partial}_{\bar{\mu}} e_{\rho\bar{\nu}} \\ &+\frac{1}{4}\partial_\mu C_{\nu\rho}\frac{1}{\square}\bar{\partial}_{\bar{\mu}}\bar{C}_{\bar{\nu}\bar{\rho}} - \frac{1}{6}e_{\mu\bar{\mu}}e_{\nu\bar{\nu}}e_{\rho\bar{\rho}} - \frac{1}{4}e_{\mu\bar{\mu}}C_{\nu\rho}\bar{C}_{\bar{\nu}\bar{\rho}}\Big].\end{aligned} \tag{3.14}$$

This action makes contact with the ones constructed in [35–37] as double copies of Chern-Simons theory. In particular, the superfield formulation of the action in [35] should also feature the 2-form fields $C_{\mu\nu}$ and $\bar{C}_{\bar{\mu}\bar{\nu}}$, although it is not stated explicitly. Let us emphasize that the action (3.14) is gauge invariant, and it yields field equations that are equivalent to local ones. Similar nonlocal actions have also appeared in topological string theory [43,44].

IV. RELATION TO KODAIRA-SPENCER GRAVITY

In this section we point out a relation between the local double copy theory constructed in Sec. II C and the topological theory given by Kodaira-Spencer gravity, which describes a complex structure deformation of a Hermitian manifold. To that end, we first rewrite the theory in complex coordinates and discuss how a truncation of that theory yields Kodaira-Spencer gravity on a 3D subspace. Finally, we show how to obtain Kodaira-Spencer gravity directly by means of a chiral double copy.

[2]In L_∞ terms this is the standard action obtained from an L_∞ inner product defined by $\langle F, G\rangle_L := \langle F, c^-G\rangle$.

A. Complex coordinates

As mentioned above, the chain complex $\mathcal{K}\otimes\bar{\mathcal{K}}$ is identical to the de Rham complex on $\mathbb{R}^6$. The two copies of the Euclidean metric on $\mathbb{R}^3$ can be combined into a background "generalized metric" H_{MN}, which is just the flat Euclidean metric of $\mathbb{R}^6$, and the $O(3,3)$ metric η_{MN},

$$H_{MN}=\begin{pmatrix}\delta_{\mu\nu} & 0\\ 0 & \delta_{\bar{\mu}\bar{\nu}}\end{pmatrix},\quad \eta_{MN}=\begin{pmatrix}\delta_{\mu\nu} & 0\\ 0 & -\delta_{\bar{\mu}\bar{\nu}}\end{pmatrix}. \tag{4.1}$$

The metric H_{MN} does not look like the one of a Hermitian manifold, which has only mixed components in holomorphic coordinates. On top of that, the coordinates x^μ and $\bar{x}^{\bar{\mu}}$ are real. To remedy this we define a set of complex coordinates given by

$$\begin{aligned}z^1&:=\frac{1}{\sqrt{2}}(x+i\bar{x}),\quad z^2:=\frac{1}{\sqrt{2}}(y+i\bar{y}),\quad z^3:=\frac{1}{\sqrt{2}}(z+i\bar{z}),\\ \bar{z}^{\bar{1}}&:=\frac{1}{\sqrt{2}}(x-i\bar{x}),\quad \bar{z}^{\bar{2}}:=\frac{1}{\sqrt{2}}(y-i\bar{y}),\quad \bar{z}^{\bar{3}}:=\frac{1}{\sqrt{2}}(z-i\bar{z}),\end{aligned} \tag{4.2}$$

which we collectively denote as $(z^a,\bar{z}^{\bar{a}})$. In these coordinates, the metric H_{MN} becomes

$$ds_H^2=dx^2+dy^2+dz^2+d\bar{x}^2+d\bar{y}^2+d\bar{z}^2=2\delta_{a\bar{b}}dz^a d\bar{z}^{\bar{b}}, \tag{4.3}$$

which is the expected form of a Hermitian flat metric in $\mathbb{C}^3$. On the other hand, the $O(3,3)$ metric becomes

$$\begin{aligned}ds_\eta^2&=dx^2+dy^2+dz^2-d\bar{x}^2-d\bar{y}^2-d\bar{z}^2\\ &=\delta_{ab}dz^a dz^b+\delta_{\bar{a}\bar{b}}d\bar{z}^{\bar{a}}d\bar{z}^{\bar{b}},\end{aligned} \tag{4.4}$$

so that we have

$$H_{AB}=\begin{pmatrix}0 & \delta_{a\bar{b}}\\ \delta_{\bar{a}b} & 0\end{pmatrix},\quad \eta_{AB}=\begin{pmatrix}\delta_{ab} & 0\\ 0 & \delta_{\bar{a}\bar{b}}\end{pmatrix}, \tag{4.5}$$

in coordinates $z^A:=(z^a,\bar{z}^{\bar{a}})$. We similarly redefine the 1-form basis by introducing $(\theta^a,\bar{\theta}^{\bar{a}})$ in the same fashion: $\theta^1:=\frac{1}{\sqrt{2}}(\theta^x+i\bar{\theta}^{\bar{x}})$ and so on. The 6D volume form can also be factorized into its (anti)holomorphic components: $\epsilon_{abc}\epsilon_{\bar{a}\bar{b}\bar{c}}$.

With these redefinitions, the elements of $\mathcal{K}\otimes\bar{\mathcal{K}}$ are given by all differential forms in $\mathbb{C}^3$,

$$\mathcal{K}\otimes\bar{\mathcal{K}}=\bigoplus_{p,q=0}^{3}\Omega^{p,q}(\mathbb{C}^3), \tag{4.6}$$

where now we split degrees according to the number of $(\theta^a,\bar{\theta}^{\bar{a}})$. The differential is the de Rham differential in six dimensions, which splits into its Dolbeault components,

$$\mathbf{d}=\theta^a\partial_a+\bar{\theta}^{\bar{a}}\bar{\partial}_{\bar{a}}=\partial+\bar{\partial}. \tag{4.7}$$

Redefining the fields according to the new holomorphic degree we write

$$\Psi=h_{1,1}+B_{2,0}+\bar{B}_{0,2}. \tag{4.8}$$

Their linearized Maurer-Cartan equations read

$$\begin{aligned}\partial B=0,&\qquad \bar{\partial}\bar{B}=0,\\ \partial h+\bar{\partial}B=0,&\qquad \bar{\partial}h+\partial\bar{B}=0,\end{aligned} \tag{4.9}$$

which are invariant under the gauge transformations

$$\delta h=\partial\bar{\lambda}+\bar{\partial}\lambda,\quad \delta B=\partial\lambda,\quad \delta\bar{B}=\bar{\partial}\bar{\lambda}. \tag{4.10}$$

If we want to continue with the exact quadratic Maurer-Cartan equation, we have to make sense of the b operators and the strong constraint. Let us start with the b operators. Using the redefinitions we have

$$b^+=H^{AB}\partial_A\frac{\partial}{\partial\theta^B}=\partial^\dagger+\bar{\partial}^\dagger,\quad b^-=\eta^{AB}\partial_A\frac{\partial}{\partial\theta^B}=\delta+\bar{\delta}, \tag{4.11}$$

where we have the adjoints of the Dolbeault operators,

$$\partial^\dagger=\delta^{a\bar{b}}\bar{\partial}_{\bar{b}}\frac{\partial}{\partial\theta^a},\quad \bar{\partial}^\dagger=\delta^{a\bar{b}}\partial_a\frac{\partial}{\partial\bar{\theta}^{\bar{b}}}, \tag{4.12}$$

which obey

$$\partial\partial^\dagger+\partial^\dagger\partial=\frac{1}{2}\square=\delta^{a\bar{b}}\partial_a\bar{\partial}_{\bar{b}}, \tag{4.13}$$

with the same commutator for the barred ones, and

$$\delta:=\delta^{ab}\partial_a\frac{\partial}{\partial\theta^b},\qquad \bar{\delta}:=\delta^{\bar{a}\bar{b}}\bar{\partial}_{\bar{a}}\frac{\partial}{\partial\bar{\theta}^{\bar{b}}}, \tag{4.14}$$

which employ the chiral Kronecker δ's. Note that here we encounter an unusual feature: the b^- operator involves new structures, which are the Kronecker δ's that, although present in $\mathbb{C}^3$, do not exist for generic Hermitian manifolds. We should think of this structure associated with $\mathbb{C}^3$ as an auxiliary structure, on par with the auxiliary metric introduced in 3D Chern-Simons theory. Using this structure, the (weak or strong) constraint reads

$$\Delta:=\eta^{AB}\partial_A\partial_B=\delta^{ab}\partial_a\partial_b+\delta^{\bar{a}\bar{b}}\bar{\partial}_{\bar{a}}\bar{\partial}_{\bar{b}}=0. \tag{4.15}$$

1. The interacting theory

Let us compute explicitly what the above field equations and gauge symmetries look like. The Lie bracket $B_2 \equiv [\cdot,\cdot]$ has the same six-dimensional form as before,

$$[F,G] = \eta^{AB}\big(\mathcal{D}_A F \partial_B G + (-1)^{FG}\mathcal{D}_A G \partial_B F\big). \tag{4.16}$$

The Maurer-Cartan equation for the 2-form $\Psi = \frac{1}{2}\theta^A\theta^B\psi_{AB}$ reads

$$\partial_{[A}\psi_{BC]} - \psi_{[A}{}^{D}\partial_{|D|}\psi_{BC]} = 0, \tag{4.17}$$

where the indices are raised with η^{AB}. The above equation is covariant under the gauge transformations

$$\delta\psi_{AB} = 2\partial_{[A}\lambda_{B]} - 2\psi_{[A}{}^{C}\partial_{|C|}\lambda_{B]} + \lambda^C\partial_C\psi_{AB}, \tag{4.18}$$

provided the strong constraint is obeyed, since the variation of the field equation reads

$$\delta_\lambda\mathcal{E}_{ABC} = [\Lambda,\mathcal{E}]_{ABC} + 6\partial^D\psi_{[AB}\partial_{|D|}\lambda_{C]}. \tag{4.19}$$

Splitting into holomorphic and antiholomorphic components, we have four field equations,

$$\begin{aligned}
2\partial_{[a}h_{b]\bar{c}} - 2h_{[a}{}^{\bar{d}}\bar{\partial}_{|\bar{d}|}h_{b]\bar{c}} + \bar{\partial}_{\bar{c}}B_{ab} - 2B_{[a}{}^{d}\partial_{|d|}h_{b]\bar{c}} + h^{d}{}_{\bar{c}}\partial_d B_{ab} - \bar{B}_{\bar{c}}{}^{\bar{d}}\bar{\partial}_{\bar{d}}B_{ab} &= 0,\\
-2\bar{\partial}_{[\bar{a}}h_{|c|\bar{b}]} - 2h^{d}{}_{[\bar{a}}\partial_{|d|}h_{|c|\bar{b}]} + \partial_c\bar{B}_{\bar{a}\bar{b}} + 2\bar{B}_{[\bar{a}}{}^{\bar{d}}\bar{\partial}_{|\bar{d}|}h_{|c|\bar{b}]} - h_c{}^{\bar{d}}\bar{\partial}_{\bar{d}}\bar{B}_{\bar{a}\bar{b}} - B_c{}^{d}\partial_d\bar{B}_{\bar{a}\bar{b}} &= 0,\\
3\partial_{[a}B_{bc]} - 3h_{[a}{}^{\bar{d}}\bar{\partial}_{|\bar{d}|}B_{bc]} - 3B_{[a}{}^{d}\partial_{|d|}B_{bc]} &= 0,\\
3\bar{\partial}_{[\bar{a}}\bar{B}_{\bar{b}\bar{c}]} + 3h^{d}{}_{[\bar{a}}\partial_{|d|}\bar{B}_{\bar{b}\bar{c}]} - 3\bar{B}_{[\bar{a}}{}^{\bar{d}}\bar{\partial}_{|\bar{d}|}\bar{B}_{\bar{b}\bar{c}]} &= 0,
\end{aligned} \tag{4.20}$$

where we used the definition $h_{\bar{b}a} \coloneqq -h_{a\bar{b}}$. Splitting the gauge transformations we obtain

$$\begin{aligned}
\delta h_{a\bar{b}} &= \partial_a\bar{\lambda}_{\bar{b}} - \bar{\partial}_{\bar{b}}\lambda_a - h_a{}^{\bar{c}}\bar{\partial}_{\bar{c}}\bar{\lambda}_{\bar{b}} - h^{c}{}_{\bar{b}}\partial_c\lambda_a + \bar{B}_{\bar{b}}{}^{\bar{c}}\bar{\partial}_{\bar{c}}\lambda_a\\
&\quad - B_a{}^{c}\partial_c\bar{\lambda}_{\bar{b}} + \bar{\lambda}^{\bar{c}}\bar{\partial}_{\bar{c}}h_{a\bar{b}} + \lambda^c\partial_c h_{a\bar{b}},\\
\delta B_{ab} &= 2\partial_{[a}\lambda_{b]} - 2h_{[a}{}^{\bar{c}}\bar{\partial}_{\bar{c}}\lambda_{b]} - 2B_{[a}{}^{c}\partial_c\lambda_{b]} + \lambda^c\partial_c B_{ab}\\
&\quad + \bar{\lambda}^{\bar{c}}\bar{\partial}_{\bar{c}}B_{ab},\\
\delta\bar{B}_{\bar{a}\bar{b}} &= 2\bar{\partial}_{[\bar{a}}\bar{\lambda}_{\bar{b}]} + 2h^{c}{}_{[\bar{a}}\partial_c\bar{\lambda}_{\bar{b}]} - 2\bar{B}_{[\bar{a}}{}^{\bar{c}}\bar{\partial}_{\bar{c}}\bar{\lambda}_{\bar{b}]}\\
&\quad + \lambda^c\partial_c\bar{B}_{\bar{a}\bar{b}} + \bar{\lambda}^{\bar{c}}\bar{\partial}_{\bar{c}}\bar{B}_{\bar{a}\bar{b}}.
\end{aligned} \tag{4.21}$$

Let us first try to interpret the standard solution to the strong constraint, which in holomorphic coordinates reads

$$\delta^{ab}\partial_a F\partial_b G + \delta^{\bar{a}\bar{b}}\bar{\partial}_{\bar{a}}F\bar{\partial}_{\bar{b}}G = 0. \tag{4.22}$$

The supergravity solution in the $(x^\mu,\bar{x}^{\bar{\mu}})$ coordinates is $x^\mu = \bar{x}^{\bar{\mu}}$, thus also identifying the corresponding derivatives. Looking at the definition (4.2) of the complex coordinates, this amounts to setting $z^a = i\bar{z}^{\bar{a}}$. This is a three-dimensional real slice of $\mathbb{C}^3$ given by a diagonal hyperplane.

Since the strong constraint is required to get rid of the obstruction (4.19) in the gauge variation of the Maurer-Cartan equation, let us see in more detail which component fields and gauge parameters are involved. Splitting (4.19) in (anti)holomorphic components we have

$$\begin{aligned}
\delta_\lambda\mathcal{E}_{abc} &= [\Lambda,\mathcal{E}]_{abc} + 6\partial^D B_{[ab}\partial_{|D|}\lambda_{c]},\\
\delta_\lambda\mathcal{E}_{ab\bar{c}} &= [\Lambda,\mathcal{E}]_{ab\bar{c}} + 2\partial^D B_{ab}\partial_D\bar{\lambda}_{\bar{c}} - 4\partial^D h_{[a|\bar{c}}\partial_{D|}\lambda_{b]},
\end{aligned} \tag{4.23}$$

with the other two components obtained by swapping barred and unbarred indices and fields. From the above expression one can see that the strong constraint would not be required if we truncated $B_{ab} = 0$ and $\lambda_a = 0$, which can also be checked by inspection of (4.20) and (4.21). The reason this is not consistent in our model is of course that we have two more equations, obtained from (4.23) by (un) barring the indices. Demanding consistency of the other equations without the strong constraint would require $\bar{B}_{\bar{a}\bar{b}} = 0$ and $\bar{\lambda}_{\bar{a}} = 0$, thus leaving no gauge symmetry whatsoever.

Nevertheless, it is still interesting to see what happens if we do forget the other two equations, and set $B_{ab} = 0$ and $\lambda_a = 0$. The system (4.20) with gauge transformations (4.21) then reduces to

$$\begin{aligned}
&\partial_a h_b{}^{\bar{c}} - \partial_b h_a{}^{\bar{c}} - h_a{}^{\bar{d}}\bar{\partial}_{\bar{d}}h_b{}^{\bar{c}} + h_b{}^{\bar{d}}\bar{\partial}_{\bar{d}}h_a{}^{\bar{c}} = 0,\\
&\delta h_a{}^{\bar{b}} = \partial_a\bar{\lambda}^{\bar{b}} - h_a{}^{\bar{c}}\bar{\partial}_{\bar{c}}\bar{\lambda}^{\bar{b}} + \bar{\lambda}^{\bar{c}}\bar{\partial}_{\bar{c}}h_a{}^{\bar{b}},
\end{aligned} \tag{4.24}$$

where we raised the antiholomorphic index with $\delta^{\bar{a}\bar{b}}$. The equations in (4.24) are the Kodaira-Spencer equation and gauge transformation for h, upon interpreting $h_a{}^{\bar{b}}$ and $\bar{\lambda}^{\bar{a}}$ as holomorphic 1- and 0-forms, respectively, taking values in antiholomorphic vector fields,

$$h \coloneqq dz^a h_a{}^{\bar{b}}\bar{\partial}_{\bar{b}}, \qquad \lambda \coloneqq \bar{\lambda}^{\bar{a}}\bar{\partial}_{\bar{a}}. \tag{4.25}$$

Of course, as derived the theory is still subject to the strong constraint and hence three dimensional, but in this form the theory is actually gauge invariant without the constraint and can hence be lifted to 6D.

B. Kodaira-Spencer gravity as a chiral double copy

We now turn to a chiral double copy construction which yields a genuine six-dimensional field theory containing the Kodaira-Spencer equation [40,41]. To this end we recall that, in the standard prescription used in the previous sections, the differential $\mathbf{d}$ and bracket B_2 on $\mathcal{K}\otimes\bar{\mathcal{K}}$ are given by

$$\begin{aligned}\mathbf{d} &= m_1\otimes 1 + 1\otimes \bar{m}_1,\\ B_2 &= b_2\otimes\bar{m}_2 - m_2\otimes\bar{b}_2,\end{aligned}\tag{4.26}$$

where (m_1, m_2, b_2) and their barred counterparts are the $\mathrm{BV}^\square$ maps in $\mathcal{K}$ and $\bar{\mathcal{K}}$, respectively. The need for the strong constraint originates from the failure of m_1 to be a derivation of the bracket b_2, since it implies that $\mathbf{d}$ fails to be a derivation of B_2 unless $\square = \bar{\square}$.

For the following construction we will treat $\mathcal{K}$ and $\bar{\mathcal{K}}$ differently: we still view $\mathcal{K}$ as the space of forms in $\mathbb{R}^3$ carrying the DGCA $(m_1 = d, m_2 = \wedge)$. On the other hand, we treat $\bar{\mathcal{K}}$ as the space of polyvectors in $\mathbb{R}^3$, forming a graded Lie algebra with bracket $\bar{b}_2$, the Schouten-Nijenhuis bracket. On $\mathcal{X} = \mathcal{K}\otimes\bar{\mathcal{K}}$ one then has a differential graded Lie algebra given by

$$\mathbf{d} = m_1\otimes 1, \qquad B_2 = m_2\otimes\bar{b}_2,\tag{4.27}$$

which is defined on $\mathbb{R}^6$, with coordinates $z^A = (z^a, \bar{z}^{\bar{a}})$. The elements of $\mathcal{X}$ are polyvector-valued forms of the following type:

$$\omega_p^q = \frac{1}{p!q!}\omega_{a_1\ldots a_p}{}^{\bar{b}_1\ldots\bar{b}_q}(z,\bar{z})\theta^{a_1}\cdots\theta^{a_p}\bar{\theta}_{\bar{b}_1}\cdots\bar{\theta}_{\bar{b}_q}.\tag{4.28}$$

The degree on $\mathcal{X}$ is the sum of the form and polyvector degrees, so that $|\omega_p^q| = p+q$. On $\mathbb{R}^6$ we can choose the following complex structure:

$$J^A{}_B = i\begin{pmatrix}\delta^a{}_b & 0\\ 0 & -\delta^{\bar{a}}{}_{\bar{b}}\end{pmatrix}.\tag{4.29}$$

In the coordinates $z^A = (z^a, \bar{z}^{\bar{a}})$ this identifies θ^a as the basis of holomorphic 1-forms and $\bar{\theta}_{\bar{a}}$ as the basis of antiholomorphic vector fields on $\mathbb{C}^3$. With this complex structure at hand, the differential $\mathbf{d}\equiv\partial = \theta^a\partial_a$ is the holomorphic Dolbeault differential, while the two-bracket B_2 acts as

$$B_2(\omega_1,\omega_2) = \frac{\partial\omega_1}{\partial\bar{\theta}_{\bar{a}}}\bar{\partial}_{\bar{a}}\omega_2 + (-1)^{\omega_1\omega_2}\frac{\partial\omega_2}{\partial\bar{\theta}_{\bar{a}}}\bar{\partial}_{\bar{a}}\omega_1,\tag{4.30}$$

where the degree in the exponent is the sum of the form degree and the polyvector field degree.

The gauge parameters Λ and fields Ψ split as

$$\begin{aligned}\Lambda &= \lambda_a\theta^a + \bar{\xi}^{\bar{a}}\bar{\theta}_{\bar{a}},\\ \Psi &= h_a{}^{\bar{b}}\theta^a\bar{\theta}_{\bar{b}} + \frac{1}{2}B_{ab}\theta^a\theta^b + \frac{1}{2}\Pi^{\bar{a}\bar{b}}\bar{\theta}_{\bar{a}}\bar{\theta}_{\bar{b}}.\end{aligned}\tag{4.31}$$

Gauge parameters thus consist of a holomorphic 1-form λ_a and an antiholomorphic vector field $\bar{\xi}^{\bar{a}}$. Fields contain a vector-valued 1-form $h_a{}^{\bar{b}}$, a 2-form B_{ab}, and a bivector $\Pi^{\bar{a}\bar{b}}$.[3] In order to interpret this set of parameters and fields we compute the gauge algebra, which is given by the bracket $B_2(\Lambda_1,\Lambda_2)$. Its 1-form and vector components read

$$\begin{aligned}B_2(\Lambda_1,\Lambda_2)_a &= \bar{\xi}_1^{\bar{b}}\bar{\partial}_{\bar{b}}\lambda_{2a} - \bar{\xi}_2^{\bar{b}}\bar{\partial}_{\bar{b}}\lambda_{1a} = \mathcal{L}_{\bar{\xi}_1}\lambda_{2a} - \mathcal{L}_{\bar{\xi}_2}\lambda_{1a},\\ B_2(\Lambda_1,\Lambda_2)^{\bar{a}} &= \bar{\xi}_1^{\bar{b}}\bar{\partial}_{\bar{b}}\bar{\xi}_2^{\bar{a}} - \bar{\xi}_2^{\bar{b}}\bar{\partial}_{\bar{b}}\bar{\xi}_1^{\bar{a}} = \mathcal{L}_{\bar{\xi}_1}\bar{\xi}_2^{\bar{a}},\end{aligned}\tag{4.32}$$

with $\mathcal{L}$ denoting the Lie derivative. This shows that the gauge transformations are the semidirect sum of Abelian holomorphic 1-form transformations and antiholomorphic diffeomorphisms. The transformations of the fields themselves are given by $\delta\Psi = \partial\Psi + B_2(\Psi,\Lambda)$, yielding

$$\begin{aligned}\delta h_a{}^{\bar{b}} &= \partial_a\bar{\xi}^{\bar{b}} + \bar{\xi}^{\bar{c}}\bar{\partial}_{\bar{c}}h_a{}^{\bar{b}} - \partial_{\bar{c}}\bar{\xi}^{\bar{b}}h_a{}^{\bar{c}} + \Pi^{\bar{b}\bar{c}}\bar{\partial}_{\bar{c}}\lambda_a\\ &= \partial_a\bar{\xi}^{\bar{b}} + \mathcal{L}_{\bar{\xi}}h_a{}^{\bar{b}} + \Pi^{\bar{b}\bar{c}}\bar{\partial}_{\bar{c}}\lambda_a,\\ \delta B_{ab} &= 2(\partial_{[a} - h_{[a}{}^{\bar{c}}\bar{\partial}_{\bar{c}})\lambda_{b]} + \bar{\xi}^{\bar{c}}\bar{\partial}_{\bar{c}}B_{ab} = 2\partial^h_{[a}\lambda_{b]} + \mathcal{L}_{\bar{\xi}}B_{ab},\\ \delta\Pi^{\bar{a}\bar{b}} &= \bar{\xi}^{\bar{c}}\bar{\partial}_{\bar{c}}\Pi^{\bar{a}\bar{b}} - \bar{\partial}_{\bar{c}}\bar{\xi}^{\bar{a}}\Pi^{\bar{c}\bar{b}} - \bar{\partial}_{\bar{c}}\bar{\xi}^{\bar{b}}\Pi^{\bar{a}\bar{c}} = \mathcal{L}_{\bar{\xi}}\Pi^{\bar{a}\bar{b}},\end{aligned}\tag{4.33}$$

where we defined the twisted derivative $\partial^h_a \coloneqq \partial_a - h_a{}^{\bar{b}}\bar{\partial}_{\bar{b}}$. If one interprets $\theta^a\partial^h_a$ as a deformed Dolbeault operator, the field $h_a{}^{\bar{b}}$ is a deformation of the complex structure. The gauge transformation of $h_a{}^{\bar{b}}$ under diffeomorphisms is the expected one, but the additional contribution from the 1-form λ_a is exotic. The field equations $\partial\Psi + \frac{1}{2}B_2(\Psi,\Psi) = 0$ decompose as

$$\begin{aligned}2\partial_{[a}h_{b]}{}^{\bar{c}} - 2h_{[a}{}^{\bar{d}}\bar{\partial}_{|\bar{d}|}h_{b]}{}^{\bar{c}} - \Pi^{\bar{c}\bar{d}}\bar{\partial}_{\bar{d}}B_{ab} &= 0,\\ 3(\partial_{[a} - h_{[a}{}^{\bar{c}}\bar{\partial}_{|\bar{c}|})B_{bc]} &= 0,\\ \partial_a\Pi^{\bar{b}\bar{c}} - h_a{}^{\bar{d}}\bar{\partial}_{\bar{d}}\Pi^{\bar{b}\bar{c}} - 2\Pi^{\bar{d}[\bar{b}}\bar{\partial}_{\bar{d}}h_a{}^{\bar{c}]} &= 0,\\ 3\Pi^{\bar{d}[\bar{a}}\bar{\partial}_{\bar{d}}\Pi^{\bar{b}\bar{c}]} &= 0.\end{aligned}\tag{4.34}$$

Interestingly, the field equation for $\Pi^{\bar{a}\bar{b}}$ is the condition for it to be a Poisson bivector compatible, in a suitable sense,

[3]This field content coincides with the one alluded to by Witten in the context of topological string theory [43].

with the deformation $h_a{}^{\bar{b}}$. In addition, the deformed Dolbeault operator

$$\partial_h \coloneqq dz^a(\partial_a - h_a{}^{\bar{b}}\bar{\partial}_{\bar{b}}) \tag{4.35}$$

is not nilpotent, with the 2-form B_{ab} sourcing the curvature for ∂_h^2. The 2-form itself has vanishing (twisted) field strength on shell.

The above is a gauge invariant dynamical system, which is novel to the best of our knowledge, that generalizes the Kodaira-Spencer equation and its gauge symmetries. The Kodaira-Spencer equation is contained as a consistent truncation, setting $\Pi^{\bar{a}\bar{b}} = 0$ and then removing the 2-form by means of the λ gauge symmetry. The resulting theory is defined by

$$\begin{aligned} &\partial_a h_b{}^{\bar{c}} - \partial_b h_a{}^{\bar{c}} - h_a{}^{\bar{d}}\bar{\partial}_{\bar{d}} h_b{}^{\bar{c}} + h_b{}^{\bar{d}}\bar{\partial}_{\bar{d}} h_a{}^{\bar{c}} = 0, \\ &\delta h_a{}^{\bar{b}} = \partial_a \bar{\lambda}^{\bar{b}} - h_a{}^{\bar{c}}\bar{\partial}_{\bar{c}}\bar{\lambda}^{\bar{b}} + \bar{\lambda}^{\bar{c}}\bar{\partial}_{\bar{c}} h_a{}^{\bar{b}}, \end{aligned} \tag{4.36}$$

which is Kodaira-Spencer gravity.[4]

V. SUMMARY AND OUTLOOK

In this paper we have constructed the double copy of 3D Chern-Simons theory, employing the same algebraic double copy recipe previously applied to pure Yang-Mills theory to obtain $\mathcal{N} = 0$ supergravity in a double field theory formulation. In this we believe to have given the first example of an explicit, local, and gauge invariant double copy construction that is exact, albeit being non-Lagrangian. An action can, however, be constructed upon partial gauge fixing and giving up locality. This result is hence an important ingredient of the larger research program of double copying general field theories.

Chern-Simons theory may also turn out to be an interesting toy model due to some similarities with the self-dual sector of Yang-Mills theory [47]: in both the double copied gravity theory features a generalized metric fluctuation and two 2-forms. The question arises whether the nonlinear field equations of self-dual gravity or even full gravity may be recovered as integrability conditions, which likely would require a novel procedure to eliminate the extra 2-form fields. Another important question is whether there might be a novel perspective to eventually obtain a local and gauge invariant action.

Finally, one of the core motivations for this program has always been to establish a precise correspondence between classical solutions of gauge theory and gravity. Since here the double copied 3D gravity theory can be written down explicitly in terms of the kinematic ingredients of 3D Chern-Simons theory, this example may be a promising starting point to attempt the double copy of solutions.

[4]See [45,46] for similar results on holomorphic Chern-Simons theories in twistor space, which are related to self-dual Yang-Mills theory and self-dual supergravities.

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