# Four loop renormalization in six dimensions using FORCER

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We employ the FORCER algorithm to renormalize a variety of six dimensional field theories to four loops. In order to achieve this we construct the FORCER master integrals in six dimensions from their four dimensional counterparts by using the Tarasov method. The $\epsilon$ expansion of the six dimensional masters are determined up to weight 9 where $d = 6 - 2\epsilon$. By applying the FORCER routine the four loop $\overline{\rm MS}$ renormalization of $\phi^3$ theory is reproduced before gauge theories are considered. The renormalization of these theories is also determined in the $\widetilde{\rm MOM}$ scheme. For instance the absence of $\zeta_4$ and $\zeta_6$ is confirmed to five loops in the $\widetilde{\rm MOM}$ renormalization of $\phi^3$ theory. We also evaluate the three loop $\beta$-function of the gauge coupling in six dimensional QCD.

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## I. INTRODUCTION

There have been significant developments in recent years to progress our knowledge of the renormalization group functions of four dimensional quantum field theories and in particular those of various sectors of the Standard Model. The main development rests in the advances in the algorithms used to calculate Feynman integrals and the implementation of these algorithms in highly efficient computer algebra and symbolic manipulation languages. The classic representation of these advances is the establishment of the five loop $\beta$-function of quantum chromodynamics (QCD) which is the field theory that relates to the strong interactions. Several groups produced the result contemporaneously [1–5] using one of two separate methods to achieve this impressive level of precision. One is the Laporta algorithm [6], which is a systematic way of solving integration by parts relations to write all the Feynman integrals of the Green's function of interest in terms of a relatively small basis set of core or master integrals. The evaluation of the latter by nonintegration by parts methods allows for the Green's function to be fully determined. While such a systematic approach will always reach its goal, it is not always the case that this will be achieved in a reasonable time frame. One other main technique that led to five loop QCD renormalization group functions [2,7] was the development of an efficient four loop integration package termed FORCER [8,9]. It is written in the symbolic manipulation language FORM [10,11]. The FORCER package is tailored specifically to evaluate four loop massless 2-point functions in $d$-dimensional spacetime. For realistic computations the $\epsilon$ expansion in $d = 4 - 2\epsilon$ dimensions can be extracted within the FORCER framework up to weight 12. The routine represents a generational advance on its precursor which was MINCER [12]. That was an algorithm for evaluating massless three loop 2-point Feynman integrals. Although it was developed in the 1980s, it was very much ahead of its time given our current knowledge of the mathematics underlying Feynman integral computations. However, the increase in experimental precision in recent years has meant that higher loop order algorithms for automatically evaluating Feynman integrals are now necessary. In fitting this requirement, FORCER implements a new integration rule for massless fields that extends the core rule central to MINCER. Moreover this diamond rule [13] represents an efficient improvement for the more algebraically demanding four loop Feynman integrals.

While FORCER is now the default package of massless four loop 2-point functions and was central to [2,7], there are theories and problems in other spacetime dimensions that it could be useful for. For instance, in six dimensions there has been interest in gauge theories with and without supersymmetry. In [14] the one loop $\beta$-functions of the two coupling non-Abelian gauge theory was studied to ascertain whether the model was unitary and perturbatively conformal. Aside from [14] there is interest in the ultraviolet completion of four dimensional theories. As an example of this we recall that four dimensional scalar $\phi^4$ theory has scalar $\phi^3$ theory as its ultraviolet completion in six dimensions. In other words $\phi^4$ theory with an $O(N)$

symmetry lies in the same universality class in $d$-dimensions at the Wilson-Fisher fixed point as $O(N)$ $\phi^3$ theory in six dimensions, [15–17]. This concept extends beyond scalar theories to include fermionic ones such as those with scalar-Yukawa interactions [18–20], scalar and fermionic quantum electrodynamics (QED) and QCD [21,22]. In the latter two cases the ultraviolet completion of six dimensional QED and QCD begins in two dimensions with the Abelian and non-Abelian Thirring model respectively. The completion of non-Abelian gauge theories to six dimensions was verified at two loops in [22]. To establish the connection between the theories in the same universality class requires the explicit values of the renormalization group functions of each theory in its critical dimension to as high a loop order as is computationally possible. Once these are available, the respective critical exponents are computed in an $\epsilon$ expansion around the critical dimension. The expressions can then be compared with the same exponents computed explicitly in $d$-dimensions with respect to a universal expansion parameter that is common to all the theories in the universality class. For example, for $O(N)$ theories this expansion parameter is usually $1/N$ where $N$ is regarded as large. The agreement of the $\epsilon$ expansion of these $d$-dimensional large $N$ exponents for each critical dimension of the theories in the same universality class establishes the ultraviolet completion. There will be other ultraviolet completions aside from the few we mentioned, but in order to concretely establish them requires high loop order computations in dimensions beyond four for which FORCER would be an indispensable integration tool. As it stands, FORCER evaluates integrals in a similar ethos to the Laporta algorithm in that a rule, based on an integration by parts construction, efficiently reduces all contributing Feynman integrals to a basis. The advantage of FORCER is that a database of integral relations does not have to be solved en route as is the case for packages that implement the algorithm of [6]. Rules such as the diamond one are already encoded to circumvent that necessity. Effectively the ultimate point of the FORCER routine is the expression of the Green's functions as $d$-dimensional integrals with the option of subsequently expanding in powers of $\epsilon$ up to weight 12 if needed. Therefore, to adapt the FORCER routine to six dimensional problems requires the $\epsilon$ expansion of the basis integrals relative to six dimensions. That is one of the main purposes of this article. We will provide the six dimensional FORCER master integrals expanded to weight 9 in $d = 6 - 2\epsilon$ dimensions. To achieve this requires a straightforward application of the Tarasov method [23,24] that relates Feynman integrals in $d$-dimensions to a set of integrals in $(d+2)$ dimensions with the same or reduced topology as that of the original $d$-dimensional integral.

Having achieved this extension, the use of the FORCER package becomes straightforward with a FORM module slotted into the automatic integration of the Green's functions of interest at the point where the four dimensional masters would be called. To verify the correctness of the masters we determine, the extension is used to reproduce the known four loop modified minimal subtraction ($\overline{\text{MS}}$) scheme renormalization group functions in scalar $\phi^3$ theory with and without $O(N)$ symmetry although we note that higher loop $\overline{\text{MS}}$ information is already available [25,26]. Having established this check, we apply the algorithm to various Abelian and non-Abelian gauge theories to verify their ultraviolet completeness at a newer level. As a by-product we will also study the renormalization group functions in several other schemes. While one of these is the canonical $\overline{\text{MS}}$ one, we will also renormalize $\phi^3$ theory with and without $O(N)$ symmetry in the $\widetilde{\text{MOM}}$ scheme as well as introduce a new scheme not unrelated to it. In four dimensional studies [27–34], the $\widetilde{\text{MOM}}$ scheme has the property that at least to five loops the core renormalization group functions of QCD do not involve the numbers $\zeta_4$ and $\zeta_6$ where $\zeta_n$ is the Riemann zeta function. It is worth recalling that the minimal momentum (mMOM) scheme of [35] also shares the same property. The definition of that scheme is based specifically on the ghost-gluon vertex. However unlike the $\widetilde{\text{MOM}}$ schemes the absence of $\zeta_4$ and $\zeta_6$ is only manifest in mMOM in the Landau gauge. Indeed a no-$\pi$ theorem has been constructed that indicates under certain conditions there should be no even zetas to all orders [30,31]. The FORCER algorithm in six dimensions is essential to verify or otherwise whether this property remains purely four dimensional or is true in six dimensions as well since the finite part of the various Green's functions are required. Indeed we can exploit the known five loop $\overline{\text{MS}}$ renormalization group functions of [25,26] to deduce the five loop $\widetilde{\text{MOM}}$ expressions for $\phi^3$ theory. Aside from establishing the FORCER masters in six dimensions, the study of these scheme properties and ultraviolet completion is the second main aim of this investigation.

The article is organized as follows. Section II summarizes the algorithm used to construct the six dimensional FORCER master integrals up to weight 9. In order to verify that known results are reproduced, we focus on such a check in Sec. III by considering $\phi^3$ theory. En route we also construct the $\widetilde{\text{MOM}}$ renormalization group functions for the $O(N)$ cubic theory. In Sec. IV the four loop renormalization group functions of QED and scalar QED are determined in both the $\overline{\text{MS}}$ and $\widetilde{\text{MOM}}$ schemes to further investigate the presence or otherwise of even zetas in renormalization group functions. This analysis is continued in Sec. V where six dimensional QCD is studied with the three loop gauge $\beta$-function being computed. We provide an overview of our efforts in Sec. VI. Finally, there are two appendices. The $\epsilon$ expansion of the FORCER masters are provided in Appendix A while Appendix B records the two $\beta$-functions of $O(N)$ $\phi^3$ theory in the $\widetilde{\text{MOM}}$ scheme at five loops.

## II. FORCER MASTERS IN SIX DIMENSIONS

As the first phase for studying six dimensional field theories at four loops is to employ the FORCER package [8,9], we need to discuss the specifics of what this entails. To do so we recall that it was primarily developed to evaluate four loop massless 2-point functions to high order in the $\epsilon$ expansion in $d = 4 - 2\epsilon$ dimensions. Indeed it extended the earlier three loop MINCER algorithm [12] that was the main working tool for many decades to compute three loop massless 2-point functions in the same $\epsilon$ expansion. In many ways MINCER was ahead of its time in that it relied heavily on integration by parts and the use of what is termed the carpet rule [36] that allowed for an efficient reduction of a class of topologies. The last step of such an integration algorithm was the substitution of a small class of core integrals whose expansion near four dimensions was deduced without the use of integration by parts. It is only since the development of the Laporta algorithm [6] that one can appreciate the prescience of the MINCER construction. The summary of how MINCER operates represents the essence of the Laporta approach where integration by parts is the engine room of the method which reduces the Feynman graphs contributing to a Green's function to a basis set of integrals now known as master integrals. Again these have to be determined by techniques other than integration by parts. While the ethos of both approaches is the same, there are several key differences. The Laporta algorithm is applicable not only to 2-point but also to higher $n$-point functions as well as the situation where the Green's functions involve masses. One of the main limitations to applying the Laporta algorithm is technological rather than procedural. By this we mean the speed of an actual reduction reduces with the increase of external legs of the Green's function, the presence of a larger number of variables, such as masses and external momenta, and increase in loop order. What has been beneficial in the years after [6] is the improvement of the reduction algorithm through pure mathematics results. The other main difference is that the MINCER algorithm includes a routine that is more efficient at reducing the topologies and subtopologies of a Feynman graph. Such a routine is necessarily connected with the fact that MINCER is restricted to 2-point functions [12,36]. These aspects reflect the tension between having a reasonably general algorithm applicable in virtually all desired setups and one which is customized to a specific set of Green's functions. Put another way, one has a choice of automatic Feynman integration evaluation tools which are extremely efficient in their respective domains.

With the need for higher precision in renormalizing quantum field theories and determining Green's functions that contribute to observables for experiments, there was a clear need to extend MINCER to the next order. This was achieved with the FORCER package [8,9]. The core approach is the same as MINCER exploiting integration routines tailored to the Feynman integrals that contribute to four loop massless 2-point functions. In particular for a subclass of topologies a new routine was required where the carpet rule of MINCER was not applicable. This rule, termed the diamond rule, was provided in [13] and moreover was encoded in an efficient way within the final symbolic manipulation routine written in FORM [10,11]. In assembling the package, several new aspects were included that were not in MINCER. In the intervening years between the two packages being developed the understanding as to what the independent master integrals were was resolved. In particular there are two three loop master integrals and fourteen at four loops in the FORCER routine. In each case several of the masters are elementary integrals such as those where there are nested bubbles. However there were others whose $\epsilon$ expansion near four dimensions had only been available to a few orders in $\epsilon$. Encoded within FORCER are the $\epsilon$ expansions of the master integrals which were compiled from [37,38] except for lower loop masters where one or two bubble insertions have been mapped to a closely related topology [8,9]. Therefore the FORCER algorithm has a range of applicability greater than MINCER. One particular feature of the package is the option to express the value of a 2-point function in terms of its masters as a function of $d$ where the masters are not expanded in powers of $\epsilon$ near four dimensions. It is this specific feature that we aim to exploit here. In other words FORCER has the potential to be adapted to the renormalization of six dimensional field theories to high loop order if that can be achieved by the evaluation of massless 2-point functions. The missing ingredient is the $\epsilon$ expansion of the $d$-dimensional four loop master integrals in $d = 6 - 2\epsilon$ dimensions.

In fact the $\epsilon$ expansion of the masters near six dimensions can be extracted from the masters near four dimensions that are already available in FORCER. The key to determining them is the Tarasov construction [23,24]. Briefly this is a method that relates an $L$ loop Feynman integral in $d$-dimensions to a sum of Feynman integrals in $(d+2)$ dimensions which has the same topology as the original lower dimensional one but with the powers of $L$ propagators increased by unity. The distribution of the increase in propagator powers is determined by the structure of the second Symanzik graph polynomial which reflects and identifies the original topology. A useful tool for extracting the specific exponent distribution for each of the master topologies was the HYPERINT package of [39] written in MAPLE. Subsequently each of the integrals in the uplift to $(d+2)$ dimensions can then be reduced using integration by parts to produce a sum of Feynman integrals, one of which is equivalent to the original topology aside from being a $(d+2)$ dimensional integral. The remaining integrals in this sum correspond to Feynman graphs with fewer propagators in the sense that they are derived from the original topology but with propagators deleted. Therefore if we take the reference dimension $d$ to be four, knowledge of its $\epsilon$

expansion means that, provided the same expansion of the graphs with a lower number of propagators is known, the only unknown is the six dimensional master that we seek. This is the procedure we have followed to determine the FORCER masters in six dimensions. It has already been used in [17] to carry out the four loop renormalization of $\phi^3$ theory. In that instance, as FORCER was not available then, the $\epsilon$ expansion for the four dimensional masters required for that computation was provided in [37]. Although the FORCER master basis was not known at the time of [17], this was not necessary as the basis of [37] was sufficient to determine the $\overline{\text{MS}}$ scheme $\phi^3$ renormalization group functions at four loops. Other schemes such as $\widetilde{\text{MOM}}$ that required the finite part of Green's functions were not considered at that time. The derivation of the relation between the four and six dimensional integrals using the Tarasov technique relied upon the Laporta algorithm [6] and its implementation in the REDUZE package [40].

While the Laporta approach could have been repeated to deduce all the FORCER masters in six dimensions, we chose a different strategy here. This was to use the $d$-dimensional aspect of the FORCER algorithm itself to effect the reduction of the $(d+2)$ dimensional integrals that the Tarasov method produces at the first step. One benefit of this approach is that the reduction of integrals in FORCER is fast and more efficient than constructing an appropriate size database of relations between quite intricate topologies using the Laporta algorithm. This is because the increase in powers of the propagators by four requires a brute force integration by parts whereas FORCER applies the custom built diamond rule and is designed solely for 2-point functions. Moreover as we wish to determine the masters to a high order in $\epsilon$, at some point the four dimensional FORCER masters would have to be imported if the Laporta reduction had been employed. In addition, employing FORCER itself as a de facto reduction tool offers an independent way to check on the previous masters. At four loops the construction followed an iterative approach. The Tarasov method was applied to the lowest level master defined as the one with the smallest number of propagators that could not be evaluated by simple integration such as those comprised as bubbles. One has to begin at this point since in the reduction step these will appear for master topologies with a larger number of propagators. Once the lowest order masters have been determined then one moves to the next level. In each case one is effectively finding the terms in the $\epsilon$ expansion by solving for the unknown coefficients of $\epsilon$ in the required master. In Appendix A we have recorded the explicit $\epsilon$ expansions of the 16 FORCER masters using the same labelling as that of [9] and to the same order in weight as the four dimensional masters that appear in Appendix C of [9]. The expressions in Appendix A should be sufficient for carrying out a five loop renormalization where the higher powers of $\epsilon$ in the four loop masters are required to extend the calculation of the lower loop graphs contributing to a Green's function. Included in our Appendix A list are two three loop masters labelled as **no** and **t105**. In other words the six dimensional masters are provided up to and including weight 9 where $\zeta_9$ would be a representative. Though, we note that the FORM module in the actual FORCER code of [9] that corresponds to the four dimensional masters includes expressions up to weight 12 which were derived from the higher order $\epsilon$ expansion of the masters provided in [38].

## III. $\phi^3$ THEORIES

We are now in a position to renormalize a variety of six dimensional theories to four loops in the $\overline{\text{MS}}$ scheme as well as the $\widetilde{\text{MOM}}$ one which has been examined in four dimensions [27–34]. Such an exercise will act partly as a check on the construction of the FORCER masters as well as confirm an underlying aspect of the $\widetilde{\text{MOM}}$ scheme that is apparent in four dimensions. In the former instance an error in a master could produce an inconsistency with the application of the renormalization group formalism to extract and encode the renormalization constants in the respective $\beta$-functions and anomalous dimensions. The $\widetilde{\text{MOM}}$ scheme and issues related to it have become of interest in recent years. Its prescription is that in theories with a cubic interaction the 2-point functions and 3-point vertex functions are renormalized by absorbing the finite part of the respective divergent Green's functions into the wave function and coupling renormalization constants. In the case of the 3-point functions the Feynman graphs are evaluated where one of the external momenta is nullified. If there are several fields leading to more than one 3-point interaction then there will be a $\widetilde{\text{MOM}}$ scheme attached to each of the vertices. We will focus in this section on several six dimensional renormalizable scalar cubic theories. The first is the basic single field instance with Lagrangian

$$L = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{g}{6}\phi^3. \tag{3.1}$$

As far as we are aware the $\widetilde{\text{MOM}}$ scheme renormalization group functions for (3.1) are not yet available although the five loop renormalization group functions are known in the $\overline{\text{MS}}$ scheme [17,25,26,41–43]. It is worth recording these for completeness since they are needed as the foundation for deriving the $\widetilde{\text{MOM}}$ five loop equivalent renormalization group functions. For completeness we recall [17,25,26,41–43]

$$\begin{aligned}
\beta^{\phi^3}_{\overline{\rm MS}}(a) &= \frac{3}{4}a^2 - \frac{125}{144}a^3 + 5[2592\zeta_3 + 6617]\frac{a^4}{20736} \\
&+ [-4225824\zeta_3 + 349920\zeta_4 + 1244160\zeta_5 - 3404365]\frac{a^5}{746496} \\
&+ [41570496\zeta_3^2 + 356380884\zeta_3 - 33912351\zeta_4 + 295089480\zeta_5 + 15746400\zeta_6 \\
&- 576843120\zeta_7 + 102052031]\frac{a^6}{6718464} + O(a^7), \\
\gamma^{\phi^3}_{\phi,\overline{\rm MS}}(a) &= -\frac{1}{12}a + \frac{13}{432}a^2 + [2592\zeta_3 - 5195]\frac{a^3}{62208} \\
&+ [10080\zeta_3 + 18144\zeta_4 - 69120\zeta_5 + 53449]\frac{a^4}{248832} \\
&+ [-3499200\zeta_3^2 - 18368532\zeta_3 - 4119579\zeta_4 + 8691624\zeta_5 - 8748000\zeta_6 \\
&+ 46294416\zeta_7 - 16492987]\frac{a^5}{20155392} + O(a^6), \qquad (3.2)
\end{aligned}$$

where $a = g^2$, and we note the parameters will always be in the same scheme as that indicated on the renormalization group function itself. In situations where otherwise there might be an ambiguous interpretation, the parameters will carry the scheme label explicitly. We note the same conventions for the coupling constant as that of [17] are used here rather than those of [27]. The expressions of (3.2) can readily be mapped to the conventions of [26] via $a \to -a$. We recall that the $\widetilde{\rm MOM}$ renormalization prescription is to remove the finite parts of both the 2- and 3-point functions at the subtraction point and absorb them into the respective wave function and coupling renormalization constants. For the 3-point function one of the two independent external momenta is set to zero when the vertex function is evaluated. In this configuration the 3-point function is equivalent to a 2-point one whence the FORCER algorithm can be applied. We have first checked that the four loop $\overline{\rm MS}$ results of (3.2) are reproduced. This also provides an initial check on the six dimensional FORCER masters in a field theory calculation. Consequently it is straightforward to deduce the $\widetilde{\rm MOM}$ scheme renormalization group functions which are

$$\begin{aligned}
\beta^{\phi^3}_{\widetilde{\rm MOM}}(a) &= \frac{3}{4}a^2 - \frac{125}{144}a^3 + [-1296\zeta_3 + 26741]\frac{a^4}{10368} \\
&+ [-1370736\zeta_3 + 2177280\zeta_5 - 2304049]\frac{a^5}{186624} \\
&+ [389670912\zeta_3^2 + 3307195440\zeta_3 + 89151840\zeta_5 - 5640570432\zeta_7 \\
&+ 2190456157]\frac{a^6}{26873856} + O(a^7), \\
\gamma^{\phi^3}_{\phi,\widetilde{\rm MOM}}(a) &= -\frac{1}{12}a + \frac{37}{432}a^2 + [-1296\zeta_3 - 4435]\frac{a^3}{31104} \\
&+ [122256\zeta_3 + 155520\zeta_5 + 135149]\frac{a^4}{559872} \\
&+ [6718464\zeta_3^2 - 51538896\zeta_3 - 108669600\zeta_5 - 185177664\zeta_7 \\
&+ 38661817]\frac{a^5}{80621568} + O(a^6). \qquad (3.3)
\end{aligned}$$

Over several years there has been interest in which elements of the $\zeta_n$ sequence appear in the renormalization group functions in four dimensions [27–34]. In the $\overline{\rm MS}$ scheme $\zeta_n$ is present at successive loop orders with $n \geq 3$ where the loop order that $\zeta_3$ first occurs depends on the underlying theory. It transpires that in the $\widetilde{\rm MOM}$ prescription in four dimensions only $\zeta_{2n+1}$ for $1 \leq n \leq 3$ appears to five loops. In other words $\zeta_4$ and $\zeta_6$ are absent. A similar feature arises in six dimensional $\phi^3$ theory as is apparent in (3.3). So the basic four dimensional property of the $\widetilde{\rm MOM}$ scheme is preserved in another spacetime dimension.

The results of (3.3) were derived from a basic property of the renormalization group equation which relates the renormalization group functions in one scheme to those in another. For a single coupling theory such as (3.1) there is a simple relation between the coupling constant in one scheme to that in another which is derived from the relation of each coupling to the bare one which formally gives

$$g_{\widetilde{\rm MOM}}(\mu) = \frac{Z_g^{\overline{\rm MS}}}{Z_g^{\widetilde{\rm MOM}}} g_{\overline{\rm MS}}(\mu). \quad (3.4)$$

The right-hand side is interpreted as being a function of $g_{\overline{\rm MS}}$ although $Z_g^{\widetilde{\rm MOM}}$ is a function of $g_{\widetilde{\rm MOM}}$. In other words

$$Z_g^{\widetilde{\rm MOM}} \equiv Z_g^{\widetilde{\rm MOM}}\left(a_{\widetilde{\rm MOM}}(a_{\overline{\rm MS}})\right) \quad (3.5)$$

with the explicit $g_{\overline{\rm MS}}$ dependence being deduced via an iterative process. Such a change of variables is necessary as otherwise the relation (3.4) would have singularities in $\epsilon$. Once the mapping is available at four loops then the five loop $\beta$-function can be found from

$$\beta^{\phi^3}_{\widetilde{\rm MOM}}(a_{\widetilde{\rm MOM}}) = \left[\beta^{\phi^3}_{\overline{\rm MS}}(a_{\overline{\rm MS}}) \frac{\partial a_{\widetilde{\rm MOM}}}{\partial a_{\overline{\rm MS}}}\right]_{\overline{\rm MS}\to\widetilde{\rm MOM}}. \quad (3.6)$$

The restriction indicates that the coupling constant, which would otherwise be in the $\overline{\rm MS}$ scheme, has to be mapped to the $\widetilde{\rm MOM}$ scheme from the inverse of (3.4). A similar process is applied to determine the five loop field anomalous dimension using the conversion function defined by

$$C^{\phi^3}_\phi(a_{\overline{\rm MS}}) = \left[\frac{Z_\phi^{\widetilde{\rm MOM}}}{Z_\phi^{\overline{\rm MS}}}\right]_{\widetilde{\rm MOM}\to\overline{\rm MS}} \quad (3.7)$$

where the coupling $a$ in conversion functions will always be an $\overline{\rm MS}$ variable, and the relation

$$\gamma^{\phi^3}_{\phi\,\widetilde{\rm MOM}}(a_{\widetilde{\rm MOM}}) = \left[\gamma^{\phi^3}_{\phi\,\overline{\rm MS}}(a_{\overline{\rm MS}}) + \beta^{\phi^3}_{\overline{\rm MS}}(a_{\overline{\rm MS}}) \frac{\partial}{\partial a_{\overline{\rm MS}}} \ln C^{\phi^3}_\phi(a_{\overline{\rm MS}})\right]_{\overline{\rm MS}\to\widetilde{\rm MOM}} \quad (3.8)$$

uses the same process as that to determine the $\beta$-function.

To assist with verifying (3.3) we record the relevant explicit expressions are

$$\begin{aligned} a_{\widetilde{\rm MOM}} &= a_{\overline{\rm MS}} - \frac{4}{3}a^2_{\overline{\rm MS}} + [-1728\zeta_3 + 8005]\frac{a^3_{\overline{\rm MS}}}{1728} \\ &+ [659232\zeta_3 - 116640\zeta_4 + 2488320\zeta_5 - 7758845]\frac{a^4_{\overline{\rm MS}}}{373248} \\ &+ [11384064\zeta_3^2 + 69803208\zeta_3 + 6620292\zeta_4 - 120916800\zeta_5 - 2332800\zeta_6 \\ &- 123451776\zeta_7 + 262216343]\frac{a^5_{\overline{\rm MS}}}{2239488} + O(a^6_{\overline{\rm MS}}) \end{aligned} \quad (3.9)$$

and

$$\begin{aligned} C^{\phi^3}_\phi(a) &= 1 + \frac{2}{9}a - \frac{511}{1728}a^2 + [-6240\zeta_3 - 2592\zeta_4 + 122099]\frac{a^3}{124416} \\ &+ [-46656\zeta_3^2 - 2058228\zeta_3 - 299862\zeta_4 + 2251152\zeta_5 + 583200\zeta_6 - 12882121]\frac{a^4}{3359232} + O(a^5). \end{aligned} \quad (3.10)$$

As it sometimes turns out to be useful, for completeness we note the coupling constant conversion function is

$$\begin{aligned} C^{\phi^3}_g(a) &= 1 + \frac{2}{3}a + [1728\zeta_3 - 5701]\frac{a^2}{3456} \\ &+ [87264\zeta_3 + 116640\zeta_4 - 2488320\zeta_5 + 4853645]\frac{a^3}{746496} \\ &+ [-155271168\zeta_3^2 - 1372972032\zeta_3 - 83529792\zeta_4 + 1456911360\zeta_5 + 37324800\zeta_6 \\ &+ 1975228416\zeta_7 - 2620417103]\frac{a^4}{71663616} + O(a^5) \end{aligned} \quad (3.11)$$

to the same order where

$$C_g^{\phi^3}(a_{\overline{\rm MS}}) = \left[\frac{Z_g^{\widetilde{\rm MOM}}}{Z_g^{\overline{\rm MS}}}\right]_{\overline{\rm MS}\to\widetilde{\rm MOM}}. \quad (3.12)$$

We can extend the five loop $\widetilde{\rm MOM}$ study of the cubic scalar theory to the case where there is an $O(N)$ symmetry. In this case there are two coupling constants since the renormalizable Lagrangian is [15,16]

$$L = \frac{1}{2}(\partial_\mu\phi^i)^2 + \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{g_1}{2}\sigma\phi^i\phi^i + \frac{g_2}{6}\sigma^3 \quad (3.13)$$

where we use the same conventions as [17]. The original single field cubic theory is clearly recovered in the limit $g_1 \to 0$ in (3.13). To determine the $O(N)$ $\phi^3$ $\widetilde{\rm MOM}$ renormalization group functions, we follow the same procedure as before by constructing the four loop conversion function but with the formalism extended to accommodate two coupling constants. For instance the relation between the coupling constants in the respective schemes is

$$g_{i\,\widetilde{\rm MOM}}(\mu) = \frac{Z_{g_i}^{\overline{\rm MS}}}{Z_{g_i}^{\widetilde{\rm MOM}}}\, g_{i\,\overline{\rm MS}}(\mu) \quad (3.14)$$

for $i = 1$ and 2 where there is no summation over $i$. Then simply differentiating with respect to the renormalization scale $\mu$ leads to

$$\beta^{O(N)}_{i\widetilde{\rm MOM}}(\mathbf{g}_{\widetilde{\rm MOM}}) = \left[\sum_{j=1}^{2}\beta^{O(N)}_{j\overline{\rm MS}}(\mathbf{g}_{\overline{\rm MS}})\frac{\partial g_i^{\widetilde{\rm MOM}}}{\partial g_j^{\overline{\rm MS}}}\right]_{\overline{\rm MS}\to\widetilde{\rm MOM}} \quad (3.15)$$

where the scheme of the coupling constants now appears as subscripts. The scheme label is included to avoid ambiguity as there is a mapping of variables. To deduce the five loop anomalous dimensions the conversion functions for the two fields are given by

$$C_\phi^{O(N)}(\mathbf{g}_{\overline{\rm MS}}) = \left[\frac{Z_\phi^{\widetilde{\rm MOM}}}{Z_\phi^{\overline{\rm MS}}}\right]_{\widetilde{\rm MOM}\to\overline{\rm MS}},$$
$$C_\sigma^{O(N)}(\mathbf{g}_{\overline{\rm MS}}) = \left[\frac{Z_\sigma^{\widetilde{\rm MOM}}}{Z_\sigma^{\overline{\rm MS}}}\right]_{\widetilde{\rm MOM}\to\overline{\rm MS}} \quad (3.16)$$

where the restriction indicates that the argument of each conversion function is expressed in terms of the $\overline{\rm MS}$ variables as the reference scheme. The explicit expressions for the $\widetilde{\rm MOM}$ anomalous dimensions are derived from

$$\gamma^{O(N)}_{\phi\widetilde{\rm MOM}}(\mathbf{g}_{\widetilde{\rm MOM}}) = \left[\gamma^{O(N)}_{\phi\overline{\rm MS}}(\mathbf{g}_{\overline{\rm MS}}) + \sum_{i=1}^{2}\beta^{O(N)}_{i\overline{\rm MS}}(\mathbf{g}_{\overline{\rm MS}})\frac{\partial}{\partial g_{i\overline{\rm MS}}}\ln\left(C_\phi^{O(N)}(\mathbf{g}_{\overline{\rm MS}})\right)\right]_{\overline{\rm MS}\to\widetilde{\rm MOM}}$$
$$\gamma^{O(N)}_{\sigma\widetilde{\rm MOM}}(\mathbf{g}^{\widetilde{\rm MOM}}) = \left[\gamma^{O(N)}_{\sigma\overline{\rm MS}}(\mathbf{g}_{\overline{\rm MS}}) + \sum_{i=1}^{2}\beta^{O(N)}_{i\overline{\rm MS}}(\mathbf{g}_{\overline{\rm MS}})\frac{\partial}{\partial g_{i\overline{\rm MS}}}\ln\left(C_\sigma^{O(N)}(\mathbf{g}_{\overline{\rm MS}})\right)\right]_{\overline{\rm MS}\to\widetilde{\rm MOM}}. \quad (3.17)$$

The final stage of the process is to recall that the $O(N)$ $\phi^3$ $\overline{\rm MS}$ five loop renormalization group functions are available from [25]. In addition we have carried out the explicit four loop renormalization of (3.13) in the $\widetilde{\rm MOM}$ scheme which allows us to determine the field conversion functions and coupling constant mappings to four loops. With these we have established the $O(N)$ five loop $\widetilde{\rm MOM}$ renormalization group functions. The main motivation for doing so is to ascertain whether there are any terms involving $\zeta_4$ or $\zeta_6$ which are absent in the single coupling case for (3.1). We find the same outcome, in keeping with the analysis of [33], in that $\zeta_4$ or $\zeta_6$ are absent from each of the four renormalization group functions for all $N$. The full expressions together with the conversion functions and coupling constant mappings are provided in [44] but we have provided the two $\beta$-functions in Appendix B. By way of example the expressions for the $\gamma^{O(N)}_{\sigma\widetilde{\rm MOM}}(g_1, g_2)$ and $\beta^{O(N)}_{1\widetilde{\rm MOM}}(g_1, g_2)$ for $N = 2$ are

$$\gamma^{O(2)}_{\sigma\widetilde{\rm MOM}}(g_1,g_2) = \left[-\frac{1}{6}g_1^2 - \frac{1}{12}g_2^2\right] + \left[-\frac{11}{216}g_1^2g_2^2 + \frac{4}{9}g_1^3g_2 + \frac{13}{108}g_1^4 + \frac{37}{432}g_2^4\right]$$
$$+ \left[-\frac{4435}{31104}g_2^6 - \frac{1247}{1296}g_1^4g_2^2 - \frac{505}{972}g_1^6 - \frac{89}{216}g_1^5g_2 - \frac{83}{324}g_1^3g_2^3 - \frac{5}{12}\zeta_3g_1^4g_2^2\right.$$
$$\left. - \frac{1}{6}\zeta_3g_1^6 - \frac{1}{24}\zeta_3g_2^6 + \frac{169}{1944}g_1^2g_2^4\right]$$

$$
\begin{aligned}
&+\left[-\frac{26969}{139968}g_1^2g_2^6-\frac{439}{648}\zeta_3g_1^6g_2^2-\frac{23}{162}\zeta_3g_1^2g_2^6-\frac{11}{432}\zeta_3g_1^3g_2^5+\frac{5}{18}\zeta_5g_2^8\right.\\
&\quad+\frac{20}{9}\zeta_5g_1^4g_2^4+\frac{20}{9}\zeta_5g_1^8+\frac{40}{9}\zeta_5g_1^6g_2^2+\frac{127}{324}\zeta_3g_1^8+\frac{139}{54}\zeta_3g_1^7g_2\\
&\quad+\frac{283}{1296}\zeta_3g_2^8+\frac{1279}{1296}\zeta_3g_1^4g_2^4+\frac{2486}{729}g_1^7g_2+\frac{24283}{46656}g_1^3g_2^5+\frac{55901}{69984}g_1^6g_2^2\\
&\quad\left.+\frac{68407}{69984}g_1^8+\frac{101983}{46656}g_1^5g_2^3+\frac{135149}{559872}g_2^8+\frac{170393}{279936}g_1^4g_2^4+3\zeta_3g_1^5g_2^3\right]\\
&+\left[-\frac{18870089}{40310784}g_1^2g_2^8-\frac{10106999}{5038848}g_1^8g_2^2-\frac{453335}{157464}g_1^{10}-\frac{119303}{186624}\zeta_3g_2^{10}\right.\\
&\quad-\frac{98207}{7776}\zeta_3g_1^5g_2^5-\frac{43223}{11664}\zeta_3g_1^6g_2^4-\frac{20011}{1944}\zeta_3g_1^9g_2-\frac{18665}{2592}\zeta_5g_1^4g_2^6\\
&\quad-\frac{13975}{10368}\zeta_5g_2^{10}-\frac{10157}{3888}\zeta_3g_1^8g_2^2-\frac{8815}{216}\zeta_5g_1^9g_2-\frac{5755}{108}\zeta_5g_1^7g_2^3\\
&\quad-\frac{4555}{1296}\zeta_3g_1^7g_2^3-\frac{2585}{1296}\zeta_5g_1^{10}-\frac{2401}{32}\zeta_7g_1^8g_2^2-\frac{1323}{32}\zeta_7g_1^6g_2^4-\frac{755}{144}\zeta_5g_1^3g_2^7\\
&\quad-\frac{539}{32}\zeta_7g_1^4g_2^6-\frac{441}{16}\zeta_7g_1^{10}-\frac{147}{64}\zeta_7g_2^{10}-\frac{95}{27}\zeta_5g_1^5g_2^5-\frac{4}{27}\zeta_3g_1^3g_2^7\\
&\quad-\frac{1}{6}\zeta_3^2g_1^5g_2^5-\frac{1}{12}\zeta_3^2g_1^2g_2^8-\frac{1}{12}\zeta_3^2g_1^6g_2^4+\frac{1}{3}\zeta_3^2g_1^3g_2^7+\frac{1}{12}\zeta_3^2g_2^{10}+\frac{5}{6}\zeta_3^2g_1^9g_2\\
&\quad+\frac{5}{12}\zeta_3^2g_1^4g_2^6+\frac{5}{36}\zeta_3^2g_1^8g_2^2+\frac{7}{12}\zeta_3^2g_1^{10}+\frac{11}{3}\zeta_3^2g_1^7g_2^3+\frac{1505}{1296}\zeta_5g_1^2g_2^8\\
&\quad+\frac{3665}{1296}\zeta_5g_1^8g_2^2+\frac{10151}{5184}g_1^3g_2^7+\frac{10255}{1728}\zeta_5g_1^6g_2^4+\frac{13667}{11664}\zeta_3g_1^2g_2^8\\
&\quad+\frac{41861}{11664}\zeta_3g_1^{10}+\frac{204311}{93312}\zeta_3g_1^4g_2^6+\frac{652231}{104976}g_1^9g_2+\frac{5288099}{20155392}g_1^6g_2^4\\
&\quad\left.+\frac{6725627}{3359232}g_1^5g_2^5+\frac{24943961}{1679616}g_1^7g_2^3+\frac{28840009}{20155392}g_1^4g_2^6+\frac{38661817}{80621568}g_2^{10}\right]+O(g_i^{12}) \qquad (3.18)
\end{aligned}
$$

and

$$
\begin{aligned}
\beta^{O(2)}_{\widetilde{1\,\mathrm{MOM}}}(g_1,g_2)=&\left[-\frac{1}{24}g_1g_2^2+\frac{1}{2}g_1^2g_2+\frac{1}{4}g_1^3\right]\\
&+\left[-\frac{337}{432}g_1^3g_2^2-\frac{59}{72}g_1^5-\frac{1}{12}g_1^2g_2^3-\frac{1}{18}g_1^4g_2+\frac{37}{864}g_1g_2^4\right]\\
&+\left[-\frac{4435}{62208}g_1g_2^6-\frac{1}{6}\zeta_3g_1^3g_2^4-\frac{1}{48}\zeta_3g_1g_2^6+\frac{1}{2}\zeta_3g_1^4g_2^3+\frac{7}{4}\zeta_3g_1^7+\frac{7}{24}\zeta_3g_1^5g_2^2\right.\\
&\quad+\frac{559}{1728}g_1^2g_2^5+\frac{2189}{2592}g_1^4g_2^3+\frac{2803}{1296}g_1^7+\frac{5075}{2592}g_1^5g_2^2+\frac{6367}{1296}g_1^6g_2\\
&\quad\left.+\frac{7079}{7776}g_1^3g_2^4-3\zeta_3g_1^6g_2\right]\\
&+\left[-\frac{3188293}{139968}g_1^7g_2^2-\frac{1584779}{46656}g_1^8g_2-\frac{843827}{139968}g_1^5g_2^4-\frac{721603}{23328}g_1^6g_2^3\right.\\
&\quad-\frac{482599}{23328}g_1^9-\frac{72569}{15552}g_1^3g_2^6-\frac{38011}{31104}g_1^2g_2^7-\frac{2579}{1296}\zeta_3g_1^7g_2^2-\frac{2339}{1296}\zeta_3g_1^3g_2^6\\
&\quad-\frac{1931}{108}\zeta_3g_1^6g_2^3-\frac{1603}{288}\zeta_3g_1^5g_2^4-\frac{1253}{54}\zeta_3g_1^9-\frac{1189}{864}\zeta_3g_1^4g_2^5-\frac{613}{24}\zeta_3g_1^8g_2
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{35}{9}\zeta_5 g_1^7 g_2^2 - \frac{5}{2}\zeta_5 g_1^4 g_2^5 + \frac{1}{48}\zeta_3 g_1^2 g_2^7 + \frac{5}{36}\zeta_5 g_1 g_2^8 + \frac{70}{3}\zeta_5 g_1^9 \\
&+ \frac{85}{18}\zeta_5 g_1^3 g_2^6 + \frac{95}{18}\zeta_5 g_1^5 g_2^4 + \frac{130}{3}\zeta_5 g_1^6 g_2^3 + \frac{283}{2592}\zeta_3 g_1 g_2^8 + \frac{325}{6}\zeta_5 g_1^8 g_2 \\
&+ \frac{135149}{1119744} g_1 g_2^8 + \frac{202163}{186624} g_1^4 g_2^5 \Bigg] \\
+ \Bigg[ &\frac{2392700219}{80621568} g_1^3 g_2^8 + \frac{3141423697}{5038848} g_1^9 g_2^2 + \frac{10824707071}{40310784} g_1^7 g_2^4 \\
&- \frac{117194495}{13436928} g_1^4 g_2^7 - \frac{2042725}{10368}\zeta_5 g_1^7 g_2^4 - \frac{713515}{864}\zeta_5 g_1^9 g_2^2 \\
&- \frac{604061}{31104}\zeta_3 g_1^4 g_2^7 - \frac{122227}{96}\zeta_7 g_1^{10} g_2 - \frac{119303}{373248}\zeta_3 g_1 g_2^{10} - \frac{44255}{864}\zeta_5 g_1^{11} \\
&- \frac{44009}{32}\zeta_7 g_1^8 g_2^3 - \frac{42133}{192}\zeta_7 g_1^7 g_2^4 - \frac{32977}{96}\zeta_7 g_1^6 g_2^5 - \frac{32179}{64}\zeta_7 g_1^9 g_2^2 \\
&- \frac{26125}{2592}\zeta_5 g_1^3 g_2^8 - \frac{13975}{20736}\zeta_5 g_1 g_2^{10} - \frac{10017}{16}\zeta_7 g_1^{11} - \frac{9485}{64}\zeta_7 g_1^5 g_2^6 \\
&- \frac{215}{144}\zeta_5 g_1^2 g_2^9 - \frac{182}{3}\zeta_7 g_1^3 g_2^8 - \frac{147}{128}\zeta_7 g_1 g_2^{10} + \frac{1}{24}\zeta_3^2 g_1 g_2^{10} + \frac{17}{24}\zeta_3^2 g_1^4 g_2^7 \\
&+ \frac{65}{12}\zeta_3^2 g_1^5 g_2^6 + \frac{287}{8}\zeta_3^2 g_1^{11} + \frac{287}{72}\zeta_3^2 g_1^3 g_2^8 + \frac{293}{12}\zeta_3^2 g_1^6 g_2^5 + \frac{1085}{12}\zeta_3^2 g_1^{10} g_2 \\
&+ \frac{1477}{96}\zeta_7 g_1^4 g_2^7 + \frac{1601}{72}\zeta_3^2 g_1^7 g_2^4 + \frac{1679}{24}\zeta_3^2 g_1^8 g_2^3 + \frac{3725}{432}\zeta_5 g_1^6 g_2^5 \\
&+ \frac{7391}{10368}\zeta_3 g_1^2 g_2^9 + \frac{15881}{432}\zeta_3 g_1^3 g_2^8 + \frac{18805}{864}\zeta_5 g_1^4 g_2^7 + \frac{29225}{108}\zeta_5 g_1^{10} g_2 \\
&+ \frac{158255}{288}\zeta_5 g_1^8 g_2^3 + \frac{190625}{5184}\zeta_5 g_1^5 g_2^6 + \frac{663761}{1296}\zeta_3 g_1^{10} g_2 + \frac{1065955}{5832}\zeta_3 g_1^7 g_2^4 \\
&+ \frac{2616893}{15552}\zeta_3 g_1^6 g_2^5 + \frac{3783533}{7776}\zeta_3 g_1^{11} + \frac{9270535}{15552}\zeta_3 g_1^8 g_2^3 \\
&+ \frac{16931831}{23328}\zeta_3 g_1^9 g_2^2 + \frac{17933611}{186624}\zeta_3 g_1^5 g_2^6 + \frac{31673845}{4478976} g_1^2 g_2^9 \\
&+ \frac{38661817}{161243136} g_1 g_2^{10} + \frac{46773469}{373248} g_1^8 g_2^3 + \frac{82651885}{419904} g_1^{11} + \frac{155899681}{419904} g_1^{10} g_2 \\
&+ \frac{351087313}{2239488} g_1^6 g_2^5 + \frac{427400711}{40310784} g_1^5 g_2^6 + 72\zeta_3^2 g_1^9 g_2^2 \Bigg] + O(g_i^{13}). \qquad (3.19)
\end{aligned}
$$

One check on the full $O(N)$ $\widetilde{\text{MOM}}$ results is that the respective expressions of (3.2) correctly emerged in the $g_1 \to 0$ limit.

Having evaluated the four loop FORCER masters to high order in powers of $\epsilon$ we can study the properties of a scheme that is not unrelated to the $\widetilde{\text{MOM}}$ one. In the $\widetilde{\text{MOM}}$ prescription the finite part of the Green's function at the subtraction point is absorbed into the respective renormalization constants. One natural extension of this prescription is to not only absorb the finite or $O(1)$ part with respect to $\epsilon$ but also the higher order terms in the $\epsilon$ expansion. By introducing such a scheme one is in effect absorbing the full structure of the underlying Feynman graphs determined as a function of the regularizing parameter. In other words one subtracts all properties of the quantum corrections. This may appear to be an unusual prescription and there is no clear expectation as to what it means for the properties of the resultant renormalization group functions. However it is an easy algebraic exercise to pursue in this toy scalar field theory in order to explore the consequences. There is the tacit assumption that whatever transpires in this example ought to have exact parallels in four dimensional theories in much the same way that the even zetas are absent in the $\widetilde{\text{MOM}}$ scheme for theories in both four and six dimensions to a specific loop order. As this prescription is the polar opposite to the $\overline{\text{MS}}$ one we will term this new scheme the maximal subtraction scheme and denote it by $\overline{\text{MaxS}}$

where the overline retains the nod to the absence of $\zeta_2$ in the $\overline{\rm MS}$ renormalization group functions. Therefore we have repeated the procedure that resulted in the $\widetilde{\rm MOM}$ renormalization group functions of (3.3). In other words we have applied the $\overline{\rm MaxS}$ prescription to the 2- and 3-point functions and deduced the respective coupling and wave function renormalization constants before constructing the coupling constant map and the wave function conversion function. Retaining the higher order terms in $\epsilon$ necessarily involved an additional quantity of intermediate algebra. However, we were able to determine that the two $\overline{\rm MaxS}$ renormalization group functions are

$$\begin{aligned}
\beta^{\phi^3}_{\overline{\rm MaxS}}(a) &= \frac{3}{4}a^2 - \frac{125}{144}a^3 + [-1296\zeta_3 + 26741]\frac{a^4}{10368} \\
&\quad + [-1370736\zeta_3 + 2177280\zeta_5 - 2304049]\frac{a^5}{186624} \\
&\quad + [389670912\zeta_3^2 + 3307195440\zeta_3 + 89151840\zeta_5 - 5640570432\zeta_7 + 2190456157]\frac{a^6}{26873856} + O(a^7), \\
\gamma^{\phi^3}_{\phi,\overline{\rm MaxS}}(a) &= -\frac{1}{12}a + \frac{37}{432}a^2 + [-1296\zeta_3 - 4435]\frac{a^3}{31104} \\
&\quad + [122256\zeta_3 + 155520\zeta_5 + 135149]\frac{a^4}{559872} \\
&\quad + [6718464\zeta_3^2 - 51538896\zeta_3 - 108669600\zeta_5 - 185177664\zeta_7 + 38661817]\frac{a^5}{80621568} + O(a^6). \qquad (3.20)
\end{aligned}$$

What is interesting is that the expressions are formally the same as the respective $\widetilde{\rm MOM}$ ones to five loops. This is not unexpected given that even zeta contributions were absent in the $\widetilde{\rm MOM}$ scheme renormalization group functions but are present in the $O(\epsilon^0)$ part of the $\widetilde{\rm MOM}$ and $\overline{\rm MaxS}$ renormalization constants. The higher order $\overline{\rm MaxS}$ $\epsilon$ terms will involve the subsequent terms of the $\zeta$-series as well as rationals. Such higher order zeta contributions cannot for instance appear in the renormalization group functions before a certain loop order. This is preserved naturally in the $\overline{\rm MaxS}$ scenario. Where the difference in the $\widetilde{\rm MOM}$ and $\overline{\rm MaxS}$ renormalization group functions would be manifest is in the ultimate step that results in (3.20). That step is to set the regularizing parameter, $\epsilon$, to zero. Prior to that the renormalization group functions are $\epsilon$ dependent with the coefficients of $\epsilon$ being in a direct relationship with the constant and $O(\epsilon)$ terms of the respective renormalization constants. In the case of the $\widetilde{\rm MOM}$ scheme the dependence on $\epsilon$ would be linear in contrast to the structure of the $\overline{\rm MaxS}$ renormalization group functions. In that instance the coefficients of $a$ would be an infinite series in $\epsilon$. However for practical reasons in our construction we restricted our analysis to weight 9 as that was the weight we determined the six dimensional FORCER masters to. What this exercise has revealed in addition to the above is that the $\widetilde{\rm MOM}$ and $\overline{\rm MaxS}$ schemes are synonymous in the critical dimension of $\phi^3$ theory. Moreover at least to five loops, the $\widetilde{\rm MOM}$ scheme is equivalent to the scheme where the full Feynman integrals themselves are completely subtracted at the renormalization point which is something we believe has not been examined previously.

## IV. ABELIAN GAUGE THEORIES

Having established the usefulness of the four loop FORCER masters in the cubic scalar theory we devote this section to extending their application to gauge theories in six dimensions. In particular our focus here will be on Abelian gauge theories as these are of interest in [21]. The aim is to extend the renormalization of both quantum electrodynamics (QED) and scalar QED (sQED) to four loops. First we recall the QED Lagrangian in six dimensions is [21]

$$L^{(6)}\Big|_{\rm QED} = i\bar{\psi}^i \not{D}\psi^i - \frac{1}{4}(\partial_\mu F_{\nu\sigma})(\partial^\mu F^{\nu\sigma}) - \frac{1}{2\alpha}(\partial_\mu\partial^\nu A_\nu)(\partial^\mu\partial^\sigma A_\sigma), \qquad (4.1)$$

where $g$ is the gauge coupling constant, $\alpha$ is the gauge parameter, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\nu$, $A_\mu$ is the photon and $\psi^i$ is the electron with $1 \leq i \leq N_f$. The linear gauge fixing term is such that the photon propagator takes the form

$$\langle A_\mu(p) A_\nu(-p)\rangle = -\frac{1}{(p^2)^2}\left[\eta_{\mu\nu} - (1-\alpha)\frac{p_\mu p_\nu}{p^2}\right]. \qquad (4.2)$$

We note that the second order pole in the propagator is not infrared pathological in six dimensions as it would be in four dimensions. Moreover it is not an obstruction to extending the three loop renormalization group functions to the next order using FORCER. Since we have followed the same algorithm that produced the renormalization group functions of the previous section for $\phi^3$ theory, we record the equivalent QED four loop results are

$$
\begin{aligned}
\beta^{\mathrm{QED}}_{\overline{\mathrm{MS}}}(g,\alpha) &= -\frac{2N_f}{15}g^3 - \frac{19N_f}{27}g^5 + 17[-111N_f + 200]\frac{N_f g^7}{12150} \\
&\quad + [170362N_f^2 + 17107200\zeta_3 N_f - 14025425N_f - 7500000]\frac{N_f g^9}{8201250} + O(g^{11}), \\
\gamma^{\mathrm{QED}}_{A,\overline{\mathrm{MS}}}(g,\alpha) &= -\frac{4N_f}{15}g^2 - \frac{38N_f}{27}g^4 + 17[-111N_f + 200]\frac{N_f g^6}{6075} \\
&\quad + [170362N_f^2 + 17107200\zeta_3 N_f - 14025425N_f - 7500000]\frac{N_f g^8}{4100625} + O(g^{10}), \\
\gamma^{\mathrm{QED}}_{\alpha,\overline{\mathrm{MS}}}(g,\alpha) &= -\gamma^{\mathrm{QED}}_{A,\overline{\mathrm{MS}}}(g,\alpha), \\
\gamma^{\mathrm{QED}}_{\psi,\overline{\mathrm{MS}}}(g,\alpha) &= [3\alpha + 5]\frac{g^2}{6} + 2[32N_f - 125]\frac{g^4}{135} \\
&\quad + [2864N_f^2 - 648000\zeta_3 N_f + 730375N_f + 1944000\zeta_3 - 1033000]\frac{g^6}{243000} \\
&\quad + [-518400\zeta_3 N_f^3 - 25824N_f^3 - 43156800\zeta_3 N_f^2 + 17496000\zeta_4 N_f^2 + 28663075N_f^2 \\
&\qquad - 1560999600\zeta_3 N_f - 52488000\zeta_4 N_f + 3061800000\zeta_5 N_f - 1179131300N_f \\
&\qquad + 1552770000\zeta_3 - 3061800000\zeta_5 + 1003751250]\frac{g^8}{16402500} + O(g^{10}), \\
\gamma^{\mathrm{QED}}_{m,\overline{\mathrm{MS}}}(g) &= -\frac{5}{3}g^2 - [68N_f + 25]\frac{g^4}{135} \\
&\quad + [13456N_f^2 + 648000N_f\zeta_3 - 818575N_f + 1215000\zeta_3 - 726875]\frac{g^6}{121500} \\
&\quad + [518400N_f^3\zeta_3 - 216384N_f^3 + 31492800N_f^2\zeta_3 - 17496000N_f^2\zeta_4 - 6336415N_f^2 \\
&\qquad - 137011500N_f\zeta_3 - 32805000N_f\zeta_4 + 656100000N_f\zeta_5 - 318912625N_f \\
&\qquad + 2574281250\zeta_3 - 4538025000\zeta_5 + 845045625]\frac{g^8}{8201250} + O(g^{10})
\end{aligned}
\tag{4.3}
$$

in the $\overline{\mathrm{MS}}$ scheme. We note that the $\beta$-function is derived from the Ward-Takahashi identity

$$\beta_1(g) = \frac{g}{2}\gamma_A(g,\alpha) \tag{4.4}$$

to four loops. The previous three loop results of [22] are recovered partially verifying our procedure. Another check is that the electron anomalous dimension has the same feature as four dimensional QED in that the only dependence on the gauge parameter $\alpha$ is in the one loop term. Interesting properties of this renormalization group function in four dimensional QED were initially discussed in [45,46] in the on-shell renormalization scheme. There it was noted that the anomalous dimension was gauge independent. Subsequently it was shown in [47] that in the $\overline{\mathrm{MS}}$ scheme the gauge parameter of a linear covariant gauge fixing appears only in the one loop term of the electron anomalous dimension in four dimensional QED. That analysis made use of a Landau-Khalatnikov-Fradkin (LKF) transformation [48,49] and this formalism should be equally applicable to establish the same property in scalar QED. In more recent years the absence of $\alpha$ in the two and higher order corrections in four dimensions was examined from the Hopf algebra point of view in [50–52]. It seems clear that the LKF transformation approach as well as the graphical and algebraic arguments could be extended to the six dimensional theory of (4.1). The expression for $\gamma^{\mathrm{QED}}_{\psi,\overline{\mathrm{MS}}}(g,\alpha)$ in (4.3) would support that expectation. Any proof that $\alpha$ only appears at one loop in the $\overline{\rm MS}$ electron anomalous dimension though should probably be constructed in a way that is applicable to all even dimensions beginning from four. This is simply because it is known that the same property is present at two loops in eight dimensional QED [22]. However such a proof is beyond the scope of the current article.

In light of the discussion concerning the location of the set of numbers $\zeta_n$ of the previous section, we recall that the $\overline{\rm MS}$ $\beta$-function has the same property as its four dimensional counterpart in that $\zeta_3$ is absent at three loops but appears for the first time at four loops which is a consequence of gauge symmetry. By contrast $\zeta_3$ is present in the three loop $\overline{\rm MS}$ $\beta$-function of $\phi^3$ theory as well as in the $\widetilde{\rm MOM}$ scheme. Therefore we have repeated the renormalization of (4.1) in the $\widetilde{\rm MOM}$ scheme to ascertain whether $\zeta_3$ first appears at three or four loops. We found the analogous expressions to (4.3) are

$$\begin{aligned}
\beta^{\rm QED}_{\widetilde{\rm MOM}}(g,\alpha) &= -\frac{2N_f}{15}g^3 - \frac{19N_f}{27}g^5 + [2592\zeta_3N_f - 2889N_f + 1700]\frac{N_fg^7}{6075} \\
&\quad + [165888\zeta_3N_f^2 - 306876N_f^2 + 4976640\zeta_3N_f - 4665600\zeta_5N_f \\
&\quad - 692335N_f - 500000]\frac{N_fg^9}{546750} + O(g^{11}), \\
\gamma^{\rm QED}_{A,\widetilde{\rm MOM}}(g,\alpha) &= -\frac{4N_f}{15}g^2 - \frac{38N_f}{27}g^4 + 2[2592\zeta_3N_f - 2889N_f + 1700]\frac{N_fg^6}{6075} \\
&\quad + [165888\zeta_3N_f^2 - 306876N_f^2 + 4976640\zeta_3N_f - 4665600\zeta_5N_f \\
&\quad - 692335N_f - 500000]\frac{N_fg^8}{273375} + O(g^{10}), \\
\gamma^{\rm QED}_{\alpha,\widetilde{\rm MOM}}(g,\alpha) &= -\gamma^{\rm QED}_{A,\widetilde{\rm MOM}}(g,\alpha), \\
\gamma^{\rm QED}_{\psi,\widetilde{\rm MOM}}(g,\alpha) &= [3\alpha + 5]\frac{g^2}{6} + 2[4N_f - 25]\frac{g^4}{27} \\
&\quad + [976N_f^2 - 675\alpha N_f + 8555N_f + 38880\zeta_3 - 20660]\frac{g^6}{4860} \\
&\quad + [54675\alpha^2N_f - 97200\zeta_3\alpha^2N_f + 103680\zeta_3\alpha N_f^2 - 144720\alpha N_f^2 - 324000\zeta_3\alpha N_f \\
&\qquad + 81000\alpha N_f + 150528N_f^3 - 1731456\zeta_3N_f^2 + 3643432N_f^2 - 72595440\zeta_3N_f \\
&\qquad + 139968000\zeta_5N_f - 59802795N_f + 82814400\zeta_3 - 163296000\zeta_5 \\
&\qquad + 53533400]\frac{g^8}{874800} + O(g^{10}), \\
\gamma^{\rm QED}_{m,\widetilde{\rm MOM}}(g,\alpha) &= -\frac{5}{3}g^2 - [N_f + 5]\frac{g^4}{27} \\
&\quad + [1215\alpha N_f - 1184N_f^2 + 5184N_f\zeta_3 - 12613N_f + 48600\zeta_3 - 29075]\frac{g^6}{4860} \\
&\quad + [194400\alpha^2N_f\zeta_3 - 103275\alpha^2N_f - 207360\alpha N_f^2\zeta_3 + 136080\alpha N_f^2 \\
&\qquad + 1069200\alpha N_f - 4992N_f^3 + 2094336N_f^2\zeta_3 - 4752992N_f^2 + 10886400N_f\zeta_3 \\
&\qquad + 9331200N_f\zeta_5 - 29410965N_f + 274590000\zeta_3 - 484056000\zeta_5 \\
&\qquad + 90138200]\frac{g^8}{874800} + O(g^{10}).
\end{aligned} \tag{4.5}$$

Clearly $\zeta_3$ arises for the first time at three loops more in keeping with $\phi^3$ theory and similar to the properties of the $\widetilde{\rm MOM}$ scheme renormalization group functions in four dimensional gauge theories. One other feature of (4.5) is that unlike $\gamma^{\rm QED}_{\psi,\overline{\rm MS}}(g,\alpha)$ there is $\alpha$ dependence in $\gamma^{\rm QED}_{\psi,\widetilde{\rm MOM}}(g,\alpha)$ beyond one loop. However it first appears at three loops, like $\widetilde{\rm MOM}$

schemes in four dimensional QED, rather than at two loops for the MOM schemes of Celmaster and Gonsalves [53,54]. The contrast in this structural difference may be attributed to the difference in the underlying renormalization prescription. For instance in the $\widetilde{\rm MOM}$ suite of schemes the subtraction for 3-point functions is carried out at a point where one of the external legs has its momentum nullified. By contrast for the MOM schemes of [53,54] the prescription is that the vertex subtraction is enacted at the fully symmetric point where the squared momenta of all three external legs are nonzero and equal. So it would appear that the properties of the vertex kinematics has a bearing on the structure of the renormalization group functions and in particular this is manifest in the electron renormalization in an Abelian theory.

It is worth examining whether these observations are peculiar to six dimensional QED or a more general feature. Therefore to explore this we have repeated the above QED analysis but for the version where the fermions are replaced by a scalar field $\phi^i$. The corresponding Lagrangian is

$$L^{\rm sQED} = \overline{D_\mu \phi^i} D^\mu \phi^i - \frac{1}{4} \partial_\mu F_{\nu\sigma} \partial^\mu F^{\nu\sigma} - \frac{1}{2\alpha} (\partial_\mu \partial^\nu A_\nu)(\partial^\mu \partial^\sigma A_\sigma) \tag{4.6}$$

which is the ultraviolet compeletion of four dimensional scalar QED where $1 \leq i \leq N_f$. Carrying out the renormalization process we find that the $\overline{\rm MS}$ renormalization group functions are

$$\begin{aligned}
\beta^{\rm sQED}_{\overline{\rm MS}}(g) &= -\frac{N_f}{60} g^3 - \frac{37 N_f}{216} g^5 - [1017 N_f + 59600] \frac{N_f g^7}{97200} \\
&\quad + [214131600 \zeta_3 N_f - 407787250 N_f - 282521 N_f^2 + 118098000 \zeta_3 \\
&\qquad - 456939750] \frac{N_f g^9}{1049760000} + O(g^{11}), \\
\gamma^{\rm sQED}_{A,\overline{\rm MS}}(g,\alpha) &= -\frac{N_f}{30} g^2 - \frac{37 N_f}{108} g^4 - [1017 N_f + 59600] \frac{N_f g^6}{48600} \\
&\quad + [214131600 \zeta_3 N_f - 407787250 N_f - 282521 N_f^2 + 118098000 \zeta_3 \\
&\qquad - 456939750] \frac{N_f g^8}{524880000} + O(g^{10}), \\
\gamma^{\rm sQED}_{\alpha,\overline{\rm MS}}(g,\alpha) &= -\gamma^{\rm sQED}_{A,\overline{\rm MS}}(g,\alpha), \\
\gamma^{\rm sQED}_{\phi,\overline{\rm MS}}(g,\alpha) &= [3\alpha - 10] \frac{g^2}{6} + [-98 N_f + 1375] \frac{g^4}{1080} \\
&\quad + [662 N_f^2 + 648000 \zeta_3 N_f - 1430875 N_f + 1458000 \zeta_3 + 516500] \frac{g^6}{972000} \\
&\quad + [12960 \zeta_3 N_f^3 - 2382 N_f^3 + 8631360 \zeta_3 N_f^2 - 3499200 \zeta_4 N_f^2 - 12293840 N_f^2 \\
&\qquad - 1435890240 \zeta_3 N_f - 7873200 \zeta_4 N_f + 2536920000 \zeta_5 N_f - 470052370 N_f \\
&\qquad + 1663578000 \zeta_3 + 2274480000 \zeta_5 - 3242042625] \frac{g^8}{104976000} \\
&\quad + O(g^{10}).
\end{aligned} \tag{4.7}$$

The lower loop expressions are in agreement with [55,56]. As another check we have computed the critical exponents for the scalar electron and the photon from (4.7) in powers of $1/N_f$ at the Wilson-Fisher fixed point of the $\beta$-function in $d = 6 - 2\epsilon$ dimensions. These were compared with the direct large $N_f$ expansion of the same quantities computed in the underlying universal theory in $d$-dimensions in [55]. Expanding the results of [55] to $O(\epsilon^3)$ we find exact agreement. From a careful comparison it is evident the properties of (4.3) that were highlighted earlier are the same for (4.7). The same situation occurs for the four loop $\widetilde{\rm MOM}$ renormalization group functions which we determined as

$$
\begin{aligned}
\beta^{\text{sQED}}_{\widetilde{\text{MOM}}}(g,\alpha) &= -\frac{N_f}{60}g^3 - \frac{37N_f}{216}g^5 + [648\zeta_3 N_f - 1701N_f - 59600]\frac{N_f g^7}{97200} \\
&\quad + [6480\zeta_3 N_f^2 - 18039N_f^2 + 2610792\zeta_3 N_f - 933120\zeta_5 N_f \\
&\qquad - 3049985N_f + 787320\zeta_3 - 3046265]\frac{N_f g^9}{6998400} + O(g^{11}), \\
\gamma^{\text{sQED}}_{A,\widetilde{\text{MOM}}}(g,\alpha) &= -\frac{N_f}{30}g^2 - \frac{37N_f}{108}g^4 + [648\zeta_3 N_f - 1701N_f - 59600]\frac{N_f g^6}{48600} \\
&\quad + [6480\zeta_3 N_f^2 - 18039N_f^2 + 2610792\zeta_3 N_f - 933120\zeta_5 N_f \\
&\qquad - 3049985N_f + 787320\zeta_3 - 3046265]\frac{N_f g^8}{3499200} + O(g^{10}), \\
\gamma^{\text{sQED}}_{\alpha,\widetilde{\text{MOM}}}(g,\alpha) &= -\gamma^{\text{sQED}}_{A,\widetilde{\text{MOM}}}(g,\alpha), \\
\gamma^{\text{sQED}}_{\psi,\widetilde{\text{MOM}}}(g,\alpha) &= [3\alpha - 10]\frac{g^2}{6} + [-4N_f + 275]\frac{g^4}{216} \\
&\quad + [270\alpha N_f - 49N_f^2 - 648\zeta_3 N_f - 1684N_f + 14580\zeta_3 + 5165]\frac{g^6}{9720} \\
&\quad + [194400\alpha^2\zeta_3 N_f - 97200\alpha^2 N_f - 25920\alpha\zeta_3 N_f^2 + 17280\alpha N_f^2 - 1296000\alpha\zeta_3 N_f \\
&\qquad + 2349000\alpha N_f - 1176N_f^3 + 222912\zeta_3 N_f^2 - 1008229N_f^2 - 160535520\zeta_3 N_f \\
&\qquad + 218116800\zeta_5 N_f - 7940520N_f + 110905200\zeta_3 + 151632000\zeta_5 \\
&\qquad - 216136175]\frac{g^8}{6998400} + O(g^{10}).
\end{aligned}
\tag{4.8}
$$

The emergence of the same $\zeta_n$ and $\alpha$ structures for both theories only reinforces the notion that the kinematics of the vertex subtraction point have an influence on the scheme dependent parts of the renormalization group functions. Although this was already apparent in the MOM schemes of [53,54], the new insight here is that the $\alpha$ dependence of the MOM and $\widetilde{\text{MOM}}$ schemes is dependent on whether there is a nullification of an external vertex leg or not. Finally we note that like $\phi^3$ theory, the Abelian gauge theory $\widetilde{\text{MOM}}$ scheme renormalization group functions are devoid of even zetas to the order we have calculated to in complete agreement with four dimensional studies [27–34]. While there are clear similarities with the zeta structure of four dimensional theories, a formal proof of the absence of even zetas in six dimensions may only be possible if, for instance, the approach using the KLF formalism for massless correlation functions of [57] could be generalized to six dimensions. To the high loop orders discussed here and other places [27–34], this may not be as straightforward as it would seem at first. For instance in the context of the explicit calculations carried out here one would have to establish that there is no primitive graph that arises as a FORCER master at very high loop order whose simple pole residue is an even power of $\pi$. However there is some evidence in four dimensions [58] that there may be one or more such primitive graphs at weight 12 whose residue involves $\zeta_{12}$. If such an integral remained in the determination of the renormalization constant of a massless correlation function then the no-$\pi$ theorem of [30–32] may need to be revisited. Similar reasoning should equally apply to six dimensions.

## V. SIX DIMENSIONAL QCD

Next we turn to the six dimensional version of QCD that has been studied at one and two loops in [14,22,59]. We will use the Lagrangian of [22] which is

$$
\begin{aligned}
L^{(6)} &= -\frac{1}{4}(D_\mu G^a_{\nu\sigma})(D^\mu G^{a\nu\sigma}) + \frac{g_2}{6}f^{abc}G^a_{\mu\nu}G^{b\mu\sigma}G^{c\nu}_{\sigma} \\
&\quad - \frac{1}{2\alpha}(\partial_\mu\partial^\nu A^a_\nu)(\partial^\mu\partial^\sigma A^a_\sigma) - \bar{c}^a\Box(\partial^\mu D_\mu c)^a + i\bar{\psi}^{iI}\not{D}\psi^{iI}
\end{aligned}
\tag{5.1}
$$

where the gauge coupling $g_1$ is embedded in the covariant derivative and field strength $G^a_{\mu\nu}$ and the indices lie in the ranges $1 \le i \le N_f$, $1 \le a \le N_A$ and $1 \le I \le N_F$ with $N_F$ and $N_A$ corresponding to the dimension of the fundamental and adjoint representations of the color group respectively and $N_f$ is the number of quarks. The operator associated with $g_2$ is sometimes referred to as a spectator. In [14,59] the Lagrangian took different forms. This is because there are three possible gauge invariant dimension six operators but only two are independent in the action which can be deduced from the Bianchi identity and integration by parts. The renormalization group functions of one formulation of the Lagrangian can be translated to those of another via [14]

$$(D^\mu G^a_{\mu\sigma})(D_\nu G^{a\nu\sigma}) = \frac{1}{2}(D_\mu G^a_{\nu\sigma})(D^\mu G^{a\nu\sigma}) + g_1 f^{abc} G^a_{\mu\nu} G^{b\mu\sigma} G^{c\nu}{}_\sigma \tag{5.2}$$

which is used to connect the respective coupling constants. The gauge fixing terms have been chosen to ensure the gluon and ghost dimensions match. The one and two loop renormalization of (5.1) was carried out in the $\overline{\text{MS}}$ scheme by a direct computation of Feynman integrals. Subsequently the one loop renormalization for a more general version of (5.1) was performed using the heat kernel expansion in [14]. Included for instance in that generalization were scalar fields as well as others that matched the field content of the supersymmetric extension. Taking the limit of the one loop results of [14] to recover the field content of (5.1) and using the relation between the coupling constants of both formulation that results from (5.2) produced agreement between the $\beta$-function at this one loop order. Given that we now have a six dimensional version of FORCER it is possible to extend results for the renormalization group functions to three loops. This required the renormalization of the gluon, ghost and quark 2-point functions and, to extract the gauge coupling $\beta$-function, the ghost-gluon vertex. Additionally we renormalized the quark mass operator. The number of graphs that were evaluated are recorded in Table I. To initiate the automatic Feynman integration process requires the compilation of the electronic representation of the graphs. This was effected with QGRAF, [60]. At the outset we need to be clear and record the fact that with FORCER it is not possible to deduce the three loop $\beta$-function of the nongauge coupling $g_2$. To do so requires the renormalization of the triple gluon vertex. The structure of the triple gluon vertex Feynman rule will involve both $g_1$ and $g_2$ in contrast to the ghost-gluon vertex that only contains $g_1$. So by determining the renormalization of $g_1$ from the ghost vertex that of $g_2$ can only be deduced from the triple gluon vertex renormalization. However as the cubic term of (5.1) involves the product of $G^a_{\mu\nu}$ nullifying any external leg of the triple gluon vertex Feynman rule, the terms involving $g_2$ vanish identically. Therefore FORCER cannot be employed. Moreover rewriting the cubic operator of (5.1) to redefine it in terms of the other two operators of (5.2) produces the same outcome. Any nullification of an external gluon momentum in the resultant triple gluon Feynman rule excludes access to the nongauge coupling constant. The only route to find the $\beta$-function for $g_2$ is to evaluate the triple gluon vertex at a nonexceptional momentum configuration such as the symmetric point one that was employed in [22]. At three loops this is clearly beyond the scope of the present article.

TABLE I. Number of Feynman diagrams computed for various Green's functions at $L$ loops.

| Green's function | $L=1$ | $L=2$ | $L=3$ | Total |
|---|---|---|---|---|
| $A^a_\mu A^b_\nu$ | 3 | 18 | 267 | 288 |
| $c^a \bar{c}^b$ | 1 | 6 | 78 | 85 |
| $\psi\bar{\psi}$ | 1 | 6 | 78 | 85 |
| $c^a \bar{c}^b A^c_\sigma$ | 2 | 33 | 702 | 737 |
| Total | 7 | 63 | 1125 | 1195 |

Having explained the background to the renormalization of (5.1), we record the $\overline{\text{MS}}$ renormalization group functions. For the gauge coupling $\beta$-function we have

$$\begin{aligned}
\beta_1(g_1,g_2) = & -[249C_A + 16N_fT_F]\frac{g_1^3}{120} \\
& + [-50682C_A^2 g_1^3 + 2439C_A^2 g_1^2 g_2 + 3129C_A^2 g_1 g_2^2 - 315C_A^2 g_2^3 - 1328C_A N_f T_F g_1^3 \\
& - 624C_A N_f T_F g_1^2 g_2 + 96C_A N_f T_F g_1 g_2^2 - 3040C_F N_f T_F g_1^3]\frac{g_1^2}{4320} \\
& + [-7464290865C_A^3 g_1^6 + 1091499579C_A^3 g_1^5 g_2 + 809468904C_A^3 g_1^4 g_2^2 \\
& - 141762510C_A^3 g_1^3 g_2^3 - 15628455C_A^3 g_1^2 g_2^4 + 1812375C_A^3 g_1 g_2^5 \\
& + 94500C_A^3 g_2^6 - 495581080C_A^2 N_f T_F g_1^6 - 66132288C_A^2 N_f T_F g_1^5 g_2 \\
& + 63346632C_A^2 N_f T_F g_1^4 g_2^2 - 1733040C_A^2 N_f T_F g_1^3 g_2^3 - 652320C_A^2 N_f T_F g_1^2 g_2^4 \\
& - 411047360C_A C_F N_f T_F g_1^6 - 37440000C_A C_F N_f T_F g_1^5 g_2 \\
& + 14592000C_A C_F N_f T_F g_1^4 g_2^2 + 987520C_A N_f^2 T_F^2 g_1^6 - 4732416C_A N_f^2 T_F^2 g_1^5 g_2 \\
& + 728064C_A N_f^2 T_F^2 g_1^4 g_2^2 + 17408000C_F^2 N_f T_F g_1^6 \\
& - 9661440C_F N_f^2 T_F^2 g_1^6]\frac{g_1}{62208000} + O(g_i^9).
\end{aligned} \tag{5.3}$$

As all the results in this section will be in the $\overline{\text{MS}}$ scheme, we do not include the scheme label on the renormalization group functions. For the renormalization of the fields, we have the gauge parameter dependent expressions

$$
\begin{aligned}
\gamma_A(g_1,g_2,\alpha) &= [20\alpha C_A - 199C_A - 16N_fT_F]\frac{g_1^2}{60} \\
&+ [130\alpha^2C_A^2g_1^3 + 1095\alpha C_A^2g_1^3 - 81412C_A^2g_1^3 + 2178C_A^2g_1^2g_2 + 5658C_A^2g_1g_2^2 \\
&\quad - 630C_A^2g_2^3 - 1568C_AN_fT_Fg_1^3 - 1248C_AN_fT_Fg_1^2g_2 + 192C_AN_fT_Fg_1g_2^2 \\
&\quad - 6080C_FN_fT_Fg_1^3]\frac{g_1}{4320} \\
&+ [-1215000\zeta_3\alpha^3C_A^3g_1^6 + 1800375\alpha^3C_A^3g_1^6 + 891000\zeta_3\alpha^2C_A^3g_1^6 - 314125\alpha^2C_A^3g_1^6 \\
&\quad - 4617000\zeta_3\alpha C_A^3g_1^6 + 82449225\alpha C_A^3g_1^6 - 1944000\zeta_3\alpha C_A^3g_1^5g_2 \\
&\quad - 7416000\alpha C_A^3g_1^5g_2 - 1842000\alpha C_A^3g_1^4g_2^2 + 5906400\alpha C_A^2N_fT_Fg_1^6 \\
&\quad + 90477000\zeta_3C_A^3g_1^6 - 6130578488C_A^3g_1^6 - 66744000\zeta_3C_A^3g_1^5g_2 \\
&\quad + 829769679C_A^3g_1^5g_2 - 2592000\zeta_3C_A^3g_1^4g_2^2 + 718180404C_A^3g_1^4g_2^2 \\
&\quad + 3888000\zeta_3C_A^3g_1^3g_2^3 - 127466010C_A^3g_1^3g_2^3 - 13886955C_A^3g_1^2g_2^4 \\
&\quad + 1812375C_A^3g_1g_2^5 + 94500C_A^3g_2^6 + 62208000\zeta_3C_A^2N_fT_Fg_1^6 \\
&\quad - 449504664C_A^2N_fT_Fg_1^6 - 61797888C_A^2N_fT_Fg_1^5g_2 + 56482632C_A^2N_fT_Fg_1^4g_2^2 \\
&\quad - 1733040C_A^2N_fT_Fg_1^3g_2^3 - 652320C_A^2N_fT_Fg_1^2g_2^4 - 82944000\zeta_3C_AC_FN_fT_Fg_1^6 \\
&\quad - 307447360C_AC_FN_fT_Fg_1^6 - 37440000C_AC_FN_fT_Fg_1^5g_2 \\
&\quad + 14592000C_AC_FN_fT_Fg_1^4g_2^2 - 734848C_AN_f^2T_F^2g_1^6 - 4732416C_AN_f^2T_F^2g_1^5g_2 \\
&\quad + 728064C_AN_f^2T_F^2g_1^4g_2^2 + 17408000C_F^2N_fT_Fg_1^6 \\
&\quad - 9661440C_FN_f^2T_F^2g_1^6]\frac{1}{31104000} + O(g_i^8), \\
\gamma_c(g_1,g_2,\alpha) &= [\alpha - 5]\frac{C_Ag_1^2}{12} \\
&+ [-55\alpha^2C_Ag_1^2 + 60\alpha C_Ag_1^2 - 19952C_Ag_1^2 + 2700C_Ag_1g_2 + 600C_Ag_2^2 \\
&\quad - 1088N_fT_Fg_1^2]\frac{C_Ag_1^2}{8640} \\
&+ [-2673000\zeta_3\alpha^3C_A^2g_1^4 + 3385125\alpha^3C_A^2g_1^4 - 891000\zeta_3\alpha^2C_A^2g_1^4 \\
&\quad + 867625\alpha^2C_A^2g_1^4 - 2187000\zeta_3\alpha C_A^2g_1^4 + 27779025\alpha C_A^2g_1^4 \\
&\quad - 1944000\zeta_3\alpha C_A^2g_1^3g_2 - 1764000\alpha C_A^2g_1^3g_2 - 336000\alpha C_A^2g_1^2g_2^2 \\
&\quad + 2457600\alpha N_fT_FC_Ag_1^4 - 90477000\zeta_3C_A^2g_1^4 - 1333712377C_A^2g_1^4 \\
&\quad + 66744000\zeta_3C_A^2g_1^3g_2 + 261729900C_A^2g_1^3g_2 + 2592000\zeta_3C_A^2g_1^2g_2^2 \\
&\quad + 91288500C_A^2g_1^2g_2^2 - 3888000\zeta_3C_A^2g_1g_2^3 - 14296500C_A^2g_1g_2^3 \\
&\quad - 1741500C_A^2g_2^4 - 62208000\zeta_3N_fT_FC_Ag_1^4 - 46076416C_AN_fT_Fg_1^4 \\
&\quad - 4334400C_AN_fT_Fg_1^3g_2 + 6864000C_AN_fT_Fg_1^2g_2^2 + 82944000\zeta_3C_FN_fT_Fg_1^4 \\
&\quad - 103600000C_FN_fT_Fg_1^4 + 1722368N_f^2T_F^2g_1^4]\frac{C_Ag_1^2}{62208000} + O(g_i^8), \\
\gamma_\psi(g_1,g_2,\alpha) &= [3\alpha + 5]\frac{C_Fg_1^2}{6} \\
&+ [75\alpha^2C_Ag_1^2 + 1830\alpha C_Ag_1^2 + 43617C_Ag_1^2 - 600C_Ag_2^2 - 8000C_Fg_1^2
\end{aligned}
$$

$$
\begin{aligned}
&+2048N_fT_Fg_1^2]\frac{C_Fg_1^2}{4320}\\
&+[-972000\zeta_3\alpha^3C_A^2g_1^4+1296375\alpha^3C_A^2g_1^4+1215000\zeta_3\alpha^2C_A^2g_1^4\\
&-1231875\alpha^2C_A^2g_1^4+2430000\zeta_3\alpha C_A^2g_1^4+29389350\alpha C_A^2g_1^4\\
&-1944000\zeta_3\alpha C_A^2g_1^3g_2-1971000\alpha C_A^2g_1^3g_2-747000\alpha C_A^2g_1^2g_2^2\\
&+2420400\alpha C_AN_fT_Fg_1^4+74601000\zeta_3C_A^2g_1^4+816942603C_A^2g_1^4\\
&-44712000\zeta_3C_A^2g_1^3g_2-1730025C_A^2g_1^3g_2+7776000\zeta_3C_A^2g_1^2g_2^2\\
&-55844625C_A^2g_1^2g_2^2+4357125C_A^2g_1g_2^3+300375C_A^2g_2^4-62208000\zeta_3C_AC_Fg_1^4\\
&-178896000C_AC_Fg_1^4+3240000C_AC_Fg_1^3g_2+4800000C_AC_Fg_1^2g_2^2\\
&+20736000\zeta_3C_AN_fT_Fg_1^4+41385824C_AN_fT_Fg_1^4+7520400C_AN_fT_Fg_1^3g_2\\
&-2580000C_AN_fT_Fg_1^2g_2^2+62208000\zeta_3C_F^2g_1^4-33056000C_F^2g_1^4\\
&-20736000\zeta_3C_FN_fT_Fg_1^4+23372000C_FN_fT_Fg_1^4\\
&+91648N_f^2T_F^2g_1^4]\frac{C_Fg_1^2}{7776000}+O(g_i^8).
\end{aligned}
\tag{5.4}
$$

Finally we note that the quark mass dimension is given by

$$
\begin{aligned}
\gamma_m(g_1,g_2)=&-\frac{5}{3}C_Fg_1^2+[-11301C_Ag_1^2+300C_Ag_2^2-200C_Fg_1^2-544N_fT_Fg_1^2]\frac{C_Fg_1^2}{1080}\\
&+[38880000\zeta_3C_A^2g_1^4-424927488C_A^2g_1^4-11664000\zeta_3C_A^2g_1^3g_2\\
&+54029025C_A^2g_1^3g_2+24506625C_A^2g_1^2g_2^2-2197125C_A^2g_1g_2^3\\
&-435375C_A^2g_2^4-97200000\zeta_3C_AC_Fg_1^4+30950400C_AC_Fg_1^4\\
&+23328000\zeta_3C_AC_Fg_1^3g_2-22356000C_AC_Fg_1^3g_2+240000C_AC_Fg_1^2g_2^2\\
&-20736000\zeta_3C_AN_fT_Fg_1^4-8608304C_AN_fT_Fg_1^4-4712400C_AN_fT_Fg_1^3g_2\\
&+1716000C_AN_fT_Fg_1^2g_2^2+38880000\zeta_3C_F^2g_1^4-23260000C_F^2g_1^4\\
&+20736000\zeta_3C_FN_fT_Fg_1^4-26194400C_FN_fT_Fg_1^4\\
&+430592N_f^2T_F^2g_1^4]\frac{C_Fg_1^2}{3888000}+O(g_i^8)
\end{aligned}
\tag{5.5}
$$

which, like the $\beta$-function, is independent of $\alpha$ as it ought to be in the $\overline{\text{MS}}$ scheme [61]. While part of the focus in previous sections examined the $\widetilde{\text{MOM}}$ scheme in six dimensions, it is not possible to repeat that for (5.1). This is primarily because interest in the $\widetilde{\text{MOM}}$ scheme concerned the absence of even zetas commencing from four loops. As we do not have a full set of three loop renormalization group functions yet, let alone at four loops, that investigation clearly has to be postponed to a later point. However any study of the $\widetilde{\text{MOM}}$ schemes similar to [27,34] may only be able to focus on the vertices that determine $\beta_1(g_1,g_2)$ since the renormalization of $g_2$ cannot be directly accessed when one of the external momenta of a gluon 3-point vertex is nullified. For similar reasons we have not constructed the equivalent of the mMOM scheme of [35] to four loops. One check on our FORCER computation is that the two loop expressions of [22] have been reproduced and we recall that [22] used the Laporta approach. Another check is that $\beta_1(g_1,g_2)$ satisfies the Slavnov-Taylor identity being given by a linear combination of the Landau gauge gluon and ghost anomalous dimensions. More specifically

$$
\beta_a(g_1,g_2)=\frac{1}{2}g_1[\gamma_A(g_1,g_2,0)+2\gamma_c(g_1,g_2,0)] \tag{5.6}
$$

as the ghost-gluon vertex of (5.1) is finite in the Landau gauge for the same reasons as its four dimensional counterpart, [62].

A separate check on the three loop contributions to the gluon, ghost, quark and quark mass anomalous dimensions lies in comparing with known critical exponents of the fields computed to several orders in the $1/N_f$ expansion. These renormalization group invariants have been computed as a function of $d$ at the Wilson-Fisher fixed point defined as the nontrivial solution of $\beta_i(g_1, g_2) = 0$ closest to the origin. Evaluating the anomalous dimensions at the Wilson-Fisher critical point produces critical exponents which will be functions of $d$ and $N_f$ as well as the group Casimirs. They can be expanded as a double Taylor series in powers of $1/N_f$, where $N_f$ is large, and $\epsilon$ where $d = d_c - 2\epsilon$. Here $d_c$ is the critical dimension of any of the quantum field theories that lie in the same universality class. In this situation these are the two dimensional non-Abelian Thirring model, QCD in four dimensions and (5.1) in six dimensions as well as the tower of theories that lie in eight dimensions and beyond. Therefore the expansion of the large $N_f$ exponents when $d_c = 6$ have to be consistent with the analogous critical anomalous dimensions evaluated at the same fixed point and expanded in the same double Taylor series. This is the background to the three loop large $N_f$ check extending the previous two loop check of [22]. It is in fact possible to carry out such an analysis even though the three loop terms of $\beta_2(g_1, g_2)$ have not been found. The evidence for this is deduced from examining the $g_2$ and $N_f$ dependence of $\gamma_A(g_1, g_2, \alpha)$, $\gamma_c(g_1, g_2, \alpha)$, $\gamma_\psi(g_1, g_2, \alpha)$ and $\gamma_m(g_1, g_2)$. It is apparent that the coupling constant associated with the spectator interaction, which is $g_2$, first appears at two loops. In addition at three loops there are no $g_2^2$ terms in $\gamma_\psi(g_1, g_2, \alpha)$ and $\gamma_m(g_1, g_2)$ meaning that their associated exponents can be expanded to $O(1/N_f^2)$ without the three loop correction to the $g_2$ critical coupling and then compared with the $\epsilon$ expansions of the large $N_f$ exponents of [63]. For $\gamma_A(g_1, g_2, \alpha)$ and $\gamma_c(g_1, g_2, \alpha)$ only the $O(1/N_f)$ $d$-dimensional exponents are known [64]. More concretely, following the large $N_f$ approach of [21,22] we set

$$g_1 = \frac{i}{2}\sqrt{\frac{15\epsilon}{T_F N_f}}x, \qquad g_2 = \frac{i}{2}\sqrt{\frac{15\epsilon}{T_F N_f}}y \tag{5.7}$$

and solve $\beta_i(g_1, g_2) = 0$ to find the Wilson-Fisher fixed point is located at

$$\begin{aligned} x &= 1 + \left[ -\frac{249}{32} C_A + \left[ \frac{475}{48} C_F + \frac{5855}{768} C_A \right] \epsilon - \left[ \frac{3145}{384} C_F + \frac{104729}{18432} C_A \right] \epsilon^2 \right] \frac{1}{T_F N_f} \\ &\quad + \left[ \frac{186003}{2048} C_A^2 - \left[ \frac{197125}{512} C_A C_F + \frac{7530655}{32768} C_A^2 \right] \epsilon \right. \\ &\quad \left. + \left[ \frac{549125}{1536} C_F^2 + \frac{12040925}{18432} C_A C_F + \frac{1150323311}{3145728} C_A^2 \right] \epsilon^2 \right] \frac{1}{T_F^2 N_f^2} + O\left( \frac{\epsilon^3}{T_F^3 N_f^3} \right), \\ y &= \frac{13}{4} + \left[ -\frac{51327}{2048} C_A + \left[ \frac{2325}{64} C_F + \frac{62385}{4096} C_A \right] \epsilon \right] \frac{1}{T_F N_f} + O\left( \frac{\epsilon^2}{T_F^2 N_f^2} \right) \end{aligned} \tag{5.8}$$

in the large $N_f$ expansion where the order symbols indicate the truncation order of both the $\epsilon$ and large $N_f$ expansions. Substituting into the various anomalous dimensions we have

$$\begin{aligned} \gamma_A(g_1^\star, g_2^\star, 0) &= \epsilon - \left[ \frac{25}{8}\epsilon + \frac{85}{24}\epsilon^2 + \frac{841}{288}\epsilon^3 \right] \frac{C_A}{T_F N_f} + O\left( \frac{\epsilon^3}{T_F^2 N_f^2} \right), \\ \gamma_c(g_1^\star, g_2^\star, 0) &= \left[ \frac{25}{16}\epsilon - \frac{85}{48}\epsilon^2 - \frac{841}{576}\epsilon^3 \right] \frac{C_A}{T_F N_f} + O\left( \frac{\epsilon^3}{T_F^2 N_f^2} \right), \\ \gamma_\psi(g_1^\star, g_2^\star, 0) &= \left[ -\frac{25}{8}\epsilon + \frac{20}{3}\epsilon^2 - \frac{179}{288}\epsilon^3 \right] \frac{C_F}{T_F N_f} + \left[ \frac{6225}{128} C_A \epsilon - \left[ \frac{5625}{64} C_F + \frac{102755}{768} C_A \right] \epsilon^2 \right. \\ &\quad \left. + \left[ \left[ \frac{199}{32} - \frac{1125}{8}\zeta_3 \right] C_A + \left[ \frac{60125}{384} + \frac{1125}{8}\zeta_3 \right] C_F \right] \epsilon^3 \right] \frac{C_F}{T_F^2 N_f^2} + O\left( \frac{\epsilon^3}{T_F^3 N_f^3} \right), \\ \gamma_m(g_1^\star, g_2^\star) &= \left[ \frac{25}{4}\epsilon - \frac{85}{12}\epsilon^2 - \frac{841}{144}\epsilon^3 \right] \frac{C_F}{T_F N_f} + \left[ -\frac{6225}{64} C_A \epsilon + \left[ \frac{3875}{32} C_F + \frac{161185}{768} C_A \right] \epsilon^2 \right. \\ &\quad \left. + \left[ \left[ \frac{98741}{1536} + \frac{1125}{4}\zeta_3 \right] C_A - \left[ \frac{5275}{192} + \frac{1125}{4}\zeta_3 \right] C_F \right] \epsilon^3 \right] \frac{C_F}{T_F^2 N_f^2} + O\left( \frac{\epsilon^3}{T_F^3 N_f^3} \right) \end{aligned} \tag{5.9}$$

for the exponents where $g_i^\star$ are the critical couplings evaluated at $x$ and $y$. These exponents are in full agreement with the $\epsilon$ expansion of the large $N_f$ exponents of [63,64] near six dimensions. One caveat to this is that for the first three exponents of (5.9) the check is restricted to the Landau gauge since that is a fixed point of the renormalization group equations.

With the present renormalization group equations we can examine one property which has parallels in the four dimensional counterpart of (5.1). It is known that aside from the Banks-Zaks fixed point of [65,66], treating the gauge parameter as a second coupling constant opens up a richer critical point phase plane [67], which has been studied more recently in four dimensions in [68]. This is the observation that there is a nonzero critical value of the gauge parameter that leads to an infrared stable fixed point in the plane of gauge coupling and parameter. Moreover one can redefine the renormalization group function for $\alpha$ in such a way that this critical value is exposed at leading order [69]. For (5.1) we recall the parallel transformation is derived by first redefining the gauge field by [69]

$$\hat{A}^a_\mu = g_1 A^a_\mu. \tag{5.10}$$

With this the relevant sector of the Lagrangian becomes

$$L^{(6)}_{\text{gluonic}} = -\frac{1}{4g_1^2}\left(\hat{D}_\mu \hat{G}^a_{\nu\sigma}\right)\left(\hat{D}^\mu \hat{G}^{a\nu\sigma}\right) - \frac{1}{2\alpha g_1^2}\left(\partial_\mu\partial^\nu \hat{A}^a_\nu\right)\left(\partial^\mu\partial^\sigma \hat{A}^a_\sigma\right) + \ldots \tag{5.11}$$

where the hatted quantities are the same as the unhatted ones but expressed as a function of $\hat{A}^a_\mu$. This identifies the transverse and longitudinal operators with separate independent coupling constants. If we compute the renormalization group function for the combination $\alpha g_1^2$ given in [69] we find

$$\hat{\gamma}_\alpha(g_1,g_2,\alpha) = \frac{2}{g_1}\beta_1(g_1,g_2) + \gamma_\alpha(g_1,g_2,\alpha) \tag{5.12}$$

which gives

$$\begin{aligned}
\hat{\gamma}_\alpha(g_1,g_2,\alpha) = & -[2\alpha+5]\frac{C_A g_1^2}{6} \\
& + [2700 C_A g_1 g_2 + 600 C_A g_2^2 - 130\alpha^2 C_A g_1^2 - 1095\alpha C_A g_1^2 \\
& - 19952 C_A g_1^2 - 1088 N_f T_F g_1^2]\frac{C_A g_1^2}{4320} \\
& + [1215000\zeta_3\alpha^3 C_A^2 g_1^4 - 1800375\alpha^3 C_A^2 g_1^4 - 891000\zeta_3\alpha^2 C_A^2 g_1^4 + 314125\alpha^2 C_A^2 g_1^4 \\
& + 4617000\zeta_3\alpha C_A^2 g_1^4 - 82449225\alpha C_A^2 g_1^4 + 1944000\zeta_3\alpha C_A^2 g_1^3 g_2 \\
& + 7416000\alpha C_A^2 g_1^3 g_2 + 1842000\alpha C_A^2 g_1^2 g_2^2 - 5906400 N_f T_F \alpha C_A g_1^4 \\
& - 90477000\zeta_3 C_A^2 g_1^4 - 1333712377 C_A^2 g_1^4 + 66744000\zeta_3 C_A^2 g_1^3 g_2 \\
& + 261729900 C_A^2 g_1^3 g_2 + 2592000\zeta_3 C_A^2 g_1^2 g_2^2 + 91288500 C_A^2 g_1^2 g_2^2 \\
& - 3888000\zeta_3 C_A^2 g_1 g_2^3 - 14296500 C_A^2 g_1 g_2^3 - 1741500 C_A^2 g_2^4 \\
& - 62208000\zeta_3 C_A N_f T_F g_1^4 - 46076416 C_A N_f T_F g_1^4 - 4334400 C_A N_f T_F g_1^3 g_2 \\
& + 6864000 C_A N_f T_F g_1^2 g_2^2 + 82944000\zeta_3 C_F N_f T_F g_1^4 - 103600000 C_F N_f T_F g_1^4 \\
& + 1722368 N_f^2 T_F^2 g_1^4]\frac{C_A g_1^2}{31104000} + O(g_i^8).
\end{aligned} \tag{5.13}$$

Therefore in the $(g_1,\alpha)$ plane there is a critical gauge parameter value of $\alpha = -\frac{5}{2}$. In four dimensions the analogous value was $(-3)$ and it was suggested that defining the combination $\alpha g_1^2$ in [67,69] as a second coupling then that coupling in effect was an accounting parameter for the longitudinal modes of the gluon in the perturbative expansion of gauge variant Green's functions. In fact such a pattern of a rational value for a critical gauge parameter extends to the next case which is the eight dimensional extension of (5.1) and was renormalized at one loop in [70].

Examining the (5.12) for eight dimensions gave $(-\frac{7}{3})$ for the critical gauge parameter, $\alpha_\star$. In fact it is straightforward to deduce the $d$ dependence of $\alpha_\star$ and we note

$$\alpha_\star(d) = -\frac{2(d-1)}{(d-2)}. \tag{5.14}$$

This is a monotonically increasing function of $d$ for $d > 2$ with limits

$$\lim_{d\to 2^+}\alpha_\star = -\infty, \qquad \lim_{d\to\infty}\alpha_\star = -2. \tag{5.15}$$

The singular behavior in strictly two dimensions may be indicative of the absence of longitudinal modes in the gluon in that dimension. One puzzle that arose in four dimensions was the relation of the integer value of $\alpha_\star(4)$ to the value for the Yennie gauge. The latter is $\alpha = 3$ and $\alpha_\star(4)$ has been referred to as the anti-Yennie gauge in, for example, [71]. This apparent connection does not translate to the six and higher dimensional generalizations of QCD since the defining criterion for the Yennie gauge is the vanishing of the ghost anomalous dimension at leading order. Clearly in six dimensions this occurs at $\alpha = 5$ while in eight dimensions $\alpha$ would be 7. The generalization is

$$\alpha_{\rm Yennie}(d) = d - 1. \tag{5.16}$$

It is straightforward to check this directly by computing the one loop correction to the ghost 2-point function that eventually produces $\gamma_c(g_i,\alpha)$ in the higher dimensional Lagrangians. The key sector of those is the ghost term given by

$$L^{(d)}_{\rm ghost} = -(\Box^{n_d}\bar{c}^a)(\partial^\mu D_\mu c)^a \tag{5.17}$$

where $n_d = \frac{1}{2}d - 2$ when $d$ is integer. Extracting the gluon propagator from the relevant sector of

$$L^{(d)}_{\rm gluon\,ke} = -\frac{1}{4}(D^{\mu_1}\ldots D^{\mu_{n_d}}G^{a\mu\nu})^2 - \frac{(-1)^{n_d}}{2\alpha}(\partial^\mu A^a_\mu)\Box^{n_d}(\partial^\nu A^a_\nu) \tag{5.18}$$

and using the ghost-gluon interaction from (5.17), the one loop ghost 2-point function can be computed in $d$-dimensions. Expanding it in powers of $\epsilon$ near each even critical dimension strictly above two produces (5.16). We note that for higher dimensions the Lagrangian can always be written in terms of one independent gauge invariant operator with two gluons. While an alternative operator to the first term of (5.18) is $G^a_{\mu\nu}\Box^{n_d}G^{a\mu\nu}$, the latter can be rewritten in terms of it using integration by parts in the Lagrangian plus additional gauge invariant operators with three or more gluon legs.

## VI. DISCUSSION

One of our main tasks was to provide the missing information that prevented the symbolic manipulation FORCER algorithm being used to study properties of quantum field theories in six dimensions. This required mapping the $\epsilon$ expansion of the known four dimensional FORCER masters to six dimensions using the Tarasov method [23,24] with the six dimensional counterparts available now at weight 9. By way of checks we reproduced the available four loop renormalization group functions of $\phi^3$ theory both with and without $O(N)$ symmetry. As a consequence of the masters being available to high order in powers of $\epsilon$ we were able to explore the $\widetilde{\rm MOM}$ scheme property of the cubic theory to five loops and verify that the absence of even zetas to this order is not restricted to four dimensions. As the scalar $\phi^3$ interaction is devoid of derivatives, it was relatively easy to explore a new scheme which was $\overline{\rm MaxS}$ whereby all the terms of the $\epsilon$ expansion of the sum of the Feynman integrals are removed from the two divergent Green's functions. This scheme had the interesting feature that in strictly six dimensions the renormalization group functions were equivalent to those of the $\widetilde{\rm MOM}$ scheme. Such a property should also hold in four dimensional theories including gauge theories. Having studied $\phi^3$ theory and extended the three loop renormalization of six dimensional QED to four loops, we were able to repeat the same exercise for scalar QED in six dimensions. One outcome of the latter was to verify the ultraviolet completion of the four dimensional version of QED to six dimensions. In essence this demonstrates the usefulness of compiling the new masters and opens the way to extend other theories to a similar level. A first step in that direction has also been provided here in that all the renormalization group functions of the six dimensional ultraviolet completion of QCD have been determined to three loops bar one. That outstanding $\beta$-function is not accessible using a 2-point function approach due to the fact that the operator associated with the coupling involves the product of three field strengths. If that could be overcome at three loops then the three loop QCD renormalization group functions computed here could be extended to four loops. There are other six dimensional theories which have been of interest recently whose ultraviolet completeness has been studied at one or two loops. For instance in [55,56] the six dimensional version of the $\mathbb{CP}(N)$ nonlinear $\sigma$ model has been studied to two loops and in the large $N_f$ expansion. For a certain configuration of the coupling constants it contains scalar QED. The $\mathbb{CP}(N)$ $\sigma$ model contains an additional scalar field and in principle it can now be examined using the FORCER construction and the ultraviolet completion studied that will necessarily build on the earlier work of [72] and the large $N$ results of [73] that allow for a connection of theories in $d$-dimensions.

One reason we mention this particular theory is by way of caution. In the course of the application of the new masters to renormalizing a variety of theories they have all carried the same feature with respect to the renormalization of the vertices. This is that when nullifying the momentum of one external leg of the 3-point function no infrared singularities arise. For $\phi^3$ theory the propagators at the nullified vertex will reduce to a contribution of $\frac{1}{(p^2)^2}$ where $p$ is the momentum of a propagator. In six dimensions, unlike four dimensions, this is infrared safe. So we have not needed to introduce any infrared rearrangement. For certain theories in six dimensions this may be necessary and is usually the case for gauge fields since they have a higher derivative kinetic term as is evident in (4.1) and (5.1). For the cases considered here no infrared rearrangement was required. For the ghost-gluon vertex of (5.1) the extra derivatives in the interaction were the redeeming feature. However for the six dimensional version of the $\mathbb{CP}(N)$ nonlinear $\sigma$ model infrared rearrangement may be necessary given the absence of derivatives in the basic interactions involving the gauge field. Some evidence of such difficulties have been implicit in this article. For instance, the mass dimension of the scalar field of (4.6) was not provided despite the fact the electron mass dimension was computed from (4.1). The reason for this has been given previously in [55,73] which is that the scalar electron mass operator mixes with other operators. One is the square of the photon field strength. The remainder include the square of the operator defining the linear covariant gauge condition and those which are total derivative operators. Of the latter the gauge variant ones will not play a role in the extraction of the eigen-anomalous dimensions of the two gauge invariant mass operators. However to extract the renormalization constants for the photon mass operator, inserting it at zero momentum in a photon 2-point will immediately produce an infrared issue which would require either an infrared rearrangement ahead of a FORCER evaluation or determining the mixing matrix by routing the external momentum in one external leg and out through the operator itself. In this latter case the mixing with gauge variant and total derivative operators would have to be included in the construction. While this is beyond the scope of the current article we note it as a reminder and illustration of the potential pitfalls in naively applying Feynman graph integration routines.

The data representing the main results here are accessible from [44].

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## APPENDIX A: SIX DIMENSIONAL FORCER MASTERS

In this appendix we record the $\epsilon$ expansion of the 16 three and four loop FORCER masters in six dimensions. The name of each topology matches exactly with those of [8,9] where the respective Feynman graphs are defined graphically. We have carried out the expansion of each master to the level of $\zeta_9$ which does not equate to the same order in $\epsilon$ for each master. Similar to [9] each is expressed in the $G$-scheme. Using the same master label as that of the `Masters.prc` file of the FORCER release we have

$$\begin{aligned}
\mathbf{haha} = {}& -\frac{1}{72\epsilon^2} + \left[-\frac{17}{108} - \frac{1}{36}\zeta_3 + \frac{5}{36}\zeta_5\right]\frac{1}{\epsilon} - \frac{407}{432} - \frac{1}{24}\zeta_4 - \frac{1}{36}\zeta_3^2 + \frac{25}{72}\zeta_6 + \frac{25}{216}\zeta_3 + \frac{85}{216}\zeta_5 \\
& + \left[-\frac{24671}{7776} - \frac{67}{48}\zeta_7 - \frac{17}{216}\zeta_3^2 - \frac{13}{144}\zeta_5 - \frac{1}{12}\zeta_3\zeta_4 + \frac{25}{144}\zeta_4 + \frac{425}{432}\zeta_6 + \frac{895}{432}\zeta_3\right]\epsilon \\
& + \left[-\frac{58795}{7776}\zeta_5 - \frac{6011}{720}\zeta_8 - \frac{3785}{288}\zeta_7 - \frac{287}{144}\zeta_3^2 - \frac{17}{72}\zeta_3\zeta_4 - \frac{5}{32}\zeta_6 + \frac{9}{10}\zeta_{5,3} + \frac{17}{2}\zeta_3\zeta_5 \right. \\
& \left. + \frac{895}{288}\zeta_4 + \frac{103955}{7776}\zeta_3 + \frac{113711}{46656}\right]\epsilon^2 \\
& + \left[-\frac{4168205}{46656}\zeta_5 - \frac{298475}{15552}\zeta_6 - \frac{178393}{7776}\zeta_3^2 - \frac{41341}{432}\zeta_9 - \frac{26929}{432}\zeta_8 - \frac{6235}{192}\zeta_7 \right. \\
& - \frac{287}{48}\zeta_3\zeta_4 - \frac{127}{54}\zeta_3^3 + \frac{21}{4}\zeta_{5,3} + \frac{385}{18}\zeta_3\zeta_6 + \frac{559}{12}\zeta_3\zeta_5 + \frac{103955}{5184}\zeta_4 \\
& \left. + \frac{2837317}{46656}\zeta_3 + \frac{12816899}{93312} + 6\zeta_4\zeta_5\right]\epsilon^3 + O(\epsilon^4),
\end{aligned}$$

$$\begin{aligned}
\mathbf{no1} = {}& \left[-\frac{1}{72}\zeta_3 + \frac{1}{216}\right]\frac{1}{\epsilon^2} + \left[-\frac{13}{216}\zeta_3 - \frac{1}{48}\zeta_4 + \frac{65}{1296}\right]\frac{1}{\epsilon} - \frac{13}{144}\zeta_4 - \frac{2}{9}\zeta_3 - \frac{1}{36}\zeta_5 + \frac{179}{648} \\
& + \left[-\frac{457}{972}\zeta_3 - \frac{97}{432}\zeta_5 - \frac{5}{12}\zeta_3^2 - \frac{5}{144}\zeta_6 - \frac{1}{3}\zeta_4 + \frac{4109}{5832}\right]\epsilon
\end{aligned}$$

$$+\left[-\frac{113837}{34992}-\frac{457}{648}\zeta_4-\frac{355}{864}\zeta_6-\frac{271}{96}\zeta_5-\frac{239}{144}\zeta_3^2-\frac{5}{4}\zeta_3\zeta_4+\frac{1079}{144}\zeta_7+\frac{32069}{11664}\zeta_3\right]\epsilon^2$$
$$+\left[-\frac{461663}{7776}-\frac{233099}{15552}\zeta_5-\frac{3745}{576}\zeta_6-\frac{1025}{36}\zeta_3\zeta_5-\frac{239}{48}\zeta_3\zeta_4+\frac{9}{10}\zeta_{5,3}+\frac{623}{288}\zeta_3^2+\frac{3583}{216}\zeta_7+\frac{5533}{240}\zeta_8+\frac{32069}{7776}\zeta_4+\frac{1051357}{23328}\zeta_3\right]\epsilon^3$$
$$+\left[-\frac{652648633}{1259712}-\frac{3871021}{93312}\zeta_5-\frac{1128935}{31104}\zeta_6-\frac{50051}{864}\zeta_7-\frac{16367}{1920}\zeta_8-\frac{4975}{72}\zeta_3\zeta_6-\frac{1238}{27}\zeta_3\zeta_5-\frac{1187}{24}\zeta_4\zeta_5-\frac{785}{108}\zeta_3^3+\frac{263}{10}\zeta_{5,3}+\frac{623}{96}\zeta_3\zeta_4+\frac{87667}{288}\zeta_9+\frac{129635}{5184}\zeta_3^2+\frac{1051357}{15552}\zeta_4+\frac{159555301}{419904}\zeta_3\right]\epsilon^4+O(\epsilon^5),$$

$$\mathbf{no2}=\left[-\frac{1}{72}\zeta_3+\frac{1}{216}\right]\frac{1}{\epsilon^2}+\left[-\frac{13}{216}\zeta_3-\frac{1}{48}\zeta_4+\frac{37}{648}\right]\frac{1}{\epsilon}-\frac{41}{144}\zeta_3-\frac{13}{144}\zeta_4-\frac{1}{16}\zeta_5+\frac{505}{1296}$$
$$+\left[-\frac{10433}{7776}\zeta_3-\frac{83}{288}\zeta_5-\frac{41}{96}\zeta_4-\frac{35}{288}\zeta_6-\frac{29}{144}\zeta_3^2+\frac{41141}{23328}\right]\epsilon$$
$$+\left[-\frac{215839}{46656}\zeta_3-\frac{10433}{5184}\zeta_4-\frac{3529}{1728}\zeta_5-\frac{985}{1728}\zeta_6-\frac{29}{48}\zeta_3\zeta_4+\frac{275}{864}\zeta_3^2+\frac{607}{96}\zeta_7+\frac{569563}{139968}\right]\epsilon^2$$
$$+\left[-\frac{1736987}{93312}-\frac{354467}{93312}\zeta_3-\frac{215839}{31104}\zeta_4-\frac{44501}{3456}\zeta_5-\frac{15185}{3456}\zeta_6-\frac{147}{8}\zeta_3\zeta_5+\frac{57}{20}\zeta_{5,3}+\frac{275}{288}\zeta_3\zeta_4+\frac{4873}{192}\zeta_7+\frac{9347}{1728}\zeta_3^2+\frac{29021}{1920}\zeta_8\right]\epsilon^3$$
$$+\left[-\frac{1681647517}{5038848}-\frac{3011489}{62208}\zeta_5-\frac{1793885}{62208}\zeta_6-\frac{354467}{62208}\zeta_4-\frac{6085}{144}\zeta_3\zeta_5-\frac{3235}{72}\zeta_3\zeta_6-\frac{783}{16}\zeta_4\zeta_5+\frac{391}{40}\zeta_{5,3}+\frac{1117}{216}\zeta_3^3+\frac{9347}{576}\zeta_3\zeta_4+\frac{107111}{432}\zeta_9+\frac{252305}{3456}\zeta_7+\frac{260863}{3840}\zeta_8+\frac{1032319}{31104}\zeta_3^2+\frac{174453625}{1679616}\zeta_3\right]\epsilon^4+O(\epsilon^5),$$

$$\mathbf{no6}=\frac{1}{864\epsilon^3}+\left[-\frac{1}{216}\zeta_3+\frac{85}{10368}\right]\frac{1}{\epsilon^2}$$
$$+\left[-\frac{7}{324}\zeta_3-\frac{1}{144}\zeta_4+\frac{1171}{41472}\right]\frac{1}{\epsilon}-\frac{7}{216}\zeta_4-\frac{1}{108}\zeta_5+\frac{43}{7776}\zeta_3+\frac{575}{1492992}$$
$$+\left[-\frac{15492679}{17915904}-\frac{191}{1296}\zeta_5-\frac{5}{36}\zeta_3^2-\frac{5}{432}\zeta_6+\frac{43}{5184}\zeta_4+\frac{45173}{93312}\zeta_3\right]\epsilon$$
$$+\left[-\frac{635174947}{71663616}-\frac{815}{2592}\zeta_6-\frac{265}{432}\zeta_3^2-\frac{5}{12}\zeta_3\zeta_4+\frac{319}{216}\zeta_7+\frac{12191}{7776}\zeta_5+\frac{45173}{62208}\zeta_4+\frac{499027}{124416}\zeta_3\right]\epsilon^2$$
$$+\left[-\frac{172136749511}{2579890176}-\frac{11483}{2592}\zeta_3^2-\frac{355}{54}\zeta_3\zeta_5-\frac{265}{144}\zeta_3\zeta_4+\frac{3}{5}\zeta_{5,3}+\frac{4877}{1440}\zeta_8+\frac{15107}{2592}\zeta_7+\frac{15185}{3888}\zeta_6+\frac{312319}{23328}\zeta_5+\frac{499027}{82944}\zeta_4+\frac{124450471}{4478976}\zeta_3\right]\epsilon^3$$
$$+\left[-\frac{13667311599401}{30958682112}+\frac{29876391323}{161243136}\zeta_3-\frac{89813}{3888}\zeta_3^2-\frac{18185}{648}\zeta_3\zeta_5-\frac{11483}{864}\zeta_3\zeta_4-\frac{517}{36}\zeta_4\zeta_5-\frac{425}{27}\zeta_3\zeta_6+\frac{11}{4}\zeta_{5,3}+\frac{310}{81}\zeta_3^3+\frac{5071}{384}\zeta_8+\frac{11917}{216}\zeta_9+\frac{395303}{6912}\zeta_5\right.$$

$$+\frac{1285387}{15552}\zeta_7+\frac{6020515}{186624}\zeta_6+\frac{124450471}{2985984}\zeta_4\Bigg]\epsilon^4+O(\epsilon^5),$$

$$\textbf{lala}=\frac{7}{103680\epsilon^3}+\frac{61}{77760\epsilon^2}+\frac{32939}{7464960\epsilon}+\frac{1}{1296}\zeta_3+\frac{277411}{17915904}$$
$$+\left[\frac{1}{864}\zeta_4+\frac{781}{155520}\zeta_3+\frac{19619333}{1074954240}\right]\epsilon$$
$$+\left[-\frac{4976176237}{12899450880}-\frac{147}{64}\zeta_7+\frac{781}{103680}\zeta_4+\frac{1999}{1728}\zeta_5+\frac{113243}{93312}\zeta_3\right]\epsilon^2$$
$$+\Bigg[-\frac{958677083731}{154793410560}-\frac{27097}{6480}\zeta_5-\frac{12547}{1620}\zeta_3^2-\frac{6189}{640}\zeta_8-\frac{245}{64}\zeta_7+\frac{27}{40}\zeta_{5,3}+\frac{45}{8}\zeta_3\zeta_5$$
$$+\frac{29965}{10368}\zeta_6+\frac{113243}{62208}\zeta_4+\frac{207922571}{11197440}\zeta_3\Bigg]\epsilon^3$$
$$+\Bigg[-\frac{124644920515261}{1857520926720}+\frac{24089428751}{134369280}\zeta_3-\frac{3018899}{41472}\zeta_5-\frac{835661}{15552}\zeta_3^2-\frac{651109}{62208}\zeta_6$$
$$-\frac{12547}{540}\zeta_3\zeta_4-\frac{4583}{48}\zeta_9-\frac{2063}{128}\zeta_8+\frac{9}{8}\zeta_{5,3}+\frac{27}{8}\zeta_4\zeta_5+\frac{75}{8}\zeta_3\zeta_5+\frac{89}{8}\zeta_3^3$$
$$+\frac{225}{16}\zeta_3\zeta_6+\frac{66137}{960}\zeta_7+\frac{207922571}{7464960}\zeta_4\Bigg]\epsilon^4+O(\epsilon^5),$$

$$\textbf{nono}=\frac{7}{20736\epsilon^3}+\frac{277}{138240\epsilon^2}+\left[\frac{1}{2880}\zeta_3+\frac{88751}{14929920}\right]\frac{1}{\epsilon}$$
$$-\frac{30421}{59719680}+\frac{1}{1920}\zeta_4+\frac{1183}{103680}\zeta_3$$
$$+\left[-\frac{333199777}{2149908480}-\frac{23}{2880}\zeta_5+\frac{1183}{69120}\zeta_4+\frac{5915}{82944}\zeta_3\right]\epsilon$$
$$+\left[-\frac{88173683267}{60197437440}-\frac{1}{48}\zeta_6+\frac{55}{4032}\zeta_3^2+\frac{511}{1280}\zeta_5+\frac{5915}{55296}\zeta_4+\frac{39770197}{104509440}\zeta_3\right]\epsilon^2$$
$$+\Bigg[-\frac{784677709121141}{75848771174400}+\frac{33133204459}{14631321600}\zeta_3-\frac{8088181}{25401600}\zeta_3^2-\frac{1321}{5760}\zeta_7+\frac{55}{1344}\zeta_3\zeta_4$$
$$+\frac{2513}{2592}\zeta_6+\frac{228563}{82944}\zeta_5+\frac{39770197}{69672960}\zeta_4\Bigg]\epsilon^3$$
$$+\Bigg[-\frac{76520147938928867}{1179869773824000}-\frac{7099705903}{3556224000}\zeta_3^2+\frac{33133204459}{9754214400}\zeta_4$$
$$+\frac{268698233469613}{18435465216000}\zeta_3-\frac{8088181}{8467200}\zeta_3\zeta_4-\frac{12819}{12800}\zeta_8+\frac{51}{400}\zeta_{5,3}$$
$$+\frac{2183}{2016}\zeta_3\zeta_5+\frac{46385}{6912}\zeta_6+\frac{23604521}{2419200}\zeta_7+\frac{151421381}{11612160}\zeta_5\Bigg]\epsilon^4$$
$$+\Bigg[-\frac{5101030757005946137661}{1337972323516416000 0}-\frac{38332734677761}{4480842240000}\zeta_3^2-\frac{7099705903}{1185408000}\zeta_3\zeta_4$$
$$+\frac{69388016609}{1016064000}\zeta_7+\frac{873900507443}{14631321600}\zeta_5+\frac{268698233469613}{12290310144000}\zeta_4$$
$$+\frac{80588867392532353}{860321710080000}\zeta_3-\frac{14767793}{1411200}\zeta_3\zeta_5-\frac{75241}{15120}\zeta_9-\frac{7501}{10080}\zeta_3^3$$
$$+\frac{95}{36}\zeta_3\zeta_6+\frac{4489}{6720}\zeta_4\zeta_5+\frac{145967}{56000}\zeta_{5,3}+\frac{23625397}{746496}\zeta_6+\frac{163411417}{6912000}\zeta_8\Bigg]\epsilon^5+O(\epsilon^6),$$

$$\begin{aligned}
\mathbf{cross} = {} & \frac{1}{69120\epsilon^2} + \frac{47}{497664\epsilon} + \frac{9241}{29859840} + \left[ -\frac{1}{288}\zeta_5 + \frac{1}{405}\zeta_3 + \frac{383903}{358318080} \right]\epsilon \\
& + \left[ -\frac{65}{3456}\zeta_5 - \frac{5}{576}\zeta_6 + \frac{1}{270}\zeta_4 + \frac{7}{1440}\zeta_3^2 + \frac{107}{38880}\zeta_3 + \frac{49253441}{4299816960} \right]\epsilon^2 \\
& + \left[ -\frac{36259}{233280}\zeta_3 - \frac{325}{6912}\zeta_6 - \frac{127}{2880}\zeta_7 + \frac{7}{480}\zeta_3\zeta_4 + \frac{91}{3456}\zeta_3^2 + \frac{107}{25920}\zeta_4 + \frac{10889}{207360}\zeta_5 + \frac{160435019}{1146617856} \right]\epsilon^3 \\
& + \left[ \frac{275319668651}{206391214080} - \frac{10696109}{5598720}\zeta_3 - \frac{36259}{155520}\zeta_4 - \frac{1651}{6912}\zeta_7 - \frac{27}{200}\zeta_{5,3} - \frac{11}{720}\zeta_3\zeta_5 \right. \\
& \left. + \frac{91}{1152}\zeta_3\zeta_4 + \frac{1153}{9216}\zeta_6 + \frac{4129}{19200}\zeta_8 + \frac{52129}{207360}\zeta_3^2 + \frac{1273231}{2488320}\zeta_5 \right]\epsilon^4 \\
& + \left[ \frac{78476268381583}{7430083706880} - \frac{68574311}{4478976}\zeta_3 - \frac{10696109}{3732480}\zeta_4 - \frac{871}{2160}\zeta_3^3 - \frac{173}{2160}\zeta_9 \right. \\
& - \frac{143}{1728}\zeta_3\zeta_5 - \frac{117}{160}\zeta_{5,3} - \frac{1}{16}\zeta_3\zeta_6 + \frac{95}{96}\zeta_4\zeta_5 + \frac{52129}{69120}\zeta_3\zeta_4 + \frac{53677}{46080}\zeta_8 \\
& \left. + \frac{1164091}{414720}\zeta_7 + \frac{1266383}{995328}\zeta_6 + \frac{2105237}{829440}\zeta_3^2 + \frac{2121787}{9953280}\zeta_5 \right]\epsilon^5 + O(\epsilon^6),
\end{aligned}$$

$$\begin{aligned}
\mathbf{bebe} = {} & \frac{1}{41472\epsilon^3} + \frac{173}{1382400\epsilon^2} + \left[ \frac{1}{28800}\zeta_3 + \frac{245651}{746496000} \right]\frac{1}{\epsilon} \\
& - \frac{46303}{4976640000} + \frac{1}{19200}\zeta_4 + \frac{2989}{5184000}\zeta_3 \\
& + \left[ -\frac{18489907021}{2687385600000} - \frac{23}{28800}\zeta_5 + \frac{2989}{3456000}\zeta_4 + \frac{13087}{3840000}\zeta_3 \right]\epsilon \\
& + \left[ -\frac{1087166778487}{17915904000000} - \frac{1}{480}\zeta_6 + \frac{29}{28800}\zeta_3^2 + \frac{13087}{2560000}\zeta_4 + \frac{25751}{1728000}\zeta_5 + \frac{318793381}{18662400000}\zeta_3 \right]\epsilon^2 \\
& + \left[ -\frac{3908335481909509}{9674588160000000} + \frac{33530592221}{373248000000}\zeta_3 - \frac{67019}{5184000}\zeta_3^2 - \frac{221}{11520}\zeta_7 + \frac{29}{9600}\zeta_3\zeta_4 \right. \\
& \left. + \frac{9283}{259200}\zeta_6 + \frac{404999}{3840000}\zeta_5 + \frac{318793381}{12441600000}\zeta_4 \right]\epsilon^3 \\
& + \left[ -\frac{155391945049543423}{64497254400000000} + \frac{2964690079}{6220800000}\zeta_5 + \frac{33530592221}{248832000000}\zeta_4 \right. \\
& + \frac{33844757157949}{67184640000000}\zeta_3 - \frac{242977}{3840000}\zeta_3^2 - \frac{67019}{1728000}\zeta_3\zeta_4 - \frac{9403}{128000}\zeta_8 + \frac{27}{4000}\zeta_{5,3} \\
& \left. + \frac{1153}{14400}\zeta_3\zeta_5 + \frac{48989}{192000}\zeta_6 + \frac{252157}{691200}\zeta_7 \right]\epsilon^4 \\
& + \left[ -\frac{472681590682990336861}{348285173760000000000} - \frac{2341117451}{18662400000}\zeta_3^2 + \frac{735800616917}{373248000000}\zeta_5 \right. \\
& + \frac{33844757157949}{44789760000000}\zeta_4 + \frac{3881209963480309}{1343692800000000}\zeta_3 - \frac{304861}{864000}\zeta_3\zeta_5 \\
& - \frac{242977}{1280000}\zeta_3\zeta_4 - \frac{967}{14400}\zeta_3^3 - \frac{643}{1800}\zeta_9 + \frac{281}{1440}\zeta_3\zeta_6 + \frac{667}{9600}\zeta_4\zeta_5 + \frac{7767}{80000}\zeta_{5,3} \\
& \left. + \frac{12090239}{4608000}\zeta_7 + \frac{61097399}{69120000}\zeta_8 + \frac{1071909607}{933120000}\zeta_6 \right]\epsilon^5 + O(\epsilon^6),
\end{aligned}$$

$$\begin{aligned}
\textbf{nostar6} = {} & \frac{1}{20736\epsilon^3} + \frac{11}{248832\epsilon^2} + \left[\frac{1}{576}\zeta_3 + \frac{3377}{2985984}\right]\frac{1}{\epsilon} + \frac{1}{384}\zeta_4 + \frac{95}{20736}\zeta_3 + \frac{924031}{35831808} \\
& + \left[-\frac{9827}{248832}\zeta_3 - \frac{13}{576}\zeta_5 + \frac{95}{13824}\zeta_4 + \frac{40074923}{143327232}\right]\epsilon \\
& + \left[\frac{3824875973}{1719926784} - \frac{1629893}{2985984}\zeta_3 - \frac{9827}{165888}\zeta_4 - \frac{1625}{6912}\zeta_5 - \frac{35}{576}\zeta_6 + \frac{19}{576}\zeta_3^2\right]\epsilon^2 \\
& + \left[\frac{930294184801}{61917364224} - \frac{152188699}{35831808}\zeta_3 - \frac{1629893}{1990656}\zeta_4 - \frac{66521}{27648}\zeta_5 - \frac{12425}{20736}\zeta_6 - \frac{341}{432}\zeta_7\right. \\
& \quad \left. + \frac{19}{192}\zeta_3\zeta_4 + \frac{8417}{20736}\zeta_3^2\right]\epsilon^3 \\
& + \left[-\frac{3901210535}{143327232}\zeta_3 + \frac{68481144449711}{743008370688} - \frac{152188699}{23887872}\zeta_4 - \frac{19005437}{995328}\zeta_5\right. \\
& \quad \left. - \frac{1472155}{248832}\zeta_6 - \frac{257761}{69120}\zeta_8 - \frac{21965}{3456}\zeta_7 + \frac{21}{40}\zeta_{5,3} + \frac{2899}{864}\zeta_3\zeta_5 + \frac{8417}{6912}\zeta_3\zeta_4 + \frac{981187}{248832}\zeta_3^2\right]\epsilon^4 \\
& + \left[-\frac{272730778249}{1719926784}\zeta_3 - \frac{3901210535}{95551488}\zeta_4 + \frac{4741980017952065}{8916100448256} - \frac{1512777571}{11943936}\zeta_5\right. \\
& \quad - \frac{138466045}{2985984}\zeta_6 - \frac{7866851}{276480}\zeta_8 - \frac{753571}{13824}\zeta_7 - \frac{47587}{2592}\zeta_9 - \frac{4307}{2592}\zeta_3^3 + \frac{621}{160}\zeta_{5,3} \\
& \quad \left. + \frac{631}{576}\zeta_4\zeta_5 + \frac{7105}{864}\zeta_3\zeta_6 + \frac{99673}{3456}\zeta_3\zeta_5 + \frac{981187}{82944}\zeta_3\zeta_4 + \frac{87305845}{2985984}\zeta_3^2\right]\epsilon^5 + O(\epsilon^6),
\end{aligned}$$

$$\begin{aligned}
\textbf{nostar5} = {} & -\frac{1}{5184\epsilon^4} - \frac{53}{62208\epsilon^3} - \frac{497}{746496\epsilon^2} + \left[-\frac{59}{5184}\zeta_3 + \frac{169241}{8957952}\right]\frac{1}{\epsilon} \\
& - \frac{3847}{62208}\zeta_3 - \frac{59}{3456}\zeta_4 + \frac{7291429}{35831808} \\
& + \left[-\frac{234397}{746496}\zeta_3 - \frac{3847}{41472}\zeta_4 - \frac{881}{1728}\zeta_5 + \frac{651732331}{429981696}\right]\epsilon \\
& + \left[\frac{151848323951}{15479341056} - \frac{17417963}{8957952}\zeta_3 - \frac{234397}{497664}\zeta_4 - \frac{6217}{2304}\zeta_5 - \frac{1615}{1296}\zeta_6 + \frac{2029}{5184}\zeta_3^2\right]\epsilon^2 \\
& + \left[\frac{10963043240449}{185752092672} - \frac{468045319}{35831808}\zeta_3 - \frac{17417963}{5971968}\zeta_4 - \frac{916561}{82944}\zeta_5 - \frac{205015}{31104}\zeta_6\right. \\
& \quad \left. - \frac{45301}{3456}\zeta_7 + \frac{2029}{1728}\zeta_3\zeta_4 + \frac{124469}{62208}\zeta_3^2\right]\epsilon^3 \\
& + \left[-\frac{36911324041}{429981696}\zeta_3 + \frac{757459450377103}{2229025112064} - \frac{468045319}{23887872}\zeta_4 - \frac{146327317}{2985984}\zeta_5\right. \\
& \quad \left. - \frac{10018315}{373248}\zeta_6 - \frac{2876173}{41472}\zeta_7 - \frac{2334341}{69120}\zeta_8 - \frac{223}{80}\zeta_{5,3} + \frac{13361}{864}\zeta_3\zeta_5 + \frac{124469}{20736}\zeta_3\zeta_4 + \frac{6561335}{746496}\zeta_3^2\right]\epsilon^4 \\
& + \left[-\frac{8331112515413}{15479341056}\zeta_3 - \frac{36911324041}{286654464}\zeta_4 - \frac{3041772121}{11943936}\zeta_5 + \frac{50829818629660001}{26748301344768}\right. \\
& \quad - \frac{526954985}{4478976}\zeta_6 - \frac{144123917}{829440}\zeta_8 - \frac{135196807}{497664}\zeta_7 - \frac{9290029}{31104}\zeta_9 - \frac{48359}{7776}\zeta_3^3 \\
& \quad - \frac{5397}{320}\zeta_{5,3} + \frac{25403}{576}\zeta_4\zeta_5 + \frac{95135}{2592}\zeta_3\zeta_6 + \frac{258347}{3456}\zeta_3\zeta_5 + \frac{6561335}{248832}\zeta_3\zeta_4 \\
& \quad \left. + \frac{424883665}{8957952}\zeta_3^2\right]\epsilon^5 + O(\epsilon^6),
\end{aligned}$$

$$\begin{aligned}
\mathbf{no} = & -\frac{1}{36\epsilon^2} + \left[-\frac{11}{72} + \frac{1}{18}\zeta_3\right]\frac{1}{\epsilon} - \frac{631}{1296} + \frac{1}{6}\zeta_3 + \frac{1}{12}\zeta_4 + \left[-\frac{797}{864} - \frac{4}{9}\zeta_5 + \frac{1}{4}\zeta_4 + \frac{14}{81}\zeta_3\right]\epsilon \\
& + \left[-\frac{215}{324}\zeta_3 - \frac{5}{4}\zeta_6 - \frac{4}{3}\zeta_5 + \frac{7}{9}\zeta_3^2 + \frac{7}{27}\zeta_4 + \frac{31829}{46656}\right]\epsilon^2 \\
& + \left[-\frac{29345}{5832}\zeta_3 - \frac{1709}{162}\zeta_5 - \frac{215}{216}\zeta_4 - \frac{67}{6}\zeta_7 - \frac{15}{4}\zeta_6 + \frac{7}{3}\zeta_3\zeta_4 + \frac{7}{3}\zeta_3^2 + \frac{1674437}{93312}\right]\epsilon^3 \\
& + \left[-\frac{252601}{11664}\zeta_3 - \frac{29345}{3888}\zeta_4 - \frac{20015}{324}\zeta_5 - \frac{17989}{360}\zeta_8 - \frac{965}{36}\zeta_6 - \frac{67}{2}\zeta_7 + \frac{36}{5}\zeta_{5,3} \right. \\
& \left. + \frac{121}{3}\zeta_3\zeta_5 + \frac{2147}{162}\zeta_3^2 + \frac{201342089}{1679616} + 7\zeta_3\zeta_4\right]\epsilon^4 \\
& + \left[-\frac{16004225}{209952}\zeta_3 - \frac{1673735}{5832}\zeta_5 - \frac{252601}{7776}\zeta_4 - \frac{17989}{120}\zeta_8 - \frac{14522}{81}\zeta_9 - \frac{10609}{54}\zeta_7 \right. \\
& - \frac{1375}{9}\zeta_6 - \frac{337}{27}\zeta_3^3 + \frac{13}{2}\zeta_4\zeta_5 + \frac{108}{5}\zeta_{5,3} + \frac{1745}{18}\zeta_3\zeta_6 + \frac{2147}{54}\zeta_3\zeta_4 + \frac{21263}{324}\zeta_3^2 \\
& \left. + \frac{2095164001}{3359232} + 121\zeta_3\zeta_5\right]\epsilon^5 + O(\epsilon^6),
\end{aligned}$$

$$\begin{aligned}
\mathbf{fastar2} = & -\frac{1}{31104\epsilon^4} - \frac{7}{46656\epsilon^3} - \frac{4061}{11197440\epsilon^2} + \left[-\frac{25}{15552}\zeta_3 + \frac{164153}{134369280}\right]\frac{1}{\epsilon} \\
& - \frac{175}{23328}\zeta_3 - \frac{25}{10368}\zeta_4 + \frac{498311}{19906560} \\
& + \left[-\frac{264677}{5598720}\zeta_3 - \frac{175}{15552}\zeta_4 - \frac{155}{1728}\zeta_5 + \frac{1468119763}{6449725440}\right]\epsilon \\
& + \left[\frac{382027011479}{232190115840} - \frac{24144799}{67184640}\zeta_3 - \frac{264677}{3732480}\zeta_4 - \frac{3425}{15552}\zeta_6 - \frac{1085}{2592}\zeta_5 + \frac{1247}{15552}\zeta_3^2\right]\epsilon^2 \\
& + \left[\frac{29537619574489}{2786281390080} - \frac{24144799}{44789760}\zeta_4 - \frac{8528651}{3317760}\zeta_3 - \frac{399797}{207360}\zeta_5 - \frac{23975}{23328}\zeta_6 \right. \\
& \left. - \frac{12503}{5184}\zeta_7 + \frac{1247}{5184}\zeta_3\zeta_4 + \frac{8729}{23328}\zeta_3^2\right]\epsilon^3 \\
& + \left[-\frac{54343033589}{3224862720}\zeta_3 + \frac{2131076904260791}{33435376680960} - \frac{8767813}{829440}\zeta_5 - \frac{8528651}{2211840}\zeta_4 - \frac{5264921}{1119744}\zeta_6 \right. \\
& \left. - \frac{1397189}{207360}\zeta_8 - \frac{87521}{7776}\zeta_7 - \frac{3}{10}\zeta_{5,3} + \frac{1087}{288}\zeta_3\zeta_5 + \frac{8729}{7776}\zeta_3\zeta_4 + \frac{10919251}{5598720}\zeta_3^2\right]\epsilon^4 \\
& + \left[-\frac{12001629023377}{116095057920}\zeta_3 - \frac{54343033589}{2149908480}\zeta_4 + \frac{146881530580013753}{401224520171520} - \frac{343024027}{13436928}\zeta_6 \right. \\
& - \frac{91051111}{1866240}\zeta_7 - \frac{22782833}{368640}\zeta_5 - \frac{9780323}{311040}\zeta_8 - \frac{2619709}{46656}\zeta_9 - \frac{37081}{23328}\zeta_3^3 - \frac{7}{5}\zeta_{5,3} \\
& \left. + \frac{1519}{192}\zeta_4\zeta_5 + \frac{7609}{432}\zeta_3\zeta_5 + \frac{70255}{7776}\zeta_3\zeta_6 + \frac{10919251}{1866240}\zeta_3\zeta_4 + \frac{819297737}{67184640}\zeta_3^2\right]\epsilon^5 + O(\epsilon^6),
\end{aligned}$$

$$\begin{aligned}
\mathbf{t155} = & -\frac{1}{15552\epsilon^4} - \frac{23}{93312\epsilon^3} - \frac{679}{5598720\epsilon^2} + \left[-\frac{31}{7776}\zeta_3 + \frac{123281}{22394880}\right]\frac{1}{\epsilon} \\
& - \frac{713}{46656}\zeta_3 - \frac{31}{5184}\zeta_4 + \frac{15165583}{268738560}
\end{aligned}$$

$$+\left[-\frac{213343}{2799360}\zeta_3-\frac{713}{31104}\zeta_4-\frac{449}{2592}\zeta_5+\frac{88785839}{214990848}\right]\epsilon$$
$$+\left[\frac{307092224429}{116095057920}-\frac{5868103}{11197440}\zeta_3-\frac{213343}{1866240}\zeta_4-\frac{10327}{15552}\zeta_5-\frac{1645}{3888}\zeta_6+\frac{983}{7776}\zeta_3^2\right]\epsilon^2$$
$$+\left[\frac{21968267826851}{1393140695040}-\frac{482659289}{134369280}\zeta_3-\frac{5868103}{7464960}\zeta_4-\frac{2356007}{933120}\zeta_5-\frac{37835}{23328}\zeta_6\right.$$
$$\left.-\frac{5669}{1296}\zeta_7+\frac{983}{2592}\zeta_3\zeta_4+\frac{22609}{46656}\zeta_3^2\right]\epsilon^3$$
$$+\left[-\frac{2458665145}{107495424}\zeta_3+\frac{300549734350057}{3343537668096}-\frac{482659289}{89579520}\zeta_4-\frac{48004847}{3732480}\zeta_5\right.$$
$$\left.-\frac{3427339}{559872}\zeta_6-\frac{242437}{20736}\zeta_8-\frac{130387}{7776}\zeta_7-\frac{3}{4}\zeta_{5,3}+\frac{6817}{1296}\zeta_3\zeta_5+\frac{22609}{15552}\zeta_3\zeta_4+\frac{6051689}{2799360}\zeta_3^2\right]\epsilon^4$$
$$+\left[-\frac{7956272660107}{58047528960}\zeta_3-\frac{3353771761}{44789760}\zeta_5-\frac{2458665145}{71663616}\zeta_4+\frac{3693436841274617}{7430083706880}\right.$$
$$-\frac{69073219}{2239488}\zeta_6-\frac{55868149}{933120}\zeta_7-\frac{5576051}{124416}\zeta_8-\frac{2293555}{23328}\zeta_9-\frac{23959}{11664}\zeta_3^3-\frac{23}{8}\zeta_{5,3}$$
$$\left.+\frac{11677}{864}\zeta_4\zeta_5+\frac{24335}{1944}\zeta_3\zeta_6+\frac{156791}{7776}\zeta_3\zeta_5+\frac{6051689}{933120}\zeta_3\zeta_4+\frac{150129569}{11197440}\zeta_3^2\right]\epsilon^5+O(\epsilon^6),$$

$$\mathbf{t124}=-\frac{1}{51840\epsilon^3}-\frac{31}{155520\epsilon^2}-\frac{2339}{1866240\epsilon}-\frac{3667}{559872}+\frac{1}{1296}\zeta_3$$
$$+\left[-\frac{2148781}{67184640}+\frac{1}{864}\zeta_4+\frac{199}{31104}\zeta_3\right]\epsilon$$
$$+\left[-\frac{1289797}{8398080}+\frac{37}{2592}\zeta_5+\frac{199}{20736}\zeta_4+\frac{70097}{1866240}\zeta_3\right]\epsilon^2$$
$$+\left[-\frac{447731989}{604661760}-\frac{17}{1080}\zeta_3^2+\frac{175}{5184}\zeta_6+\frac{215}{1944}\zeta_5+\frac{70097}{1244160}\zeta_4+\frac{4467643}{22394880}\zeta_3\right]\epsilon^3$$
$$+\left[-\frac{104687174539}{29023764480}-\frac{3131}{25920}\zeta_3^2-\frac{17}{360}\zeta_3\zeta_4+\frac{13157}{51840}\zeta_7+\frac{16205}{62208}\zeta_6+\frac{146803}{233280}\zeta_5\right.$$
$$\left.+\frac{4467643}{14929920}\zeta_4+\frac{92262887}{89579520}\zeta_3\right]\epsilon^4$$
$$+\left[-\frac{1236480137413}{69657034752}-\frac{129191}{186624}\zeta_3^2-\frac{3131}{8640}\zeta_3\zeta_4-\frac{1829}{3240}\zeta_3\zeta_5+\frac{3}{400}\zeta_{5,3}+\frac{40901}{27648}\zeta_6\right.$$
$$\left.+\frac{123043}{172800}\zeta_8+\frac{1191553}{622080}\zeta_7+\frac{18642809}{5598720}\zeta_5+\frac{23369059}{4423680}\zeta_3+\frac{92262887}{59719680}\zeta_4\right]\epsilon^5$$
$$+\left[-\frac{368047815281083}{4179422085120}+\frac{1047221158789}{38698352640}\zeta_3-\frac{13947499}{3732480}\zeta_3^2-\frac{129191}{62208}\zeta_3\zeta_4\right.$$
$$-\frac{82493}{19440}\zeta_3\zeta_5-\frac{3901}{4320}\zeta_4\zeta_5-\frac{1727}{1296}\zeta_3\zeta_6+\frac{89}{1600}\zeta_{5,3}+\frac{2161}{9720}\zeta_3^3+\frac{735227}{155520}\zeta_9$$
$$\left.+\frac{11097827}{2073600}\zeta_8+\frac{23369059}{2949120}\zeta_4+\frac{70103593}{8957952}\zeta_6+\frac{80333051}{7464960}\zeta_7+\frac{1165361683}{67184640}\zeta_5\right]\epsilon^6+O(\epsilon^7),$$

$$
\begin{aligned}
\mathbf{t145} = & -\frac{5}{124416\epsilon^3} - \frac{1873}{7464960\epsilon^2} - \frac{80371}{89579520\epsilon} - \frac{566551}{358318080} - \frac{19}{31104}\zeta_3 \\
& + \left[-\frac{7463}{1866240}\zeta_3 - \frac{19}{20736}\zeta_4 + \frac{30715831}{4299816960}\right]\epsilon \\
& + \left[-\frac{600293}{22394880}\zeta_3 - \frac{7463}{1244160}\zeta_4 - \frac{341}{10368}\zeta_5 + \frac{358030103}{3439853568}\right]\epsilon^2 \\
& + \left[\frac{1513492357733}{1857520926720} - \frac{17437313}{89579520}\zeta_3 - \frac{600293}{14929920}\zeta_4 - \frac{130957}{622080}\zeta_5 - \frac{1255}{15552}\zeta_6 + \frac{493}{15552}\zeta_3^2\right]\epsilon^3 \\
& + \left[\frac{118328890166123}{22290251120640} - \frac{1445143903}{1074954240}\zeta_3 - \frac{17437313}{59719680}\zeta_4 - \frac{8077207}{7464960}\zeta_5 - \frac{16619}{20736}\zeta_7 \right. \\
& \quad \left. - \frac{3011}{5832}\zeta_6 + \frac{493}{5184}\zeta_3\zeta_4 + \frac{189527}{933120}\zeta_3^2\right]\epsilon^4 \\
& + \left[-\frac{36938283067}{4299816960}\zeta_3 + \frac{1692801725704513}{53496602689536} - \frac{1445143903}{716636160}\zeta_4 - \frac{172335787}{29859840}\zeta_5 \right. \\
& \quad \left. - \frac{3182639}{622080}\zeta_7 - \frac{738479}{279936}\zeta_6 - \frac{91817}{41472}\zeta_8 - \frac{3}{32}\zeta_{5,3} + \frac{3617}{2592}\zeta_3\zeta_5 + \frac{189527}{311040}\zeta_3\zeta_4 + \frac{12287453}{11197440}\zeta_3^2\right]\epsilon^5 \\
& + \left[-\frac{23912983818461}{464380231680}\zeta_3 - \frac{36938283067}{2866544640}\zeta_4 - \frac{11522465077}{358318080}\zeta_5 \right. \\
& \quad + \frac{63870314319138931}{356644017930240} - \frac{188849819}{7464960}\zeta_7 - \frac{8795191}{622080}\zeta_8 - \frac{3902891}{279936}\zeta_6 \\
& \quad - \frac{3258785}{186624}\zeta_9 - \frac{6583}{11664}\zeta_3^3 - \frac{19}{32}\zeta_{5,3} + \frac{151}{54}\zeta_4\zeta_5 + \frac{25895}{7776}\zeta_3\zeta_6 + \frac{1386853}{155520}\zeta_3\zeta_5 \\
& \quad \left. + \frac{12287453}{3732480}\zeta_3\zeta_4 + \frac{283280153}{44789760}\zeta_3^2\right]\epsilon^6 + O(\epsilon^7), \\
\mathbf{t105} = & -\frac{1}{648\epsilon^3} - \frac{17}{3888\epsilon^2} - \frac{121}{23328\epsilon} - \frac{7}{324}\zeta_3 + \frac{3025}{139968} + \left[-\frac{119}{1944}\zeta_3 - \frac{7}{216}\zeta_4 + \frac{186539}{839808}\right]\epsilon \\
& + \left[-\frac{2935}{11664}\zeta_3 - \frac{119}{1296}\zeta_4 - \frac{7}{12}\zeta_5 + \frac{6682285}{5038848}\right]\epsilon^2 \\
& + \left[-\frac{97013}{69984}\zeta_3 - \frac{2935}{7776}\zeta_4 - \frac{455}{324}\zeta_6 - \frac{119}{72}\zeta_5 + \frac{113}{324}\zeta_3^2 + \frac{203468495}{30233088}\right]\epsilon^3 \\
& + \left[\frac{5727067633}{181398528} - \frac{3107395}{419904}\zeta_3 - \frac{97013}{46656}\zeta_4 - \frac{7735}{1944}\zeta_6 - \frac{6485}{1296}\zeta_5 - \frac{245}{27}\zeta_7 + \frac{113}{108}\zeta_3\zeta_4 + \frac{1921}{1944}\zeta_3^2\right]\epsilon^4 \\
& + \left[\frac{153930181235}{1088391168} - \frac{92052137}{2519424}\zeta_3 - \frac{3107395}{279936}\zeta_4 - \frac{159719}{7776}\zeta_5 - \frac{138575}{11664}\zeta_6 - \frac{99647}{4320}\zeta_8 \right. \\
& \quad \left. - \frac{4165}{162}\zeta_7 - \frac{6}{5}\zeta_{5,3} + \frac{53}{6}\zeta_3\zeta_5 + \frac{1921}{648}\zeta_3\zeta_4 + \frac{40817}{11664}\zeta_3^2\right]\epsilon^5 \\
& + \left[-\frac{2571366511}{15116544}\zeta_3 + \frac{4014789658453}{6530347008} - \frac{92052137}{1679616}\zeta_4 - \frac{4418665}{46656}\zeta_5 - \frac{3351145}{69984}\zeta_6 \right. \\
& \quad - \frac{1693999}{25920}\zeta_8 - \frac{123547}{972}\zeta_9 - \frac{35311}{486}\zeta_7 - \frac{1315}{486}\zeta_3^3 - \frac{17}{5}\zeta_{5,3} + \frac{89}{4}\zeta_4\zeta_5 + \frac{901}{36}\zeta_3\zeta_5 \\
& \quad \left. + \frac{3295}{162}\zeta_3\zeta_6 + \frac{40817}{3888}\zeta_3\zeta_4 + \frac{1194619}{69984}\zeta_3^2\right]\epsilon^6 + O(\epsilon^7). \qquad \text{(A1)}
\end{aligned}
$$

Here $\zeta_{5,3}$ denotes the multiple zeta that was discovered in [74]. In [17] the $\epsilon$ expansion for the set of masters that was required to renormalize $\phi^3$ to four loops in six dimensions was provided but in the same master basis as [37]. We have checked that those masters derived from FORCER in this paper are in agreement up to the order in $\epsilon$ given in [17]. To assist with this comparison we note the mapping of the overlapping masters in each basis is **haha** $\leftrightarrow M_{61}$, **no1** $\leftrightarrow M_{63}$, **no2** $\leftrightarrow M_{62}$, **no6** $\leftrightarrow M_{51}$, **cross** $\leftrightarrow M_{36}$, **nono** $\leftrightarrow M_{45}$ and **bebe** $\leftrightarrow M_{35}$.

## APPENDIX B: $O(N)$ $\phi^3$ $\beta$-FUNCTIONS

In this appendix we record the two $\beta$-functions in $O(N)$ $\phi^3$ theory for all $N$ in the $\widetilde{\text{MOM}}$ scheme. We have

$$
\begin{aligned}
\beta^{O(N)}_{1\,\widetilde{\text{MOM}}}(g_1,g_2) = {}& \left[-\frac{1}{24}g_1g_2^2-\frac{1}{24}Ng_1^3+\frac{1}{2}g_1^2g_2+\frac{1}{3}g_1^3\right] \\
&+\left[-\frac{163}{216}g_1^3g_2^2-\frac{67}{108}g_1^5-\frac{43}{432}Ng_1^5-\frac{13}{36}g_1^4g_2-\frac{11}{864}Ng_1^3g_2^2-\frac{1}{12}g_1^2g_2^3\right. \\
&\left.+\frac{11}{72}Ng_1^4g_2+\frac{37}{864}g_1g_2^4\right] \\
&+\left[-\frac{13217}{31104}Ng_1^5g_2^2-\frac{4747}{31104}N^2g_1^7-\frac{4435}{62208}g_1g_2^6-\frac{233}{1296}Ng_1^4g_2^3-\frac{91}{20736}N^2g_1^5g_2^2\right. \\
&-\frac{3}{2}\zeta_3Ng_1^6g_2-\frac{1}{2}\zeta_3g_1^7-\frac{1}{6}\zeta_3g_1^3g_2^4-\frac{1}{6}\zeta_3g_1^5g_2^2-\frac{1}{48}\zeta_3g_1g_2^6+\frac{1}{2}\zeta_3g_1^4g_2^3 \\
&+\frac{9}{8}\zeta_3Ng_1^7+\frac{11}{48}\zeta_3Ng_1^5g_2^2+\frac{101}{1728}N^2g_1^6g_2+\frac{122}{81}g_1^6g_2+\frac{169}{7776}Ng_1^3g_2^4 \\
&+\frac{251}{15552}Ng_1^7+\frac{257}{162}Ng_1^6g_2+\frac{559}{1728}g_1^2g_2^5+\frac{749}{864}g_1^3g_2^4+\frac{3121}{2592}g_1^4g_2^3 \\
&\left.+\frac{10657}{3888}g_1^7+\frac{10985}{3888}g_1^5g_2^2\right] \\
&+\left[-\frac{4216621}{559872}g_1^5g_2^4-\frac{2553617}{559872}Ng_1^7g_2^2-\frac{1951993}{93312}g_1^6g_2^3-\frac{1717723}{139968}g_1^7g_2^2\right. \\
&-\frac{1279273}{279936}g_1^3g_2^6-\frac{907375}{186624}Ng_1^6g_2^3-\frac{676499}{46656}Ng_1^8g_2-\frac{383461}{1119744}N^2g_1^7g_2^2 \\
&-\frac{229721}{15552}g_1^8g_2-\frac{111733}{69984}N^2g_1^9-\frac{43283}{2592}g_1^9-\frac{38011}{31104}g_1^2g_2^7 \\
&-\frac{26969}{559872}Ng_1^3g_2^6-\frac{20323}{1296}\zeta_3Ng_1^9-\frac{9953}{1296}\zeta_3g_1^5g_2^4-\frac{6761}{93312}N^2g_1^6g_2^3 \\
&-\frac{5189}{93312}N^3g_1^9-\frac{4045}{432}\zeta_3Ng_1^8g_2-\frac{1295}{108}\zeta_3g_1^6g_2^3-\frac{749}{432}\zeta_3g_1^3g_2^6 \\
&-\frac{677}{373248}N^3g_1^7g_2^2-\frac{535}{432}\zeta_3g_1^4g_2^5-\frac{341}{72}\zeta_3g_1^8g_2-\frac{209}{72}\zeta_3Ng_1^6g_2^3 \\
&-\frac{155}{18}\zeta_5Ng_1^7g_2^2-\frac{119}{1728}\zeta_3Ng_1^4g_2^5-\frac{115}{8}\zeta_3g_1^7g_2^2-\frac{25}{18}\zeta_5Ng_1^5g_2^4-\frac{25}{36}\zeta_5N^2g_1^9 \\
&-\frac{23}{648}\zeta_3Ng_1^3g_2^6-\frac{14}{27}\zeta_3N^2g_1^8g_2-\frac{5}{2}\zeta_5g_1^4g_2^5-\frac{1}{48}\zeta_3N^2g_1^6g_2^3-\frac{1}{1728}\zeta_3N^3g_1^8g_2 \\
&+\frac{1}{48}\zeta_3g_1^2g_2^7+\frac{1}{5184}\zeta_3N^3g_1^7g_2^2+\frac{5}{36}\zeta_5g_1g_2^8+\frac{5}{576}\zeta_3N^3g_1^9+\frac{7}{1296}\zeta_3N^2g_1^5g_2^4 \\
&+\frac{25}{9}\zeta_5g_1^9+\frac{35}{3}\zeta_5Ng_1^9+\frac{40}{3}\zeta_5g_1^7g_2^2+\frac{47}{5184}\zeta_3N^2g_1^7g_2^2+\frac{65}{3}\zeta_5Ng_1^8g_2 \\
&+\frac{65}{6}\zeta_5g_1^8g_2+\frac{70}{3}\zeta_5g_1^6g_2^3+\frac{85}{18}\zeta_5g_1^3g_2^6+\frac{131}{62208}g_1^4g_2^5+\frac{145}{18}\zeta_5g_1^5g_2^4 \\
&+\frac{283}{2592}\zeta_3g_1g_2^8+\frac{631}{216}\zeta_3g_1^9+\frac{889}{144}\zeta_3Ng_1^7g_2^2+\frac{967}{93312}N^2g_1^5g_2^4+\frac{1213}{62208}N^3g_1^8g_2 \\
&+\frac{3349}{2592}\zeta_3N^2g_1^9+\frac{5423}{5184}\zeta_3Ng_1^5g_2^4+\frac{56263}{23328}N^2g_1^8g_2+\frac{99455}{69984}Ng_1^9
\end{aligned}
$$

$$+\frac{100885}{186624}Ng_1^4g_2^5+\frac{135149}{1119744}g_1g_2^8+\frac{818105}{1119744}Ng_1^5g_2^4+10\zeta_5Ng_1^6g_2^3\Big]$$
$$+\Big[\frac{4787589055}{20155392}Ng_1^9g_2^2-\frac{44097997}{4478976}N^2g_1^8g_2^3-\frac{43577579}{13436928}g_1^4g_2^7$$
$$-\frac{23131849}{8957952}N^3g_1^{11}-\frac{19776671}{6718464}Ng_1^{11}-\frac{18870089}{161243136}Ng_1^3g_2^8$$
$$-\frac{17762039}{3359232}N^2g_1^{10}g_2-\frac{9924857}{2519424}Ng_1^5g_2^6-\frac{6134743}{2239488}Ng_1^4g_2^7$$
$$-\frac{1465675}{124416}\zeta_3Ng_1^5g_2^6-\frac{818735}{2592}\zeta_5Ng_1^9g_2^2-\frac{626087}{31104}\zeta_3g_1^4g_2^7$$
$$-\frac{475265}{10368}\zeta_5N^2g_1^9g_2^2-\frac{431969}{1119744}N^2g_1^7g_2^4-\frac{362675}{20736}\zeta_5Ng_1^7g_2^4$$
$$-\frac{310787}{3359232}N^3g_1^9g_2^2-\frac{254395}{1728}\zeta_5g_1^7g_2^4-\frac{251831}{6718464}N^4g_1^{11}-\frac{151631}{15552}\zeta_3N^2g_1^{11}$$
$$-\frac{119303}{373248}\zeta_3g_1g_2^{10}-\frac{46865}{384}\zeta_7Ng_1^9g_2^2-\frac{42193}{746496}N^3g_1^8g_2^3-\frac{39625}{3456}\zeta_3N^2g_1^8g_2^3$$
$$-\frac{37885}{1296}\zeta_5Ng_1^{11}-\frac{34315}{864}\zeta_5Ng_1^6g_2^5-\frac{30863}{96}\zeta_7g_1^9g_2^2-\frac{27905}{864}\zeta_3g_1^{10}g_2$$
$$-\frac{27461}{32}\zeta_7g_1^8g_2^3-\frac{23915}{26873856}N^4g_1^9g_2^2-\frac{23303}{96}\zeta_7g_1^{10}g_2-\frac{22055}{432}\zeta_5Ng_1^{10}g_2$$
$$-\frac{19439}{48}\zeta_7Ng_1^{10}g_2-\frac{17101}{96}\zeta_7g_1^6g_2^5-\frac{14935}{1296}\zeta_5g_1^9g_2^2-\frac{13975}{20736}\zeta_5g_1g_2^{10}$$
$$-\frac{9605}{2592}\zeta_5N^2g_1^7g_2^4-\frac{8395}{576}\zeta_5N^2g_1^{11}-\frac{7259}{32}\zeta_7Ng_1^{11}-\frac{7231}{32}\zeta_7g_1^{11}$$
$$-\frac{5593}{384}\zeta_7Ng_1^7g_2^4-\frac{5453}{32}\zeta_7g_1^5g_2^6-\frac{5095}{15552}\zeta_3N^3g_1^{10}g_2-\frac{4137}{16}\zeta_7Ng_1^8g_2^3$$
$$-\frac{3045}{16}\zeta_7g_1^7g_2^4-\frac{1535}{144}\zeta_5g_1^3g_2^8-\frac{1361}{144}\zeta_3^2Ng_1^7g_2^4-\frac{1323}{16}\zeta_7Ng_1^6g_2^5$$
$$-\frac{995}{15552}\zeta_3N^2g_1^5g_2^6-\frac{575}{20736}\zeta_5N^2g_1^5g_2^6-\frac{441}{8}\zeta_7N^2g_1^{10}g_2-\frac{395}{576}\zeta_5Ng_1^4g_2^7$$
$$-\frac{283}{10368}\zeta_3N^3g_1^8g_2^3-\frac{245}{36}\zeta_3^2N^2g_1^{11}-\frac{215}{144}\zeta_5g_1^2g_2^9-\frac{182}{3}\zeta_7g_1^3g_2^8$$
$$-\frac{147}{128}\zeta_7g_1g_2^{10}-\frac{109}{12}\zeta_3^2g_1^{10}g_2-\frac{83}{48}\zeta_3^2Ng_1^5g_2^6-\frac{55}{64}\zeta_5N^3g_1^{10}g_2$$
$$-\frac{25}{31104}\zeta_3N^4g_1^{10}g_2-\frac{3}{2}\zeta_3^2N^2g_1^8g_2^3-\frac{1}{48}\zeta_3^2Ng_1^3g_2^8+\frac{1}{12}\zeta_3^2Ng_1^4g_2^7$$
$$+\frac{1}{24}\zeta_3^2g_1g_2^{10}+\frac{7}{144}\zeta_3^2N^3g_1^{11}+\frac{11}{12}\zeta_3^2N^2g_1^{10}g_2+\frac{13}{24}\zeta_3^2g_1^4g_2^7+\frac{13}{72}\zeta_3^2N^2g_1^7g_2^4$$
$$+\frac{41}{4}\zeta_3^2Ng_1^6g_2^5+\frac{47}{12}\zeta_3^2g_1^6g_2^5+\frac{55}{373248}\zeta_3N^4g_1^9g_2^2+\frac{71}{8}\zeta_3^2g_1^5g_2^6+\frac{145}{36}\zeta_3^2g_1^3g_2^8$$
$$+\frac{151}{41472}\zeta_3N^4g_1^{11}+\frac{185}{72}\zeta_3^2N^2g_1^9g_2^2+\frac{245}{24}\zeta_3^2g_1^9g_2^2+\frac{295}{256}\zeta_5N^3g_1^{11}+\frac{343}{18}\zeta_3^2g_1^{11}$$
$$+\frac{385}{7776}\zeta_3N^2g_1^6g_2^5+\frac{435}{64}\zeta_5Ng_1^5g_2^6+\frac{481}{221184}N^2g_1^5g_2^6+\frac{485}{12}\zeta_3^2g_1^7g_2^4$$
$$+\frac{515}{5184}\zeta_5N^3g_1^9g_2^2+\frac{575}{12}\zeta_3^2Ng_1^{10}g_2+\frac{635}{1728}\zeta_5N^2g_1^6g_2^5+\frac{683}{186624}\zeta_3N^3g_1^7g_2^4$$

$$
\begin{aligned}
&+\frac{1093}{41472}\zeta_3N^3g_1^9g_2^2+\frac{1247}{24}\zeta_3^2g_1^8g_2^3+\frac{1421}{128}\zeta_7Ng_1^5g_2^6+\frac{1477}{96}\zeta_7g_1^4g_2^7\\
&+\frac{1505}{5184}\zeta_5Ng_1^3g_2^8+\frac{1715}{128}\zeta_7N^2g_1^{11}+\frac{2009}{128}\zeta_7N^2g_1^9g_2^2+\frac{3143}{144}\zeta_3^2Ng_1^{11}\\
&+\frac{3671}{10368}\zeta_3Ng_1^4g_2^7+\frac{3709}{144}\zeta_3^2Ng_1^9g_2^2+\frac{4805}{108}\zeta_5N^2g_1^{10}g_2+\frac{7391}{10368}\zeta_3g_1^2g_2^9\\
&+\frac{9995}{432}\zeta_5g_1^4g_2^7+\frac{13667}{46656}\zeta_3Ng_1^3g_2^8+\frac{19549}{7776}\zeta_3N^2g_1^{10}g_2+\frac{21775}{108}\zeta_5g_1^{10}g_2\\
&+\frac{24413}{2239488}N^4g_1^{10}g_2+\frac{26855}{288}\zeta_5g_1^8g_2^3+\frac{37405}{432}\zeta_5g_1^6g_2^5+\frac{37705}{1728}\zeta_5N^2g_1^8g_2^3\\
&+\frac{60365}{2592}\zeta_5g_1^5g_2^6+\frac{72995}{1296}\zeta_5g_1^{11}+\frac{106975}{93312}\zeta_3N^3g_1^{11}+\frac{159395}{864}\zeta_5Ng_1^8g_2^3\\
&+\frac{197227}{53747712}N^3g_1^7g_2^4+\frac{262375}{3888}\zeta_3g_1^{11}+\frac{684313}{3456}\zeta_3Ng_1^9g_2^2+\frac{704323}{5832}\zeta_3g_1^7g_2^4\\
&+\frac{740725}{15552}\zeta_3Ng_1^6g_2^5+\frac{782275}{15552}\zeta_3Ng_1^8g_2^3+\frac{843907}{23328}\zeta_3g_1^3g_2^8\\
&+\frac{1002293}{373248}\zeta_3N^2g_1^7g_2^4+\frac{1132363}{15552}\zeta_3g_1^6g_2^5+\frac{1723973}{93312}\zeta_3N^2g_1^9g_2^2\\
&+\frac{3392747}{1679616}N^3g_1^{10}g_2+\frac{4175995}{15552}\zeta_3Ng_1^{10}g_2+\frac{4781087}{186624}\zeta_3Ng_1^7g_2^4\\
&+\frac{5235413}{23328}\zeta_3Ng_1^{11}+\frac{5594599}{46656}\zeta_3g_1^5g_2^6+\frac{5964659}{23328}\zeta_3g_1^9g_2^2\\
&+\frac{6586807}{13436928}N^2g_1^6g_2^5+\frac{8422631}{15552}\zeta_3g_1^8g_2^3+\frac{31673845}{4478976}g_1^2g_2^9\\
&+\frac{38661817}{161243136}g_1g_2^{10}+\frac{86076149}{559872}g_1^{10}g_2+\frac{107275895}{6718464}Ng_1^8g_2^3\\
&+\frac{107949341}{20155392}N^2g_1^9g_2^2+\frac{119287091}{2239488}Ng_1^6g_2^5+\frac{124107581}{6718464}g_1^5g_2^6\\
&+\frac{130062553}{5038848}N^2g_1^{11}+\frac{229662773}{8957952}Ng_1^7g_2^4+\frac{324365779}{6718464}g_1^6g_2^5\\
&+\frac{373459423}{3359232}Ng_1^{10}g_2+\frac{428987345}{3359232}g_1^9g_2^2+\frac{447498265}{3359232}g_1^8g_2^3\\
&+\frac{551172643}{2519424}g_1^7g_2^4+\frac{602892577}{20155392}g_1^3g_2^8+\frac{608352707}{5038848}g_1^{11}+12\zeta_3^2Ng_1^8g_2^3\Bigg]+O(g_i^{13}) \qquad \text{(B1)}
\end{aligned}
$$

and

$$
\begin{aligned}
\beta^{O(N)}_{\widetilde{2\,\mathrm{MOM}}}(g_1,g_2) = {}& \left[-\frac{1}{8}Ng_1^2g_2+\frac{1}{2}Ng_1^3+\frac{3}{8}g_2^3\right]\\
&+\left[-\frac{149}{144}Ng_1^4g_2-\frac{125}{288}g_2^5-\frac{1}{4}Ng_1^5-\frac{1}{24}Ng_1^3g_2^2+\frac{7}{288}Ng_1^2g_2^3\right]\\
&+\Bigg[-\frac{5063}{10368}N^2g_1^6g_2-\frac{361}{1296}Ng_1^2g_2^5-\frac{25}{48}Ng_1^7-\frac{17}{8}\zeta_3Ng_1^6g_2-\frac{17}{288}N^2g_1^5g_2^2\\
&\quad-\frac{7}{4}\zeta_3Ng_1^3g_2^4-\frac{3}{4}\zeta_3N^2g_1^7-\frac{1}{16}\zeta_3g_2^7+\frac{1}{4}\zeta_3Ng_1^2g_2^5+\frac{1}{4}\zeta_3N^2g_1^6g_2+\frac{3}{2}\zeta_3Ng_1^7
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{35}{16}\zeta_3 N g_1^4 g_2^3+\frac{65}{72}N^2 g_1^7+\frac{173}{6912}N^2 g_1^4 g_2^3+\frac{889}{288}N g_1^5 g_2^2+\frac{941}{864}N g_1^3 g_2^4\\
&+\frac{1171}{1152}N g_1^4 g_2^3+\frac{22051}{5184}N g_1^6 g_2+\frac{26741}{20736}g_2^7\Big]\\
+\Big[&-\frac{3333025}{373248}N g_1^4 g_2^5-\frac{2304049}{373248}g_2^9-\frac{726439}{46656}N g_1^8 g_2-\frac{690361}{62208}N g_1^6 g_2^3\\
&-\frac{473453}{62208}N g_1^3 g_2^6-\frac{384881}{10368}N g_1^7 g_2^2-\frac{274939}{62208}N^2 g_1^8 g_2-\frac{245249}{373248}N^2 g_1^6 g_2^3\\
&-\frac{176951}{20736}N g_1^5 g_2^4-\frac{16649}{1728}N^2 g_1^9-\frac{11933}{1728}\zeta_3 N g_1^4 g_2^5-\frac{11291}{62208}N^2 g_1^7 g_2^2\\
&-\frac{10987}{62208}N^2 g_1^4 g_2^5-\frac{5429}{20736}N^3 g_1^8 g_2-\frac{4501}{432}\zeta_3 N g_1^7 g_2^2-\frac{3173}{864}\zeta_3 g_2^9\\
&-\frac{3023}{144}\zeta_3 N g_1^8 g_2-\frac{1439}{216}\zeta_3 N g_1^6 g_2^3-\frac{1091}{216}\zeta_3 N g_1^9-\frac{773}{192}\zeta_3 N g_1^3 g_2^6\\
&-\frac{593}{48}\zeta_3 N^2 g_1^7 g_2^2-\frac{167}{20736}N^3 g_1^7 g_2^2-\frac{65}{12}\zeta_5 N^2 g_1^8 g_2-\frac{25}{6}\zeta_5 N^2 g_1^6 g_2^3\\
&-\frac{25}{36}\zeta_3 N^3 g_1^9-\frac{17}{6}\zeta_3 N g_1^5 g_2^4-\frac{15}{2}\zeta_5 N g_1^9-\frac{5}{2}\zeta_5 N g_1^2 g_2^7-\frac{1}{16}\zeta_3 N^2 g_1^5 g_2^4\\
&-\frac{1}{192}\zeta_3 N^3 g_1^6 g_2^3+\frac{1}{192}\zeta_3 N^3 g_1^7 g_2^2+\frac{5}{2}\zeta_5 N g_1^4 g_2^5+\frac{5}{2}\zeta_5 N g_1^5 g_2^4+\frac{5}{3}\zeta_5 N^2 g_1^9\\
&+\frac{17}{432}\zeta_3 N^2 g_1^4 g_2^5+\frac{35}{6}\zeta_5 g_2^9+\frac{49}{192}\zeta_3 N^3 g_1^8 g_2+\frac{50}{3}\zeta_5 N g_1^3 g_2^6+\frac{115}{4}\zeta_5 N g_1^8 g_2\\
&+\frac{187}{432}\zeta_3 N g_1^2 g_2^7+\frac{673}{124416}N^3 g_1^6 g_2^3+\frac{997}{216}\zeta_3 N^2 g_1^9+\frac{2713}{1728}\zeta_3 N^2 g_1^6 g_2^3\\
&+\frac{6913}{864}\zeta_3 N^2 g_1^8 g_2+\frac{7663}{10368}N^3 g_1^9+\frac{34279}{31104}N^2 g_1^5 g_2^4+\frac{40835}{5184}N g_1^9\\
&+\frac{175907}{93312}N g_1^2 g_2^7+5\zeta_5 N g_1^6 g_2^3+20\zeta_5 N^2 g_1^7 g_2^2+30\zeta_5 N g_1^7 g_2^2\Big]\\
+\Big[&\frac{2190456157}{53747712}g_2^{11}-\frac{934040489}{53747712}N g_1^2 g_2^9-\frac{125156905}{26873856}N^3 g_1^{10} g_2\\
&-\frac{21280913}{1119744}N^2 g_1^5 g_2^6-\frac{20862691}{746496}N^2 g_1^9 g_2^2-\frac{2572613}{17915904}N^3 g_1^6 g_2^5\\
&-\frac{1663367}{559872}N^3 g_1^8 g_2^3-\frac{1233275}{15552}N g_1^{11}-\frac{1099369}{31104}\zeta_3 N^2 g_1^8 g_2^3\\
&-\frac{710845}{2592}\zeta_5 N g_1^8 g_2^3-\frac{704519}{10368}\zeta_3 N g_1^5 g_2^6-\frac{577133}{31104}\zeta_3 N g_1^2 g_2^9\\
&-\frac{511715}{10368}\zeta_5 N^2 g_1^8 g_2^3-\frac{486025}{20736}\zeta_5 N g_1^6 g_2^5-\frac{364729}{2239488}N^4 g_1^{10} g_2\\
&-\frac{333119}{15552}\zeta_3 N^2 g_1^{11}-\frac{235985}{6912}\zeta_5 N^3 g_1^{10} g_2-\frac{230725}{7776}\zeta_3 N g_1^{11}\\
&-\frac{219565}{10368}\zeta_3 N^2 g_1^5 g_2^6-\frac{115265}{1296}\zeta_5 N^2 g_1^6 g_2^5-\frac{115045}{1728}\zeta_5 N g_1^3 g_2^8\\
&-\frac{100565}{41472}\zeta_3 N^3 g_1^8 g_2^3-\frac{71675}{10368}\zeta_3 N^3 g_1^9 g_2^2-\frac{50585}{288}\zeta_5 N g_1^9 g_2^2\\
&-\frac{47375}{432}\zeta_5 N^2 g_1^{10} g_2-\frac{38829}{128}\zeta_7 N g_1^6 g_2^5-\frac{35805}{128}\zeta_7 N g_1^8 g_2^3-\frac{26453}{128}\zeta_7 N g_1^4 g_2^7
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{25775}{6912}\zeta_5N^2g_1^4g_2^7-\frac{24521}{32}\zeta_7Ng_1^9g_2^2-\frac{18985}{432}\zeta_5N^2g_1^7g_2^4-\frac{16135}{128}\zeta_7N^2g_1^{10}g_2\\
&-\frac{13433}{128}\zeta_7g_2^{11}-\frac{12817}{32}\zeta_7Ng_1^{10}g_2-\frac{12755}{3072}N^3g_1^{11}-\frac{9261}{32}\zeta_7N^2g_1^7g_2^4\\
&-\frac{5733}{32}\zeta_7Ng_1^3g_2^8-\frac{2583}{8}\zeta_7N^2g_1^9g_2^2-\frac{2377}{186624}N^4g_1^9g_2^2-\frac{2023}{4}\zeta_7Ng_1^7g_2^4\\
&-\frac{1975}{864}\zeta_5N^3g_1^7g_2^4-\frac{1545}{64}\zeta_5Ng_1^4g_2^7-\frac{931}{32}\zeta_7Ng_1^5g_2^6-\frac{595}{54}\zeta_5N^2g_1^{11}\\
&-\frac{441}{32}\zeta_7N^3g_1^{11}-\frac{395}{864}\zeta_3N^4g_1^{11}-\frac{145}{2592}\zeta_5N^4g_1^{10}g_2-\frac{125}{124416}\zeta_3N^4g_1^8g_2^3\\
&-\frac{121}{48}\zeta_3^2Ng_1^2g_2^9-\frac{91}{4}\zeta_7N^2g_1^{11}-\frac{25}{12}\zeta_3^2Ng_1^7g_2^4-\frac{19}{18}\zeta_3^2N^2g_1^8g_2^3\\
&-\frac{17}{144}\zeta_3^2N^3g_1^{10}g_2-\frac{15}{4}\zeta_3^2N^2g_1^5g_2^6-\frac{11}{12}\zeta_3^2N^2g_1^{11}-\frac{7}{2}\zeta_3^2Ng_1^{11}-\frac{5}{3}\zeta_3^2N^3g_1^9g_2^2\\
&+\frac{1}{3}\zeta_3^2N^3g_1^8g_2^3+\frac{5}{12}\zeta_3^2N^2g_1^4g_2^7+\frac{13}{18}\zeta_3^2N^2g_1^{10}g_2+\frac{29}{4}\zeta_3^2g_2^{11}\\
&+\frac{35}{10368}\zeta_3N^4g_1^9g_2^2+\frac{55}{3}\zeta_3^2N^2g_1^9g_2^2+\frac{85}{648}\zeta_5N^3g_1^6g_2^5+\frac{89}{24}\zeta_3^2N^3g_1^{11}\\
&+\frac{95}{8}\zeta_3^2Ng_1^3g_2^8+\frac{125}{8}\zeta_3^2Ng_1^5g_2^6+\frac{147}{16}\zeta_7N^3g_1^{10}g_2+\frac{287}{72}\zeta_3^2N^2g_1^6g_2^5\\
&+\frac{415}{1728}\zeta_5N^4g_1^{11}+\frac{441}{8}\zeta_7N^2g_1^6g_2^5+\frac{441}{16}\zeta_7Ng_1^2g_2^9+\frac{515}{8}\zeta_3^2Ng_1^9g_2^2\\
&+\frac{705}{8}\zeta_5Ng_1^{11}+\frac{811}{144}\zeta_3^2Ng_1^6g_2^5+\frac{1195}{576}\zeta_5N^3g_1^9g_2^2+\frac{1235}{432}\zeta_3N^3g_1^{11}\\
&+\frac{1883}{10368}\zeta_3N^3g_1^7g_2^4+\frac{2051}{32}\zeta_7Ng_1^{11}+\frac{2317}{144}\zeta_3^2Ng_1^4g_2^7+\frac{2731}{144}\zeta_3^2Ng_1^{10}g_2\\
&+\frac{2855}{62208}\zeta_3N^3g_1^6g_2^5+\frac{3277}{2592}\zeta_3N^2g_1^4g_2^7+\frac{4627}{128}\zeta_7N^2g_1^8g_2^3+\frac{4693}{41472}\zeta_3N^4g_1^{10}g_2\\
&+\frac{7045}{36}\zeta_5N^2g_1^9g_2^2+\frac{9403}{144}\zeta_3^2Ng_1^8g_2^3+\frac{11465}{6912}\zeta_5g_2^{11}+\frac{15529}{31104}N^4g_1^{11}\\
&+\frac{16865}{1728}\zeta_5N^3g_1^{11}+\frac{40805}{864}\zeta_5Ng_1^5g_2^6+\frac{44449}{8957952}N^4g_1^8g_2^3+\frac{48415}{5184}\zeta_5N^3g_1^8g_2^3\\
&+\frac{70135}{1728}\zeta_5N^2g_1^5g_2^6+\frac{81655}{5184}\zeta_5Ng_1^2g_2^9+\frac{93265}{1296}\zeta_5Ng_1^{10}g_2+\frac{116125}{648}\zeta_3N^2g_1^7g_2^4\\
&+\frac{205339}{165888}N^3g_1^7g_2^4+\frac{279349}{10368}\zeta_3N^3g_1^{10}g_2+\frac{292661}{17496}N^2g_1^6g_2^5\\
&+\frac{498613}{7776}\zeta_3N^2g_1^{10}g_2+\frac{515255}{3888}\zeta_3N^2g_1^9g_2^2+\frac{518057}{13824}\zeta_3N^2g_1^6g_2^5\\
&+\frac{825475}{1728}\zeta_5Ng_1^7g_2^4+\frac{1057981}{3888}\zeta_3Ng_1^{10}g_2+\frac{1206983}{10368}\zeta_3Ng_1^3g_2^8\\
&+\frac{1215581}{15552}\zeta_3Ng_1^7g_2^4+\frac{2205067}{10368}\zeta_3Ng_1^8g_2^3+\frac{2369663}{13824}\zeta_3Ng_1^4g_2^7\\
&+\frac{2895059}{5184}\zeta_3Ng_1^9g_2^2+\frac{7655545}{124416}\zeta_3g_2^{11}+\frac{11130787}{186624}Ng_1^7g_2^4\\
&+\frac{11441425}{4478976}N^3g_1^9g_2^2+\frac{15400453}{5971968}N^2g_1^4g_2^7+\frac{16013395}{62208}\zeta_3Ng_1^6g_2^5\\
&+\frac{28704085}{373248}N^2g_1^{11}+\frac{34251571}{497664}Ng_1^3g_2^8+\frac{94907947}{2239488}N^2g_1^7g_2^4
\end{aligned}
$$

$$+\frac{128402411}{1492992}Ng_1^5g_2^6+\frac{213879685}{3359232}N^2g_1^8g_2^3+\frac{287841901}{6718464}Ng_1^{10}g_2$$
$$+\frac{331932757}{1119744}Ng_1^9g_2^2+\frac{470815561}{6718464}Ng_1^4g_2^7+\frac{588615251}{3359232}N^2g_1^{10}g_2$$
$$+\frac{759191107}{2239488}Ng_1^8g_2^3+\frac{2002332845}{26873856}Ng_1^6g_2^5+15\zeta_3^2N^2g_1^7g_2^4\Bigg]+O(g_i^{13}). \tag{B2}$$

It is evident from both expressions that no terms involve $\zeta_4$ or $\zeta_6$. The situation is the same for the other two renormalization group functions $\gamma^{O(N)}_{\underset{\phi\,\mathrm{MOM}}{\sim}}(g_1,g_2)$ and $\gamma^{O(N)}_{\underset{\sigma\,\mathrm{MOM}}{\sim}}(g_1,g_2)$ as is apparent by using a search tool on the respective expressions recorded in [44].


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