Locally scale invariant Chern-Simons actions in 3 + 1 dimensions and their emergence from (4 + 2)-dimensional 2*T* physics

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The traditional Chern-Simons (CS) terms in $3+1$ dimensions that modify general relativity (GR), quantum chromodynamics (QCD), and quantum electrodynamics (QED), typically lack scale invariance. However, a locally scale invariant and geodesically complete framework for the Standard Model (SM) coupled to GR was previously constructed by employing a tailored form of local scale (Weyl) symmetry. This refined SM + GR model closely resembles the conventional SM in subatomic realms where gravitational effects are negligible. Nevertheless, it offers an intriguing prediction: the emergence of new physics beyond the traditional SM and GR near spacetime singularities, characterized by intense gravity and substantial deviations in the Higgs field. In this study, we expand upon the enhanced SM + GR by incorporating Weyl invariant CS terms for gravity, QCD, and QED in $3+1$ dimensions, thereby integrating CS contributions within the locally scale-invariant and geodesically complete paradigm. Additionally, we establish a holographic correspondence between the new CS terms in $3+1$ dimensions and novel $(4+2)$-dimensional CS-type actions within 2T physics. We demonstrate that the Weyl transformation in $3+1$ dimensions arises from $4+2$ general coordinate transformations, which unify the hidden extra $1+1$ large (not curled up) dimensions with the evident $3+1$ dimensions. By leveraging the newfound local conformal symmetry, the augmented and geodesically complete SM + GR + CS introduces innovative tools and perspectives for exploring classical field theory aspects of black hole and cosmological singularities in $3+1$ dimensions, while the $(4+2)$-dimensional connection unveils deeper facets of spacetime.

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I. INTRODUCTION

Chern-Simons (CS) terms that modify general relativity (GR) [1] find their theoretical underpinnings in both field theory [2–4] and string theory [5]. In the realm of field theory, the emergence of the gravitational CS term stems from the stress tensor trace anomaly, while in particle physics, it arises from the chiral anomaly involving axial currents [3,4]. In string theory, the CS term serves to rectify anomalous symmetries [5]. Thus, the prevalence of CS terms in GR, QCD, and QED is well-founded from a theoretical standpoint.

Initially conceived as a $(2+1)$-dimensional topological theory [6], Chern-Simons gravity later found extension to $3+1$ dimensions by embedding the three-dimensional Chern-Simons topological current into a four-dimensional spacetime manifold. The conventional CS term within the $(3+1)$-dimensional gravity action as formulated by [1] is represented by

$$S^{3+1}_{\mathrm{CS-GR}} = \int d^4x \tilde{R}R \times (\text{scalar field}),$$

$$\tilde{R}R \equiv \frac{1}{2}\epsilon^{\mu_1\mu_2\mu_3\mu_4} R^{\lambda}{}_{\sigma\mu_1\mu_2} R^{\sigma}{}_{\lambda\mu_3\mu_4}. \quad (1)$$

This term supplements the standard Einstein-Hilbert term, kinetic terms for the scalar field, and other matter fields and their interactions. Consequently, the introduction of the GR field equations is accompanied by modifications through the inclusion of an additional Cotton-like C-tensor, constructed from derivatives of the Ricci tensor and the dual of the Riemann tensor. This C-tensor emerges when the Chern-Simons action $S^{3+1}_{\mathrm{CS-GR}}$ is varied with respect to its

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spacetime metric $g_{\mu\nu}$. Similar modifications are evident in QCD when the curvature tensor $R^{\lambda}{}_{\sigma\mu\nu}$ is replaced by the Yang-Mills field strength tensor $F^{a}_{\mu\nu}$ and similarly in QED when $R^{\lambda}{}_{\sigma\mu\nu}$ is replaced by the electromagnetic field strength tensor $F_{\mu\nu}$.

The CS corrections to GR, QCD, and QED yield significant physical implications. For instance, CS-modified gravity [7–11] has been instrumental in elucidating inflationary leptogenesis and baryogenesis through the gravitational anomaly. An intriguing aspect of this mechanism is the prediction of amplitude birefringent gravitational waves, potentially leaving a distinctive imprint on the gravitational wave background and thus on the cosmic microwave background. Investigating Chern-Simons gravity could prove crucial in addressing inquiries regarding the evolutionary trajectory of the Universe. Similarly, in QCD the postulation of an axion field aims to circumvent the breakdown of CP symmetry in strong interactions [12], while in QED the decay of the pion into two photons, among other phenomena, is encapsulated by a CS term within an effective action. Some additional discussions of the CS terms include [13–17].

A. Geodesically complete spacetime

This paper delves into how Chern-Simons (CS) terms influence physics in regions of intense gravity. However, before discussing this topic, it is essential to recognize that our conventional tools in modern physics have been insufficient in describing gravity near black hole (BH) and cosmological singularities. Beyond the anticipated but not fully comprehended quantum gravity effects, there lies the issue of spacetime's geodesic incompleteness, which arises in the Standard Model coupled to general relativity. It appears that this incompleteness has not garnered the warranted attention, perhaps due to the anticipation that quantum gravity would resolve singularity problems. Nonetheless, the tools employed to explore potential quantum gravity theories, such as string theory, also exhibit geodesic incompleteness.[1]

[1]To illustrate, gravitational background fields in particle or string theory exhibit geodesic incompleteness akin to their field theory counterparts. The presence of a dilaton or other background fields fails to rectify this issue. Similar to particle geodesics, strings propagating in such backgrounds encounter singularities of the background gravitational metric within a finite amount of proper time. What happens beyond this duration remains elusive as the standard theory offers no insights. This illustrates the problem of geodesic incompleteness in particle theory, field theory, and string theory at the classical level. There is a conspicuous absence of enlightening discussions on how quantum string theory or other quantum gravity theories might resolve this issue. However, refer to [18] for a proposed modification of string theory which is linked to the geodesic completion mechanism discussed here.

This highlights that resolving geodesic incompleteness, as outlined below, would inherently furnish new methodologies for investigating hitherto unknown physical phenomena within black holes and in the vicinity of the early universe, across both classical and quantum domains.

A geodesically complete framework for the Standard Model coupled to general relativity (SM + GR) in $3+1$ dimensions naturally emerged from 2T physics in $4+2$ dimensions [18–41]. The higher-dimensional theory, denoted as $(\mathrm{SM}+\mathrm{GR})_{4+2}$ (refer to Sec. III and Appendix), has a high degree of gauge symmetry which, in a gauge-fixed rendition, yields a holographic image in $3+1$ dimensions that encapsulates all the gauge-invariant information of the $4+2$ formalism. Termed the "holographic conformal shadow" (see Sec. III B), this image serves as the foundation for the conformally improved Standard Model coupled to gravity [33], denoted herein as $\mathrm{i}(\mathrm{SM}+\mathrm{GR})_{3+1}$ (refer to Sec. II). Geodesic completion emerged as an incidental property of the holographic conformal shadow, albeit not the primary objective of the 2T physics formalism. Even without delving into the details of 2T physics, one can outline the critical attributes of the enhanced $\mathrm{i}(\mathrm{SM}+\mathrm{GR})_{3+1}$, leading to geodesic completeness and foreseeing hitherto unimaginable physics in regions of intense gravity, particularly in the proximity of gravitational singularities [36,38,40].

There are two primary components for geodesic completeness:

(i) The first involves a local scale (Weyl) symmetry which is inherently mandated by any relativistic field theory coupled to gravity in $3+1$ dimensions, provided it is derived from 2T physics. This local scale symmetry arises from residual effects of general coordinate transformations in $4+2$ dimensions that mix the hidden extra $1+1$ dimensions with the evident $3+1$ dimensions (refer to Sec. III C). Scale invariance prohibits dimensionful parameters, implying that if 2T physics is the viable approach, then all dimensionful parameters in nature must originate from gauge fixing and/or spontaneous breaking of the emergent Weyl symmetry. In $\mathrm{i}(\mathrm{SM}+\mathrm{GR})_{3+1}$, a single real field $\phi(x)$ serves as the sole source, generating all dimensionful parameters, including the gravitational Newton constant G_N, the cosmological constant Λ, the Higgs mass or equivalently its vacuum value v (VEV), and the masses of quarks, leptons, and gauge bosons that are related to the Higgs field (refer to Sec. II C).

(ii) While the local scale invariance in $\mathrm{i}(\mathrm{SM}+\mathrm{GR})_{3+1}$ has a distinct origin from Weyl's original concept (see Sec. III C), they may appear indistinguishable from the perspective of $3+1$ dimensions. However, in the enhanced $\mathrm{i}(\mathrm{SM}+\mathrm{GR})_{3+1}$, it manifests with a uniquely specialized structure. This insight emerged in 2008 with the introduction of gravity in 2T physics [31,32] and was further elaborated upon directly in $3+1$ dimensions [33]. This structure entails a coordinated interplay between the singlet field $\phi(x)$ and the Higgs doublet $H(x)$. Both must serve as conformally coupled scalars to maintain local scale invariance. As a consequence, instead of the traditional SM + GR's Einstein-Hilbert term $(16\pi G_N)^{-1}R(g)$, this must be adjusted to $\frac{1}{12}(\phi^2 - 2H^\dagger H)R(g)$ in the enhanced $\mathrm{i(SM+GR)}_{3+1}$. The scalars ϕ, H must possess opposite signs in their nonminimal coupling to the curvature R to establish the existence of a region of spacetime where the resulting dynamical gravitational strength $G(x)$ is positive while requiring the Higgs field to have the correct sign kinetic energy term in the action. Then ϕ must have the wrong sign kinetic term, but this ghostlike ϕ causes no issues with unitarity thanks to the local gauge symmetry that compensates for the ghost-like unphysical gauge degree of freedom (refer to Sec. II). This distinctive structure with a relative minus sign for conformally coupled scalars could theoretically have been recognized directly in $3+1$ dimensions prior to 2T physics, but it remained overlooked.

Together, these two components imply that the strength of gravity, denoted as $G(x)$, is not uniform across the Universe. Instead, the strength of gravity varies dynamically with spacetime as determined by the scalar fields ϕ and H and is given by

$$(16\pi G(x))^{-1} = \frac{1}{12}\left(\phi^2(x) - 2H^\dagger H(x)\right). \tag{2}$$

In the context of $\mathrm{i(SM+GR)}_{3+1}$, the familiar low-energy physics operates within a spacetime region where the local scale-invariant ratio $(2H^\dagger H)/\phi^2$ is negligible. In this regime, ϕ is of the order of the Planck scale 10^{19} GeV, and the Higgs field H is around 246 GeV, including its vacuum expectation value (VEV). To study physics in this low-energy regime, one can adopt a gauge where $\phi(x)$ is constant, $\phi(x) = \phi_0 \sim 10^{19}$ GeV, treating the Higgs field $H \sim 246$ GeV as negligible at low energies in the expression for the almost constant gravitational strength $G(x) \simeq G_N$ in Eq. (2). In this gauge, only the Higgs field remains as a physical spin-0 degree of freedom while all dimensional physical constants emerge as proportional to ϕ_0 (see Sec. II C). This elucidates how and why $\mathrm{i(SM+GR)}_{3+1}$ closely resembles the traditional SM + GR in all observed low-energy physics aspects. Thus, $\mathrm{i(SM+GR)}_{3+1}$ stands as a valid theory for all experimentally observed low-energy physics to date, instilling confidence in its predictions outlined below in obscure or unknown spacetime regions or energy scales.

It is of interest to contrast the physical predictions of $\mathrm{i(SM+GR)}_{3+1}$ versus traditional SM + GR in domains beyond low energy physics where the theory is uncertain as follows.

(i) The field equations derived from $\mathrm{i(SM+GR)}_{3+1}$ indicate that near gravitational singularities, the Weyl gauge-invariant ratio $(2H^\dagger H)/\phi^2$ approaches 1, suggesting that $(\phi^2 - 2H^\dagger H)$ can vanish, resulting in a divergent effective gravitational strength $G(x)$ precisely at the singularity. On one side of the singularity, where the gauge-invariant ratio is less than one, $(2H^\dagger H)/\phi^2 < 1$, gravity is attractive $G(x) > 0$; on the other side, where the ratio exceeds one, $(2H^\dagger H)/\phi^2 > 1$, gravity is repulsive $G(x) < 0$. Consequently, $\mathrm{i(SM+GR)}_{3+1}$ predicts geodesically complete spacetime configurations where gravitational singularities separate gravity regions from antigravity regions. By contrast, traditional SM + GR only encompasses the gravity side of the singularity, contributing to its geodesic incompleteness.[2]

(ii) In $\mathrm{i(SM+GR)}_{3+1}$, it has been established that the majority of generic classical cosmological field solutions [34] describe fields propagating analytically through singularities from gravity regions to antigravity regions and vice versa. These typical solutions encompass the majority of phase space for on shell cosmological field configurations [34]. The divergence of $G(x)$ at spacetime regions or points

[2]The geodesic incompleteness in SM + GR becomes apparent when considering its accommodation within $\mathrm{(SM+GR)}_{3+1}$, albeit with incomplete gauge choices. An example of a geodesically incomplete gauge is the E-gauge [33] where fields $(\phi_E, H_E, g_{E\mu\nu})$ are denoted by the letter "E" to differentiate them from other gauges. In this gauge, the fields obey $\frac{1}{12}(\phi_E^2(x) - 2H_E^\dagger(x)H_E(x)) = (16\pi G_N)^{-1}$ to reproduce the Einstein frame featuring the standard Einstein-Hilbert term $(16\pi G_N)^{-1}R(g_E)$, with a spacetime-independent positive Newton constant G_N. However, the E-gauge exhibits geodesic incompleteness, since it is valid only in spacetime patches where the gauge-invariant expression $(1 - 2H^\dagger H/\phi^2)$ is positive. Notably, this expression's sign cannot be altered by local scale transformations, leading to regions where the Weyl invariant expression $(1 - 2H^\dagger H/\phi^2)$ is negative being omitted. Furthermore, the E-gauge fails to describe events at spacetime singularities where $(1 - 2H^\dagger H/\phi^2) = 0$. Conversely, it has been demonstrated [34] that in other gauges, the generic solutions of the equations of motion for ϕ and H continuously span all signs of the gauge invariant $(1 - 2H^\dagger H/\phi^2)$, encompassing both gravity and antigravity patches. This underscores the evident geodesic incompleteness of the traditional SM + GR framework in the Einstein frame. Similarly, any Jordan frame where the effective gravitational strength $G(x)$ is positive exhibits geodesic incompleteness. This incompleteness extends to the string frame that emerges in the low-energy limit of string theory, as it too constitutes a Jordan frame with solely positive effective gravitational strength $G(x)$. While such incomplete frames are attainable via Jordan-type gauge fixing in $\mathrm{i(SM+GR)}_{3+1}$ [18,33], their incompleteness issue is rectified by incorporating the antigravity regions beyond singularities predicted in $\mathrm{i(SM+GR)}_{3+1}$ as outlined in Sec. II.

where $(2H^{\dagger}H)/\phi^2 = 1$ does not hinder the continuity of fields or information flow through gravitational singularities. Similarly, geodesics connecting gravity and antigravity regions remain continuous at the singularity. Local scale invariance significantly contributes to establishing the continuity of information flow across the singularity [35–37].

Thus, the gravity and antigravity regions connected at gravitational singularities represent geodesic completions of each other. These prevalent features in classical field solutions of the geodesically complete $\mathrm{i(SM+GR)}_{3+1}$ signify a paradigm shift in understanding physical phenomena and information flow at singularities from one side to the other.

Conversely, the geodesic incompleteness of the traditional $\mathrm{SM+GR}$ framework in the Einstein or Jordan frames, including the String frame, leads to the inability to explore physics close to singularities at the classical field theory level.

(iii) Assuming that the geodesically complete field solutions in $\mathrm{i(SM+GR)}_{3+1}$ (similar to those discussed in [34]) dominate the phase space, a semiclassical path integral or WKB approach provides a fresh perspective on the flow and conservation of quantum probability across both gravity and antigravity sides of gravitational singularities [35]. This necessitates including observers on both sides of the singularities. Consequently, the information loss problem in black holes undergoes a radical transformation both technically and conceptually. It becomes evident that information lost by observers on one side of the singularity is gained by observers on the opposite side. By incorporating all observers, unitarity remains continuously preserved as the black hole evolves, whether it evaporates or not.

Some analysis of the features of $\mathrm{i(SM+GR)}_{3+1}$ described above is evident in applications to the big bang [35–37], black holes [21], and the interpretation of the antigravity regime [39]. Additionally, a recent discovery of intriguing behavior of the Higgs field inside black holes will be discussed in [40].[3]

As mentioned, $\mathrm{i(SM+GR)}_{3+1}$ represents the holographic conformal shadow of its 2T physics counterpart, $\mathrm{(SM+GR)}_{4+2}$. The latter incorporates gauge degrees of freedom that reveal the underlying $(4+2)$-dimensional spacetime, akin to revealing the SO(3,1) Lorentz symmetry of electromagnetism and its underlying relativistic spacetime through the inclusion of gauge degrees of freedom in the gauge potential A_{μ}. Similarly, the gauge degrees of freedom in 2T physics unveil spacetime symmetries such as SO(4,2) in the flat limit and its extension to coordinate invariance in curved $(4+2)$-dimensional spacetime. These spacetime symmetries remain gauge invariant under the $\mathrm{Sp}(2,R)$ gauge symmetry underlying 2T physics (See Appendix). Therefore, even after gauge fixing to $3+1$ dimensions, the $4+2$ spacetime symmetries persist as properties of the physical systems emerging in all shadows of 2T physics. For instance, the familiar SO(4,2) conformal symmetry in $3+1$ dimensions and the hidden SO(4,2) symmetry of planetary motion or the hydrogen atom, among other cases, are elucidated by the evident $(4+2)$-dimensional Lorentz symmetry in 2T physics. Particularly, the Weyl symmetry in $\mathrm{i(SM+GR)}_{3+1}$ is a remnant of general coordinate reparametrizations in $4+2$ dimensions that has not yet been gauge-fixed in the holographic conformal shadow (discussed in Sec. III C).

Since 2T physics underpins $\mathrm{i(SM+GR)}_{3+1}$, it is pertinent to briefly describe its origins and the theory's relationship with 1T physics. The following provides essential insights into the concepts of 2T physics. Further details can be found as a concise summary in Appendix.

At the core of 2T physics there is a gauge symmetry in phase space (X^M, P_M), postulating that all fundamental physical laws must treat position and momentum equally. This generalization extends Einstein's general coordinate invariance to phase space rather than just position space. This phase space gauge symmetry imposes constraints on the phase space for particle motion (see Sec. III and Appendix). Physical states that are invariant under this gauge symmetry reside within the subset of phase space that satisfy these constraints. Surprisingly, all solutions of these constraints yield nontrivial physical systems devoid of unitarity and causality issues, provided the full phase space (X^M, P_M), including gauge degrees of freedom, possesses two timelike dimensions—no less and no more.

Therefore, the appearance of an additional timelike and spacelike dimension arises naturally from phase space gauge symmetry, rather than being artificially imposed. Meanwhile, a plethora of gauge-invariant physical states consistent with these constraints exists in emergent gauge-invariant effective phase spaces in $3+1$ dimensions. These are termed holographic shadows, possessing one less timelike and one less spacelike dimension. Hence, 2T physics in $4+2$ dimensions (or more generally $d+2$) exhibits a rich gauge-invariant physical sector mirroring the

[3] Similar effects can be anticipated within the conventional framework of supergravity (SUGRA), as it presents the potential for geodesic completeness—a feature previously overlooked before the advent of 2T physics: the curvature term in SUGRA is given by $[(16\pi G_N)^{-1} - \frac{1}{6}K(\varphi,\bar{\varphi})]R(g)$ where $K(\varphi,\bar{\varphi})$ is the Kähler potential. Hence, SUGRA exhibits a sign-changing gravitational strength $G(x)$ similar to Eq. (2) when $\phi(x) \to \phi_0$. Previous SUGRA literature [42] fixated solely on the positive $G(x)$ spacetime patch, and constrained it to remain positive through a Weyl transformation to the Einstein frame, inadvertently resulting in geodesic incompleteness. SUGRA was elevated to a Weyl-symmetric version that incorporates a complex superfield version of $\phi(x)$ [33,43]. Hence, SUGRA is geodesically complete, akin to $\mathrm{i(SM+GR)}_{3+1}$, provided the presence of the antigravity regions that complete the spacetime are acknowledged.

familiar one-time physics (1T physics) in $3+1$ dimensions [or more generally $(d-1)+1$].

The holographic conformal shadow is just one among many shadows that are multidual to one another. There is a hidden SO(4,2) symmetry (and their curved space generalizations) which prevails in all shadows. The hidden symmetry, that is connected to partly hidden $4+2$ dimensions, represents previously unexpected non-linear symmetry properties of actions for all shadows in $3+1$ dimensions. This was discovered solely through 2T physics methods. What sets apart the holographic conformal shadow is its explicit display of the Poincare symmetry linearly in flat $3+1$ dimensions, along with the SO(4,2) conformal symmetry of massless systems in a recognizable nonlinear form.

2T physics remains consistent with all experimentally tested aspects of 1T physics. However, it transcends 1T physics by predicting hidden spacetime symmetries and multi-duality relationships among a myriad of 1T systems. These concealed properties permeate all facets of 1T physics, yet 1T physics alone lacks the capability to systematically predict them. This is where 2T physics furnishes new insights into the underlying $(4+2)$-dimensional system, testable with both 1T theory and experiments. Indeed, some of the simpler predictions have been theoretically verified in the early stages of 2T physics [21–27].

B. Plan of this paper

The previous research outlined above provides the foundational context for the development of Chern-Simons (CS) terms for gravity, QCD, and QED, which are both locally scale-invariant and geodesically complete at singularities. This is crucial to integrate the physical implications of these CS terms into the locally scale-invariant theory in $3+1$ dimensions. Furthermore, it is reasonable to anticipate that such CS terms in $3+1$ dimensions correspond to the holographic conformal shadow of analogous CS-type terms in $(4+2)$-dimensional 2T physics.

In this paper, we leverage the robust methodologies offered by 2T physics to investigate Chern-Simons theory. We employ 2T methods to elevate Chern-Simons theories into frameworks that exhibit local conformal scale invariance. The primary objectives of this endeavor are (1) to ascertain the necessary modifications for Chern-Simons theory to achieve local Weyl symmetry in $3+1$ dimensions, and (2) to derive the Weyl symmetric $3+1$ actions from the higher-dimensional perspective of 2T physics in $4+2$ dimensions.

To address these questions, we commence with the conformally improved Standard Model coupled to general relativity $\mathrm{i(SM+GR)}_{3+1}$ [33] as derived from 2T versions of these theories in $4+2$ dimensions [30,31]. A pivotal prediction of 2T physics is that the SM + GR, when reconciled with the constraints of 2T physics, exhibits a unique form of hidden local conformal scale (Weyl) symmetry. Geodesic completeness follows from this conformally invariant structure.

Given that the entire action for $\mathrm{i(SM+GR)}_{3+1}$ is inherently locally scale invariant (see Sec. II), our objective is to determine the structural forms that the Chern-Simons terms must adopt in $3+1$ dimensions to uphold the local scale symmetry of the complete $\mathrm{i(SM+GR)}_{3+1}$ model. We extend the conformally improved $\mathrm{i(SM+GR)}_{3+1}$ with Chern-Simons terms for gravity, QCD, and QED that remain explicitly invariant under local Weyl transformations in $3+1$ dimensions. We demonstrate that maintaining the local scale symmetry of the $\mathrm{i(SM+GR)}_{3+1}$ model necessitates the Pontryagin density $\tilde{R}R$ in our modified $3+1$ gravitational Chern-Simons term in (1) to linearly couple to a function $f_{\mathrm{GR}}(s_i/\phi)$ of the ratio of spin-0 fields. Analogous scenarios arise in QCD and QED when $R^{\lambda}{}_{\sigma\mu\nu}$ is replaced by the Yang-Mills field strength $F^a_{\mu\nu}$ (for QCD) and the electromagnetic field strength $F_{\mu\nu}$ (for QED).

Subsequently, we demonstrate that our conformally improved Chern-Simons terms in $3+1$ dimensions can be derived as holographic images of Chern-Simons actions in $4+2$ dimensions for gravity, QCD, and QED. We explain that the local scale transformation in $3+1$ dimensions is a remnant of $4+2$ general coordinate transformations that intertwine the extra $1+1$ large dimensions with the evident $3+1$ dimensions, treated at the same footing in 2T physics. With the newfound local conformal symmetry, the enhanced, expanded, and geodesically complete SM + GR + CS can be utilized to explore new physics beyond the traditional paradigm near black hole and cosmological singularities in $3+1$ dimensions at the classical level, while the $4+2$ connection unveils deeper aspects of spacetime.

The structure of the rest of this paper is as follows: In Sec. II, we elevate the gravitational Chern-Simons term to a locally Weyl symmetric version directly in $3+1$ dimensions. Next, in Sec. III, we illustrate how this Weyl-symmetric Chern-Simons term emerges as a holographic image of a 2T Chern-Simons field theoretical term in $4+2$ dimensions. We expound on the interpretation of the local scale (Weyl) symmetry of the $\mathrm{(SM+GR+CS)}_{3+1}$ theory as a vestige of general coordinate transformations that mix the extra $1+1$ dimensions with the $3+1$ dimensions. We also proceed with an examination of the QCD and QED versions of Chern-Simons theory. In Sec. IV, we delve into the role of parity and CP violation in the Chern-Simons actions. Finally, in Sec. V, we conclude with final remarks and discuss avenues for future research. In the Appendix A, we summarize the structure, scope, and some of the methodologies of 2T physics.

II. LOCALLY SCALE INVARIANT SM, GR AND CS ACTIONS IN 3+1

In this section, we will promote the standard formalism of Chern-Simons gravity defined in Eq. (1) to a locally scale invariant formalism directly in $3+1$ dimensions. Our strategy is to begin with the $\mathrm{i(SM+GR)}_{3+1}$ that is locally Weyl symmetric [33] before incorporating the Chern-Simons correction.

A. SM and GR actions

The existing $(4+2)$-dimensional model $(\mathrm{SM+GR})_{4+2}$ [30,31] successfully yields $\mathrm{i(SM+GR)}_{3+1}$ without the CS terms as a $(3+1)$-dimensional holographic conformal image. This theory is known to agree with the conventional Standard Model in successfully fitting all known aspects of particle physics down to 10^{-18} meters as measured by the LHC at CERN [44]. Since the entire action, including the CS terms, must preserve the local Weyl symmetry, our task is to promote the CS term in (1), and the similar QCD and QED terms, to locally scale invariant versions.

In the absence of the Chern-Simons correction, the action for $\mathrm{i(SM+GR)}_{3+1}$ is given by [33]

$$S^{3+1}_{\mathrm{SM+GR}} = \int d^4x\sqrt{-g}\begin{pmatrix} \mathcal{L}_{\mathrm{SM}} + \frac{1}{12}(\phi^2 - 2H^\dagger H)R(g) \\ +\frac{1}{2}g^{\mu\nu}(\partial_\mu\phi\partial_\nu\phi - 2\partial_\mu H^\dagger\partial_\nu H) - V(\phi,H) \end{pmatrix}. \tag{3}$$

The first term, $\mathcal{L}_{\mathrm{SM}}(A_\mu^{\gamma,W,Z,g},\psi^{q,l},\nu_R,\chi,g_{\mu\nu},H,\phi)$, encapsulates the familiar degrees of freedom within the Standard Model, excluding the scalar field terms that are displayed in (3). $\mathcal{L}_{\mathrm{SM}}$ encompasses the gauge fields $A_\mu^{\gamma,W,Z,g}$ corresponding to the photon, $W^\pm$, Z, and gluons g; the fermionic fields representing quarks and leptons $\psi^{q,l}$; right-handed neutrinos ν_R; and candidates for dark matter χ. These entities interact through $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ gauge symmetric Yukawa and gauge couplings with the doublet H and singlet ϕ (that could possibly couple only to ν_R or χ), and incorporating the difference between $\mathrm{SU}(2)\times\mathrm{U}(1)$ covariant versus ordinary derivatives in the Higgs fields's kinetic energy term $(-2D_\mu H^\dagger D^\mu H + 2\partial_\mu H^\dagger\partial^\mu H)$. Moreover, all fields in $\mathcal{L}_{\mathrm{SM}}$ are subject to minimal coupling with the gravitational metric $g_{\mu\nu}$. The term $\mathcal{L}_{\mathrm{SM}}$ is written separately from the scalar sector displayed in the rest of the action (3) because fermion, gauge boson, and Yukawa terms in $\mathrm{i(SM+GR)}_{3+1}$ are already invariant under local Weyl rescalings when minimally coupled to gravity. However for the gravitational sector and all spin 0 fields, including the Higgs field, local scale symmetry requires the special structures that are exhibited in the action (3) and discussed below.

The full action $S^{3+1}_{\mathrm{SM+GR}}$ is invariant under local Weyl rescalings of the form,

$$g_{\mu\nu}\to\Omega^2 g_{\mu\nu},\qquad \phi\to\Omega^{-1}\phi,\qquad H\to\Omega^{-1}H,$$
$$\psi^{q,l}\to\Omega^{-3/2}\psi^{q,l},\qquad A_\mu^{\gamma,W,Z,g}\to\Omega^0 A_\mu^{\gamma,W,Z,g}, \tag{4}$$

for an arbitrary local parameter $\Omega(x)$.

The geodesic completeness and predictions of new physics beyond the Standard Model emerge from the Weyl invariant special structures in this action as discussed in Sec. I A. These include the following features:

(i) Local scale invariance prohibits dimensionful parameters such as the Higgs mass, cosmological constant Λ, or the Newton constant G_N. Consequently, the Einstein-Hilbert term cannot appear in the action (3). Instead, there is an effective, spacetime dependent gravitational strength $G(x)$ that is determined by the scalar singlet $\phi(x)$ and the doublet Higgs boson $H(x)$, as in Eq. (2), $[16\pi G(x))^{-1} = \frac{1}{12}(\phi^2(x) - 2H^\dagger H(x)]$.

(ii) In the curvature and kinetic energy terms of the scalars ϕ and H, there is a relative minus sign. Hence ϕ seems to be a ghostlike field, but there is no issue with unitarity because the local scale symmetry eliminates the ghost by either gauge fixing it to a constant [the c-gauge, $\phi(x)=\phi_0$] or by compensating for it in other Weyl gauges. Without the ghostlike $\phi(x)$ there would be no way to have an underlying local scale symmetry as well as the presence of a patch of spacetime where the gravitational strength $G(x)$ is positive as explained in [33] and in Sec. II C. This relative minus sign structure between ϕ and H first emerged in the context of 2T physics as outlined in Sec. I A.

(iii) For the Standard Model, the potential $V(\phi,H)$ is the most general renormalizable purely quartic expression,

$$V_0(\phi,H) = \frac{\lambda}{4}(2H^\dagger H - \alpha^2\phi^2)^2 + \frac{\lambda'}{4}\phi^4, \tag{5}$$

where α,λ,λ' are dimensionless couplings. However, due to renormalization effects and the coupling to gravity, $V(\phi,H)$ may admit additional contributions beyond V_0 that would vanish when GR is decoupled from the SM and would be imperceptibly tiny at low energies. Such modifications of $V(\phi,H)$

could become substantial in strong gravity regions as discussed in [40]. To ensure the principle of local scale invariance (required if derived from 2T physics) the full potential must be homogeneous of degree 4 under scale transformations of the fields. Then the most general potential consistent with Weyl symmetry is given by $V(\phi, H) = \phi^4 v(\sqrt{2H^\dagger H/\phi^2})$ where $v(z)$ is any dimensionless function of z, as long as the deviations from the Standard Model potential V_0 in (5) are imperceptibly small at low energies as compared to the Planck energy scale.

The local scale symmetry (4) can be used to remove 1 field degree of freedom, such as locally rescaling $\phi(x)$ or $(\phi^2 - 2H^\dagger H)$, or some other combination of fields, to be spacetime independent constants in various patches of spacetime. In choosing such gauges, one should be mindful that certain gauge choices are valid only within geodesically incomplete patches of spacetime. Examples of geodesically incomplete gauges are those that lead to the Einstein, Jordan, or String frames when they are limited to only the gravity side of gravitational singularities (see footnote 2).

In this paper we can replace the doublet H by its $\mathrm{SU}(2)\times\mathrm{U}(1)$ unitary gauge fixed version $H^\dagger(x) = (0, s(x)/\sqrt{2})$, so we substitute everywhere $2H^\dagger H = s^2$ and $2\partial_\mu H^\dagger \partial_\nu H = \partial_\mu s \partial_\nu s$. Moreover, we suppress the term $\mathcal{L}_{\mathrm{SM}}$ because it plays no role in our discussion. Hence, our starting point for new CS terms is the action in Eq. (3) that consists of two conformally coupled scalars (ϕ, s) interacting with gravity and each other, while obeying a local scale symmetry.

In addition to the Higgs boson $s(x)$, that is the only established physical spin 0 elementary field, there may be additional spin 0 fields that may play a role in the fundamental theory. The Weyl symmetry permits such additional scalars or pseudoscalars that we denote by $s_i, i = 1, \ldots, N$. We can allow only one ghostlike ϕ since the Weyl symmetry is capable to ensure unitarity only with 1 ghostlike degree of freedom. In addition to the Higgs boson s (renamed s_1), the s_i can include scalar or pseudoscalar dark matter candidates (e.g., the axion) or other spin 0 fields as motivated by string theory (e.g., the dilaton or axion), supersymmetry, supergravity, or grand unified theories (GUTs). In the simplest Weyl invariant couplings, all spin 0 fields are treated as conformally coupled scalars. However, there are more general Weyl invariant couplings for multiple spin 0 fields when there are additional scalars beyond ϕ and the Higgs boson H, as discussed in [33]. Here we will deal with the case of conformally coupled scalars for simplicity by replacing the Higgs boson s in Eq. (3) with s_i and summing over i in the kinetic and R terms in (3). Thus, we will at times use the more general notation (ϕ, s_i) to include all possible conformally coupled spin 0 fields s_i.

Likewise, the potential $V(\phi, s_i)$ can be modified to a general function of the fields that is homogeneous of degree 4, which can be written as $V(\phi, s_i) = \phi^4 v(s_i/\phi)$ where $v(z_i)$ is any dimensionless function of its Weyl invariant arguments $z_i \equiv s_i/\phi$. Other Weyl invariant ratios, such as s_i/s_j, are not independent since they can be written in terms of z_i/z_j.

B. CS action in GR

We are now ready for the new Chern-Simons terms. In the gauge symmetric version of the physically correct $\mathrm{i(SM + GR)}_{3+1}$, we consider consistently a Weyl invariant version of $S^{3+1}_{\mathrm{CS-GR}}$. We will argue that the Chern-Simons term $S^{3+1}_{\mathrm{CS-GR}}$ in (1) for gravity is promoted to be scale invariant under Weyl transformations if the Pontryagin density $\tilde{R}R$ linearly couples to a dimensionless function of the ratios of the scalar fields, $f_{\mathrm{GR}}(s_i/\phi)$ as follows:

$$S^{3+1}_{\mathrm{CS-GR}} = \int d^4x \tilde{R}R f_{\mathrm{GR}}(s_i/\phi). \tag{6}$$

Below we will first argue that if $f_{\mathrm{GR}}(s_i/\phi)$ is a function of only the ratios s_i/ϕ, or a constant, then $S^{3+1}_{\mathrm{CS-GR}}$ is invariant under *global* scale transformations. We will then argue that this is also sufficient for $S^{3+1}_{\mathrm{CS-GR}}$ to be invariant under *local* scale transformations as well.

The same reasoning applies also in the cases of QCD and QED by simply replacing $\tilde{R}R$ by the QCD field strengths $\tilde{F}^a F^a$ or QED field strengths $\tilde{F}F$. Therefore, we will refrain from repeating the arguments for QCD and QED until the discussion in Sec. III D.

Under the local Weyl transformations (4), the determinant of the metric and the curvatures transform as follows:

$$\begin{aligned}
&\sqrt{-g} \to \Omega^4(x)\sqrt{-g}, \qquad R(g) \to \Omega^{-2}(x)(R(g) + \text{derivatives of } \Omega) \\
&R_{\mu\nu}(g) \to \Omega^0(x)(R_{\mu\nu}(g) + \text{derivatives of } \Omega) \\
&R_{\mu\nu\lambda\sigma}(g) \to \Omega^2(x)(R_{\mu\nu\lambda\sigma}(g) + \text{derivatives of } \Omega) \\
&R^\mu{}_{\nu\lambda\sigma}(g) \to \Omega^0(x)(R^\mu{}_{\nu\lambda\sigma}(g) + \text{derivatives of } \Omega) \\
&\text{hence:}\ (\tilde{R}R) \to 1(\tilde{R}R + \text{derivatives of } \Omega).
\end{aligned} \tag{7}$$

However, the derivative terms in $(\tilde{R}R) \to (\tilde{R}R +$ derivatives of $\Omega)$ do not seem to vanish unless $\Omega(x)$ is independent of x. Hence, the Chern-Simons action $S^{3+1}_{\text{CS-GR}}$ in Eq. (6) seems to be at best invariant under *global* Weyl transformations (i.e., transformations where $\Omega = \text{constant}$), provided the function f_{GR} is invariant under global rescalings. This is satisfied by requiring f_{GR} to be any dimensionless general function of only Weyl invariant ratios $z_i = s_i/\phi$ as indicated in (6).

We will now consider whether $S^{3+1}_{\text{CS-GR}}$ can be made invariant under *local* Weyl transformations for any local $\Omega(x)$. This is necessary because in the absence of local symmetry, the ghostlike field $\phi(x)$ would remain as a dynamical degree of freedom that would ruin the unitarity of the theory since it would not be possible to remove the ghost by mapping it to a constant $\phi(x) \to \phi_0$ via a local gauge transformation. Therefore, it is essential to determine whether $S^{3+1}_{\text{CS-GR}}$ can be made invariant beyond global scale transformations, or whether it is necessary to introduce additional terms to improve its symmetry properties.

For this purpose, we consider the Weyl tensor $C_{\mu_1\mu_2\mu_3\mu_4}$ defined by

$$C_{\mu_1\mu_2\mu_3\mu_4} = \begin{pmatrix} R_{\mu_1\mu_2\mu_3\mu_4} + \frac{1}{6}(g_{\mu_1\mu_3}g_{\mu_2\mu_4} - g_{\mu_1\mu_4}g_{\mu_2\mu_3})R \\ -\frac{1}{2}(R_{\mu_1\mu_3}g_{\mu_2\mu_4} - R_{\mu_1\mu_4}g_{\mu_2\mu_3} + R_{\mu_2\mu_4}g_{\mu_1\mu_3} - R_{\mu_2\mu_3}g_{\mu_1\mu_4}) \end{pmatrix}, \tag{8}$$

where $R_{\mu\nu}$ is the Ricci tensor and R is the curvature scalar. The Weyl tensor has the following properties under permutation of indices and tracing:

$$\begin{aligned} &(a)\ C_{\mu_1\mu_2\mu_3\mu_4} = C_{\mu_3\mu_4\mu_1\mu_2} \\ &(b)\ C_{\mu_1\mu_2\mu_3\mu_4} = -C_{\mu_2\mu_1\mu_3\mu_4} = -C_{\mu_1\mu_2\mu_4\mu_3} \\ &(c)\ C_{\mu_1\mu_2\mu_3\mu_4} + C_{\mu_1\mu_3\mu_4\mu_2} + C_{\mu_1\mu_4\mu_2\mu_3} = 0, \\ &\quad\ \text{hence for any } \mu \text{ and } \nu \text{ we have}: C_{\mu\mu_2\mu_3\mu_4}\varepsilon^{\nu\mu_2\mu_3\mu_4} = 0. \\ &(d)\ \text{Traceless for any pair of indices, example}: C_{\mu_1\mu_2\mu_3\mu_4}g^{\mu_2\mu_3} = 0. \end{aligned} \tag{9}$$

The Riemann tensor $R_{\mu_1\mu_2\mu_3\mu_4}$ shares the properties (a, b, c), but the last one (d) is valid only for $C_{\mu_1\mu_2\mu_3\mu_4}$.

Under the local Weyl rescaling of the metric given in (7), $R_{\mu_1\mu_2\mu_3\mu_4}, R_{\mu\nu}$, and R not only rescale by an overall factor but also have derivative terms of $\Omega(x)$. The derivative terms seem to prevent $S^{3+1}_{\text{CS-GR}}$ from being locally Weyl invariant. By contrast, all the derivative terms cancel in the combination $C_{\mu_1\mu_2\mu_3\mu_4}$ given in (8). Therefore, the Weyl tensor only rescales by an overall factor under a *local* Weyl transformation,

$$C_{\mu_1\mu_2\mu_3\mu_4} \to \Omega^2(x)C_{\mu_1\mu_2\mu_3\mu_4}. \tag{10}$$

Moreover, when one of the indices is raised $C^{\mu_1}{}_{\mu_2\mu_3\mu_4} = g^{\mu_1\nu}C_{\nu\mu_2\mu_3\mu_4}$, it remains fully invariant under *local* Weyl transformations,

$$C^{\mu_1}{}_{\mu_2\mu_3\mu_4} \to \Omega^0(x)C^{\mu_1}{}_{\mu_2\mu_3\mu_4}. \tag{11}$$

It is now evident that a locally Weyl invariant version of the CS term in gravity is given by replacing $R^\lambda{}_{\sigma\mu\nu}$ with $C^\lambda{}_{\sigma\mu\nu}$ in Eq. (6),

$$S^{3+1}_{\text{CS-GR}} = \int d^4x \tilde{C}C f_{\text{GR}}(s_i/\phi), \qquad \tilde{C}C = \frac{1}{2}\epsilon^{\mu_1\mu_2\mu_3\mu_4}C^\lambda{}_{\sigma\mu_1\mu_2}C^\sigma{}_{\lambda\mu_3\mu_4}, \tag{12}$$

where $f_{\text{GR}}(s_i/\phi)$ is any general function of the ratio of the fields, including a constant.

It seems that $\tilde{C}C$ contains a few additional terms as compared to $\tilde{R}R$, thus providing the desired *local* Weyl invariance property of the action. However, using the permutation and trace properties of the Weyl tensor listed in (9), it can be shown that

$$\tilde{C}C = \tilde{R}R. \tag{13}$$

Hence, the local Weyl invariance of $\tilde{R}R$ can be made manifest by rewriting it in terms of only the Weyl tensor. Therefore, the action as written originally in Eq. (6) is actually Weyl invariant under *local* transformations. However, this is possible only if $f_{\text{GR}}(s_i/\phi)$ is any dimensionless function of the ratio of the conformal scalar or pseudoscalar fields as indicated.

In conclusion, the full action for the conformally invariant $(\text{SM} + \text{GR})_{4+2}$ augmented with the gravitational Chern-Simons correction is given by

$$S^{3+1}_{\mathrm{SM+GR+CS}} = S^{3+1}_{\mathrm{SM+GR}} + S^{3+1}_{\mathrm{CS-GR}}. \quad (14)$$

A similar CS term $S^{3+1}_{\mathrm{CS-QCD}}$ associated with QCD and the axion can be considered assuming the axion exists (see Secs. III D and IV). Also in QED, the decay of the pion into two photons, and other similar processes involving hadrons, are captured by similar terms in an effective Lagrangian approach.

C. Emergence of dimensionful parameters

One of the virtues of this formalism is its unified approach to explain mass generation in the Standard Model. In our theory, all dimensionful parameters of the Standard Model and gravity are initially absent, including the Newton constant, Higgs mass, cosmological constant, and quark, lepton, and gauge boson masses. However—just as in spontaneously broken gauge symmetries—all of the dimensionful parameters in the $\mathrm{SM+GR+CS}$ are generated from the same single source: namely the field ϕ after gauge fixing the local Weyl symmetry. This demonstrates a higher degree of unification as to the source of all dimensionful parameters.

The gauge in which we essentially recover the usual renormalizable field theory for the Standard Model in flat space is known as the c-gauge [33]. In the c-gauge, the source of mass arises when the field ϕ is fixed to a constant $\phi(x) \to \phi_0 = \sqrt{12/(16\pi G)}$ such that ϕ_0 is associated with the emergence of the Planck scale, $\phi_0 \sim 10^{19}$ GeV. Explicitly, for the gravitational constant G_N, the cosmological constant Λ, and the Higgs vacuum expectation value (VEV) $\langle |H| \rangle = v/\sqrt{2}$ that minimizes the potential V_0, are obtained by replacing $\phi(x)$ by the constant ϕ_0 in the action (3),

$$\frac{1}{16\pi G_N} = \frac{\phi_0^2}{12}, \qquad \frac{\Lambda}{16\pi G_N} = \frac{\lambda'}{4}\phi_0^4, \qquad v^2 = \alpha^2\phi_0^2. \quad (15)$$

Then, the electroweak symmetry breaking that is mediated by the VEV of the Higgs doublet becomes also driven by the constant value ϕ_0 of the gauge fixed $\phi(x)$, so the masses of all quarks, leptons, and gauge bosons at low energies are also driven by the same unique ϕ_0 source.

In the c-gauge, we will use the letter "c" to label all fields such as, s_{ic}, $g_c^{\mu\nu}$ etc. and $\phi_c(x) \equiv \phi_0$, to emphasize that these gauge fixed fields are different than the fields in other gauges, such as the E-gauge, the string gauge, etc. In the c-gauge, the action given in Eq. (14) reduces identically to the usual Standard Model except for the curvature term that deviates from the Einstein-Hilbert action and takes the form $\frac{1}{12}(\phi_0^2 - s_c^2(x))R(g_c)$. This contains the spacetime dependent effective gravitational strength $[16\pi G_c(x))^{-1} = \frac{1}{12}(\phi_0^2 - s_c^2(x)]$. At experiments conducted at accelerators, since energies are relatively tiny as compared to Planck energy, the dimensionless ratio $s_c^2(x)/\phi_0^2$ is of order 10^{-34}. Accordingly, $G_c(x)$ is approximated by the constant G_N in (15) with great accuracy. Then, low energy physical phenomena are not sensitive to the deviation from the traditional $\mathrm{SM+GR}$ that has a spacetime independent gravitational constants G_N. We see that, in the c-gauge, the familiar Standard Model as tested and measured at low energies emerges from Eq. (14) almost identically.

One can compute in any other gauge. Recalling that Weyl gauge invariant quantities are physical, agreement with the usual Standard Model conventions and interpretations persists in regimes when the gauge invariant ratio $s^2(x)/\phi^2(x) = s_c^2(x)/\phi_0^2 \ll 1$ is tiny.

While at low energies the locally scale invariant theory $\mathrm{i(SM+GR)}_{3+1}$ is practically identical to the usual SM and GR, its physical aspects are quite different at the neighborhood of singularities such as the big bang and black holes. In such spacetime regions, the Weyl symmetry repairs the geodesical incompleteness of spacetime and introduces new physics and new perspectives of spacetime beyond the usual Standard Model and general relativity [33,35,38,40]. The emergence of novel features becomes conspicuous in the vicinity, at, and beyond singularities. The new aspects apply also with supersymmetry [33,43] and in a formalism that repairs geodesic incompleteness in string theory [18]. Our work in this paper extends geodesic completeness to be valid in the presence of the Chern-Simons correction to GR as well.

III. 2T PHYSICS APPROACH

The goal in this section is to construct a $(4+2)$-dimensional version of Chern-Simons gravity $S^{4+2}_{\mathrm{CS-GR}}$ [Eq. (6)] which reduces to the $(3+1)$-dimensional formulation of Chern-Simons gravity $S^{3+1}_{\mathrm{CS-GR}}$ that has a local conformal scale invariance. Since this process involves tools and techniques from 2T physics, we include an Appendix A that gives brief reviews of the main concepts, features and tools of 2T physics. The interested reader may consult the Appendix to better understand the contents of this section.

In this section, we will recall the 2T physics formulation of $(\mathrm{SM+GR})_{4+2}$, construct the $(4+2)$-dimensional Chern-Simons term $S^{4+2}_{\mathrm{CS-GR}}$, and illustrate how, by gauge fixing and solving constraints, to obtain the $(3+1)$-dimensional holographic shadow, $S^{4+2}_{\mathrm{SM+GR+CS}} \to S^{3+1}_{\mathrm{SM+GR+CS}}$ that was discussed in the previous sections.

A. SM, GR and CS actions in 4+2

We will use the same gauge choice (i.e., the conformal shadow gauge) that was used to derive the 2T version of the Standard Model coupled to general relativity in $3+1$ dimensions from its $(4+2)$-dimensional counterpart, $S^{4+2}_{\mathrm{SM+GR}} \to S^{3+1}_{\mathrm{SM+GR}}$. The computations for the reduction $S^{4+2}_{\mathrm{SM+GR}} \to S^{3+1}_{\mathrm{SM+GR}}$ for a $(d+2)$-dimensional metric were given in [30–32]. Here, we will apply similar techniques to the reduction $S^{4+2}_{\mathrm{CS-GR}} \to S^{3+1}_{\mathrm{CS-GR}}$, specializing $d+2$ to $4+2$ dimensions.

We start with the $4+2$ action for 2T gravity coupled to Klein-Gordon Φ, S_i, Dirac $\Psi^{L,R}_\alpha$, and Yang-Mills A^a_M type matter fields in the absence of the Chern-Simons correction [31],

$$S^{4+2}_{\mathrm{SM+GR}} = \gamma \int d^{4+2}X\{\delta(W)\mathcal{L}_1(\Phi, S_i, \Psi^{q,l}_\alpha, A^a) + \delta'(W)\mathcal{L}_2(W, \Phi, S_i)\},$$
$$\mathcal{L}_1 \equiv \sqrt{G}\left[\frac{1}{12}(\Phi^2 - S_i^2)R(G) + \frac{1}{2}(\nabla\Phi)^2_G - \frac{1}{2}(\nabla S_i)^2_G - V(\Phi, S_i) + \cdots\right],$$
$$\mathcal{L}_2 \equiv \frac{1}{12}\sqrt{G}\left[(\Phi^2 - S_i^2)(4 - \nabla^2 W) + G^{MN}\nabla_M W \nabla_N(\Phi^2 - S_i^2)\right]. \tag{16}$$

Here X^M are the $4+2$ spacetime coordinates, $G_{MN}(X)$ is the $4+2$ gravitational metric, and $R(G)$ is its Riemann curvature scalar in $4+2$ from which the $(3+1)$-dimensional shadow metric $g_{\mu\nu}(x)$ and its curvature scalar $R(g)$ are derived. Similarly, (Φ, S_i) with their potential $V(\Phi, S_i)$, are the $(4+2)$-dimensional fields whose $(3+1)$-dimensional shadows $[\phi(x), s_i(x)]$ and $V(\phi, s_i)$ are derived. γ is an overall normalization constant proportional to the inverse of Planck's constant $\hbar$. The "$\cdots$" in $\mathcal{L}_1$ indicates additional fermions as $\mathrm{SO}(4,2) = \mathrm{SU}(2,2)$ spinors $\Psi^{q,l}_\alpha(X)$ and Yang-Mills gauge fields $A^a_M(X)$ as SO(4,2) vectors from which the $(3+1)$-dimensional shadow fermions $\psi^{L,R}(x)$ and shadow gauge fields $A^a_\mu(x)$ in the $(\mathrm{SM+GR})_{3+1}$ are derived.

Finally, $W(X)$ is an auxiliary scalar field that appears in $\mathcal{L}_{W,\Phi,S}$ and in the delta function $\delta(W)$ and its derivative $\delta'(W)$ in Eq. (16). The origins of this W field as one of the generators of $\mathrm{Sp}(2,R)$ in phase space is explained in Appendix A 3. The restriction to the $\mathrm{Sp}(2,R)$ gauge invariant sector by the vanishing of all three generators of $\mathrm{Sp}(2,R)$ is partially implemented by the delta function and its derivatives. The vanishing of the other two $\mathrm{Sp}(2,R)$ generators in the local field theory context, including interactions, is described in Appendix A 3.

We now introduce the $(4+2)$-dimensional Chern-Simons term that can be added to the action (16) as an effective term induced by anomalous quantum effects. This is the starting term in $4+2$ dimensions from which $S^{3+1}_{\mathrm{CS-GR}}$ in Eq. (6) emerges as a $(3+1)$-dimensional "shadow." To accomplish this goal, it must have the following form:

$$S^{4+2}_{\mathrm{CS-GR}} = \gamma \int d^{4+2}X\delta(W(X)) \quad (\tilde{R}RA)$$
$$(\tilde{R}RA) \equiv \frac{1}{2}\epsilon^{M_1M_2M_3M_4M_5M_6} R^M_{NM_1M_2} R^N_{MM_3M_4} A^{\mathrm{GR}}_{M_5M_6}, \tag{17}$$

where $\epsilon^{M_1M_2M_3M_4M_5M_6}$ is the $4+2$ Levi-Civita tensor, and $R^M_{NM_1M_2}(X)$ is the Riemann tensor derived from the metric $G_{MN}(X)$ in $4+2$ dimensions. The $A^{\mathrm{GR}}_{M_5M_6}(X)$ is an antisymmetric tensor constructed from the other $(4+2)$-dimensional field degrees of freedom in the 2T theory.

One would like to choose some A^{GR}_{MN} whose shadow would reproduce the $f_{\mathrm{GR}}(s_i/\phi)$ in $S^{3+1}_{\mathrm{CS-GR}}$. Moreover, the $(3+1)$-dimensional curvature tensor $R^\mu{}_{\nu\mu_1\mu_2}(g(x))$ constructed from the shadow metric $g_{\mu\nu}(x)$ in $3+1$ dimensions must emerge as the shadow of its $(4+2)$-dimensional parent $R^M_{NM_1M_2}(G(X))$ in $4+2$ dimensions. We introduce the following A^{GR}_{MN} constructed from the derivatives of W and Φ, where the field $W(X)$ is the one that appears in the delta function in the 2T action:

$$A^{\mathrm{GR}}_{MN} = \frac{1}{4}\left(\frac{\partial W}{\partial X^M}\frac{\partial(\ln\Phi)}{\partial X^N} - \frac{\partial W}{\partial X^N}\frac{\partial(\ln\Phi)}{\partial X^M}\right) f_{\mathrm{GR}}\left(\frac{S_i}{\Phi}\right). \tag{18}$$

The form in Eq. (18) is motivated on the basis of engineering dimensions as follows. The 2T gauge symmetry [30–32] of the $(4+2)$-dimensional action requires the following engineering dimensions for the coordinates X^M, the fields W, Φ, and S, the metric G_{MN}, the Yang-Mills gauge field A_M, the Christoffel connection $\Gamma^\alpha_{\beta\gamma}$, and the Riemann tensor $R^M_{NM_1M_2}$:

$$\dim(X^M) = -1, \quad \dim(W) = -2, \quad \dim(\Phi) = +1, \quad \dim(S_i) = +1,$$
$$\dim(G_{MN}) = 0, \quad \dim(A_M) = +1, \quad \dim(\Gamma^\alpha_{\beta\gamma}) = +1, \quad \dim(R^M_{NM_1M_2}) = +2. \tag{19}$$

Given that the action must have dimension 0, $A^{\mathrm{GR}}_{M_5M_6}$ should have engineering dimension 0, after taking into account that in the volume element, $d^{4+2}X\delta(W(X))$, the factor $d^{4+2}X$ has dimension -6 while the delta function $\delta(W(X)$ has dimension $+2$. Accordingly, the $\dim(A^{\mathrm{GR}}_{MN}) = 0$ is consistent with the assigned dimensions since $\dim(\partial W/\partial X^M) = -1$ and $\dim(\partial(\ln\Phi)/\partial X^M) = +1$.

B. Holographic conformal shadow

We will now outline the process by which the $(4+2)$-dimensional metric, fields, Riemann tensors, and antisymmetric tensor $[G_{MN}, \Phi(X), S_i(X), R^M_{NM_1M_2}, A^{\mathrm{GR}}_{MN}]$ are reduced to their $(3+1)$-dimensional shadow counterparts $[g_{\mu\nu}(x), \phi(x), s_i(x), R^\lambda_{\sigma\mu_1\mu_2}(g(x)), f_{\mathrm{GR}}(s_i/\phi)]$.

There are many ways to parametrize X^M in $4+2$ dimensions to reduce the $4+2$ field theory to the emergent $3+1$ Chern-Simons action. One approach is to make a convenient choice of coordinate transformations $X^M \to (w, u, x^\mu)$ so that $W(X) = W(w, u, x^\mu) = w$ is one of the independent coordinates.[4] The emergent $3+1$ shadows $[g_{\mu\nu}(x), \phi(x), s(x)]$ will be functions of only x^μ and tensor indices run only in $3+1$ directions, $\mu, \nu = 0, 1, 2, 3$. The tangent basis for this curved space (w, u, x^μ) is given by $\partial_M = (\partial_w, \partial_u, \partial_\mu)$ where (w, u) are the coordinates associated with the extra $1+1$ dimensions beyond the $3+1$ coordinates x^μ. The volume element becomes

$$d^{4+2}X\delta(W(X)) = dw\, du\, d^4x\, \delta(w). \tag{20}$$

Our strategy is to use the fact that the 2T action is manifestly invariant under general coordinate transformations in $4+2$ dimensions to gauge fix components of the metric as functions of (w, u, x^μ) as shown below. In flat spacetime, before gauge fixing, the 2T action in $4+2$ has a global SO(4,2) symmetry which is linearly realized on $4+2$ coordinates X^M and is manifest in the action. This is the Lorentz symmetry in $4+2$ dimensions which treats all coordinates on an equal footing. In the field theoretic formulation in the presence of gravity, the SO(4, 2) symmetry is elevated to general coordinate invariance and Yang-Mills type gauge symmetries in $4+2$ dimensions. After gauge fixing components of the metric G_{MN}, there remains a general coordinate symmetry that allows us to arbitrarily reparametrize the subspace (u, x^μ) without altering the gauge fixed metric G_{MN}. With this freedom, we can choose a parametrization which makes use of the fact that w is an independent coordinate to ensure that the conformal shadow will only depend on the shadow $g_{\mu\nu}(x)$ degrees of freedom in the end.

As outlined in Appendix A 3, the equations of motion derived from the 2T action imposes certain kinematic constraints (called B and C) and a dynamical constraint (called A), for the metric and scalar fields (G_{MN}, Φ, S_i). The kinematic B and C constraints, combined with gauge conditions for the metric $G_{MN}(X)$ and its inverse $G^{MN}(X)$ are solved by the following tensor configuration as functions of (w, u, x^μ) [32]:

$$G_{MN}(X) = \begin{array}{c} M/N \\ w \\ u \\ \mu \end{array} \begin{array}{c} \begin{array}{ccc} w & u & \nu \end{array} \\ \begin{pmatrix} 0 & -1 & 0 \\ -1 & -4w & 0 \\ 0 & 0 & G_{\mu\nu}(X) \end{pmatrix} \end{array}, \quad G^{MN}(X) = \begin{array}{c} M/N \\ w \\ u \\ \mu \end{array} \begin{array}{c} \begin{array}{ccc} w & u & \nu \end{array} \\ \begin{pmatrix} 4w & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & G^{\mu\nu}(X) \end{pmatrix} \end{array}, \tag{21}$$

where

$$G_{\mu\nu}(X) = e^{-4u}\tilde{g}_{\mu\nu}(x, we^{4u}), \qquad \Phi(X) = e^{2u}\tilde{\phi}(x, we^{4u}), \qquad S_i(X) = e^{2u}\tilde{s}_i(x, we^{4u}). \tag{22}$$

Because of the delta function in the volume element in Eq. (20), we can consider a Kaluza-Klein type series expansion in powers of w,

$$\begin{aligned} G_{\mu\nu}(X) &= e^{-4u}\left(g_{\mu\nu}(x) + we^{4u}g_{1\mu\nu}(x) + \frac{1}{2}(we^{4u})^2 g_{2\mu\nu}(x) + \cdots\right) \\ \Phi(X) &= e^{2u}\left(\phi(x) + we^{4u}\phi_1(x) + \frac{1}{2}(we^{4u})^2\phi_2(x) + \cdots\right) \\ S_i(X) &= e^{2u}\left(s_i(x) + we^{4u}s_{1i}(x) + \frac{1}{2}(we^{4u})^2 s_{2i}(x) + \cdots\right). \end{aligned} \tag{23}$$

The lowest modes in this expansion are the $(3+1)$-dimensional shadows of the $(4+2)$-dimensional fields, $[G_{MN}(X), \Phi(X), S_i(X)) \to (g_{\mu\nu}(x), \phi(x), s_i(x))]$. The additional fields in the w-expansion are called prolongations of the shadow for each field.

[4] In flat space, we begin with $W_{\mathrm{flat}} \equiv (X^2)_{\mathrm{flat}} = X^M X^N \eta_{MN}$ where η_{MN} is a flat metric in $4+2$ dimensions. We can always parametrize the six coordinates X^M in terms of another set of six coordinates (w, u, x^μ) such that $X^M X^N \eta_{MN} = w$. In curved space, $W(X)$ is a general scalar field that, by the 2T gauge transformations [32], can be gauge fixed to $W(X) = w$.

It was established in [32] that the prolongations are not independent but are all determined by the shadows $[g_{\mu\nu}(x), \phi(x), s_i(x)]$ up to gauge freedom. The dynamics of the shadows $[g_{\mu\nu}(x), \phi(x), s_i(x)]$ (given by the A equations in Appendix A 3) as derived from the parent 2T action $(\mathrm{SM}+\mathrm{GR}+\mathrm{CS})_{4+2}$ are also reproduced by the $(3+1)$-dimensional shadow action $(\mathrm{SM}+\mathrm{GR}+\mathrm{CS})_{3+1}$. The holographic conformal shadow action in $3+1$ looks like the familiar relativistic field theory with one additional requirement: namely, that it is conformally scale invariant in the special form exhibited in (3) and (6).

We now turn to the similar derivation of the conformal shadow for the case of the CS action $S^{4+2}_{\mathrm{CS-GR}} \to S^{3+1}_{\mathrm{CS-GR}}$. Using the expansions in powers of w for $[G_{MN}(X), \Phi(X), S_i(X)]$ given above, we need to evaluate the derivatives of these fields in order to compute $R^M_{NM_1M_2}$ and $A^{\mathrm{GR}}_{M_5M_6}$. In particular, for $A^{\mathrm{GR}}_{M_5M_6}$ one must evaluate $\partial_M W$ and $\partial_M(\ln\Phi) = \partial_M \ln\phi(x, we^{4u})$ by using the chain rule. Since $W(X) = W(w,u,x) = w$ is a function of only the coordinate w, the derivative $\partial_M W$ vanishes for directions M except for $M = w$,

$$\partial_M W(X) = \frac{\partial w}{\partial X^M}\frac{\partial}{\partial w} W(w,u,x) = \delta^w_M. \tag{24}$$

Similarly,

$$\begin{aligned}\partial_N \ln(\Phi) &= \left(\frac{\partial w}{\partial X^N}\frac{\partial}{\partial w} + \frac{\partial u}{\partial X^N}\frac{\partial}{\partial u} + \frac{\partial x^\mu}{\partial X^N}\frac{\partial}{\partial x^\mu}\right)\ln\left(e^{2u}\tilde\phi(x, we^{4u})\right)\\ &= \left(2\delta^u_N + \frac{\tilde\phi'(x,we^{4u})}{\tilde\phi(x,we^{4u})}e^{4u}(\delta^w_N + 4w\delta^u_N) + \delta^\mu_N \frac{\partial_\mu\tilde\phi(x,we^{4u})}{\tilde\phi(x,we^{4u})}\right),\end{aligned} \tag{25}$$

where $\tilde\phi'(x, we^{4u})$ is the derivative of $\tilde\phi$ with respect to the argument we^{4u}. The resulting expression for A^{GR}_{MN} is

$$A^{\mathrm{GR}}_{MN} = \frac{1}{4}\left(\delta^w_M\partial_N(\ln\Phi) - \delta^w_N\partial_M(\ln\Phi)\right) f_{\mathrm{GR}}\left(\frac{S_i}{\Phi}\right).$$

Components of the asymmetric tensor A^{GR}_{MN} that do not contain $M = w$ or $N = w$ vanish because of the factor δ^w_M. Moreover, the component A_{ww} also vanishes because of the antisymmetry in $M \leftrightarrow N$,

$$A^{\mathrm{GR}}_{ww} = 0, \qquad A^{\mathrm{GR}}_{uu} = 0, \qquad A^{\mathrm{GR}}_{u\nu} = -A^{\mathrm{GR}}_{\mu u} = 0 \quad \text{and} \quad A^{\mathrm{GR}}_{\mu\nu} = 0. \tag{26}$$

The vanishing condition $W(X) = w \to 0$ due to the delta function should be implemented only after evaluating all the derivatives ∂_w with respect to w, as done above. The remaining components of A^{GR}_{MN} in the limit of $w \to 0$ are

$$\begin{aligned}A^{\mathrm{GR}}_{wu} &= -(A^{\mathrm{GR}}_{uw}) = \frac{1}{4}\left(2 + 4we^{4u}\frac{\tilde\phi'(x,we^{4u})}{\tilde\phi(x,we^{4u})}\right) f_{\mathrm{GR}}\left(\frac{\tilde s_i(x,we^{4u})}{\tilde\phi(x,we^{4u})}\right) \xrightarrow{w\to 0} \frac{1}{2} f_{\mathrm{GR}}\left(\frac{s_i(x)}{\phi(x)}\right),\\ A^{\mathrm{GR}}_{w\nu} &= -(A^{\mathrm{GR}}_{\mu w}) = \frac{1}{4}\frac{\partial_\mu\tilde\phi(x,we^{4u})}{\tilde\phi(x,we^{4u})} f_{\mathrm{GR}}\left(\frac{\tilde s_i(x,we^{4u})}{\tilde\phi(x,we^{4u})}\right) \xrightarrow{w\to 0} \frac{1}{4}\frac{\partial_\mu\phi(x)}{\phi(x)} f_{\mathrm{GR}}\left(\frac{s_i(x)}{\phi(x)}\right).\end{aligned} \tag{27}$$

Inserting this form of $A^{\mathrm{GR}}_{M_5M_6}$ in the $S^{4+2}_{\mathrm{CS-GR}}$ action in Eq. (17), we obtain

$$\begin{aligned}(\tilde R R A) &= \frac{1}{2}\varepsilon^{M_1M_2M_3M_4M_5M_6} R^M_{NM_1M_2} R^N_{MM_3M_4} A^{\mathrm{GR}}_{M_5M_6}\\ &= \frac{1}{2}\varepsilon^{M_1M_2M_3M_4wu} R^M_{NM_1M_2} R^N_{MM_3M_4} 2A^{\mathrm{GR}}_{wu} - \frac{1}{2}\varepsilon^{M_1M_2M_3M_4\mu_5 w} R^M_{NM_1M_2} R^N_{MM_3M_4} 2A^{\mathrm{GR}}_{\mu_5 w}.\end{aligned} \tag{28}$$

Because of antisymmetry properties of the Levi-Civita tensor, in the first term $M_1M_2M_3M_4$ must all point in the $3+1$ directions $\mu_1\mu_2\mu_3\mu_4$, while in the second term at least one of $M_1M_2M_3M_4$ must point in the u direction. Therefore, we can simplify

$$(\tilde R R A) = \frac{1}{2}\varepsilon^{\mu_1\mu_2\mu_3\mu_4}\left(R^M_{N\mu_1\mu_2} R^N_{M\mu_3\mu_4} 2A_{wu} - R^M_{N\mu_1\mu_2} R^N_{M\mu_3 u} 8A^{\mathrm{GR}}_{\mu_5 w}\right). \tag{29}$$

The next task is to compute the components of the Riemann tensors $R^M_{NM_1M_2}(G), R^N_{MM_3M_4}(G)$ for the $4+2$ metric G_{MN} given in Eq. (21) and express the result in terms of the shadow $g_{\mu\nu}(x)$ and the prolongations $g_{1\mu\nu}(x), g_{2\mu\nu}(x)$, etc. as fields in $3+1$ dimensions. Borrowing from Eqs. (6.10)–(6.11) in [32], we list the nonvanishing components of $R^M_{NM_1M_2}(G)$ with one upper index in the limit $w\to 0$,

$$\begin{aligned}
R^u{}_{\rho\mu_1 w}(G) &= \frac{e^{4u}}{4}\left(g^\sigma_{1\mu_1}g_{1\sigma\rho} - 2g_{2\mu_1\rho}\right),\\
R^u{}_{\rho\mu_1\mu_2}(G) &= \frac{1}{2}\left(\nabla_{\mu_1}g_{1\mu_2\rho} - \nabla_{\mu_2}g_{1\mu_1\rho}\right),\\
R^\lambda{}_{w\mu_1 w}(G) &= \frac{e^{8u}}{4}\left(g^{\lambda\sigma}_1 g_{1\sigma\mu_1} - 2g^\lambda_{2\mu_1}\right),\\
R^\lambda{}_{w\mu_1\mu_2}(G) &= \frac{e^{4u}}{2}\left(\nabla_{\mu_1}g^\lambda_{1\mu_2} - \nabla_{\mu_2}g^\lambda_{1\mu_1}\right),\\
R^\lambda{}_{\rho\mu_1 w}(G) &= \frac{e^{4u}}{2}g^{\lambda\sigma}\left(\nabla_\sigma g_{1\rho\mu_1} - \nabla_\rho g_{1\sigma\mu_1}\right)\\
R^\lambda{}_{\rho\mu_1\mu_2}(G) &= \left(R^\lambda{}_{\rho\mu_1\mu_2}(g) - g^\lambda_{1[\mu_1}g_{\mu_2]\rho} - \delta^\lambda_{[\mu_1}g_{1\mu_2]\rho}\right).
\end{aligned} \tag{30}$$

These expressions include the prolongations of the shadow that survive the $w\to 0$ limit after taking ∂_w derivatives. Here ∇_μ is the covariant derivative with respect to the $3+1$ shadow metric $g_{\mu\nu}(x)$ and $R^\lambda{}_{\rho\mu_1\mu_2}(g)$ is its Riemann tensor, but the shadow $R^\lambda{}_{\rho\mu_1\mu_2}(g)$ is only the first part of $R^\lambda{}_{\rho\mu_1\mu_2}(G)$ as seen in the last term of Eq. (30). Moreover, any upper index on the prolongations, such as $g^\lambda_{1\mu_1}$, $g^{\lambda\sigma}_1$ or $g^\lambda_{2\mu_1}$, was obtained by using the inverse shadow metric $g^{\lambda\sigma}$, such as $g^\lambda_{1\mu_1}\equiv g^{\lambda\sigma}g_{1\sigma\mu_1}$, etc.

Inserting these results in Eq. (29), we see that in the sums over M and N, only the last curvature $R^\lambda{}_{\rho\mu_1\mu_2}(G)$ contributes, so the second term in (29) drops out, and the first term yields $(\tilde{R}RA) = \frac{1}{2}\varepsilon^{\mu_1\mu_2\mu_3\mu_4}R^\lambda{}_{\rho\mu_1\mu_2}(G)R^\rho{}_{\lambda\mu_3\mu_4}(G)f_{\rm GR}(s_i/\phi)$. In this expression, $R^\lambda{}_{\rho\mu_1\mu_2}(G)$ given in (30) includes contributions from the prolongation $g_{1\mu\nu}$ and $g_{2\mu\nu}$ in Eq. (30). However, these pieces drop out in the sums over μ_1,μ_2,μ_3,μ_4 due to the complete antisymmetric nature of $\varepsilon^{\mu_1\mu_2\mu_3\mu_4}$ versus the symmetric nature of $g_{1\mu\nu}$ and $g_{2\mu\nu}$. In the shadow action $S^{3+1}_{\rm CS-GR}$, only the shadow fields $[g_{\mu\nu}(x),\phi(x),s_i(x)]$ survive while all prolongations drop out. Thus, the final result contains only the shadow $R^\lambda{}_{\rho\mu_1\mu_2}(g)$ piece of $R^\lambda{}_{\rho\mu_1\mu_2}(G)$,

$$(\tilde{R}RA) = \frac{1}{2}\varepsilon^{\mu_1\mu_2\mu_3\mu_4}R^\lambda{}_{\rho\mu_1\mu_2}(g) \qquad R^\rho{}_{\lambda\mu_3\mu_4}(g)f_{\rm GR}(s_i/\phi). \tag{31}$$

At this point, the density $(\tilde{R}RA)$ is independent of u as well as w. Therefore, in the volume element in Eq. (20), the u and w integrations are performed, and the resulting constant coefficient is canceled against γ as done in every term of the full action [31], to obtain the conformal shadow action in $3+1$ dimensions,

$$\begin{aligned}
S^{4+2}_{\rm CS-GR} &= \gamma\int d^{4+2}X\delta(W(X))\frac{1}{2}\epsilon^{M_1M_2M_3M_4M_5M_6}R^M_{NM_1M_2}R^N_{MM_3M_4}A^{\rm GR}_{M_5M_6}\\
\overset{\rm Shadow}{\Rightarrow} S^{3+1}_{\rm CS-GR} &= \int d^4x\frac{1}{2}\varepsilon^{\mu_1\mu_2\mu_3\mu_4}R^\lambda{}_{\rho\mu_1\mu_2}(g)R^\rho{}_{\lambda\mu_3\mu_4}(g)f_{\rm GR}(s_i/\phi).
\end{aligned} \tag{32}$$

This proves that the $3+1$ Chern-Simons action term in Eq. (6) indeed emerges as a holographic image of the $4+2$ action term in Eq. (17).

It should be emphasized that the $(3+1)$-dimensional shadow degrees of freedom are self-sufficient to describe the gauge invariant physical phenomena in $3+1$ dimensions. However, as seen in Eq. (30) there are prolongations of the shadow that describe nontrivial phenomena occurring in the extra $1+1$ dimensions. The prolongations of the metric $g_{1\mu\nu}(x)$ and $g_{2\mu\nu}(x)$ are fully determined by the shadow $g_{\mu\nu}$ as discussed in [31]. Therefore, the shadow determines all the prolongations of the geometry shown in (30) and similarly for the prolongations of all other field degrees of freedom Φ, S_i, Ψ, A^a_M.

The fact that there are (Kaluza-Klein type) prolongations in the extra dimensions as seen in Eq. (30) is also noted for all shadows (beyond the conformal shadow discussed in this section). This is also a common feature in the classical and quantum phase spaces in various shadows of a single particle in $4+2$ dimensions as displayed in [19,21,26]. All of this, including the multidualities among the shadows, are indirect indications of the existence of the extra dimensions. All prolongations are determined by the shadows, so unlike the usual Kaluza-Klein setting the prolongations are not independent degrees of freedom. However, there may be ways to analyze physical effects associated with the prolongations, thus probing the extra dimensions directly.

C. The emergent Weyl symmetry in 3+1

We seize this moment to elucidate the $4+2$ genesis of local conformal scale (Weyl) symmetry within the conformal shadow. Recall that the $4+2$ parent actions, denoted as $S^{4+2}_{\rm CS-GR}$ and $S^{4+2}_{\rm SM+GR}$, are invariant under general coordinate transformations in $4+2$ dimensions, albeit lacking Weyl symmetry. During the process of gauge fixing and solving the kinematic constraints, a portion of the general coordinate reparametrizations is also fixed. Nevertheless, within the conformal shadow, alongside the residual reparametrization symmetry, there persists an additional symmetry stemming from reparametrizations that mix the extra $1+1$ dimensions with the $3+1$ dimensions. This residual $4+2$ symmetry manifests as the $3+1$ Weyl transformations delineated in Eq. (7).

To illustrate this, let us examine the solutions of the kinematic B and C constraints for $G_{MN}(w,u,x)$, $\Phi(w,u,x)$, and $S_i(w,u,x)$ as provided in Eqs. (22) and (23). We apply a specific general coordinate transformation that involves mixing u with x^μ as follows: $u\to(u-\lambda(x)/2)$ where $\lambda(x)$ represents an arbitrary function of the $3+1$ spacetime coordinates x^μ. Under this transformation, the fields transform as follows: $\Phi(w,u,x)\to\Phi(w,(u-\frac{1}{2}\lambda(x)),x)$, and similarly for $S_i(w,u,x)$. While $G_{MN}(w,u,x)$ typically transforms as a tensor, under this particular transformation, the gauge-fixed form in (21) remains unchanged. Instead, only $g_{\mu\nu}(w,u,x)$ transforms into $g_{\mu\nu}(w,(u-\frac{1}{2}\lambda(x)),x)$. By substituting the w series expansion of Eq. (23), we obtain

$$\Phi(w,u,x)\to\Phi\left(w,\left(u-\frac{1}{2}\lambda(x)\right),x\right)=e^{2u-\lambda(x)}\left(\phi(x)+we^{4u-2\lambda(x)}\phi_1(x)+\frac{1}{2}(we^{4u-2\lambda(x)})^2\phi_2(x)+\cdots\right). \tag{33}$$

This is equivalent to transforming the shadow and prolongation fields $(\phi(x),\phi_1(x),\phi_2(x),\cdots)$ including all higher modes as follows:

$$\begin{aligned}
&\phi(x)\to\phi(x)e^{-\lambda(x)}, \qquad \phi_1(x)\to\phi_1(x)e^{-3\lambda(x)}, \qquad \phi_2(x)\to\phi_2(x)e^{-5\lambda(x)},\cdots\\
&s(x)\to s(x)e^{-\lambda(x)}, \qquad s_1(x)\to s_1(x)e^{-3\lambda(x)}, \qquad s_2(x)\to s_2(x)e^{-5\lambda(x)},\cdots\\
&g_{\mu\nu}(x)\to g_{\mu\nu}(x)e^{2\lambda(x)}, \qquad g_{1\mu\nu}(x)\to g_{1\mu\nu}e^{0\lambda(x)}, \qquad g_{2\mu\nu}(x)\to g_{2\mu\nu}e^{-2\lambda(x)},\cdots
\end{aligned} \tag{34}$$

These Weyl-type local scale transformations in $3+1$ dimensions are assured to be symmetries of the parent actions $S^{4+2}_{\text{CS-GR}}$ and $S^{4+2}_{\text{SM+GR}}$ as they correspond to specific general coordinate transformations in $4+2$ dimensions. It becomes apparent that the first column in Eq. (34) represents the Weyl transformation of the shadow fields $(\phi,s,g_{\mu\nu})$, constituting the local scale symmetry of the shadow actions $S^{3+1}_{\text{CS-GR}}$ and $S^{3+1}_{\text{SM+GR}}$. This demonstrates that the local Weyl symmetry in $3+1$ relativistic field theory in Eq. (4) is synonymous with a distinct general coordinate transformation in $4+2$ dimensions, $\Omega(x)=e^{\lambda(x)}$, entwining the extra $1+1$ dimensions u, w with the $3+1$ dimensions x^μ while keeping $w=0$.

D. (4 + 2)-dimensional QCD and QED CS actions

In this section, we discuss the QCD and QED formalisms of Chern-Simons theory. In [30], it was demonstrated that Weyl symmetric QCD action emerges from the action $S^{4+2}_{\text{SM+GR}}$ given by Eq. (16) in the absence of the Chern-Simons corrections. Therefore, constructing a locally scale invariant formalism of Chern-Simons theory in QCD amounts to determining the structural forms required for the modified QCD Chern-Simons term to preserve the local scale invariance of the full action.

It is known from 1T physics that the QCD formalism of Chern-Simons theory is structurally similar to its gravitational counterpart. We will exploit this correspondence to construct the QCD Chern-Simons term. For QCD the pseudoscalar axion field $a(x)$, which is one of the possible fields in $s_i(x)$ in the $\text{i(SM}+\text{GR)}_{3+1}$, couples to the QCD instanton density $\tilde{F}F$ instead of the Pontryagin density $\tilde{R}R$. By extension of the arguments presented in Sec. II, we expect that a locally scale invariant QCD Chern-Simons term in $3+1$ dimensions will involve a function $f_{\text{QCD}}(a/\phi)$ linearly coupled to the instanton density $\tilde{F}F$,

$$\begin{aligned}
S^{3+1}_{\text{CS-QCD}}&=\int d^4x f_{\text{QCD}}(a/\phi)\tilde{F}F,\\
\tilde{F}F&=\frac{1}{2}\epsilon^{\mu_1\mu_2\mu_3\mu_4}F^a_{\mu_1\mu_2}F^b_{\mu_3\mu_4}\eta_{ab},
\end{aligned} \tag{35}$$

where η_{ab} is the Killing metric for the gauge group SU(3), and $F^a_{\mu_1\mu_2}$ is the QCD Yang-Mills field strength. To match to previous work on this case [12], we should take a linear function of the axion field a,

$$f_{\text{QCD}}(a/\phi)=\frac{a}{\phi}c, \tag{36}$$

where c is a dimensionless constant. The field $\phi(x)$ turns into a constant $\phi(x)\to\phi_0$ when the local Weyl symmetry is gauge fixed to produce the familiar term [12] in the previously Weyl noninvariant $\text{SM}+\text{GR}$. This structure shows the hidden conformal symmetry including the Weyl symmetry and its relation to the underlying $(4+2)$-dimensional spacetime.

We now consider how this $(3+1)$-dimensional QCD Chern-Simons term can be derived from a $(4+2)$-dimensional parent term. By extension of the arguments presented in Sec. III, the $(4+2)$-dimensional parent is expected to take the form,

$$S^{4+2}_{\mathrm{CS-QCD}} = \int d^{4+2}X\delta(W(X))(\tilde{F}FA),$$
$$(\tilde{F}FA) \equiv \frac{1}{2}\epsilon^{M_1M_2M_3M_4M_5M_6}\eta_{ab}F^a_{M_1M_2}F^b_{M_3M_4}A^{\mathrm{QCD}}_{M_5M_6}. \quad (37)$$

The $F^a_{M_1M_2}$ is given by

$$F^a_{M_1M_2} = \frac{\partial A^a_{M_2}}{\partial X^{M_1}} - \frac{\partial A^a_{M_1}}{\partial X^{M_2}} + gf^a{}_{bc}A^b_{M_1}A^c_{M_2}, \quad (38)$$

where $A^a_M(X)$ is the $4+2$ parent of the $3+1$ Yang-Mills gauge field $A^a_\mu(x)$. The $A^{\mathrm{QCD}}_{M_5M_6}$ in Eq. (37) is given by

$$A^{\mathrm{QCD}}_{M_5M_6} = \frac{1}{2}\left(\frac{\partial W}{\partial X^{M_5}}\frac{\partial(\ln\Phi)}{\partial X^{M_6}} - \frac{\partial W}{\partial X^{M_6}}\frac{\partial(\ln\Phi)}{\partial X^{M_5}}\right)f_{\mathrm{QCD}}\left(\frac{\mathcal{A}}{\Phi}\right), \quad (39)$$

where $\mathcal{A}(X)$ is the $4+2$ parent of the $3+1$ pseudoscalar axion field $a(x)$ that appears in S^{3+1}_{QCD}. We can follow a similar procedure as outlined in Sec. III to reduce the $(4+2)$-dimensional fields, field strength tensors, and antisymmetric tensor $(\Phi, \mathcal{A}, F^a_{M_1,M_2}, A^{\mathrm{QCD}}_{MN})$ to their $(3+1)$-dimensional counterparts $[\phi, a, F^a_{\mu_1\mu_2}, f_{\mathrm{QCD}}(a/\phi)]$. This procedure yields the action in Eq. (35) as a holographic image of Eq. (37).

Similar considerations apply to anomalous terms in QED where instead of the axion a, we have the neutral pion π^0 (or similar relevant hadrons) and instead of the QCD field strength $F^a_{M_1M_2}(X)$, we have the QED field strength $F_{M_1M_2}(X)$.

IV. P AND *CP* VIOLATION

We will now discuss the parity (P) and charge parity (*CP*) conjugation properties of our locally scale invariant Chern-Simons action. In 1T physics, the gravitational, QCD and QED Chern-Simons actions terms are expected to satisfy certain transformation properties under parity and *CP* transformations. The goal of this section is to determine how the functions $f_{\mathrm{GR}}, f_{\mathrm{QCD}}$ and f_{QED} need to be restricted for consistency with the established 1T transformation properties.

We previously obtained the following Weyl invariant $3+1$ Chern-Simons action terms for gravity, QCD and QED in Eqs. (6), (35):

$$S^{3+1}_{\mathrm{CS-GR}} = \int d^4x f_{\mathrm{GR}}(s_i/\phi)\tilde{R}R, \qquad \tilde{R}R = \frac{1}{2}\epsilon^{\mu_1\mu_2\mu_3\mu_4}R^\lambda{}_{\sigma\mu_1\mu_2}R^\sigma{}_{\lambda\mu_3\mu_4},$$
$$S^{3+1}_{\mathrm{CS-QCD}} = \int d^4x f_{\mathrm{QCD}}(a/\phi)(\tilde{F}F)_{\mathrm{QCD}}, \qquad (\tilde{F}F)_{\mathrm{QCD}} = \frac{1}{2}\epsilon^{\mu_1\mu_2\mu_3\mu_4}F^a_{\mu_1\mu_2}F^b_{\mu_3\mu_4}\eta_{ab},$$
$$S^{3+1}_{\mathrm{CS-QED}} = \int d^4x f_{\mathrm{QED}}(\pi^0/\phi)(\tilde{F}F)_{\mathrm{QED}}, \qquad (\tilde{F}F)_{\mathrm{QED}} = \frac{1}{2}\epsilon^{\mu_1\mu_2\mu_3\mu_4}F_{\mu_1\mu_2}F_{\mu_3\mu_4}. \quad (40)$$

In $3+1$ dimensions, $\tilde{R}R, (\tilde{F}F)_{\mathrm{QCD}}, (\tilde{F}F)_{\mathrm{QED}}$ flip signs under parity (P) and charge plus parity (*CP*) transformations. Hence, the P and *CP* transformation characteristics of the aforementioned CS action terms hinge crucially on the nature of the functions $f_{\mathrm{GR}}(s_i/\phi)$, $f_{\mathrm{QCD}}(a/\phi)$, and $f_{\mathrm{QED}}(\pi^0/\phi)$. Notably, $f_{\mathrm{QED}}(\pi^0/\phi)$ is a recognized P and *CP*-odd function, as detailed below, rendering $S^{3+1}_{\mathrm{CS-QED}}$ invariant under P and *CP* transformations. Moreover, it is contended that $f_{\mathrm{QCD}}(a/\phi)$ also exhibits P and *CP* odd behavior, contingent upon the existence of the axion field $a(x)$ associated with the Peccei-Quinn symmetry in the strong interactions [12]. In such an instance, $S^{3+1}_{\mathrm{CS-QCD}}$ retains P and *CP* invariance, akin to $S^{3+1}_{\mathrm{CS-QED}}$. However, in the event of the axion's absence, theoretical assumptions regarding the P and *CP* properties of f_{QCD} become uncertain. It could potentially function as a dimensionless quantity dependent on all s_i/ϕ, notwithstanding experimental observations indicating its nonexistence or extreme rarity, leaving the underlying reasons for its minute magnitude unresolved, akin to other unexplained hierarchies. Lastly, the function $f_{\mathrm{GR}}(s_i/\phi)$ remains unconstrained by either theoretical postulations or experimental evidence, rendering its P and *CP* properties undetermined.

In more detail, the action $S^{3+1}_{\mathrm{CS-QED}}$ is not part of the fundamental renormalizable action, but it arises in quantum loop corrections and is sometimes included in effective actions involving certain hadrons that decay into two photons. The coefficient f_{QED} that arises from the chiral triangle anomaly was computed reliably in perturbative QCD, and the result agrees quantitatively with the measurement of the decay of the neutral pion into two photons. Therefore, we conclude that $f_{\mathrm{QED}}(\pi^0/\phi)$ in the Weyl invariant $\mathrm{i(SM+GR)}_{3+1}$, when taken at low energies in the c-gauge $\phi(x) \to \phi_0$, is already known in the form $f_{\mathrm{QED}}(\pi^0/\phi_0) = (c/\phi_0)\pi^0(x)$ where the dimensionful constant (c/ϕ_0) is given by $\alpha/(16\pi f_\pi)$ in terms of measured quantities: namely, the dimensionless $\alpha = 1/137$ is the electromagnetic fine structure constant, and the dimensionful f_π is the pion decay constant. This fixes the unknown dimensionless coefficient c. Therefore, before fixing the Weyl gauge we can write the fully determined dimensionless and scale invariant f_{QED} as

$$f_{\mathrm{QED}}\left(\frac{\pi^0}{\phi}\right) = \alpha \frac{\phi_0}{16\pi f_\pi}\frac{\pi^0(x)}{\phi(x)}. \qquad (41)$$

In the c-gauge, we observe that $\phi_0/\phi_c(x) = 1$, yielding $f_{\mathrm{QED}} = \alpha \frac{1}{16\pi f_\pi}\pi_c^0(x)$ wherein $\pi_c^0(x)$ represents the measured and interpreted pion field at low energies. Significantly, the gauge-invariant form outlined in (41) proves instrumental in bridging the low-energy description within the c-gauge to all regions encompassed by the geodesically complete theory, facilitated by the Weyl gauge-invariant ratios $\pi^0(x)/\phi(x) = \pi_c^0(x)/\phi_0$. This correlation enables computations of $\pi^0(x)/\phi(x)$ in proximity to gravitational singularities to be directly linked to the observed low-energy behavior.

In QCD, $S^{3+1}_{\mathrm{CS-QCD}}$ is not part of the fundamental renormalizable action unless f_{QCD} is a constant. However, since it violates P and CP in the strong interactions it is not introduced as a fundamental constant. Nevertheless, f_{QCD} arises in quantum loop corrections when the electroweak interactions are coupled to quarks although the nonperturbative nature of strong interactions renders the computation of f_{QCD} elusive, relegating it to an enigmatic effective term. In $\mathrm{i(SM+GR)}_{3+1}$ $f_{\mathrm{QCD}}(s_i/\phi)$ must be a general dimensionless function due to the Weyl symmetry. Given the presence of P and CP violations in the weak interactions that are part of the SM, it is hard to understand why such f_{QCD} must be unnaturally minute to reconcile the lack of P and CP violation in experiments involving the neutron's electric dipole moment. This discrepancy raises the fundamental issue of elucidating why f_{QCD} should be tuned to such diminutive proportions while f_{QED} in QED remains non-negligible.

However, under the auspices of a Peccei-Quinn symmetry in the strong interactions [12], f_{QCD} assumes linearity with respect to the axion field $a(x)$. In the realm of Weyl symmetric theory, it adopts the form $f_{\mathrm{QCD}}(a/\phi) = c \times (a(x)/\phi(x))$, where ϕ is symmetric under P and CP while $a(x)$ represents the pseudoscalar axion field, exhibiting oddness under both P and CP. This configuration yields the density $f_{\mathrm{QCD}} \times (\tilde{F}F)_{\mathrm{QCD}}$ that conserves both P and CP symmetries. This formulation resolves the strong CP problem by leveraging the PQ symmetry to assert the vanishing vacuum expectation value of the axion, thereby ensuring conservation of P and CP in QCD, akin to QED's conservation involving the pion.

In the absence of PQ symmetry or its associated axion, the negligible size of f_{QCD} is possible, but it is unexplained like a few other unnatural hierarchy problems in the Standard Model.

In the case of gravity, a more general function $f_{\mathrm{GR}}(s_i/\phi)$ that is not purely odd or purely even under P and CP is permitted within existing experimental constraints. Moreover, since GR is already nonrenormalizable, the action $S^{3+1}_{\mathrm{CS-GR}}$ is not prevented from appearing in the action like Eq. (14) with its own independent parameters, rather than being computed in quantum loops. If $f_{\mathrm{GR}}(s_i/\phi)$ is odd under P and CP, then the $f_{\mathrm{GR}}(s_i/\phi)\tilde{R}R$ CS density is symmetric under P and CP. On the other hand, if there are only two spin 0 fields (ϕ, s), recalling that both ϕ and the Higgs boson s are both P and CP even, then $f_{\mathrm{GR}}(s/\phi)$ is automatically even under both P and CP. In that case, the CS term $f_{\mathrm{GR}}(s/\phi)\tilde{R}R$ violates both P and CP.

Future phenomenological studies are needed to constrain the exact forms of the functions $f_{\mathrm{GR}}(s/\phi)$, $f_{\mathrm{QCD}}(a/\phi)$. More generally, $f_{\mathrm{GR}}(s_i/\phi)$ and $f_{\mathrm{QCD}}(s_i/\phi)$ may involve several more fundamental spin-0 fields beyond the Higgs boson (and the axion) if more fields are present in nature.

V. DISCUSSION AND OUTLOOK

We have successfully modified the Chern-Simons terms in gravity, QCD, and QED to accommodate local conformal scale (Weyl) invariance. Commencing with the conformally improved $\mathrm{i(SM+GR)}_{3+1}$ [33], we extended this framework by formulating Weyl symmetric CS terms.

First, we devised a Weyl invariant formulation of the Chern-Simons term in gravity directly in $3+1$ dimensions. We found that the local scale symmetry of the $(\mathrm{SM}+\mathrm{GR})_{3+1}$ model could be upheld by linearly coupling the Pontryagin density $\tilde{R}R$ in the Chern-Simons term to a function of the ratio of conformally coupled scalar fields, $f_{\mathrm{GR}}(s_i/\phi)$, as in Eqs. (6) and (12). This form of $f_{\mathrm{GR}}(s_i/\phi)$ proved indispensable in achieving local conformal scale invariance. Similarly, we discerned that the Chern-Simons terms in QCD and QED could attain local Weyl symmetry if the density $\tilde{F}F$ linearly couples to a function $f_{\mathrm{QCD}}(s_i/\phi)$ or $f_{\mathrm{QED}}(s_i/\phi)$ as in Eqs. (40).

The physical interpretation at low energies unfolds in the c-gauge, where the $\mathrm{i(SM+GR)}_{3+1}$ model virtually coincides with the customary SM + GR as discussed in Sec. II C. In the c-gauge of $(\mathrm{SM}+\mathrm{GR}+\mathrm{CS})_{3+1}$, we denote all fields with the letter "c" such as $[\phi_c(x), s_{ci}(x), g_{c\mu\nu}]$ to differentiate them from other gauges. The c-gauge is delineated by the gauge choice $\phi_c(x) = \phi_0$ where ϕ_0 remains constant across all x^μ within our spacetime patch as observers situated outside all gravitational singularities. Within this spacetime patch, the c-gauge scalars $s_{ci}(x)$ represent the spin 0 fields (e.g., the Higgs boson, axion, etc.) as measured from our cosmological and accelerator physics perspectives, conforming to their formulation in the standard

SM + GR. The invariance of field ratios under Weyl gauge transformations enables us to express $s_i(x)/\phi(x) = s_{ci}(x)/\phi_0$. On the left-hand side the fields are in any gauge, on the right-hand side, we can measure both $s_{ci}(x)$ and ϕ_0 as explained in Sec. II C. This relationship facilitates the linkage of the low-energy interpretation of the scalars $s_{ci}(x)$ from the low-energy gravity patch to the geodesically complete full spacetime in any gauge encompassing the vicinity of every singularity and antigravity patches behind each singularity.

Thus, computations can be executed in any convenient gauge within the complete spacetime of $(\mathrm{SM}+\mathrm{GR}+\mathrm{CS})_{3+1}$, and the gauge-invariant information $s_i(x)/\phi(x)$ can be translated into the low-energy language of the c-gauge [refer to the example in Eq. (41)]. Other gauge-invariant ratios s_i/s_j can be reexpressed in terms of s_i/ϕ.

It is evident that at this stage, only the Weyl invariant Higgs field $s/\phi = \sqrt{2H^\dagger H/\phi^2}$ is confirmed to exist in nature. Thus, we ascertain that at least two scalar fields (ϕ, H) are constituents of the Weyl invariant theory $(\mathrm{SM}+\mathrm{GR}+\mathrm{CS})_{3+1}$. This theory adeptly encapsulates all known facets of particle physics and gravity in any gauge. This comprehensiveness stems from our understanding that in our low-energy spacetime patch, the gauge-invariant $s(x)/\phi(x) \sim 10^{-17}$ is minuscule, and in this limit, the Weyl invariant theory closely mirrors the remarkably accurate conventional SM + GR.

In Sec. III, we determined the $(4+2)$-dimensional counterparts for the $3+1$ CS terms. Recognizing that the locally conformally invariant $\mathrm{i}(\mathrm{SM}+\mathrm{GR})_{3+1}$ serves as a holographic image of a $(4+2)$-dimensional field theoretical action, we delved into the $(4+2)$-dimensional counterparts of the $3+1$ CS terms within the 2T physics formalism. Successfully formulating the actions $S^{4+2}_{\mathrm{CS-GR}}$ for gravity, QCD, and QED, we demonstrated how the emergent $3+1$ actions $S^{3+1}_{\mathrm{CS-GR}}$ are derived as a holographic shadow of 2T physics.

In the process, we clarified how Weyl symmetry in $3+1$ dimensions in the complete action $(\mathrm{SM}+\mathrm{GR}+\mathrm{CS})_{3+1}$ arises from a more fundamental general coordinate invariance in the $(4+2)$-dimensional parent theory $(\mathrm{SM}+\mathrm{GR}+\mathrm{CS})_{4+2}$. It is crucial to note that Weyl symmetry was not among the gauge symmetries of the $(4+2)$-dimensional $S^{4+2}_{\mathrm{SM+GR+CS}}$ actions; rather, it emerged in the $3+1$ holographic conformal shadow from general coordinate transformations in $4+2$ dimensions that intertwine the apparent $3+1$ dimensions with the concealed $1+1$ large (not curled up) dimensions.

Thus, the genesis of Weyl symmetry within our work bears the hallmark of 2T physics. While reminiscent of the original Weyl transformations, it diverges conceptually since Weyl's concepts bore no relation to the additional hidden dimensions pertinent to $4+2$ spacetime. Furthermore, Weyl's geometry [45] included a physical vector gauge field degree of freedom that is absent in our scenario.

The incorporation of local scale symmetry renders our improved $(\mathrm{SM}+\mathrm{GR}+\mathrm{CS})_{3+1}$ action geodesically complete by incorporating additional spacetime behind gravitational singularities. This completeness is valid in the presence of Chern-Simons corrections to GR, QCD, and QED. The symmetric formalism furnishes analytical control through singularities and steers physical interpretation by tethering it to the low-energy interpretation of the fields. New physics manifests as the factor $[\phi^2(x) - s^2(x)]$ multiplying curvature in the action (14) can vanish and change sign in regions of spacetime where the gauge-invariant Higgs field $\sqrt{2H^\dagger H}/|\phi|$ burgeons to approach the value 1.

By examining the equations of motion, it becomes apparent that spacetime singularities manifest precisely when $\sqrt{2H^\dagger H}/|\phi| = 1$. In regions of spacetime where singularities may arise, strong gravity dominates, driven by the remarkable behavior of the effective spacetime-dependent gravitational strength $G(x)$ satisfying $16\pi G(x) = 12(\phi^2(x) - s^2(x))^{-1}$ which diverges. It is noteworthy that in proximity to such singularities, the magnitude of $\sqrt{2H^\dagger H}/|\phi|$ can approach unity even if the gauge-dependent fields ϕ and H are both either individually small, individually large, or intermediate. In previous studies in a cosmological setting, it was revealed that at the singularity both ϕ and H remarkably vanished at the big bang singularity where $\sqrt{2H^\dagger H}/|\phi| = 1$ [36], indicating that the electroweak $\mathrm{SU}(2)\times\mathrm{U}(1)$ symmetry is restored at the singularity. This and similar manifestations of new physics are completely unexpected in the conventional SM + GR.

Extending the leads of $(\mathrm{SM}+\mathrm{GR}+\mathrm{CS})_{3+1}$ and $(\mathrm{SM}+\mathrm{GR}+\mathrm{CS})_{4+2}$ toward an enhanced version of string theory that is capable of reproducing these Weyl symmetric and geodesically complete field theories at their low-energy limits stands as a promising avenue for future research. An essential requirement for such an improved string theory is the emergence of the dimensionful string tension from a field, mirroring how all dimensionful parameters arise from ϕ within the field theory framework. Progress in this direction was initiated in [18]. Accomplishing this objective would lay the groundwork for establishing a coherent quantum treatment of these innovative theories that ensures geodesic completeness.

Another intriguing avenue for investigation involves delving into the realms of field theory beyond the conformal shadows discussed in this study. This exploration not only facilitates the development of duality relationships among shadows but also offers avenues for crafting new computational tools within 1T-physics at a fundamental field theoretic level. It is conceivable that frameworks such as the AdS/CFT duality could find elucidation through this approach alongside the potential discovery of novel dualities. Numerous striking examples of multidualities already exist in classical and quantum physics as exemplified by [19–21]. The objective here is to develop analogous methodologies in the context of field theory for shadows derived from $(\mathrm{SM}+\mathrm{GR}+\mathrm{CS})_{4+2}$. Preliminary explorations of such field theory dualities and hidden symmetries related to the $4+2$ dimensions are illustrated with a few simple examples in [26]. These endeavors not only shed light on the existence and nature of the additional $1+1$ large dimensions but also underscore the significance of constructing the parent theory in $4+2$ dimensions as demonstrated in this paper for $(\mathrm{SM}+\mathrm{GR}+\mathrm{CS})_{4+2}$. Future investigations in this domain are bound to unveil deeper insights into the fabric of spacetime.

APPENDIX: AN OVERVIEW OF 2T PHYSICS

2T physics originated in 1998 from a fundamental gauge symmetry, $\mathrm{Sp}(2,R)$, manifest in the phase space (X^M, P_M) within a $(4+2)$-dimensional framework (more generally, $d+2$) [18–41]. The basic notion is the postulate that the fundamental rules of physics should be invariant under phase space transformations that generalize Einstein's general coordinate transformations. This was accomplished with the $\mathrm{Sp}(2,R)$ gauge symmetry that encapsulates three generators that are functions of phase space. These are represented as symmetric tensors $Q_{ij}(X,P)$ where $i=1,2$, and they conform to the closure condition under Poisson brackets, forming the $\mathrm{Sp}(2,R)$ Lie algebra. A plethora of such structures exists with an infinite variety of configurations [28,29]. Among them, the simplest set, delineated below, serves as a foundational example,

$$Q_{11}=\frac{1}{2}X\cdot X,\quad Q_{12}=\frac{1}{2}X\cdot P=Q_{21},\quad Q_{22}=\frac{1}{2}P\cdot P. \tag{A1}$$

The dot product in these elemental expressions can possess an arbitrary signature within any flat geometry, without impacting the Lie algebra. However, for physical viability, only the $4+2$ signature (or more generally, $d+2$ with $d\geq 2$) remains physically pertinent, as elucidated below.

1. 2T physics in the phase space of a single particle

Consider the action governing the worldline dynamics of a single spinless particle, imbued with $\mathrm{Sp}(2,R)$ gauge invariance in phase space,

$$L=\dot{X}(\tau)\cdot P(\tau)-\frac{1}{2}A^{ij}(\tau)Q_{ij}(X(\tau),P(\tau)), \tag{A2}$$

where $A^{ij}(\tau)$ represents the 2×2 symmetric Yang-Mills type $\mathrm{Sp}(2,R)$ gauge fields on the worldline and $Q_{ij}(X,P)$ denotes the 2×2 symmetric generators of $\mathrm{Sp}(2,R)$ transformations acting on the "matter" variables (X^M, P_M) [19–21].

Turning our attention to the gauge-invariant physical sector of the theory, corresponding to Sp(2,R) singlets invariant under Sp(2,R) transformations, we impose the condition that all generators of Sp(2,R) vanish: $Q_{ij}(X,P)=0$. This requirement, enforced by the equation of motion for A^{ij}, necessitates specific conditions on the phase space, exemplified by the simple form of Q_{ij} in Eq. (A1), $X^2=P^2=X\cdot P=0$. Importantly, for nontrivial solutions to exist, the flat metric in these dot products, η_{MN}, must possess two or more timelike directions.[5] However, if timelike directions exceed two, the gauge symmetry becomes insufficient for eliminating negative probability ghosts. Thus, to ensure nontrivial and ghost-free physical solutions that remain Sp(2,R) gauge-invariant, while adhering to unitary and causal scenarios, the phase space—including gauge degrees of freedom—must contain precisely two timelike dimensions—no more and no less [20].

Compared with 1T physics, which typically involves one constraint (e.g., $p^2+m^2=0$ for a freely moving relativistic particle, or generalizations), 2T physics imposes two additional constraints. This augmented gauge symmetry allows removal of one timelike and one spacelike dimension from the phase space via gauge fixing, and resolving two out of the three constraints. Consequently, 2T physics encompasses one extra timelike and one extra spatial dimension compared to 1T physics. The gauge-invariant physical sector in $4+2$ dimensions (more generally $d+2$) resembles 1T-physics in $3+1$ dimensions [more generally $(d-1)+1$].

In this formalism of theoretical physics, intriguing connections emerge between seemingly disparate phenomena. Among these connections are the shadows—alternate descriptions or formulations—of physical systems with different Hamiltonians as described below.

The solutions satisfying the simplest constraints [such as Eq. (A1)] are termed holographic shadows. These shadows encapsulate all 1T physics systems describable by Hamiltonians derived from a single particle's phase space. Thus, all 1T physics manifestations are unified into the trio of 2T physics equations, $X^2=P^2=X\cdot P=0$, highlighting a profound unification of all 1T physics Hamiltonian systems as emerging only from the $\mathrm{Sp}(2,R)$ constraints.

Shadow actions, representing gauge-fixed solutions of Eq. (A1), delineate 1T physics systems with varied Hamiltonians within the 1T formalism. The profusion of shadows arises from the myriad ways of embedding

[5]For 0 times, the solutions are $X^M=P^M=0$. For 1 time, both X^M and P^M must be lightlike and parallel to each other; hence, there is no angular momentum. Therefore, both cases describe trivial motions. With 2 or more times, there are an infinite number of solutions.

$(3+1)$-dimensional phase space (x^μ, p_μ) into $(4+2)$-dimensional phase space (X^M, P_M), leading to diverse gauge-fixing methods and resolutions of $\mathrm{Sp}(2,R)$ constraints. Consequently, multiple manifestations of the same $4+2$ phenomena emerge as shadows in the parlance of 1T physics. Remarkably, besides the three constraints $X^2 = P^2 = X \cdot P = 0$, no additional equations are required to capture and unify all single-particle 1T physics dynamics for a single spinless particle. This encompasses all Hamiltonians, incorporating any associated parameters such as mass and interaction parameters, which emerge as moduli within the diverse embeddings of $(3+1)$-dimensional phase space into $(4+2)$-dimensional phase space.

Evidently, due to the dot product's $\mathrm{SO}(d,2)$ invariance, all shadows possess global $\mathrm{SO}(d,2)$ spacetime symmetry, with conserved generators represented by $L^{MN} = (X^M P^N - X^N P^M)$. $\mathrm{SO}(d,2)$ transformation rules for the full, overt, or covert $\mathrm{SO}(d,2)$ symmetry in each shadow action are generated by the $\mathrm{SO}(d,2)$ generators L^{MN} where (X^M, P^N) are replaced by their gauge-fixed versions, $X^M(x^\mu, p_\mu)$ and $P^N(x^\mu, p_\mu)$, expressed in $(3+1)$-dimensional phase space (x^μ, p_μ) of the given shadow.

The gauge group $\mathrm{Sp}(2,R)$ is generalized into a supergroup with the inclusion of extensions to phase space accommodating spin [24] and/or supersymmetry [25]. In the realm of local field theory, local fields in 2T physics—be they scalars, vectors, tensors, or spinors—extend not only as functions of a larger spacetime X^M but also encompass tensor or spinor components along the additional $1+1$ directions. The momentum P_M transitions into covariant derivatives acting upon such fields.

These shadows exhibit multidualities under $\mathrm{Sp}(2,R)$ gauge transformations, transitioning between fixed gauges. For instance, a spinless particle that obeys the $\mathrm{Sp}(2,R)$ constraints in a flat $(d+2)$-dimensional spacetime, manifests as various shadows in $3+1$ dimensions, including but not limited to those in the following list [19–27]: a free massless relativistic particle, a free massive relativistic particle, a free massive non-relativistic particle, a particle in anti–de Sitter space AdS_d, a particle in de Sitter space dS_d, a particle in any maximally symmetric space (e.g., $\mathrm{AdS}_{d-n} \times \mathrm{S}^n$), a particle in the Robertson-Walker spacetime (for both open and closed universes), BTZ black holes (for $d=3$), the nonrelativistic Hydrogen atom, the nonrelativistic harmonic oscillator, amongst others, and a twistor reformulation of all of these systems.

Remarkably, each $3+1$ shadow fully encapsulates holographically the $\mathrm{Sp}(2,R)$ gauge-invariant physical essence of the parent $(4+2)$-dimensional theory. Consequently, all shadows exhibit gauge equivalence under $\mathrm{Sp}(2,R)$ gauge transformations, taking the form of canonical transformations within 1T physics that include transformations of time, Hamiltonian, and three positions and momenta. These canonical transformations among shadows are construed as multiduality transformations within the framework of 1T physics [21]. Thus, 2T physics not only unveils hidden extra $1+1$ large dimensions, treated equivalently to the overt $3+1$ dimensions but also unveils numerous unsuspected multidualities, serving as novel tools for duality-based computations within 1T physics while deepening understanding of spacetime's fundamental nature.

Despite their diverse expressions as Hamiltonians in phase space in $(d-1)+1$ dimensions, all shadow actions share a fundamental underlying hidden spacetime symmetry in $d+2$ dimensions, revealing the extra $1+1$ dimensions. Specifically, shadows arising from $(d+2)$-dimensional flat spacetime encompass a vast array of $[(d-1)+1]$-dimensional spacetimes. At the quantum level, all emerging 1T dynamical shadow systems represent physically distinct 1T manifestations of the same unifying mathematical unitary representation of $\mathrm{SO}(d,2)$ common to all shadows. This unique representation of $\mathrm{SO}(d,2)$, known as the singleton representation, emerges directly from the covariant quantization treatment of the 2T system in $d+2$ dimensions, consisting of the vanishing of three Hermitian generators of $\mathrm{Sp}(2,R)$, namely $X^2 = P^2 = (X \cdot P + P \cdot X) = 0$. This singleton representation is distinguished solely by its unique Casimir eigenvalues which are determined exclusively by the number of dimensions d [see Eq. (9) in [22]]. In essence, the Hilbert spaces for the $[(d-1)+1]$-dimensional shadows are expressed as unitarily equivalent bases of the same singleton representation of $\mathrm{SO}(d,2)$. Across various 1T shadows, their quantum Hilbert bases differ only by the choice of a subset of simultaneous observables, all of which are functions of the generators of $\mathrm{SO}(d,2)$, namely $L^{MN} = X^M P^N - X^N P^M$, while the Casimir eigenvalues remain unaltered for all shadows. Remarkably, this prediction is corroborated by computations directly in 1T physics, where the quantum spectra of systems such as the hydrogen atom, harmonic oscillator, and others confirm the presence of the hidden $\mathrm{SO}(d,2)$ symmetry, with the anticipated eigenvalues of the Casimir operators [22,23].

Of particular importance is the "conformal shadow," where the embedding of $3+1$ phase space into $4+2$ phase space aligns with the formalism of relativistic $3+1$ spacetime, encompassing conformal symmetry SO(4,2). In this shadow, the partially concealed nonlinear special conformal transformations in $3+1$ spacetime x^μ emerge from the explicit linear Lorentz transformations SO(4,2) in $4+2$ spacetime X^M, including the extra $1+1$ spacetime dimensions as gauge degrees of freedom [19–21]. The significance of the conformal shadow lies in its direct correlation with relativistic field theory in 1T physics. Through this connection, 2T physics stands as a robust descriptor of nature across all energy or distance scales known to date, while predicting new hidden symmetries and dualities discernible with tools of 1T physics.

According to 2T physics, all 1T physics systems have hidden symmetries that relate to the presence of extra $1+1$ dimensions. The key to revealing the hidden spacetime structure of a given 1T physics system lies in constructing the $(4+2)$-dimensional parent that underlies its evident $3+1$ spacetime structure (not necessarily expressed in relativistically covariant notation). Viewing the latter as a shadow of 2T physics opens up a pathway to unification. The parent structure predicts multi-dual shadows, allowing us to unify disparate 1T systems. In doing so, we expand our understanding of spacetime and gain a powerful tool for predicting dualities and hidden symmetries within 1T-physics.

This pursuit of unification and discovery lies at the heart of 2T physics. In this spirit, we delved into the $3+1$ Chern-Simons action, denoted as $S^{3+1}_{\mathrm{CS-GR}}$, in Sec. II and followed up by constructing the parent action $S^{4+2}_{\mathrm{CS-GR}}$ in Sec. III. In turn, starting with the parent $S^{4+2}_{\mathrm{SM+GR+CS}}$ it is in principle possible to derive various holographic shadows in the context of field theory, beyond the conformal shadow, study the multidualities among them for their inherent intereting properties and hopefully develop useful new tools for computations based on dualities.

2. Field theory in 2T physics

In the framework of 2T field theory formalism, the architecture of the 2T action is designed to enforce the vanishing of $\mathrm{Sp}(2,R)$ generators (and their extensions) through equations of motion derived from the 2T action principle (refer to Sec. III and Appendix A 3). This 2T physics field theory encompasses interactions ranging from Yang-Mills gauge symmetry in $\mathrm{d}+2$ dimensions to general coordinate reparametrization symmetry and even supergravity in $\mathrm{d}+2$ dimensions.

When exploring classical or quantum mechanics within the $(4+2)$-dimensional theory, as well as in the context of field theory, a myriad of $(3+1)$-dimensional holographic representations can be obtained, as discussed in the previous section, by gauge fixing and solving two (out of three) of the $\mathrm{Sp}(2,R)$ constraint equations of motion. In the field theory version, these are termed “kinematic equations” as discussed in Appendix A 3. These kinematic constraints are independent of field interactions, while the remaining third “dynamical constraint” encompasses the mentioned interactions. Essentially, the solution space of the kinematic constraints suggests that for a given action in 2T physics field theory in $4+2$ dimensions, numerous emergent field theory actions (referred to as shadows) in $3+1$ dimensions are in principle feasible.

Among these diverse shadows, particular attention is drawn to the *holographic conformal* shadow in the current discourse. This shadow conspicuously exhibits the attributes of special or general relativity, serving as the conduit from 2T physics to relativistic field theory in 1T physics. It is within this shadow that all relativistic field theories manifest, including the empirically successful Standard Model (SM) [30], general relativity (GR) [31,32], and their supersymmetric extensions. The coupled system in 1T physics $\mathrm{i}(\mathrm{SM}+\mathrm{GR})_{3+1}$ [33] is deemed the conformal shadow of the parent field theory $(\mathrm{SM}+\mathrm{GR})_{4+2}$ as elaborated in Sec. III. It is noteworthy that the parent $(\mathrm{SM}+\mathrm{GR})_{4+2}$ theoretically yields a multitude of shadows, including the conformal shadow $\mathrm{i}(\mathrm{SM}+\mathrm{GR})_{3+1}$, which are multidual field theories to one another. Various simple examples of such dual shadows within the realm of field theory were expounded upon in [26,27]. This aspect of field theoretic duality within 2T physics stands open to broader exploration, offering avenues for the development of novel duality-based computational methodologies.

3. $\mathrm{Sp}(2,R)$ gauge invariant sector and holographic reduction of $4+2$ to $3+1$

In this section, we elucidate the significance of the field $W(X)$ introduced in Sec. III in the delta function $\delta(W(X))$. As discussed in [30–32], the presence of $W(X)$ and its inclusion in the delta function stem from the foundational formulation of 2T physics, which hinges on a broader gauge symmetry in phase space (X^M, P_M) rather than solely position space X^M. In a flat space scenario, $W(X)$ is denoted by $W_{\mathrm{flat}} \equiv (X^2)_{\mathrm{flat}} = X^M X^N \eta_{MN}$ where η_{MN} represents a flat metric in $4+2$ dimensions. This X^2 stands as one of the generators of the fundamental $\mathrm{Sp}(2,R)$ gauge symmetry in flat phase space as displayed in Eq. (A1). The other two generators, in the absence of interactions in flat space and after quantum ordering, are $\frac{1}{2}(X\cdot P + P\cdot X)_{\mathrm{flat}}$ and $(P^2)_{\mathrm{flat}}$. These three quadratic phase space quantities collectively form the $\mathrm{Sp}(2,R)$ Lie algebra, closing under both Poisson brackets and quantum commutators.

In interacting field theory within curved space, $(X^2)_{\mathrm{flat}}$ is elevated to the field $W(X)$, with momenta transformed into derivatives $P_M \to -i\partial_M$, and interactions integrated across all fields, incorporating a metric $G_{MN}(X)$ in curved space. Within the gauge invariant sector of 2T physics, known as the singlet sector of $\mathrm{Sp}(2,R)$ and deemed physical, all three generators of $\mathrm{Sp}(2,R)$ must vanish. The delta function $\delta(W(X))$ imposes the vanishing of one of these three $\mathrm{Sp}(2,R)$ gauge symmetry generators, $W(X)=0$, thus partially enforcing the underlying phase space $\mathrm{Sp}(2,R)$ gauge invariance and consequently eliminating part of the gauge degrees of freedom. This process selectively emphasizes $\mathrm{Sp}(2,R)$ gauge invariants while minimizing gauge degrees of freedom. It is essential to note that the other two generators of $\mathrm{Sp}(2,R)$ must also vanish to fully concentrate solely on the gauge invariant physical degrees of freedom.

We enforce the vanishing of these generators by solving specific constraint equations of motion derived from varying the 2T field theory action which incorporates the delta function $\delta(W(X))$. For instance, upon varying the action in Eq. (16) for each field (e.g., the Φ field), a linear combination of the delta function and its derivatives is obtained [31],

$$A_\Phi \delta(W) + B_\Phi \delta'(W) + C_\Phi \delta''(W) = 0. \qquad \text{(A3)}$$

Given that $\delta(W), \delta'(W)$, and $\delta''(W)$ are linearly independent, this yields three equations, $A_\Phi|_{W=0} = B_\Phi|_{W=0} = C_\Phi|_{W=0} = 0$, and similarly for all fields $\Phi, S_i, \Psi_\alpha, A_M^a, G_{MN}$ in addition to those displayed for Φ. These three differential equations (and associated equations for tensors and spinors restricting their directions) enforce the vanishing of all $\mathrm{Sp}(2,R)$ generators when applied to the fields $G_{MN}, \Phi, S_i, \Psi_\alpha, A_M^a$ [30–32]. The solutions to these equations yield the physical $\mathrm{Sp}(2,R)$ gauge invariant sector of the theory. The B and C conditions (and their tensorial and spinorial counterparts) are termed the "kinematic equations," while the A condition is labeled the "dynamical equation." Notably, only the A equation depends on interactions. The solutions of the kinematic equations, independent of interactions, yield the $(3+1)$-dimensional "shadows" for each field and their "prolongations" into $4+2$ dimensions. The shadow represents the analog of the bottom mode $(g_{\mu\nu}, \phi, s_i, \psi, A_\mu^a)$ in a Kaluza-Klein type expansion, whereas the prolongations are conceptually akin to higher Kaluza-Klein type modes as illustrated in the series expansions Eq. (23).

However, unlike the conventional Kaluza-Klein case, within the framework of 2T physics, the A, B, C equations restrict the prolongations, rendering them not independent modes. Instead, the prolongations are entirely determined by the shadows of each field $(g_{\mu\nu}, \phi, s_i, \psi, A_\mu^a)$ in $3+1$ dimensions—except for unphysical unfixed prolongation gauge degrees of freedom independent of the shadow fields. Thus, the $\mathrm{Sp}(2,R)$ invariant physical sector emerges as a $(3+1)$-dimensional field theory directly treatable within $3+1$ dimensions.

Applying this approach to the $(\mathrm{SM}+\mathrm{GR})_{4+2}$ action presented in Eq. (16) yields the conformal shadow $\mathrm{i}(\mathrm{SM}+\mathrm{GR})_{3+1}$ outlined in Eq. (3). This emergent $(3+1)$-dimensional theory differs from the conventional formulation of the Standard Model + general relativity (SM + GR) in several key aspects [30–32]. Notably, it must adhere to local scale invariance, with a requisite relative sign between the ϕ and s_i fields [33], as depicted in Eq. (3). These additional properties are necessitated by the underlying $\mathrm{Sp}(2,R)$ gauge symmetry, including the local scale (Weyl) symmetry emerging from general coordinate reparametrization that amalgamates the extra $1+1$ dimensions with the $3+1$ dimensions [refer to Eqs. (7) and (34)]. This necessitates that all scalar fields in relativistic field theory, including the Higgs boson, must adhere to the principles of conformal coupling. If there exists more than one s-type scalar, the Weyl symmetry can manifest in a broader and more nonlinear fashion [33].

The mandated Weyl symmetry illustrates how 2T physics constrains the behavior of the Higgs boson in this new emergent action, diverging in certain aspects from the conventional Standard Model plus gravity paradigm. Although imperceptible at low energies in particle accelerators, this deviation becomes significantly pronounced in regions of spacetime characterized by strong gravity, such as the interiors of black holes, where the physics of $\mathrm{i}(\mathrm{SM}+\mathrm{GR})_{3+1}$ diverges markedly from the predictions of the standard theory [38,40].

While our discussion in this paper primarily revolves around the conformal shadow, it is noteworthy that other shadows stemming from the same $4+2$ action, contingent upon alternative $3+1$ embeddings in $4+2$ dimensions, harbor distinct interpretations within 1T physics. For some simple examples of other shadows in the realm of field theory, refer to [26,27]. In essence, these shadows represent specialized instances of potential $(3+1)$-dimensional field theories that are mutually dual. The exploration and development of such multidualities in field theory, alongside their application in deriving novel computational methodologies within 1T physics, remain vibrant areas of investigation within the domain of 2T physics.

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