All-loop soft theorem for pions

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(Received 13 January 2024; accepted 24 June 2024; published 7 August 2024)

In this paper we discuss a generalization of the Adler zero to loop integrands in the planar limit of the $SU(N)$ nonlinear sigma model (NLSM). The Adler zero for integrands is violated starting at the two-loop order and is only recovered after integration. Here we propose a soft theorem satisfied by loop integrands with any number of loops and legs. This requires a generalization of NLSM integrands to an off shell framework with certain deformed kinematics. Defining an *algebraic soft limit*, we identify a simple nonvanishing soft behavior of integrands, which we call the *algebraic soft theorem*. We find that the proposed soft theorem is satisfied by the "surface" integrand of Arkani-Hamed and Figueiredo [arXiv:2403.04826], which is obtained from the shifted $\mathrm{Tr}\varphi^3$ surfacehedron integrand. Finally, we derive an on shell version of the algebraic soft theorem that takes an interesting form in terms of self-energy factors and lower-loop integrands in a mixed theory of pions and scalars.

DOI: 10.1103/PhysRevD.110.045009

I. INTRODUCTION

The Adler zero is a fundamental property of scattering amplitudes of pions.[1] Their dynamics in the chiral limit is captured by the $U(N)$ nonlinear sigma model (NLSM). The latter is defined as the effective field theory of the Goldstone bosons which appear as a consequence of the spontaneous chiral symmetry breaking according to the pattern $U(N)\times U(N)\to U(N)$. In what follows, we restrict ourselves to the NLSM Lagrangian corresponding to the leading order within the derivative expansion, i.e. to the order $O(\partial^2)$. Let us note, that NLSM is a universal effective field theory describing the low-energy limit of all the theories with the same symmetry breaking pattern.[2] The most prominent examples of such theories are quantum chromodynamics (QCD) in the chiral and large N_c limits, where it describes the dynamics of pseudoscalar nonet[3] (for $N=3$), or e.g. the $U(N)$ linear sigma model.

[1]Here and in what follows we use the term "pions" in a more abstract sense for the Goldstone bosons of $U(N)$ NLSM, as is the common lore in the amplitude community.

[2]The individual theories differ only by the values and interpretations of the low-energy constants of the corresponding NLSM.

[3]Beyond the chiral limit, the pseudoscalars become pseudo-Goldstone bosons and the corresponding low-energy effective theory is known as the chiral perturbation theory.

The shift symmetry of NLSM ensures that both tree- and loop-level amplitudes vanish when any momentum is taken soft. At n points, NLSM tree-level amplitudes can be then written as a sum of *flavor-ordered* amplitudes [1,2],

$$\mathcal{A}_n^{\mathrm{tree}}=\sum_{\sigma}\mathrm{Tr}(T^{a_1}T^{\sigma(a_2)}\ldots T^{\sigma(a_n)})A_n(1\sigma(23\ldots n)),\qquad(1)$$

where σ runs over all permutations of labels $\{2,\ldots,n\}$. Not only does the full amplitude $\mathcal{A}_n$ enjoy the Adler zero but also the ordered amplitudes satisfy

$$\lim_{p_i\to 0}A_n=0.\qquad(2)$$

This *soft theorem* and consistent factorization fix tree-level NLSM amplitudes uniquely and lead to the formulation of on shell recursion relations and many other important insights [3–17]. NLSM amplitudes also appear in other contexts; they satisfy Bern-Carrasco-Johansson relations [18], can be calculated in the Cachazo-He-Yuan (CHY) formalism [19] or ambitwistor strings [20], and satisfy various uniqueness conditions [21–23]. Recently exciting progress has been made on obtaining NLSM amplitudes from the surfacehedron picture [24–28].

In parallel, some work has gone into extending these efforts to *loop integrands*. These are rational functions of the external kinematics $\{p_i\}$ and off shell loop momenta $\{\ell_i\}$. They can be obtained as sums of Feynman diagrams prior to integration or using generalized unitarity. Unlike on shell amplitudes, integrands are *a priori* not unique objects. To define global loop variables and meaningfully talk about a single rational function, we need to work in the planar limit $N \to \infty$. This allows us to define a *planar integrand* $I_n^{L-\text{loop}}$ as the kinematic coefficient of the single-trace structure analogous to (1).

Integrands are not invariant under field redefinitions. When calculating from Feynman diagrams using flavor-ordered Feynman rules [2] we get different integrands for different parametrizations. However, upon integration over loop momenta, all these integrands yield the same amplitude,

$$A_n^{L-\text{loop}} = \int d^4\ell_1 d^4\ell_2 \ldots d^4\ell_L I_n^{L-\text{loop}}. \tag{3}$$

This ambiguity of the loop integrand is not new; the same situation arises in planar $\mathcal{N}=4$ super Yang-Mills. There a *preferred* integrand could be found [29–32] which is invariant under a hidden *dual conformal symmetry* [33,34]. This symmetry is also built into its construction using on shell diagrams and the positive Grassmannian [35] which provide triangulations of an underlying amplituhedron geometry [36–44]. By analogy, a natural question to ask is if there exists a consistent *soft theorem* for NLSM integrands that would allow to distinguish a preferred object that satisfies it.

In this article we provide a candidate for one such soft theorem, valid to all loops and multiplicities, which we call the *algebraic soft theorem*. Correspondingly, we identify a new recursive decomposition of NLSM tree-level off shell correlators that manifests the new soft theorem, which itself is a natural generalization of the Adler zero to off shell momenta. We further provide evidence that this construction can be extended to the loop level, giving a number of explicit examples. Following the statement of the algebraic soft theorem for all loops and multiplicities, we identify one solution; the (off shell) NLSM surface integrand obtained from a certain shift of the Trφ^3 surfacehedron [24–28]. Finally we derive an on shell version of the algebraic soft theorem, revealing a rich soft structure of NLSM on shell integrands.

II. FAILURE OF ADLER ZERO FOR THE LOOP INTEGRAND

At one loop the structure of integrands is simple enough to analyze the whole space of rational functions. The general decomposition gives us the sum of boxes, triangles, bubbles, tadpoles and polynomials in loop momentum. All integrands are fixed from cuts via generalized unitarity (cut constructible part, CC), except for tadpoles and polynomials in loop momenta which do not have any physical cut,

$$I_n^{1-\text{loop}} = (\text{CC}) + (\text{tadp.}) + (\text{poly. in } \ell_k).$$

In [45] we gave a prescription to fix tadpoles and polynomials by demanding that the planar integrand *vanishes in the soft limit*, i.e. we upgraded the Adler zero to an integrand-level statement. When extending this analysis to two loops, we realize that there exists *no integrand* which satisfies all cuts and exhibits the Adler zero. This is already true at four points. We conclude that the Adler zero is not a property of planar NLSM integrands beyond one loop.

III. A HIDDEN SOFT THEOREM AT TREE LEVEL

We return back to tree-level amplitudes where we want to analyze the structure of the Adler zero in more detail. The two lowest-order NLSM amplitudes are

$$A_4 = X_{13} + X_{24},$$
$$A_6 = -\frac{1}{2}\frac{(X_{13}+X_{24})(X_{46}+X_{15})}{X_{14}} + X_{13} + \text{cyc.}, \tag{4}$$

where we introduced *planar variables* [46]

$$X_{ij} \equiv (p_i + p_{i+1} + \cdots + p_{j-1})^2. \tag{5}$$

Note that $X_{ii+1} = p_i^2 = 0$ in the massless case while $X_{ii} = 0$ identically. It will be important to highlight a particular fact about the soft behavior of NLSM amplitudes that was first appreciated from the CHY perspective [47]; the coefficient of the leading term in the Adler zero is controlled by amplitudes in an extended theory coupling biadjoint scalars φ to NLSM pions π. Some simple examples are

$$M_3(1^\varphi, 2^\varphi, 3^\varphi) = 1,$$
$$M_5(1^\varphi, 2^\pi, 3^\pi, 4^\varphi, 5^\varphi) = 1 - \frac{X_{13}+X_{24}}{X_{14}} + \frac{X_{24}+X_{35}}{X_{25}}. \tag{6}$$

Our ultimate goal is to formulate a consistent soft theorem for NLSM loop integrands. The first step we take in this direction is to promote amplitudes A_n, M_n to off shell correlators $\boldsymbol{A}_n$, $\boldsymbol{M}_n$. To do this, we implement a *minimal prescription*; we require the algebraic form of the off shell correlators in terms of X_{ij} to be the same as that of the on shell amplitudes, i.e.

$$\boldsymbol{A}_n \equiv A_n, \qquad \boldsymbol{M}_n \equiv M_n, \tag{7}$$

as functions of the planar variables. Notably, the correlators could in principle explicitly depend on variables $X_{ii+1} = p_i^2 \neq 0$ but in the minimal prescription they do not. Thus $\boldsymbol{A}_n$, $\boldsymbol{M}_n$ can be interpreted as amputated Green's functions and the on shell amplitudes can be obtained by a simple limit

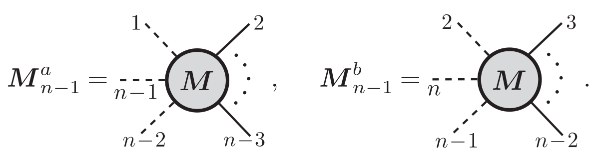
$$\boldsymbol{A}_n \overset{X_{ii+1}=0}{\longrightarrow} A_n, \qquad \boldsymbol{M}_n \overset{X_{ii+1}=0}{\longrightarrow} M_n. \tag{8}$$

From the on shell perspective, this might look like taking a step in the wrong direction. We traded unique on shell amplitudes for some *a priori* ambiguous correlators. Indeed, these objects are generally not invariant under field redefinitions. Different parametrizations of the Lagrangian would produce different functions, all of which would agree only once the on shell limit is taken. However, in this sense correlators are already more analogous to loop integrands which are the objects we are interested in.

We now come to one of the central observations of this letter. To any number of points, the minimal prescription correlators (7) can be recursively decomposed as [48]

$$\boldsymbol{A}_n = X_{1n-1}\boldsymbol{M}^a_{n-1} + X_{2n}\boldsymbol{M}^b_{n-1} + \boldsymbol{R}_n. \tag{9}$$

This representation breaks manifest cyclicity by making the nth leg special. We will show that this way of writing the correlator manifests an off shell generalization of the Adler zero for $p_n \to 0$ and naturally implies the latter in the on shell limit. The correlators $\boldsymbol{M}^{a,b}_{n-1}$ appearing in (9) always involve three adjacent biadjoint scalars (dashed lines), whereas the remaining legs are associated with pions (solid lines),



(10)

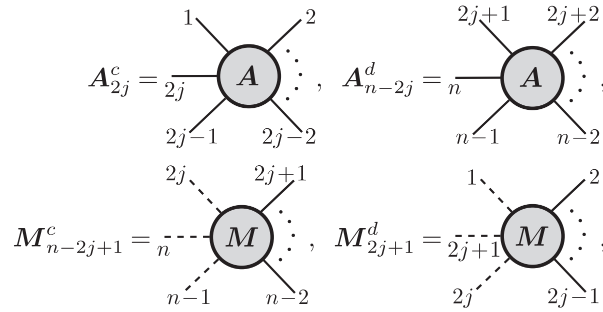
The suppressed external lines correspond to pions and the labeling indicates that the middle scalar has off shellness $(p_{n-1} + p_n)^2 = X_{1n-1}$ for $\boldsymbol{M}^a$ and $(p_n + p_1)^2 = X_{2n}$ for $\boldsymbol{M}^b$, respectively. The *remainder* $\boldsymbol{R}_n$ in (9) can itself be decomposed further in terms of lower-point correlators in the extended theory,

$$\boldsymbol{R}_n = \sum_{j=2}^{n/2-1} \boldsymbol{S}(X_{1,2j}, X_{2j,n})\boldsymbol{A}^c_{2j}\boldsymbol{M}^c_{n-2j+1} + \sum_{j=1}^{n/2-2} \boldsymbol{S}(X_{2j+1,n}, X_{1,2j+1})\boldsymbol{M}^d_{2j+1}\boldsymbol{A}^d_{n-2j}. \tag{11}$$

Here we have introduced the *soft factor*,

$$\boldsymbol{S}(X_A, X_B) = \frac{X_A - X_B}{X_A}, \tag{12}$$

with the crucial property that $\boldsymbol{S} = 0$ for $X_A = X_B \neq 0$. The distribution of external φ and π particles for the correlators in (11) is



(13)

where again only mixed correlators with three adjacent biadjoint scalars appear. Note that for general off shell momenta

$$X_{\text{in}} - X_{i1} = p_n \cdot (p_1 + \cdots + p_{i-1} - (p_i + \cdots + p_{n-1})). \tag{14}$$

Thus, in terms of the variables X_{ij}, the limit $p_n \to 0$ can be reformulated by defining an *algebraic soft limit*,

$$\text{Algebraic soft limit: replace label } n \text{ by label 1 in } X_{ij}.$$

Since the soft factors in the remainder (11) are of the form (14), they ensure that it vanishes in the algebraic soft limit,

$$\boldsymbol{R}_n \overset{n\mapsto 1}{\mapsto} 0. \tag{15}$$

With this in mind, we can also take the algebraic soft limit of the NLSM correlator (9) to obtain

$$\boldsymbol{A}_n \overset{n\mapsto 1}{\mapsto} X_{1n-1}\boldsymbol{M}^a_{n-1} + X_{12}\boldsymbol{M}^b_{n-1}, \tag{16}$$

where $\boldsymbol{M}^{a,b}$ are now evaluated on soft kinematics $p_n = 0$. On-shell, when $X_{1n-1} = X_{12} = 0$, the above establishes the Adler zero. Off shell X_{1n-1}, X_{12} are nonzero and (16) quantifies exactly how the Adler zero fails. More specifically it tells us that the leading coefficient of the (non-vanishing) soft theorem for the off shell minimal prescription NLSM correlators $\boldsymbol{A}_n$ is once again controlled by the mixed correlators $\boldsymbol{M}^{a,b}$. The *algebraic soft theorem* (16) serves as a criterion to select a preferred form of the correlators $\boldsymbol{A}_n$ and $\boldsymbol{M}_n$, ones which satisfy (9) and (11). The minimal prescription correlators $\boldsymbol{A}_n$ agree with the results obtained from the minimal parametrization Lagrangian [2]. They are also identical to the objects obtained from shifting $\text{Tr}\varphi^3$ surfacehedron amplitudes [28].

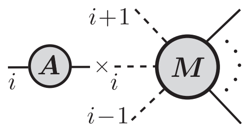
With an eye on later extensions to loop-level we propose a useful interpretation of the algebraic soft theorem (16). Defining the NLSM *two-point* correlator,


$$= \boldsymbol{A}_2(i, i+1) \equiv X_{ii+1} = p_i^2, \tag{17}$$

we can identify the prefactors of $\boldsymbol{M}^{a,b}$ in (16),

$$X_{1n-1} = \boldsymbol{A}_2(n-1, 1), \qquad X_{12} = \boldsymbol{A}_2(1, 2).$$

Thus the soft theorem (16) can be graphically represented as

$$\boldsymbol{A}_n \overset{n \mapsto 1}{\longmapsto} \sum_{i=n-1,1} \tag{18}$$


where, since we are on soft kinematics $p_n = 0$, the index i is cyclic mod $n-1$.

Thus the NLSM correlator factorizes in the soft limit into a *self-energy* and an extended theory correlator. There are two such contributions from legs $i = n-1$, 1 adjacent to the momentum p_n that was taken soft, a characteristic shared with the soft gluon theorem for Yang-Mills amplitudes [50]. We will see that these soft properties also apply to loop-level correlators.

To conclude, we give two simple examples. The four-point NLSM correlator reads

$$\boldsymbol{A}_4 = X_{13}\boldsymbol{M}_3^a + X_{24}\boldsymbol{M}_3^b, \tag{19}$$

where $\boldsymbol{M}_3^{a,b} = 1$ and we see that trivially $\boldsymbol{R}_4 = 0$ at four points. At six points we have

$$\boldsymbol{A}_6 = X_{15}\boldsymbol{M}_5^a + X_{26}\boldsymbol{M}_5^b + \boldsymbol{R}_6, \tag{20}$$

with $\boldsymbol{M}_5^{a,b}$ formally as in (6), and remainder

$$\boldsymbol{R}_6 = \frac{X_{14} - X_{46}}{X_{14}}(X_{13} + X_{24}) + \frac{X_{36} - X_{13}}{X_{36}}(X_{35} + X_{46}).$$

Both (19) and (20) can be seen to satisfy the algebraic soft theorem (16). Taking the on shell limit (8) they reproduce the known NLSM amplitudes (4).

IV. THE ALGEBRAIC SOFT THEOREM FOR LOOP INTEGRANDS

We now wish to define a preferred off shell integrand, or amputated loop-level correlator $\mathbf{I}_n$. Following the tree-level philosophy we want to look for an off shell soft theorem, possibly a generalization of (16), that would be satisfied by $\mathbf{I}_n$. At the tree level, a crucial ingredient to ensuring the validity of the algebraic soft theorem was the rigid recursive structure of the correlator following from (9) and (11). A generalization of this structure remains valid to all loop orders [51] provided we add one more layer of generalization; we allow our integrand to depend on the planar variables $X_{ii} \neq 0$. Following (5) these parameters are identically zero even for off shell kinematics, so we have to consider them as formal deformations. We refer to this integrand with explicit dependence on X_{ii} as a *surface integrand* and denote it $\mathcal{I}_n$. We can then define an on shell integrand I_n as a sequence of limits,

$$\mathcal{I}_n \overset{X_{ii}=0}{\longrightarrow} \mathbf{I}_n \overset{X_{ii+1}=0}{\longrightarrow} I_n. \tag{21}$$

The first step in this sequence, i.e. going from $\mathcal{I}_n$ to $\boldsymbol{I}_n$, is more subtle than indicated as it will involve *amputation* of external legs, a procedure we will touch upon shortly. We formulate now the main statement of this section. To any number of loops and legs there exist surface functions $\boldsymbol{\mathcal{U}}_{n-1}^{a,b}$ such that the NLSM surface integrand $\mathcal{I}_n$ satisfies the *algebraic soft theorem*

$$\mathcal{I}_n \overset{n \mapsto 1}{\mapsto} X_{1n-1}\boldsymbol{\mathcal{U}}_{n-1}^a + X_{12}\boldsymbol{\mathcal{U}}_{n-1}^b. \tag{22}$$

While a fully satisfactory interpretation of the odd-point functions $\boldsymbol{\mathcal{U}}^{a,b}$ is currently lacking, their analytic form suggests (see example) that they are appropriate generalizations of the mixed theory correlators $\boldsymbol{M}^{a,b}$ to loop level. We leave a detailed investigation for future work. When searching for the integrand that satisfies (22) we find a solution: the NLSM surface integrand obtained from shifting the $\mathrm{Tr}\varphi^3$ surfacehedron integrand [27,28]. Whether the shifted surfacehedron integrand is the unique solution to the soft theorem constraint (22) remains an open question, but some other natural candidates do fail that condition. For example, the minimal parametrization integrand violates (22) as it does not make use of the X_{ii} variables.

It is instructive to show why turning on X_{ii} is required to satisfy the soft theorem (22). Let us consider the following soft factor

$$\boldsymbol{S}(X_{1n}, X_{11}) = \frac{X_{1n} - X_{11}}{X_{1n}}.$$

For the on shell integrand I_n this quantity is not well-defined as $X_{1n} = 0$. For the off shell integrand $\boldsymbol{I}_n$ we have $X_{11} = 0$ so the factor becomes $X_{1n}/X_{1n} = 1$ and remains unchanged when taking the algebraic soft limit. Finally, for the surface integrand $\mathcal{I}_n$, taking the limit $n \mapsto 1$, we map $X_{1n} \mapsto X_{11}$ in the numerator and the soft factor vanishes as it did in the tree-level case. Exactly these types of cancellations are needed for the surface integrand $\mathcal{I}_n$ to satisfy (22).

V. TWO- AND FOUR-POINT EXAMPLES

Here we present results for the simplest two- and four-point one-loop surface integrands. At loop level, integrands depend on the planar loop variables,

$$X_{iz} = (\ell + p_1 + \cdots + p_{i-1})^2, \tag{23}$$

where ℓ and z denote the loop momentum and its associated label. We start with the two-point surface integrand which can be decomposed similarly to (9),

$$\mathcal{I}_2(1,2) = X_{11}\mathcal{U}_1(1) + X_{22}\mathcal{U}_1(2) + \mathcal{R}_2, \tag{24}$$

with the one-loop *tadpole* and remainder

$$\mathcal{U}_1(1) = -\frac{1}{X_{1z}}, \quad \mathcal{R}_2 = \boldsymbol{S}(X_{2z}, X_{1z}) + \boldsymbol{S}(X_{1z}, X_{2z}). \tag{25}$$

Since $\mathcal{R}_2$ is again composed of soft factors it vanishes in the algebraic soft limit, i.e. $\mathcal{R}_2 \mapsto 0$ for $2 \mapsto 1$. This ensures that $\mathcal{I}_2$ satisfies the algebraic soft theorem (22). At four points we observe that the integrand still follows the expected structure,

$$\mathcal{I}_4 = X_{13}\mathcal{U}_3^a + X_{24}\mathcal{U}_3^b + \mathcal{R}_4, \tag{26}$$

where now

$$\mathcal{U}_3^a = \mathcal{U}_3(1,2,3) = \frac{X_{13}+X_{2z}}{X_{1z}X_{3z}} - \frac{1}{X_{1z}} - \frac{1}{X_{2z}} - \frac{1}{X_{3z}} - \frac{1}{X_{12}}\mathcal{I}_2(1,2) - \frac{1}{X_{23}}\mathcal{I}_2(2,3),$$

and $\mathcal{U}_3^b = \mathcal{U}_3(2,3,1)$. The remainder $\mathcal{R}_4$ is given by

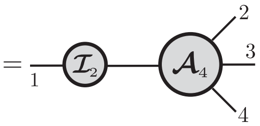
$$\mathcal{R}_4 = \boldsymbol{S}(X_{4z}, X_{1z})\left(1 - \frac{X_{13}+X_{24}}{X_{14}} - \frac{X_{24}+X_{3z}}{X_{2z}}\right) + \boldsymbol{S}(X_{12}, X_{24})\mathcal{I}_2(1,2)\mathcal{M}_3 + \boldsymbol{S}(X_{14}, X_{11})\mathcal{A}_4\mathcal{U}_1(1) + (1 \leftrightarrow 4, 2 \leftrightarrow 3),$$

with surface three- and four-point functions $\mathcal{M}_3 = 1$, $\mathcal{A}_4 = X_{13} + X_{24}$ identical to their correlator counterparts. The notation $(i \leftrightarrow j, \ldots)$ indicates the same set of terms with labels permuted accordingly.

Again, the soft factor structure of $\mathcal{R}_4$ ensures the validity of the algebraic soft theorem (22) for $\mathcal{I}_4$. The above examples of $\mathcal{I}_2$ and $\mathcal{I}_4$ clearly reveal a recursive structure of the surface integrands, closely following the tree-level pattern of (9) and (11), which will be the subject of future work [51].

Two remarks about the general structure of the surface correlators $\mathcal{I}_n$ are in order. Firstly, although not at all manifest, the correlators are cyclic in the external labels $\{1, \ldots, n\}$. Secondly, the correlators include contributions from self-energy corrections on external legs. For example, in our four-point example, the integrand contains a structure

$$\mathcal{I}_4 \subset \mathcal{I}_2 \frac{1}{X_{12}} \mathcal{A}_4 = \tag{27}$$



The surface integrands $\mathcal{I}_n$ therefore correspond to deformations (recall $X_{ii} \neq 0$) of *partially amputated* (i.e. without free propagators on external legs) NLSM Green's functions. A fact that will be of prime importance as we want to transition to the on shell limit next.

VI. BACK TO ON SHELL INTEGRANDS

We will now extract an on shell integrand I_n from the surface object $\mathcal{I}_n$. The latter includes self-energy corrections on external legs, since their inclusion was necessary to ensure the validity of the algebraic soft theorem (22). We will *amputate* these contributions, effectively just dropping them from $\mathcal{I}_n$, to make the on shell limit well-defined. In the same step we turn off the deformation parameters $X_{ii} \equiv 0$ to obtain the amputated off shell correlator $\boldsymbol{I}_n$. Taking the on shell limit $X_{ii+1} \to 0$ we arrive at the on shell integrand I_n.

Let us consider the simplest example of the one-loop two-point function (24). In this case there is nothing to amputate and we can immediately take $X_{ii}, X_{ii+1} \to 0$. Thus we obtain the on shell self-energy

$$I_2(1,2) = \boldsymbol{S}(X_{2z}, X_{1z}) + \boldsymbol{S}(X_{1z}, X_{2z}). \tag{28}$$

At four-points, the one-loop on shell integrand I_4 extracted from the surface function $\mathcal{I}_4$ in (26) reads

$$I_4 = X_{13}U_3^a + X_{24}U_3^b + R_4 + I_2(1,2)M_3 + I_2(3,4)M_3 + A_4U_1(1) + A_4U_1(4). \tag{29}$$

The process of amputation and going on shell has left us with the lower-point on shell objects

$$U_3^a = U_3(1,2,3) = \frac{X_{13}+X_{2z}}{X_{1z}X_{3z}} - \frac{1}{X_{1z}} - \frac{1}{X_{2z}} - \frac{1}{X_{3z}},$$

as well as $U_3^b = U_3(2,3,1)$ and $U_1(1) = \mathcal{U}_1(1)$. In (29) we have separated the self-energy and tadpole contributions from the on shell remainder R_4. This was done because, in going on shell, the soft factors corresponding to these terms in (27) get partially amputated and thus their soft properties are spoiled. For example, in the on shell limit,

$$\mathcal{R}_4 \supset \boldsymbol{S}(X_{12}, X_{24})\mathcal{I}_2(1,2)\mathcal{M}_3 \mapsto I_2(1,2)M_3. \tag{30}$$

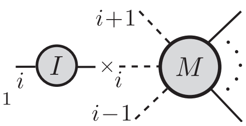
Excluding these terms from the definition of the on shell remainder R_4 leaves us with

$$R_4 = \boldsymbol{S}(X_{4z}, X_{1z})\left(1 - \frac{X_{24} + X_{3z}}{X_{2z}}\right) + (1 \leftrightarrow 4, 2 \leftrightarrow 3).$$

This ensures that it still satisfies the soft limit $R_4 \xrightarrow{p_4 \to 0} 0$ in close analogy with (15). However, the integrand (29) does not exhibit the Adler zero, precisely as a consequence of the partial amputation (30). Instead,

$$\lim_{p_4 \to 0} I_4 = I_2(1,2)M_3 + I_2(3,1)M_3, \qquad (31)$$

as all other terms in (29) manifestly vanish. Just as for the tree-level correlator (16) the result splits into the *self-energy* I_2 and *mixed amplitudes* M_3. This generalizes to the one-loop n-point case where the on shell soft theorem takes the form

$$\lim_{p_n \to 0} I_n = \sum_{i=n-1,1}$$


$, \qquad (32)$

with the kinematic configuration of self-energies I_2 and amplitudes M_{n-1} as previously in (18).

VII. ALL-LOOP ON SHELL SOFT THEOREM

At higher loops ($L \geq 2$) the full *all-loop, all-point* structure of the on shell soft theorem emerges. The soft theorem takes the form of a sum over contributions, each factorizing into three parts: a tadpole factor T_1, a self-energy I_2 and a $(n-1)$-point integrand U_{n-1}. The higher-loop integrands I_2 and U_{n-1} are analogs of the on shell objects derived from surface functions as shown previously. The tadpole factor T_1 is constructed from tadpole functions U_1 as we will discuss shortly. More precisely,

$$\lim_{p_n \to 0} I_n^{(L)} = \sum_{k_T + k_I + k_U = L} T_1^{(k_T)} \times \sum_{i=n-1,1} I_2^{(k_I)} \times U_{n-1}^{(k_U)}, \qquad (33)$$

where we sum over all triplets of integers $(k_T, k_I, k_U) \geq 0$ such that $k_T + k_I + k_U = L$. Also included is the usual sum over adjacent legs p_i, with $i = n-1$, 1. We note that the tadpole factors T_1 are identical for both of these contributions. The kinematic dependence of I_2 and U_{n-1} in (33) is exactly as in the previously established soft theorems (18) and (32) and we have $T_1 = T_1(1)$. Let us first comment on the cases where any of the $k_j = 0$. We define

$$T_1^{(0)} \equiv 1, \qquad I_2^{(0)} \equiv A_2 = 0, \qquad U_{n-1}^{(0)} \equiv M_{n-1}. \qquad (34)$$

This ensures that (33) correctly reproduces the already established results for the cases $L = 0$, 1. Indeed, at tree level ($L = 0$) only one triplet $(k_T, k_I, k_U) = (0,0,0)$ contributes. In particular, due to (34), the result will be proportional to $A_2 = 0$. In this way we recover the Adler zero for tree amplitudes. At one loop ($L = 1$) there is only one nontrivial contribution from the triplet $(k_T, k_I, k_U) = (0,1,0)$ which gives the soft theorem (32).

Finally, let us state the definition of the tadpole factor. For $L \geq 1$ it is given by

$$T_1^{(L)} = \sum_{K=1}^{L} 2^K \sum_{k_1 + \cdots + k_K = L} U_1^{(k_1)} U_1^{(k_2)} \ldots U_1^{(k_K)}. \qquad (35)$$

Similarly to the sum in the soft theorem (33), here we sum over all possible ways to distribute the L loops among the K factors of $U_1^{(k_j)}$ in (35). For the cases where any of the $k_j = 0$ we define $U_1^{(0)} \equiv 0$.

VIII. CONCLUSION AND OUTLOOK

In this letter we proposed a consistent soft theorem for planar NLSM loop integrands with any number of loops and legs, called the *algebraic soft theorem*. While tree-level amplitudes enjoy the Adler zero, this property is generically violated for loop integrands and only restored after integration. To find a simple nonvanishing soft theorem for integrands we generalized to off shell surface correlators. Correspondingly, we identified a new recursive structure of tree-level correlators and defined a simple kinematical operation, an *algebraic soft limit*. In this setup, both tree- and loop-level surface correlators satisfy the surprisingly simple algebraic soft theorem. They also agree with the functions obtained from the surfacehedron picture [27,28]. We then derived the on shell version of the algebraic soft theorem, which is nonzero (due to the violation of the Adler zero by *amputation*) but nicely factorizes into three parts; tadpole functions, self-energies and lower-point integrands in the mixed scalar-pion theory.

NLSM amplitudes belong to a larger family of mixed scalar-pion amplitudes which are hidden in their soft structure. This connection was first established for tree-level amplitudes using the CHY formalism [19]. Similarly, the algebraic soft theorem is governed by the same mixed amplitudes and their loop-level analogs, whose precise interpretation we leave for future work [49].

Let us briefly mention an apparently natural generalization, namely introducing the masses for the pions. This might be achieved by means of adding a mass term to the Lagrangian in the same way as within the chiral perturbation theory. Also in this case some sort of the soft theorems can be obtained, however, since a general mass term of this type breaks the $U(N)$ symmetry explicitly, the notion of stripped amplitudes used in this paper cannot be used, and all the formalism should be formulated completely differently. This modification is beyond the scope of the present paper and will be published elsewhere.

Further questions involve adding higher-derivative operators and a more general understanding of the surface integrands in other theories. It would also be interesting to explore integrands for theories without ordering. Natural starting points are loop amplitudes for theories with extended soft limits such as Dirac-Born-Infeld theory or the special Galileon [3,5,19,20,51–60].

ACKNOWLEDGMENTS

We thank Nima Arkani-Hamed, Taro Brown, Carolina Figueiredo, Song He, Henrik Johansson, Umut Oktem and Shruti Paranjape for very useful discussions. This work is supported by GA-ČR 24-11722S, GAUK-327422, MEYS LUAUS23126, Operační program Jan Amos Komenský CZ.02.01.01/00/22_008/0004632, DOE Grant No. SC0009999 and the funds of the University of California.

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