Conformal anomalies and renormalized stress tensor correlators for nonconformal theories

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We analyze the proposal of defining the Weyl anomaly for classically nonconformal theories as $g^{mn}\langle T_{mn}\rangle - \langle g^{mn}T_{mn}\rangle$, originally put forward by Duff, in the case of a scalar field with quartic self-interaction in 4d. We work in the context of dimensional regularization in curved background to two loops (first order in the coupling). We review the original regularized but not renormalized prescription and its ambiguities; we argue that it cannot be extended to the interacting theory as it fails to provide a finite result. We then propose an alternative prescription via renormalized expectation values. At one loop, our candidate reproduces the local heat kernel result, while its extension to interacting theories contains nonlocal contributions.

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I. INTRODUCTION

Since their discovery by Capper and Duff in [1–3], Weyl anomalies have been a central topic in quantum field theory; see [4–7] for a number of references and general reviews. These anomalies parametrize the trace of the energy momentum tensor induced by quantum corrections for classically Weyl invariant theories, and provide strong constraints as well as powerful ordering principles in the space of quantum field theories, such as the celebrated c theorem in two dimensions [8] and the a theorem in four [9].

The Weyl anomaly shares many similarities with the chiral anomaly, but also important differences. The latter manifests itself in a nonzero divergence for the axial current (thereby spoiling its conservation), it is topological and one-loop exact. Importantly for the scope of the present paper, when the axial symmetry is explicitly broken by the addition of a mass term, the divergence of the current is simply the sum of an explicit breaking contribution (proportional to the mass) and of the anomalous term. The Weyl anomaly comes in two types [10]: one topological and one built from the Weyl tensor. This second type of contributions is in general coupling dependent, although explicitly studying this effect is difficult, since in perturbation theory the underlying Weyl symmetry is generically broken by beta functions.[1]

Furthermore, the quantum trace of the stress tensor for a generic quantum field theory (QFT), when the Weyl symmetry is explicitly broken is much less understood than the conformal field theory case. A better understanding of quantum contributions to the stress tensor has potential applications to QFT in the presence of gravity and in the context of cosmology. The anomalous trace of the stress tensor is also important in the study of the renormalization group (RG), since conformal symmetry is broken along the flow. Ambitiously, an interpolating function for the anomaly coefficients can be found and provide insights for the strong a theorem in four dimensions or its attempted generalization in six [12]. Furthermore, a cancellation of some would-be anomaly coefficients has been observed in [13,14] in the case of certain Poincaré supergravities. This cancellation has not yet been explained, and is somewhat mysterious since the graviton and the gravitino do not possess two-derivative classically Weyl invariant actions.[2]

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[1]An explicit example, albeit somewhat exotic, of a Lagrangian model with type-B anomaly coefficients with explicit coupling dependence is given by the $6d$ four-derivative vector discussed in [11].

[2]Classically Weyl invariant theories of gravity and supergravity typically involve higher derivative fields. For those, Weyl anomalies are well defined, at least at one loop; see [15–18] and references therein.

In [5,7] Duff proposed, following the structure of chiral anomalies, to identify the quantum breaking of the Weyl symmetry as

$$\mathcal{A} = g^{(4)mn}\langle T_{mn}\rangle_{\text{reg}} - \langle g^{mn}T_{mn}\rangle_{\text{reg}}, \tag{1}$$

where the expectation values are taken in the regularized but not renormalized theory. The reason behind this definition [5,7] is that the anomaly should be a physical (measurable) quantity and therefore independent of the renormalization prescription. It should capture the purely quantum contribution to the stress tensor trace, and for this reason (1) is usually referred to as Weyl anomaly, although its interpretation is less clear. Notice that for standard classically Weyl-invariant theories the second term in (1) vanishes and the definition reduces to the one used in the original works [1–3] in the context of dimensional regularization. In this work, we will refer to $\mathcal{A}$ and its alternative prescriptions as anomalies, even when it is understood that the Weyl symmetry is already broken at the classical level.

An efficient way of computing the anomaly proper is via the heat kernel (HK) expansion, which retains manifest covariance with respect to the geometry. In this case the anomaly is identified with the HK coefficient of the kinetic operator Δ, so that for a conformal scalar[3]

$$g^{mn}\langle T_{mn}\rangle = a_4(\Delta). \tag{2}$$

The identification of (1) with the heat kernel coefficient is often assumed also to the case in which there is explicit breaking of Weyl symmetry, see, e.g., [5,6,13,14],

$$\mathcal{A}_{\text{hk}} = a_4(\Delta). \tag{3}$$

It is, however, *a priori* not clear which diagrammatic expression it corresponds to and how it extends beyond the quadratic (free) level. For a free scalar with generic (non-Weyl invariant) curvature coupling $\Delta = -\Box + \xi R$, the heat kernel prescription gives [6,19]

$$\begin{aligned}\mathcal{A}_{\text{hk}} &= a_4(-\Box + \xi R) \\ &= \frac{1}{180(4\pi)^2}\left[-\frac{1}{2}\mathbb{E}_4 + 6(1-5\xi)\Box R + \frac{3}{2}\text{Weyl}^2 \right. \\ &\quad \left. + \frac{5}{2}(6\xi - 1)^2 R^2\right],\end{aligned} \tag{4}$$

which features the appearance of an R^2 term, absent in the anomaly proper and showing that this quantity cannot be obtained from functional differentiation.

The definition (1) was studied in dimensional regularization in [20]. It was explicitly discussed that the definition is finite and local but presents an ambiguity on the nature of the subtraction term that can be represented by writing explicitly

$$\mathcal{A}^{(D)}_{\text{reg}} = g^{(4)mn}\langle T_{mn}\rangle_\varepsilon - \langle g^{(D)mn}T_{mn}\rangle_\varepsilon \quad (\varepsilon \to 0). \tag{5}$$

Indeed, one can subtract the trace in $D = 4$ or $D = 4 - 2\varepsilon$ dimensions. In particular, [20] focused on analyzing the case of a free scalar field with generic curvature coupling. We added the subscript "reg" to emphasize that it is built of regularized quantities. After that, [21] proposed an all-loop modification of $\mathcal{A}$ based on dimensional regularization, which effectively extends the prescription $\mathcal{A}^{(4-2\varepsilon)}$ to the interacting case.[4]

In this paper, we focus on the prototypical example of a QFT that breaks Weyl symmetry explicitly, namely a scalar in four dimensions with quartic self-interaction. The breaking of Weyl symmetry is achieved via the nonconformal quadratic coupling with the curvature as well as by nonzero beta functions. We argue that $\mathcal{A}_{\text{reg}}$ as in (1) [or rather the concrete prescriptions $\mathcal{A}^{(D)}_{\text{reg}}$ (5)] does not extend beyond free level. We thus modify the prescription (1) by promoting the expectation values to renormalized (finite) ones, and consider

$$\mathcal{A}_{\text{ren}} = g^{mn}\langle [T_{mn}]\rangle - \langle [\Theta]\rangle, \qquad \Theta = g^{mn}T_{mn}, \tag{6}$$

where Θ is the four-dimensional trace of the classical energy-momentum tensor and the square brackets in the expectation values denote the renormalized composite operators. We construct these renormalized operators in dimensional regularization following the well-established tradition of [23–26] and references therein. In particular, we work in perturbation theory to first order in the coupling with a formal expansion around a flat background $g_{mn} = \delta_{mn} + h_{mn}$ and focus on the contributions to $\mathcal{A}_{\text{ren}}$ of first and second order in h. We evaluate the former fully, while the latter suffer from the complication of three-propagator subdiagrams which need to be expanded to a nontrivial order in ε. We show that, at free (one-loop) level, (6) provides a local result that reproduces the heat kernel prescription (4). To circumvent the technical difficulties in evaluating (6) to the first order in the coupling at order h^2, we consider the spacetime integral of $\mathcal{A}_{\text{ren}}$. This is the generally covariantized analog of setting the momentum of the stress tensor to zero, thereby reducing the integrals to two-propagator diagrams. At two loops we obtain a result that is nonlocal, and we argue that this is indeed expected in the general case.

In our calculations all of the nonlocalities and departures from the anomaly proper disappear at the conformal value of the curvature coupling $\xi = \frac{1}{6}$, thus the construction might look in this case artificial. However, this value is not stable

[3]The generalization to the case of multiple fields or different spin is immediate; see, e.g., [6].

[4]Another perspective on Weyl anomalies for nonconformal theories is given in [22].

under quantum corrections, which induce an RG flow for this parameter away from the conformal point [27,28]. Despite these effects being relevant at a higher order than the ones considered in this paper, our setting is therefore generic.

The paper is organized as follows. In Sec. II we give a general review of the formal setting: action, regularization, and renormalization in curved background in perturbation theory. In Sec. III we review the regularized calculations of [20]. In Sec. IV we construct the renormalized anomaly (6), commenting on its nonlocal structure and the two-loop result to first order in the interaction. Section V concludes with a summary, a comparison with recent literature, and outlook. Appendix A summarizes notation and conventions; Appendix B reports lengthy formulas for Feynman vertices; Appendix C discusses some aspects of the renormalization of the action in curved spacetime that are relevant for our discussion.

II. SETTING AND NOTATIONAL REMARKS

We consider the scalar action in D dimensions[5] with a quartic self-interaction and to a geometrical background given by

$$S_\varphi = \int d^D x \sqrt{g}\left[\frac{1}{2}\left(\nabla_m\varphi\nabla^m\varphi + \xi R\varphi^2\right) + \frac{\lambda}{4!}\varphi^4\right], \tag{7}$$

where ξ is the dimensionless curvature coupling and λ is classically dimensionless only in $D = 4$. Weyl invariance of the kinetic term is achieved for $\xi = \xi_D \coloneqq \frac{1}{4}\frac{D-2}{D-1}$.

We note the equation-of-motion operator,

$$E_\varphi = \varphi\frac{\delta}{\delta\varphi}S = \varphi(-\Box + \xi R)\varphi + \frac{\lambda}{3!}\varphi^4, \tag{8}$$

and the stress tensor and its D-dimensional trace are

$$\begin{aligned} T^\varphi_{mn} &= \nabla_m\varphi\nabla_n\varphi - \frac{1}{2}g_{mn}\nabla_a\varphi\nabla^a\varphi - g_{mn}\frac{\lambda}{4!}\varphi^4 \\ &\quad + \xi\varphi^2\left(R_{mn} - \frac{1}{2}g_{mn}R\right) - \xi(\nabla_m\nabla_n\varphi^2 - g_{mn}\Box\varphi^2), \\ \Theta^{(D)} &= g^{(D)mn}T^\varphi_{mn} \\ &= (D-1)(\xi-\xi_D)\Box\varphi^2 - \frac{D-2}{2}E[\varphi] + \frac{(D-4)}{4!}\lambda\varphi^4. \end{aligned} \tag{9}$$

The latter indeed shows that the stress tensor is classically traceless on shell for $\xi = \xi_D$ at $D = 4$ or when $\lambda = 0$. In particular, we note the value of the classical trace in $D = 4$ dimensions,

$$\Theta \equiv \Theta^{(4)} = 3\left(\xi - \frac{1}{6}\right)\Box\varphi^2 - E_\varphi. \tag{10}$$

As we are going to review in the next subsection, the equation of motion operator has vanishing expectation value in dimensional regularization, both in the bare and in the renormalized theory. Since we will be only considering such one-point functions, we will often drop it.

A. Regularization

We adopt the framework of dimensional regularization with $d = 4 - 2\varepsilon$, which is standard in both flat and curved spacetime [6,23–26,29]. For simplicity and ease of exposition we understand the energy scale μ and reinstate it only in final expressions.

We are interested in regularized and then renormalized expectation values of T^φ_{mn} and $\Theta^{(D)}$ which we compute via the path integral,

$$\begin{aligned} \langle T^\varphi_{mn}\rangle_\varepsilon &= \int \mathcal{D}\varphi e^{-S}T^\varphi_{mn}, \\ \langle\Theta^{(D)}\rangle_\varepsilon &= \int \mathcal{D}\varphi e^{-S}\Theta^{(D)}, \\ \int \mathcal{D}\varphi e^{-S} &= 1, \end{aligned} \tag{11}$$

where the subscript ε indicates the use of bare dimensionally regularized correlators. Fundamental in our discussion is the observation that

$$\langle\Theta^{(d)}\rangle_\varepsilon = \langle g^{(d)mn}T^\varphi_{mn}\rangle_\varepsilon = g^{(d)mn}\langle T^\varphi_{mn}\rangle_\varepsilon, \tag{12}$$

namely when considering the expectation value of the $D = d$ dimensional trace $\Theta^{(D=d)}$, the contraction with the metric can be equivalently taken before or after path integration (or equivalently before and after expanding in ε). This is possible because for regularized correlators the rule $g^{(d)mn}g^{(d)}_{mn} = d = 4 - 2\varepsilon$ is valid inside and outside the path integral symbol. We stress that this holds true only for $\langle\Theta^{(D=d)}\rangle_\varepsilon$. For $\langle\Theta^{(D=4)}\rangle_\varepsilon$ it is not the case.

Another important feature of dimensional regularization is that $\langle E[\varphi]\rangle = 0$, since

$$\begin{aligned} \langle E[\varphi]\rangle &= \int \mathcal{D}\varphi\, e^{-S}\varphi(x)\frac{\delta}{\delta\varphi(x)}S \\ &= -\int \mathcal{D}\varphi\frac{\delta}{\delta\varphi(x)}\left(\varphi(x)e^{-S}\right) = 0, \end{aligned} \tag{13}$$

which vanishes as a functional boundary term.[6]

[5]We use lowercase d to denote the dimensionally regularized value $d = 4 - 2\varepsilon$. We introduce an auxiliary dimension D to be able to distinguish the two different cases $D = 4$ and $D = d$ more explicitly.

[6]In (13) we used the standard value $\frac{\delta}{\delta\varphi(x)}\varphi(x) = \delta[x-x] = 0$ of dimensional regularization [24–26,29].

We wish to evaluate the correlators in (11) in perturbation theory in λ. To be able to use the well-developed diagrammatic technology, we perform a formal expansion on a flat background $g_{mn} = \delta_{mn} + h_{mn}$ and work order by order in h_{mn}. In particular, we will need

$$\begin{aligned} S_\varphi &= S^{(0)}_{\varphi^2} + S^{(1)}_{\varphi^2} + S^{(2)}_{\varphi^2} + \cdots + S^{(0)}_{\varphi^4} + S^{(1)}_{\varphi^4} + \cdots, \\ T^\varphi_{mn} &= T^{\varphi^2(0)}_{mn} + T^{\varphi^2(1)}_{mn} + \cdots + T^{\varphi^4(0)}_{mn} + T^{\varphi^4(1)}_{mn} + \cdots, \\ \Theta &= \Theta^{\varphi^2(0)} + \Theta^{\varphi^2(1)} + \cdots, \end{aligned} \tag{14}$$

where the superscript (n) indicates the power of h, and φ^2, φ^4 distinguish the free vs interaction contributions. In particular, $S^{(0)}_{\varphi^2} + S^{(0)}_{\varphi^4}$ is the flat-space scalar action,[7]

$$S^{(0)}_{\varphi^2} = \frac{1}{2}\int d^d x \partial_m\varphi\partial_m\varphi, \qquad S^{(0)}_{\varphi^4} = \frac{\lambda}{4!}\int d^d x\varphi^4 \tag{15}$$

and we adopt the following notation for the interactions with the background metric:

$$\begin{aligned} S^{(1)}_{\varphi^2} &= \int đp\, đq\, đ\ell (2\pi)^d \delta[p+q+\ell]\varphi(p) \\ &\quad \times \varphi(q) h_{mn}(\ell) V^{\varphi^2(1)}_{mn}(p,q,\ell), \\ S^{(2)}_{\varphi^2} &= \int đp\, đq\, đk\, đ\ell (2\pi)^d \delta[p+q+k+\ell]\varphi(p)\varphi(q) \\ &\quad \times h_{mn}(\ell) h_{rs}(k) V^{\varphi^2(2)}_{mnrs}(p,q,\ell,k), \end{aligned} \tag{16}$$

and so on analogously for all terms $S^{(n)}_{\varphi^m}$. Similarly, for the stress tensor we write[8]

$$\begin{aligned} T^{\varphi^2(0)}_{mn} &= \int đp\, đq\, e^{i(p+q)x}\varphi(p)\varphi(q) W^{\varphi^2(0)}_{mn}(p,q), \\ T^{\varphi^2(1)}_{mn} &= \int đp\, đq\, đ\ell\, e^{i(p+q+\ell)x}\varphi(p)\varphi(q) h_{ac}(\ell) \\ &\quad \times W^{\varphi^2(1)}_{mnac}(p,q,\ell), \end{aligned} \tag{17}$$

and so on.[9] Explicit expressions for the relevant vertices are reported in Appendix B.

B. Renormalization

Renormalizing the theory on curved geometry requires the familiar infinite rescaling of the parameters in the action (7) as well as additional terms to cancel purely gravitational infinities. One therefore considers the total action

$$\begin{aligned} S &= S_\varphi + S_{\rm grav}, \\ S_{\rm grav} &= \int d^d x\sqrt{g}[-\alpha\mathbb{E}_4 + \gamma {\rm Weyl}^2 + \rho R^2]. \end{aligned} \tag{18}$$

The gravitational contribution is quadratic in the curvature and contains the Euler density, the square of the Weyl tensor, and the square of the Ricci scalar (explicit expressions in Appendix A). We have an expansion in h_{mn} analogous to (14),

$$\begin{aligned} S_{\rm grav} &= S^{(2)} + S^{(3)} + \cdots, \\ S^{(2)} &= \int đp\, h_{mn}(p) h_{rs}(-p) V^{(2)}_{mn,rs}(p), \\ S^{(3)} &= \int đp\, đq\, đk (2\pi)^d \delta[p+q+k] h_{mn}(p) h_{rs}(q) h_{ac}(k) \\ &\quad \times V^{(3)}_{mnrsac}(p,q,k). \end{aligned} \tag{19}$$

A finite theory is then obtained by setting

$$\begin{aligned} &\lambda \to \lambda + \sum_{i\geq 1}\frac{\lambda^{(i)}}{\varepsilon^i}, \quad \xi \to \xi + \sum_{i\geq 1}\frac{\xi^{(i)}}{\varepsilon^i}, \quad \varphi \to \left(1 + \sum_{i\geq 1}\frac{z^{(i)}}{\varepsilon^i}\right)\varphi, \\ &\alpha \to \sum_{i\geq 1}\frac{\alpha^{(i)}}{\varepsilon^i}, \quad \gamma \to \sum_{i\geq 1}\frac{\gamma^{(i)}}{\varepsilon^i}, \quad \rho \to \sum_{i\geq 1}\frac{\rho^{(i)}}{\varepsilon^i}. \end{aligned} \tag{20}$$

We work in minimal subtraction scheme so that the Weyl tensor is intended as the four-dimensional one and α, γ, ρ are only poles. The values of the counterterms in (20) have been computed in the literature long ago and we will quote the relevant ones momentarily. In Appendix C we comment on some aspects of their calculation in the spirit of the present work.

We will need the renormalized stress tensor $[T_{mn}]$ and the renormalized stress tensor trace $[\Theta]$. The square brackets denote renormalized composite operators. Constructing renormalized composite operators in dimensional regularization is standard also in the curved background context, see [23–26,29]. Here we summarize the relevant results.

A finite (renormalized) stress-tensor operator is obtained by differentiation of the renormalized full action (18) with the renormalized values (20), so that[10]

[7]We write all indices lowered to emphasize the contraction with the flat metric.

[8]The vertex functions for T_{mn} and $\Theta^{(D)}$ do not involve a momentum-conserving delta function, as they are external vertices injecting momentum in the graph.

[9]In general, one needs to introduce Feynman rules also for Θ, but in this particular example it is not necessary.

[10]An operator is renormalized by requiring that its insertion produces finite correlators. Given any finite correlator, an additional stress-tensor insertion is realized by differentiation with respect to the metric thus without introducing any additional divergence.

$$[T_{mn}] = -\frac{2}{\sqrt{g}}\frac{\delta}{\delta g^{mn}}[S] = -\frac{2}{\sqrt{g}}\frac{\delta}{\delta g^{mn}}[S_\varphi + S_{\mathrm{grav}}]$$
$$= [T^\varphi_{mn} + T^{\mathrm{grav}}_{mn}], \qquad T^{\mathrm{grav}}_{mn} = -\frac{2}{\sqrt{g}}\frac{\delta}{\delta g^{mn}}S_{\mathrm{grav}}. \tag{21}$$

The expansion (14) is therefore complemented by

$$T^{\mathrm{grav}}_{mn} = T^{0(1)}_{mn} + T^{0(2)}_{mn} + \cdots,$$
$$T^{0(1)}_{mn} = \int đ\ell\, e^{i\ell x} h_{ac}(\ell) W^{0(1)}_{mnac}(\ell),$$
$$T^{0(2)}_{mn} = \int đ\ell\, đk\, e^{i(\ell+k)x} h_{ac}(\ell) h_{rs}(k) W^{0(1)}_{mnacrs}(\ell, k). \tag{22}$$

Following (18) and (20), these terms are pure poles and are responsible for the anomaly proper.

To construct[11] a finite operator associated with the four-dimensional stress-tensor trace Θ, we start with the renormalized operator $[\varphi^2(x)]$, which is given by

$$[\varphi^2] = Z_2\varphi_0^2 + Z_g R, \qquad Z_2 = 1 + \sum_{i\geq 1}\frac{z_2^{(i)}}{\varepsilon^i}, \qquad Z_g = \sum_{i\geq 1}\frac{z_g^{(i)}}{\varepsilon^i}. \tag{23}$$

We then define the renormalized operator associated with (10) as[12]

$$[\Theta] = 3\left(\xi - \frac{1}{6}\right)\Box[\varphi^2] - E_\varphi, \tag{24}$$

where ξ is the renormalized (finite) value.

The counterterms (20) have been computed in the literature with a combination of diagrammatic and heat kernel methods. To the relevant order, the counterterms are [23,24,28]

$$\xi^{(1)} = \frac{6\xi-1}{12(4\pi)^2}\lambda, \quad \xi^{(2)} = 0, \quad \alpha^{(1)} = \frac{1}{720(4\pi)^2}, \quad \alpha^{(2)} = 0,$$
$$\gamma^{(1)} = \frac{1}{240(4\pi)^2}, \quad \gamma^{(2)} = 0, \quad \rho^{(1)} = \frac{(6\xi-1)^2}{144(4\pi)^4},$$
$$\rho^{(2)} = -\frac{(6\xi-1)^2}{288(4\pi)^2}\lambda, \quad \lambda^{(1)} = 0, \quad z^{(1)} = 0,$$
$$z_2^{(1)} = \frac{\lambda}{2(4\pi)^2}, \quad z_g^{(1)} = \frac{6\xi-1}{6(4\pi)^2}. \tag{25}$$

[11] Insertions of $\Theta^{(d)}$ in arbitrary correlators are produced by differentiation with respect to the conformal factor of the metric, so $\Theta^{(d)}$ does not require additional subtractions, consistently with (12). In contract, no shortcut is available for $\Theta^{(4)}$.

[12] The equation of motion operator does not require any additional subtraction [23,24,26] and it has vanishing expectation value.

In particular, at the order in which we are working there is no renormalization of the coupling λ nor is there wave function renormalization.

III. THE REGULARIZED EXPRESSION

A. Ambiguity and one-loop (free theory) results

In this subsection we review the one-loop calculation of [20] (cf. also [30]) and we extend some of the results and discussions.

In the context of dimensional regularization we can interpret (1) in different ways, i.e., there is an intrinsic ambiguity.

$$\mathcal{A}^{(D)}_{\mathrm{reg}} = g^{(4)mn}\langle T_{mn}\rangle_\varepsilon - \langle\Theta^{(D)}\rangle_\varepsilon \quad (\varepsilon \to 0), \tag{26}$$

with $D = 4$ or $D = d = 4 - 2\varepsilon$. The origin of this ambiguity can be appreciated by looking at the explicit expression (9): the difference between $\Theta^{(4)}$ and $\Theta^{4-2\varepsilon}$ is of order ε and thus they produce different terms when combined with the poles of loop integrals.

The case $D = d$ has a computational advantage: one can compute $\mathcal{A}^{(d)}_{\mathrm{reg}}$ with the knowledge of the divergent part of $\langle T_{mn}\rangle$ only, without the need to consider the more complicated finite pieces. Indeed, as a consequence of (12) we can write (26) for $D = d$ as

$$\mathcal{A}^{(d)}_{\mathrm{reg}} = (g^{(4)mn} - g^{(d)mn})\langle T_{mn}\rangle_\varepsilon \quad (\varepsilon \to 0). \tag{27}$$

This expression shows two important features. First, only terms in $\langle T_{mn}\rangle_\varepsilon$ proportional to the metric g_{mn} contribute: everything else cancels in the difference, as, e.g., $g^{(D)mn}R_{mn} = R$ for any D. Second, only the pole of $\langle T_{mn}\rangle_\varepsilon$ contributes, as can be seen by writing

$$\langle T_{mn}\rangle_\varepsilon = \frac{1}{\varepsilon}(P_{mn} + g_{mn}Q_{mn}) + F_{mn} + \mathcal{O}(\varepsilon), \tag{28}$$

where P_{mn} denotes tensor structures that are not proportional to g_{mn}. From (27), we thus have

$$\mathcal{A}^{(d)}_{\mathrm{reg}} = \lim_{\varepsilon\to 0}\left[\frac{4}{\varepsilon}Q^m_m - \frac{4-2\varepsilon}{\varepsilon}Q^m_m\right] = 2Q^m_m, \tag{29}$$

where P_{mn} and F_{mn} have dropped since $g^{(4)mn}P_{mn} = g^{(d)mn}P_{mn}$ and $g^{(4)mn}F_{mn} = g^{(d)mn}F_{mn} + \mathcal{O}(\varepsilon)$. We notice that this argument does not rely on perturbative expansion in h: if the full covariant expression for the (local) pole of $\langle T_{mn}\rangle_\varepsilon$ is known (as is, e.g., using the heat kernel expansion), this immediately gives the covariant result.

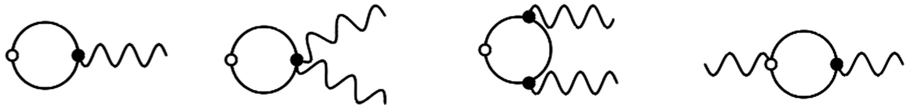

FIG. 1. Diagrammatic representation of $\langle\!\langle T_{mn}(x)\rangle\!\rangle_{\varepsilon\mathcal{O}(h^1)} + \langle\!\langle T_{mn}(x)\rangle\!\rangle_{\varepsilon\mathcal{O}(h^2)}$ in (30). Black dots denote interaction vertices V, white dots denote stress-energy tensor vertices W. Solid lines represent scalar propagators and wavy lines the metric perturbation h.

To perform this calculation diagrammatically, we expand in $g_{mn} = \delta_{mn} + h_{mn}$. Using the definitions of Sec. II B we have to consider the following terms[13]:

$$\langle T_{mn}(x)\rangle_\varepsilon = -\langle\!\langle T^{(0)}_{mn} S^{(1)}_{\varphi^2}\rangle\!\rangle_\varepsilon - \langle\!\langle T^{(0)}_{mn} S^{(2)}_{\varphi^2}\rangle\!\rangle_\varepsilon + \frac{1}{2}\langle\!\langle T^{(0)}_{mn} S^{(1)}_{\varphi^2} S^{(1)}_{\varphi^2}\rangle\!\rangle_\varepsilon - \langle\!\langle T^{(1)}_{mn} S^{(1)}_{\varphi^2}\rangle\!\rangle_\varepsilon + \mathcal{O}(h^3), \tag{30}$$

whose diagrammatic representation is pictured in Fig. 1. The associated integrals are listed in Appendix B. They can be expanded in ε and the anomaly can be obtained $\mathcal{A}^{(d)}$ by direct application of (27) [i.e. (29)]. Owing to locality, the full g dependence can be reconstructed by demanding general covariance. We refer to [20,30] for details and we simply state the result,

$$\mathcal{A}^{(d)}_{\text{reg}} = \frac{1}{180(4\pi)^2}\left[-\frac{1}{2}\mathbb{E}_4 + 6\Big(1 - 10(6\xi - 1)^2\Big)\Box R + \frac{3}{2}\text{Weyl}^2 + \frac{5}{2}(6\xi - 1)^2 R^2\right]. \tag{31}$$

The prescription $\mathcal{A}^{(4)}_{\text{reg}}$ was only briefly described in [20,30] and here we provide some more details. At first sight, this case seems to require the full evaluation of the finite parts of the correlators. However, we notice the seemingly trivial rewriting

$$\begin{aligned}\mathcal{A}^{(4)}_{\text{reg}} &= g^{(4)mn}\langle T_{mn}\rangle_\varepsilon - \langle\Theta^{(d)}\rangle_\varepsilon + \langle\Theta^{(d)}\rangle_\varepsilon - \langle\Theta^{(4)}\rangle_\varepsilon \quad (\varepsilon \to 0)\\ &= \mathcal{A}^{(d)}_{\text{reg}} + \langle\Delta\rangle,\end{aligned} \tag{32}$$

where we defined

$$\langle\Delta\rangle \coloneqq \langle\Theta^{(d)}\rangle_\varepsilon - \langle\Theta^{(4)}\rangle_\varepsilon \quad (\varepsilon \to 0). \tag{33}$$

It is clear that the splitting in the second line (32) is meaningful, namely that the two terms $\mathcal{A}^{(d)}_{\text{reg}}$ and $\langle\Delta\rangle$ are separately finite: the former is discussed above, the latter follows from

$$\Delta \coloneqq \Theta^{(d)} - \Theta^{(4)} = \varepsilon \cdot \frac{4\xi - 1}{2}\Box\varphi^2 + \varepsilon E_\varphi. \tag{34}$$

Thus, computing $\langle\Delta_\varepsilon\rangle$, the second term vanishes [cf. (10)] and the first one gives a finite and local result, which is proportional to $\Box R$ on dimensional and covariance grounds. Indeed we find $\langle\Delta\rangle = -15(4\xi - 1)(6\xi - 1)\Box R$ and as a result

$$\mathcal{A}^{(4)}_{\text{reg}} = \frac{1}{180(4\pi)^2}\left[-\frac{1}{2}\mathbb{E}_4 + 6(1 - 5\xi)\Box R + \frac{3}{2}\text{Weyl}^2 + \frac{5}{2}(6\xi - 1)^2 R^2\right]. \tag{35}$$

We can see that a generic value of ξ features the appearance of R^2 in the anomaly with either prescription. Since this is not compatible with the Wess-Zumino consistency conditions, it follows that the quantity $\mathcal{A}^{(D)}_{\text{reg}}$ is not a functional derivative of an effective action as already noticed in the original paper [5]. This observation was anticipated in the free-scalar calculation of [20,30] and is also discussed in [21], where the authors introduce an all-loop definition for the conformal anomaly in dimensional regularization which effectively extends the prescription $A^{(d)}_{\text{reg}}$. As a consequence, all four coefficients in (35) are physical and the difference between the two possible choices, $\mathcal{A}^{(4)}_{\text{reg}}$ and $\mathcal{A}^{(d)}_{\text{reg}}$, cannot be reabsorbed by the introduction of counterterms in the action.[14] In fact, finite counterterms cancel between the two terms in (26). The ambiguity $D = 4$ vs $D = d$ in the subtraction $\mathcal{A}^{(D)}_{\text{reg}}$ is not discussed in [5] where the quantity $\mathcal{A}_{\text{reg}}$ was first introduced, and to our knowledge it is not discussed anywhere else besides the references above. Finally, we observe that the heat kernel identification (4) coincides with the prescription $\mathcal{A}^{(4)}_{\text{reg}}$. This is a nontrivial result that, to our knowledge, was so far discussed only in [20,30].

This concludes our review of the calculation of [20], which hopefully clarifies some incorrect comments reported elsewhere.[15] In the next section we verify our

[13]We denote by $\langle\!\langle\cdots\rangle\!\rangle_\varepsilon$ the expectation value taken with respect to the flat-space free theory.

[14]This point is overlooked in [20] and corrected in [30].

[15]In the conclusions of both arXiv v1, v2, and of the journal version, Ref. [22] comments that $A^{(d)}_{\text{reg}}$ as in (31) is "obtained using dimensional regularization and a perturbative expansion around flat space, together with a dose of intuition to use the right amount of onshellness" to simplify the stress-tensor trace. Similar statements appear in the Introduction. It should be clear from the discussion above that this remark is incorrect in two ways: (i) operatively the result (31) does not directly depend on $\Theta^{(d)}$ but relies on the epsilon expansion of the $\langle T_{mn}\rangle_\varepsilon$ only; (ii) computing $\langle\Theta^{(d)}\rangle_\varepsilon$ directly neglecting the equation of motion (e.o.m.) operator E, as done for $\langle\Theta\rangle_\varepsilon$ in $\mathcal{A}^{(4)}_{\text{reg}}$, is not a problem because $\langle E\rangle = 0$ in dimensional regularization as in (13). Unfortunately, the authors of [22] did not share their impression with those of [20] prior to publication. For completeness, we note that $\mathcal{A}^{(4)}_{\text{reg}}$ is not discussed in [22].

claim that $\mathcal{A}_{\mathrm{reg}}^{(D)}$ is divergent in the limit where the regulator is removed in an interacting theory.

B. Failure at two loops (first order in the coupling)

We will now argue that the definition (26) does not provide a finite quantity at higher loop order by showing an explicit two-loop divergence proportional to $\Box R$. As discussed at the end of the previous subsection, it is an unambiguous quantity, in contrast to the anomaly proper.

It is enough to consider the term of order $\mathcal{O}(h^1)$. The relevant contribution is

$$\begin{aligned}\langle T_{mn}(x)\rangle_\varepsilon|_{\mathcal{O}(h^1,\lambda^1)} &= \langle\langle T_{mn}^{(0)}(x)S_{\varphi^2}^{(1)}S_{\varphi^4}^{(0)}\rangle\rangle_\varepsilon \\ &= \lambda\int d^dx\int đq\, e^{iqx}h_{rs}(x)\int đp\,\frac{1}{p^2(q-p)^2}W_{mn}^{\varphi^2(1)}(p,q-p,-q)V_{rs}^{\varphi^2(1)}(-p,p-q,q) \\ &= -\lambda\frac{(6\xi-1)^2}{(4\pi)^4\varepsilon^2}\int d^dx\int đq\, e^{iqx}h_{rs}(x)\frac{(\delta_{mn}q^2-q_mq_n)(\delta_{rs}q^2-q_rq_s)}{72}+\mathcal{O}(\varepsilon^{-1}).\end{aligned} \tag{36}$$

A direct calculation on the lines of (29) immediately shows that the presence of a double pole proportional to the metric renders the anomaly $\mathcal{A}_{\mathrm{reg}}^{(d)}$ divergent, hence the definition (27) is insufficient to accommodate for interactions.

For ease of exposition we do not discuss the analogous calculation for $\mathcal{A}_{\mathrm{reg}}^{(4)}$, as the idea is essentially the same and its divergent nature is implicit in the results of the next sections.

IV. RENORMALIZED CONSTRUCTION

Having established the insufficiency of the regularized prescription, we turn to the definition (6) based on renormalized correlators,

$$\mathcal{A}_{\mathrm{ren}} = g^{mn}\langle[T_{mn}]\rangle - \langle[\Theta]\rangle, \tag{37}$$

which by construction works to arbitrary loop order and does not have any ambiguity once a renormalization scheme is chosen. We work in minimal subtraction.

Let us see the consequences of this definition in practice. Here we focus on the $\mathcal{O}(h^1,\lambda^0)+\mathcal{O}(h^1,\lambda^1)$ contribution to parallel the discussion of the previous section. As we shall see, we do not need the more complex $\mathcal{O}(h^2,\lambda^0)$ term to fully obtain $\mathcal{A}_{\mathrm{ren}}$ in the free case. The contribution $\mathcal{O}(h^2,\lambda^1)$ is even more complicated and will be considered in a simplified setting in a later section.

The first term of (37) can be computed from the renormalized effective action on a curved background. Here we work in series of h, so

$$\Gamma[g] = \int đ^4p\, đ^4q\, h_{mn}(p)h_{rs}(q)(2\pi)^4\delta[p+q]\Gamma_{mnrs}(p,q) + \cdots, \tag{38}$$

$$\begin{aligned}\langle T^{mn}(x)\rangle &= -\frac{2}{\sqrt{g(x)}}\frac{\delta}{\delta g_{mn}(x)}\Gamma[g] \\ &= \int đk\, e^{ikx}[-4h_{rs}(k)\Gamma_{mnrs}(-k,k)+\cdots],\end{aligned} \tag{39}$$

from which the trace can be readily computed and expanded in h, $g^{mn}\langle T_{mn}\rangle = g_{mn}\langle T^{mn}\rangle = (\delta_{mn}+h_{mn})\langle T^{mn}\rangle$. The second term of (37) is essentially given by the diagrammatic evaluation of $\langle[\varphi^2]\rangle$ following the definition (34).

In particular, to the lowest order in the metric perturbation, we obtain

$$\begin{aligned}g^{mn}\langle[T_{mn}]\rangle_{\mathcal{O}(h^1)} &= \int đp\, e^{ipx}h_{mn}(p)\cdot(p_mp_n-\delta_{mn}p^2)p^2\cdot\left[\frac{11-60\xi+15(6\xi-1)^2\log\frac{p^2}{\bar\mu^2}}{180(4\pi)^2}-\lambda\frac{[3(6\xi-1)\log\frac{p^2}{\bar\mu^2}-1]^2}{216(4\pi)^4}\right] \\ &\quad+\mathcal{O}(\lambda^2),\end{aligned} \tag{40}$$

$$\langle[\Theta]\rangle_{\mathcal{O}(h^1)} = \int đp\, e^{ipx}h_{mn}(p)\cdot(p_mp_n-\delta_{mn}p^2)p^2\cdot(6\xi-1)\frac{\left[2(4\pi)^2+\lambda\log\frac{p^2}{\bar\mu^2}\right]\left[3(6\xi-1)\log\frac{p^2}{\bar\mu^2}-1\right]}{72(4\pi)^4}+\mathcal{O}(\lambda^2), \tag{41}$$

where $\bar\mu^2 \coloneqq \mu^2e^{\gamma_E/4\pi}$. As a result we can recognize the covariant structure

$$\mathcal{A}_{\mathrm{ren}} = \frac{5\xi-1}{30(4\pi)^2}\Box R+\lambda\frac{\left[3(6\xi-1)\log\frac{\Box}{\bar\mu^2}-1\right]\left[6(6\xi-1)\log\frac{\Box}{\bar\mu^2}-1\right]}{216(4\pi)^4}\Box R+\mathcal{O}(h^2,\lambda^2). \tag{42}$$

In (40) and (41) there are nonlocal terms both in the free and in the interacting contributions. We notice that when $\lambda = 0$ these exactly cancel, while they survive in the interacting case. The free contribution agrees with the regularized value $\mathcal{A}^{(4)}_{\mathrm{reg}}$ in (35) (and thus with the heat kernel prescription $\mathcal{A}_{\mathrm{hk}}$). In fact, as we shall see in the next section, this agreement can be argued on general grounds at least in the massless case and we do not need an explicit calculation to obtain in general result

$$\mathcal{A}_{\mathrm{ren}} = \mathcal{A}^{(4)}_{\mathrm{reg}} \quad \text{(free theories)}. \tag{43}$$

Effectively, this means that the renormalized definition extends the HK prescription to arbitrary loop number.

A. Some properties of the renormalized anomaly

In this section we make some general consideration on the renormalized $\mathcal{A}_{\mathrm{ren}}$, comparing it between free and interacting theory. We focus on massless theories for simplicity.

We begin with the renormalized anomaly of the free theory. Interestingly, it reproduces the result of the regularized prescription $\mathcal{A}^{(4)}_{\mathrm{reg}}$. In particular, it is local: the nonlocal contributions cancel in the difference. We can indeed see this result on general grounds. Denoting by (0) bare quantities, there are only one-loop simple-pole geometrical counterterms and

$$S = S^{(0)} + S_{\mathrm{ct1}}, \qquad [T_{mn}] = T^{(0)}_{mn} - \frac{2}{\sqrt{g}}\frac{\delta S_{\mathrm{ct1}}}{\delta g^{mn}},$$

$$T^{(0)}_{mn} = -\frac{2}{\sqrt{g}}\frac{\delta S^{(0)}}{\delta g^{mn}}, \qquad \Theta = g^{(4)mn}T^{(0)}_{mn},$$

$$\begin{aligned}&[\Theta] = \{g^{(4)mn}T^{(0)}_{mn}\} + \theta_{\mathrm{ct}g},\\ &\theta_{\mathrm{ct}g} = Z_{\Box R}\Box R + Z_{R^2}R^2 + Z_{W^2}\mathrm{Weyl}^2 + Z_{\mathbb{E}_4}\mathbb{E}_4.\end{aligned} \tag{44}$$

The brackets $\{\cdots\}$ denote the renormalized composite operator without the contribution proportional to the identity operator, which is $\theta_{\mathrm{ct}g}$ and contains only poles. We dropped irrelevant e.o.m. terms. In the case of the massless scalar, only $Z_{\Box R} \neq 0$ in minimal subtraction, cf. (24). The crucial point, as we are going to see, is that $\{g^{(4)mn}T^{(0)}_{mn}\} = g^{(4)mn}T^{(0)}_{mn}$ for free theories, but typically not when interactions are present [cf. (24) and (25)].

We now consider

$$\mathcal{A}_{\mathrm{ren}} = g^{(4)mn}\langle[T_{mn}]\rangle - \langle[\Theta]\rangle. \tag{45}$$

The indication of the dimension $g^{(4)mn}$ in $\mathcal{A}_{\mathrm{ren}}$ is naturally redundant, as the expressions in the right-hand side are finite and renormalized, so they are in four dimensions, but we keep it for clarity. Explicitly, $\mathcal{A}_{\mathrm{ren}}$ becomes

$$\begin{aligned}\mathcal{A}_{\mathrm{ren}} &= g^{(4)mn}\langle T^{(0)}_{mn}\rangle_\varepsilon - \langle g^{(4)mn}T^{(0)}_{mn}\rangle_\varepsilon - \frac{2}{\sqrt{g}}g^{(4)mn}\frac{\delta S_{\mathrm{ct1}}}{\delta g^{mn}}\\ &\quad - \theta_{\mathrm{ct}g} \quad (\varepsilon \to 0)\\ &= \mathcal{A}^{(4)}_{\mathrm{reg}} - \lim_{\varepsilon\to 0}\left(\frac{2}{\sqrt{g}}g^{(4)mn}\frac{\delta S_{\mathrm{ct1}}}{\delta g^{mn}} + \theta_{\mathrm{ct}g}\right).\end{aligned} \tag{46}$$

We have used that the difference of the first two terms in the first line is of order ε, therefore it produces a finite and local result in the $\varepsilon \to 0$ limit which is exactly $\mathcal{A}^{(4)}_{\mathrm{reg}}$. The fact that we are still taking the trace in four dimensions[16] implies that, from the counterterms in S_{ct1}, we only have a divergent contribution proportional to

$$g^{(4)mn}\frac{\delta S_{\mathrm{ct1}}}{\delta g^{mn}} \sim \frac{1}{\varepsilon}g^{(4)mn}\frac{\delta}{\delta g^{mn}}\int\sqrt{g}R^2 \propto \frac{1}{\varepsilon}\Box R. \tag{47}$$

The other contributions vanish when taking the four-dimensional trace. By definition, $\theta_{\mathrm{ct}g}$ contains only poles. By construction $\mathcal{A}_{\mathrm{ren}}$ is finite, so the divergent pieces must cancel.

This argument relies on the fact that $\{g^{(4)mn}T^{(0)}_{mn}\} = g^{(4)mn}T^{(0)}_{mn}$, which is true at free level. Including interactions produces additional "wave function" renormalization factors that induce new terms in perturbation theory [cf. (23) and (24)]. In contrast, $[T_{mn}]$ does not require additional subtractions beyond the standard action renormalization of the action (which involves only noncomposite operators), so that a cancellation of the nonlocalities in $\mathcal{A}_{\mathrm{ren}}$ seems unlikely on general grounds. Our calculation in the scalar model supports this; cf. (42) and the following section.

B. Integrated anomaly

Computing $\mathcal{A}_{\mathrm{ren}}$ to two loops and third order in the metric perturbation h requires considering the ε expansion to high order of integrals of formidable complexity. To simplify the problem, we consider the integrated quantity

$$\begin{aligned}\mathbb{A} &= \int d^4x\sqrt{g(x)}\mathcal{A}_{\mathrm{ren}}(x)\\ &= \int d^4x\sqrt{g(x)}g^{(4)mn}(x)\langle[T_{mn}](x)\rangle,\end{aligned} \tag{48}$$

which remarkably, in the present example, does not depend on $\langle[\Theta]\rangle$ since that is a total derivative. The correlator is the finite, renormalized one, and we emphasized that the trace is taken in $D = 4$ for additional clarity.

[16]We could have extended the trace to $d = 4 - 2\varepsilon$ dimensions. In this case $g^{(4)mn}\langle T^{(0)}_{mn}\rangle$ is replaced by $g^{(d)mn}\langle T^{(0)}_{mn}\rangle = \langle g^{(d)mn}T^{(0)}_{mn}\rangle = \langle\Theta^{(d)}\rangle$ and $g^{(d)mn}\frac{\delta S_{\mathrm{ct1}}}{\delta g^{mn}}$ contains in addition to (47) a finite piece, which gives rise to the anomaly. The conclusion is the same.

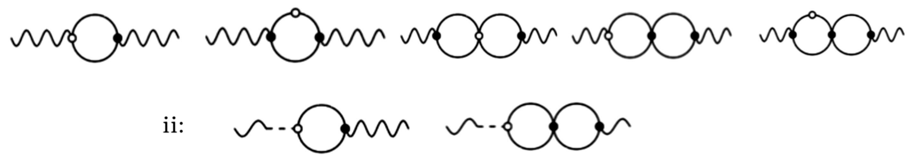

FIG. 2. Diagrammatic representation of the bare Feynman integrals in (51). Black dots are vertices V from the action; the white dot is a vertex W from the stress tensor. Wavy lines are metric perturbations h. Metric perturbations with a dashed line represent the external h_{rs} that does not come from the correlator.

We focus here on the $\mathcal{O}(h^2)$ contribution to (48)

$$(\mathbb{A})_{\mathcal{O}(h^2)} = \delta_{mn}\int d^4x \langle [T_{mn}]\rangle_{\mathcal{O}(h^2)} + \left(\frac{1}{2}\delta_{rs}\delta_{mn} - \delta_{r(m}\delta_{n)s}\right)\int d^4x\, h_{rs}\langle [T_{mn}]\rangle_{\mathcal{O}(h^1)}. \tag{49}$$

As the expectation values are computed in dimensional regularization, they have the structure

$$\langle T_{mn}\rangle_{\mathcal{O}(h^n)} = \lim_{\varepsilon\to 0}\Big[\langle T^{(0)}_{mn}\rangle_{\mathcal{O}(h^n)} + \langle T^{\mathrm{ct}}_{mn}\rangle_{\mathcal{O}(h^n)}\Big], \tag{50}$$

where the second term indicates counterterm contributions. We find it convenient to extend the dimensionality of the integrals and of all the delta symbols in (49) from 4 to $d = 4 - 2\varepsilon$ dimensions. This is consistent because the correlators are already finite, so this choice does not influence the limit $\varepsilon \to 0$. We can thus rewrite (49) as

$$\begin{aligned}(\mathbb{A})_{\mathcal{O}(h^2)} &= \lim_{\varepsilon\to 0}\int d^dx\, \delta_{mn}\Big[\langle T^{(0)}_{mn}\rangle_{\mathcal{O}(h^2)} + \langle T^{\mathrm{ct}}_{mn}\rangle_{\mathcal{O}(h^2)}\Big] + \lim_{\varepsilon\to 0}\int d^dx\, h_{rs}(x)\left(\frac{1}{2}\delta_{rs}\delta_{mn} - \delta_{r(m}\delta_{n)s}\right)\Big[\langle T^{(0)}_{mn}\rangle_{\mathcal{O}(h^1)} + \langle T^{\mathrm{ct}}_{mn}\rangle_{\mathcal{O}(h^1)}\Big],\\ &= \lim_{\varepsilon\to 0}\delta_{mn}\int đk\, h_{rs}(k)h_{ac}(-k)T^{\mathrm{i}}_{mnrsac}(k) + \lim_{\varepsilon\to 0}\left(\frac{1}{2}\delta_{rs}\delta_{mn} - \delta_{r(m}\delta_{n)s}\right)\int đk\, h_{rs}(k)h_{ac}(-k)T^{\mathrm{ii}}_{mnac}(k).\end{aligned} \tag{51}$$

Proceeding in this way has the advantage of setting to zero external momenta *before* expanding in ε, thus reducing the diagrams to manageable two-propagator integrals and avoiding IR divergent logs. In the second step, we wrote the integrands in momentum space and indeed implemented the momentum conservation arising from the integration over x. The diagrammatic representation is in Fig. 2 and we refer to Appendix B for the expressions of the corresponding integrals.

The values in (25) of the counterterms make these expressions finite, providing a consistency check. As a result, we obtain the integrated anomaly $\mathbb{A}$:

$$\begin{aligned}(\mathbb{A})_{\mathcal{O}(h^2)} = \frac{1}{180(4\pi)^2}\int \frac{d^4k}{(2\pi)^4} h_{rs}(k)h_{ac}(-k) \times \Big\{&\frac{3}{4}\delta_{ar}\delta_{cs}k^4 - \frac{3}{2}\delta_{ar}k^2k_sk_c\\ &+ \left(90\xi^2 - 30\xi + \frac{9}{4} - \frac{5}{6}\frac{\lambda}{(4\pi)^2}(6\xi-1)\left(1 - 3(6\xi-1)\log\frac{k^2}{\mu}\right)\right)(\delta_{rs}\delta_{ac}k^4 - 2\delta_{rs}k_ak_ck^2)\\ &+ \left(90\xi^2 - 30\xi + 3 - \frac{5}{6}\frac{\lambda}{(4\pi)^2}(6\xi-1)\left(1 - 3(6\xi-1)\log\frac{k^2}{\mu}\right)\right)k_ak_ck_rk_s\Big\}.\end{aligned} \tag{52}$$

We recognize the covariant structure

$$\mathbb{A} = \frac{1}{180(4\pi)^2}\int d^4x\sqrt{g}\left\{\frac{3}{2}\mathrm{Weyl}^2 + (6\xi-1)\left(\frac{5}{2}(6\xi-1) - \frac{5\lambda}{6(4\pi)^2}\right)R^2 + \frac{5\lambda(1-6\xi)^2}{2(4\pi)^2}R\log\frac{\Box}{\bar{\mu}^2}R\right\} + \mathcal{O}(h^3,\lambda^2). \tag{53}$$

The Euler term $\mathbb{E}_4$ is a total derivative in four dimensions; hence it disappears after integration, as well as the manifest total derivative $\Box R$. Interestingly, the c coefficient is in any case undeformed by λ at first order.[17]

[17] This carries resemblance with the analysis of [31,32] based on the analysis of stress-tensor correlators.

We see from (52) and (53) that the departure from conformality brings nonlocalities in $\mathcal{A}_{\rm ren}$ together with an explicit dependence on the energy scale μ besides the implicit one induced by the renormalization of the parameters.

V. CONCLUDING REMARKS

In this paper we have explored the characterization of the quantum contributions to nonconformal theories $\mathcal{A}$ (1) proposed by Duff in [5,7]. We studied in dimensional regularization the explicit example of a scalar field with a generic curvature coupling and a quartic self-interaction.

The free case was studied in [20,30] using regularized but not renormalized correlators. In particular, an ambiguity in the definition of $\mathcal{A}^{(D)}_{\rm reg}$ in (26) was pointed out, corresponding to the dimensionality of the subtraction term, $D=4$ vs $D=4-2\varepsilon$. We have reviewed and completed the calculation, spelling out some aspects that were misunderstood in the previous literature [20,22]. We explicitly showed that the prescription $\mathcal{A}^{(4)}_{\rm reg}$ reproduces the result of the heat kernel identification $\mathcal{A}_{\rm hk}=a_4$, which is advocated in [5,7] to be preferred. On the other hand, the prescription $\mathcal{A}^{(4-2\varepsilon)}$ is singled out in the analysis of [21]. There, in the context of dimensional regularization, a different notion of $\mathcal{A}$ valid to all-loop order is introduced, which is by construction finite local and reduces to $\mathcal{A}^{(4-2\varepsilon)}$ for free theories. It is naturally of interest to understand which prescription is more appropriate to capture the sought effects.

In either case, $\mathcal{A}$ of the form (1) produces a quantity that contains R^2, thus violating the Wess-Zumino (WZ) consistency conditions. This implies that it cannot be obtained as a functional derivative of an effective action and it is not subject to the same counterterm ambiguity of the anomaly proper: a finite counterterm would cancel in the difference between the two terms in $\mathcal{A}$. As an additional consequence, also the coefficient of $\Box R$ is physical.[18] Similar comments appear also in [21,30].

We have then extended the analysis of [20,30] to include interaction at lowest order in the coupling. We have shown that the regularized prescription is insufficient, as it gives a divergent result once the regulator is removed. We thus considered the definition $\mathcal{A}_{\rm ren}$ built of renormalized correlators. We have argued that it is a good candidate to extend the identification of $\mathcal{A}$ with the heat kernel coefficient in the presence of interactions, at least for generic massless theories. This identification is nontrivial, in that it suggests a firmer diagrammatic understanding of the HK prescription (6) in a way that can be extended to higher loops, and deserves to be investigated in greater generality.

This definition, however, displays nonlocalities at higher loops. We have shown this explicitly in (42) and (53). We explained the appearance of the nonlocalities as a consequence of the fact that, in constructing finite composite operators, the stress tensor does not require any additional renormalization, while the operator associated to its trace does. It is this imbalance that produces uncanceled nonlocal terms from two loops on.

Given this discussion, it seems that the situation regarding the characterization of quantum violation of Weyl invariance, when the classical symmetry is absent, is far from clear. As Weyl (conformal) invariance is absent along the RG flow, this has the potential application of shedding light on the space of QFTs and providing insights in the local version of the a theorem. Similarly, Einstein gravity and supersymmetric generalization thereof lack classical Weyl invariance, therefore the significance of the cancellation of the c anomalies in the total heat kernel coefficients in $N>4$ supergravities is unclear [13,14].

On a practical level, it would be interesting to extend our calculation to higher loop to see the appearance of the beta functions as well as including mass terms. Other field theory models would provide additional concrete examples and would, e.g., allow one to test the identification of $\mathcal{A}$ with the heat kernel coefficient more thoroughly. To make a clearer connection with the a theorem [9], it would be interesting to compute $\mathcal{A}_{\rm ren}$ without the spacetime integration considered in Sec. IV B that hides the contribution from $\mathbb{E}_4$; more advanced diagrammatic techniques are needed in order to overcome the computational complexity. With this in mind, it would also be of interest understanding how to connect the notions of anomaly discussed above with [34–36].

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APPENDIX A: NOTATION AND CONVENTIONS

We work in Euclidean signature. Dimensional regularization is considered in $d=4-2\varepsilon$ dimensions. The metric is expanded in a perturbation around a flat background as $g_{mn}(x)=\delta_{mn}+h_{mn}(x)$.

Flat-space Fourier transforms and integrals follow the convention

$$f(x)=\int đp\, e^{ipx}f(p), \qquad \int đp\, e^{ipx}=\delta(x), \qquad đp=\frac{d^dp}{(2\pi)^d}. \tag{A1}$$

The four-dimensional Euler density and Weyl curvature tensor are given, respectively, by

$$\mathbb{E}_4=\mathrm{Riem}^2-4\mathrm{Ric}^2+R^2, \qquad \mathrm{Weyl}^2=\mathrm{Riem}^2-2\mathrm{Ric}^2+\frac{1}{3}R^2. \tag{A2}$$

[18] Aspects of physicality of the coefficient of $\Box R$, in particular in relation to the RG flow, are also discussed in [33].

Quantum expectation values are denoted as follows:

$\langle\ldots\rangle$: renormalized (finite) expectation values;

$\langle\ldots\rangle_{\mathcal{O}(h^n)}$: renormalized (finite) expectation values of order n in the metric perturbation;

$\langle\ldots\rangle_{\mathrm{reg}}$: regularized but not renormalized correlators (only used in general discussion);

$\langle\ldots\rangle_{\varepsilon}$: regularized correlators in dimensional regularization;

$\langle\!\langle\ldots\rangle\!\rangle_{\varepsilon}$: bare correlators taken in the free theory, in flat space, in dimensional regularization.

APPENDIX B: FEYNMAN RULES AND DIAGRAM INTEGRALS

The propagator for the field φ in momentum space reads

$$G(p,q)=\langle\!\langle\varphi(p)\varphi(q)\rangle\!\rangle_{\varepsilon}=\frac{(2\pi)^d\delta^{(d)}[p+q]}{p^2}. \qquad \text{(B1)}$$

The action vertices as defined in (14) and following, are

$$\begin{aligned}
V^{(2)}_{mn,rs}(p)&=\frac{\gamma}{2}p^4\delta_{m(r}\delta_{s)n}-\gamma p^2 p_{(m}\delta_{n)(r}p_{s)}+p_m p_n p_r p_s\left(\rho+\frac{\gamma}{3}\right)\\
&\quad+\left(p^2 p_r p_s\delta_{mn}+p^2 p_m p_n\delta_{rs}\right)\left(\frac{\gamma}{6}-\rho\right)-p^4\delta_{mn}\delta_{rs}\left(\frac{\gamma}{6}-\rho\right)\\
V^{\varphi^2(1)}_{mn}(p,q,\ell)&=\frac{1}{2}p_{(m}q_{n)}-\frac{1}{4}\delta_{mn}pq+\frac{\xi}{2}\left(\delta_{mn}\ell^2-\ell_m\ell_n\right)\\
V^{\varphi^2(2)}_{mn,rs}(p,q,\ell,k)&=-\frac{1}{16}\delta_{mn}\delta_{rs}pq+\frac{1}{8}pq\delta_{m(r}\delta_{s)n}-\frac{1}{4}p_{(m}\delta_{n)(r}q_{s)}-\frac{1}{4}q_{(m}\delta_{n)(r}p_{s)}+\frac{1}{8}\delta_{mn}q_{(r}p_{s)}+\frac{1}{8}\delta_{rs}q_{(m}p_{n)}\\
&\quad+\xi\bigg[\frac{1}{8}\delta_{mn}\delta_{rs}(k^2+k\ell+\ell^2)-\frac{1}{8}\delta_{m(r}\delta_{s)n}(2k^2-3k\ell+2\ell^2)-\frac{1}{8}\delta_{rs}(\ell_m\ell_n+2k_m k_n)\\
&\quad-\frac{1}{8}\delta_{mn}(k_r k_s+2\ell_r\ell_s)-\frac{1}{4}\delta_{mn}k_{(r}\ell_{s)}-\frac{1}{4}\delta_{rs}k_{(m}\ell_{n)}+\frac{1}{2}k_{(m}\delta_{n)(r}k_{s)}+\frac{1}{2}\ell_{(m}\delta_{n)(r}\ell_{s)}\\
&\quad+\frac{1}{2}\ell_{(m}\delta_{n)(r}k_{s)}+\frac{1}{4}k_{(m}\delta_{n)(r}\ell_{s)}\bigg]\\
V^{\varphi^4(1)}_{mn}&=\frac{\lambda}{2\cdot 4!}\delta_{mn}.
\end{aligned}$$

We also use $V^{(3)}_{mnrsac}$, but its expression is lengthy and uninformative so we do not report it.

The stress tensor vertices as defined in (17) and following are

$$W^{\varphi^2(1)}_{mn}(p,q)=-2V^{\varphi^2(1)}_{mn}(p,q,-p-q),\qquad W^{\varphi^4(0)}_{mn}=-\frac{\lambda}{4!}\delta_{mn},\qquad W^{\varphi^4(1)}_{mnac}-\frac{\lambda}{4!}\delta_{a(m}\delta_{n)c}. \qquad \text{(B2)}$$

The integrals corresponding to the diagrams of Fig. 1 referring to Eq. (30) are

$$\begin{aligned}
-\langle\!\langle T^{\varphi^2(0)}_{mn}S^{(1)}_{\varphi^2}\rangle\!\rangle_{\varepsilon}&=-2\int đq\,e^{iqx}h_{rs}(q)\int đp\frac{1}{p^2(p-q)^2}W^{\varphi^2(0)}_{mn}(p,q-p)V^{\varphi(1)}_{rs}(-p,p-q,q)\\
\frac{1}{2}\langle\!\langle T^{\varphi^2(0)}_{mn}S^{(1)}_{\varphi^2}S^{(1)}_{\varphi^2}\rangle\!\rangle_{\varepsilon}&=4\int đk\,đ\ell\,e^{i(k+\ell)x}h_{ac}(\ell)h_{rs}(k)\int đp\frac{1}{p^2(p-\ell)^2(p+k)^2}\\
&\quad\cdot W^{\varphi^2(0)}_{mn}(\ell-p,p+k)V^{\varphi(1)}_{rs}(p,-p-k,k)V^{\varphi(1)}_{ac}(p-\ell,-p,\ell)\\
-\langle\!\langle T^{\varphi^2(0)}_{mn}S^{(2)}_{\varphi^2}\rangle\!\rangle_{\varepsilon}&=-2\int đk\,đ\ell\,e^{i(k+\ell)x}h_{ac}(\ell)h_{rs}(k)\int đp\frac{1}{p^2(p-k-\ell)^2}\\
&\quad\cdot W^{\varphi^2(0)}_{mn}(p,k+\ell-p)V^{\varphi^2(2)}_{acrs}(-p,p-k-\ell,\ell,k)\\
-\langle\!\langle T^{\varphi^2(1)}_{mn}S^{(1)}_{\varphi^2}\rangle\!\rangle_{\varepsilon}&=-2\int đk\,đ\ell\,e^{i(k+\ell)x}h_{ac}(\ell)h_{rs}(k)\int đp\frac{1}{p^2(p-k)^2}\cdot W^{\varphi^2(1)}_{mnac}(p,k-p,\ell)V^{\varphi^2(1)}_{rs}(-p,p-k,k).
\end{aligned} \qquad \text{(B3)}$$

The bare integrals corresponding to the diagrams of Fig. 2 referring to Eq. (51) are

$$\begin{aligned} T^{\text{i bare}}_{mnrsac}(k) = &-2\int đp\,\frac{W^{\varphi^2(1)}_{mnrs}(-p,-k+p,k)V^{\varphi^2(1)}_{ac}(p,k-p,-k)}{p^2(p-k)^2} \\ &-8\int đp\,\frac{W^{\varphi^2(1)}_{mn}(-p,p,0)V^{\varphi^2(1)}_{rs}(-p,p-k,k)V^{\varphi^2(1)}_{ac}(p,k-p,-k)}{p^4(p-k)^2} \\ &+12\int đp\,\frac{V^{\varphi^2(1)}_{rs}(-p,p-k,k)}{p^2(p-k)^2}\int đq\,\frac{V^{\varphi^2(1)}_{ac}(q,-q+k,k)}{q^2(q-k)^2}W^{mn}(p,-q,-p+k,-k+q) \\ &+\lambda\int đp\,\frac{W^{\varphi^2(1)}_{mnrs}(-p,p-k,k)}{p^2(p-k)^2}\int đq\,\frac{V^{\varphi^2(1)}_{ac}(q,-q+k,-k)}{q^2(q-k)^2} \\ &-2\lambda\int đp\,\frac{V^{\varphi^2(1)}_{rs}(-p,p-k,k)}{p^2(p-k)^2}\int đq\,\frac{V^{\varphi^2(1)}_{ac}(q,-q+k,-k)W^{\varphi^2(0)}_{mn}(-q,q)}{q^4(q-k)^2} \end{aligned} \tag{B4}$$

$$\begin{aligned} T^{\text{ii bare}}_{mnac}(k) = &\;4\int đp\,\frac{V^{\varphi^2(1)}_{mn}(-p,p-k,k)V^{\varphi^2(1)}_{mn}(p,k-p,-k)}{p^2(p-k)^2} \\ &-2\lambda\int đp\,\frac{V^{\varphi^2(1)}_{mn}(-p,-k+p,k)}{p^2(p-k)^2}\int đq\,\frac{V^{\varphi^2(1)}_{ac}(k-q,q,-k)}{q^2(q-k)^2}, \end{aligned} \tag{B5}$$

where we have only given the bare integrals corresponding to the diagrams displayed; the counterterm contributions can be easily derived.

APPENDIX C: REMARKS ON ACTION RENORMALIZATION ON CURVED BACKGROUND

In the notation explained in Sec. III, the bare theory induces purely gravitational infinities that need to be canceled by counterterms in S_{grav} as in (18) with (20) and (25). To determine the counterterms, it is enough to compute the effective action to second and third order in the h expansion.

For the two-point function we have

$$\begin{aligned} \Gamma_{\mathcal{O}(h^2)} = \int đq\, h_{mn}(-q)h_{rs}(q)\bigg\{&\int đp\,\frac{1}{p^2(q-p)^2}V^{\varphi^2(1)}_{mn}(p,q-p,-q)V^{\varphi^2(1)}_{rs}(-p,p-q,q) \\ &-\frac{\lambda}{2}\int đp\,\frac{1}{p^2(q-p)^2}V^{\varphi^2(1)}_{mn}(p,q-p,-q)\int đk\,\frac{1}{k^2(k-p)^2}V^{\varphi^2(1)}_{rs}(-k,k-q,q)\bigg\}. \end{aligned} \tag{C1}$$

Performing the calculation to two loop (first order in λ) fixes $\xi^{(1)}, \xi^{(2)}$ through subdiagrams, and the resulting divergences give the counterterms $\gamma^{(1)}$, $\gamma^{(2)}$, $\rho^{(1)}$, and $\rho^{(2)}$. In contrast, α is not captured because $\mathbb{E}_4$ does not have a quadratic term in the expansion on a flat background in the general dimension.

The bare three-point function is

$$\begin{aligned} \Gamma_{\mathcal{O}(h^3)} = \int đp\,đq\, h_{mn}(-p)h_{rs}(q)h_{ac}(p-q)\bigg\{&\frac{4}{3}\int đ\ell\,\frac{1}{\ell^2(\ell-p)^2(\ell-q)^2} \\ &\times V^{\varphi^2(1)}_{mn}(\ell,p-\ell,p)V^{\varphi^2(1)}_{rs}(-\ell,\ell-q,q)V^{\varphi^2(1)}_{ac}(\ell-p,q-\ell,p-q) \\ &-2\int đ\ell\,\frac{1}{\ell^2(\ell-p)^2}V^{\varphi^2(1)}_{mn}(\ell,p-\ell,-p)V^{\varphi^2(2)}_{rsac}(-\ell,\ell-p,q,p-q)\bigg\}. \end{aligned} \tag{C2}$$

As observed in [37], the three-point function does capture the coefficient of $\mathbb{E}_4$. Despite the fact that it is a total derivative in $D = 4$ and vanishes in $D < 4$, in the spirit of analytically continuing for generic (complex) D, it is indeed relevant in the ε expansion. In fact, the contribution disappears only by using identities that are not valid for $D > 4$.

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