Higher spin mode stability for STU black hole backgrounds

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This article studies mode stability of a four-dimensional rotating black hole with four pairwise equal U(1) charges derived in the framework of supergravity or the low energy string theory which is known now as STU black hole. We investigate bosonic perturbations in a proposed equation of the Teukolsky type for probe fields of different spins through the transformation technique devised by Whiting in 1989. Finally, we introduce connection relations inspired by the work of Duztaş [Phys. Rev. D **94**, 044025 (2016)] to prove the absence of unstable modes that solve the torsion-modified Dirac equation appropriate for this black hole background.

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I. INTRODUCTION

The stability of rotating black holes has been the subject of active research in the field of gravitation for many years because of its fundamental importance. In a series of recent papers (see [1] and references therein) the stability of Kerr black holes has been conclusively proved, at least for sufficiently small angular momentum. This demonstration marked an important triumph of general relativity, as it proves consistency of the theory with the observed prevalence in nature of black holes of this kind. An important first step in investigations of black hole stability is to prove the absence of exponentially increasing modes in the master equation, derived by Teukolsky for the Kerr metric [2]. Mode stability for the aforementioned solution was proved by Whiting over three decades ago [3] by virtue of transformations mapping the physical modes into functions from an auxiliary space, in which the corresponding analysis becomes much simpler. This transformation results in a wavelike equation from which an auxiliary metric can be inferred. The space associated with this auxiliary metric lacks an ergoregion, and may thus be used to evade the difficulties that arise from its presence. We also point out that the so-called superradiance instability caused by the ergoregion of a rotating black hole takes place for fields (particles) of an integer spin, but it does not occur in the fermionic case.

The present work aims to show mode stability in the pair-wise equal charged STU black hole background. Such black holes have been previously investigated in Refs. [4–10] among several others. References [4,5], in particular, have successfully made use of Whiting's procedure to demonstrate mode stability for massless scalars ($s = 0$) in STU black holes of four and five dimensions respectively. Separability of the torsion modified Dirac equation that we shall use to investigate spin $\pm 1/2$ perturbations of the metric has been proved in a recent work by two of the authors [9]. As far as we know proper perturbations of the considered black hole space-time or field equations for spin $s > 1/2$ at this background have not been studied yet, but taking into account the wave equations for $s = 0$ and $s = 1/2$, we have conjectured equations which can be treated as the generalization of the decoupled Teukolsky equations for the case of the pair-wise equal charged geometry [9]. Here we show that instead of the pair of decoupled equations a unique equation can be written and the former follows from the latter after separation of variables. Having this equation we consider perturbations of both bosonic and fermionic nature. The former ones will be studied through the implementation of Whiting's procedure, as was done in the aforementioned papers. Fermionic perturbations, however, require a separate treatment. This necessity stems from the form of Whiting's differential transformations which, as we shall show, are not appropriate for half-integer spin. To circumvent this difficulty, we will use connection relations similar to those introduced in [11] in their study of mode stability for neutrino perturbations. As shall be explained in details below, this demonstration is based on the assumption that the expansion coefficient $Y^{(\mathrm{in})}$, relating to ingoing modes at infinity, can be treated as an analytic function of a and ω, with a well-defined nonrotating limit

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as $a \to 0$. As we will show, the connection relations, which follow from the field equations and separability of the theory, are not consistent with the presence of unstable modes, thus allowing us to rule out their existence.

The paper is structured as follows: after this introductory section, we present, in Sec. II, a Teukolsky-like master equation for the STU background, which is assumed to describe decoupled parts of perturbations in the metric for different values of s. Although the derivation of the Teukolsky equation for the STU space-time is rather straightforward and follows Teukolsky's original approach [2], whether or not it can be used to describe physical perturbations with spin higher than $s > 1/2$, including gravitational perturbations ($s = \pm 2$), remains an open question. The equation used in this work has been conclusively shown to be valid for $s = 0, \pm 1/2$. Beyond these values, we make a quite reasonable assumption that there should be a unique extension of the Teukolsky equation to include higher spin ($s > 1/2$) minimally coupled fields. This assumption should, of course, be verified by direct calculations in the future. In Sec. III, we use these equations to perform an analysis with the use of Whiting's transformations, which allow for the introduction of an energy-like functional with a positive integrand, which we then use to prove that the modes are bounded. Although this procedure seems to work without explicit specification of s, which appears as a parameter in the perturbation equations, we notice that one of Whiting's transformations involves a power of a differential operator, and this power is not integer for fermions. Thus, in Sec. IV we try a different route by following a procedure first introduced in [11] to derive connection relations which contradict the possibility of unstable modes. Finally, we conclude in Sec. V, summarizing what has been done throughout the paper and pointing out perspectives meant for further investigations.

II. TEUKOLSKY EQUATION FOR A ROTATING STU BLACK HOLE

In this work, we investigate small perturbations of an STU black hole [12]. We consider only the pair-wise equal charges case, for which separability of the torsion-modified Dirac equation has been demonstrated recently [9]. The metric in the pairwise equal charges scenario is of the form [6,12]:

$$ds^2 = -\frac{X}{\Delta_0^{1/2}}(dt - a\sin^2\theta d\varphi)^2 + \Delta_0^{1/2}\left(\frac{dr^2}{X} + d\theta^2\right) + \frac{\sin^2\theta}{\Delta_0^{1/2}}(adt - ((r + 2ms_1^2)(r + 2ms_2^2) + a^2)d\varphi)^2, \tag{1}$$

where $X = r^2 - 2mr + a^2$, $\Delta_0^{1/2} = (r + 2ms_1^2)(r + 2ms_2^2) + a^2\sin^2\theta$, while $s_i = \sinh\delta_i$, $i = 1, 2$ are defined in terms of the U(1) charges δ_i, following the notational conventions of Ref. [9], we also refer reader to [6,12] for the explicit form of the Lagrangian for the considered model and explicit solutions for all the fields.

Gravitational perturbations of Kerr metric are uniquely described by the well-known Teukolsky master equation [2]. In addition to these perturbations, which are purely bosonic by nature and correspond to spin $s = \pm 1, \pm 2$, the master equation also allows one to study the massless scalar ($s = 0$) and fermionic ($s = \pm 1/2$) cases, as pointed out by Teukolsky himself [2]. Moreover, even spin $s = \pm 3/2$ can be examined. Teukolsky equation was crucial in the proof of separability for perturbations of arbitrary spin. As far as we know, gravitational perturbations on a rotating STU black hole background have not been studied yet. In any case, it is tempting to have a corresponding analog of the Teukolsky equation, since it provides a general framework in which gravitational perturbations and massless fields, as well as separability, may be investigated.

In a recent work by two of the authors [9], it was demonstrated that a torsion-modified Dirac equation on a rotating pair-wise equal charges STU black hole background [6,12] is separable. We note that separability of a massless scalar equation was proven a decade ago for more general setting of four nonequal charges [10]. The decoupled (separated) equations of motion for spin $s = 0, \pm 1/2$ can be written in the form [9]:

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\mathcal{S}_s}{\partial\theta}\right) + \left(a^2\omega^2\cos^2\theta + 2sa\omega\cos\theta - \frac{k^2 + s^2 + 2ks\cos\theta}{\sin^2\theta} + E_{k,s}\right)\mathcal{S}_s = 0, \tag{2}$$

$$X^{-s}\frac{\partial}{\partial r}\left(X^{s+1}\frac{\partial\mathcal{R}_s}{\partial r}\right) + \left(\frac{K^2(r) - 2is(r - m)K(r)}{X} + 2is\omega F'(r) - \lambda_{k,s}\right)\mathcal{R}_s = 0, \tag{3}$$

where $K(r) = \omega F(r) + ka$, $F(r) = r_1 r_2 + a^2$, $r_i = r + 2ms_i^2$, $i = 1, 2$, $\lambda_{k,s} = E_{k,s} + a^2\omega^2 + 2a\omega k - s(s+1)$ and here ω and k are the angular frequency and magnetic quantum number respectively. The functions $\mathcal{S}_s(\theta)$ and $\mathcal{R}_s(r)$ in the above equations are the angular and radial wave functions respectively, while $\lambda_{k,s}$ is the separation constant.

Since proper gravitational perturbations or field equations for higher spins $s > 1/2$ in the STU background have not been studied, we do not yet have a first principle derivation for a Teukolsky-like equation for higher spin fields in this geometry.

We may, however, exploit the formal similarity between (1) and the Kerr metric, together with our first-principle knowledge of the cases $s = 0, \pm 1/2$, to make a reasonable conjecture about the form of the master equation for higher spin. The ensuing equations are similar in form to those found by Teukolsky for the Kerr background, to which they are reduced in the limit in which the U(1) charges are set to zero. The generalization amounts to the shifts introduced by the boosts represented by δ_i, as could be reasonably expected given the form of (1). Thus, knowing that the decoupled equations (2) and (3) are valid for $s = 0, \pm 1/2$, we assume that this system of equations still holds for higher values of s, corresponding to some minimally coupled physical fields of higher spin, including gravitational perturbations ($s = \pm 2$). As we have pointed out, our assumption should be verified by direct calculations in future investigations.

The conjectured generalization of the Teukolsky equation therefore takes the form

$$\left(\frac{F^2}{X} - a^2\sin^2\theta\right)\frac{\partial^2\Psi_s}{\partial t^2} - \frac{2a(X-F)}{X}\frac{\partial^2\Psi_s}{\partial t\partial\varphi} + \left(\frac{a^2}{X} - \frac{1}{\sin^2\theta}\right)\frac{\partial^2\Psi_s}{\partial\varphi^2}$$
$$- X^{-s}\frac{\partial}{\partial r}\left(X^{s+1}\frac{\partial\Psi_s}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi_s}{\partial\theta}\right) + 2s\left(\frac{a(r-m)}{X} - i\frac{\cos\theta}{\sin^2\theta}\right)\frac{\partial\Psi_s}{\partial\varphi}$$
$$+ 2s\left(\frac{(r-m)F}{X} - r_1 - r_2 + ia\cos\theta\right)\frac{\partial\Psi_s}{\partial t} + (s^2\cot^2\theta - s)\Psi_s = 0, \tag{4}$$

where Ψ_s is the Teukolsky wave function for spin s, and $r_i = r + 2ms_i^2$ are "shifted" radial coordinates. If Ψ_s is assumed to be in the form

$$\Psi_s(t, r, \theta, \varphi) = e^{i(\omega t + k\varphi)}\mathcal{R}_s(r)\mathcal{S}_s(\theta), \tag{5}$$

the conjectured Teukolsky equation (4) can be easily decoupled to the system of equations (2) and (3) with $\lambda_{k,s}$ as the separation constant. It is worth noting that contribution of nonzero spin into the Teukolsky equation (4) is given through multiplication by powers of s.

Equation (4) can be rewritten in a more concise form, convenient for further application. To obtain it, we take the wave function Ψ_s (or rather its radial part R_s) in the form

$$\Psi_s = X^{-s/2}\bar{\Psi}_s, \quad \Leftrightarrow \quad \mathcal{R}_s(r) = X^{-s/2}\bar{\mathcal{R}}_s(r). \tag{6}$$

Thus, the above relation allows one to rewrite the Teukolsky equation (4) in the following way:

$$-\frac{\partial}{\partial r}\left(X\frac{\partial\bar{\Psi}_s}{\partial r}\right) + \frac{1}{X}\left(F\frac{\partial}{\partial t} + a\frac{\partial}{\partial\varphi} + s(r-m)\right)^2\bar{\Psi}_s - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\bar{\Psi}_s}{\partial\theta}\right)$$
$$-\frac{1}{\sin^2\theta}\left(a\sin^2\theta\frac{\partial}{\partial t} + \frac{\partial}{\partial\varphi} + is\cos\theta\right)^2\bar{\Psi}_s - 2s(r_1 + r_2 - 2ia\cos\theta)\frac{\partial\bar{\Psi}_s}{\partial t} = 0. \tag{7}$$

We stress once more that, although Eqs. (4) and (7) have been derived and are thus directly applicable only to the cases $s = 0, \pm 1/2$, we expect that features for minimally coupled fields of higher spins, including gravitational perturbations, can be properly accounted by them as well. Our expectations are also based on the fact that, in the Kerr limit ($s_1 = s_2 = 0$), both of these equations reduce to the corresponding forms of Teukolsky for Kerr spacetime.

III. STABILITY OF TEUKOLSKY EQUATION SOLUTIONS

Teukolsky equation showed its advantage at the attempt to solve a quite general puzzling question, namely whether there were unstable solutions (modes). For the Kerr black hole the matter is more subtle due to the presence of an ergoregion outside the black hole. Whiting proposed a procedure which allowed to overcome this difficulty, allowing him to demonstrate the absence of unstable modes of the Teukolsky equation for the Kerr black hole [3]. In this work we will show the absence of unstable modes (of both bosonic and fermionic nature) for a rotating STU black hole with pair-wise equal charges. For convenience, we work with Eq. (7).

Following the notation of [4,5], we introduce new variables instead of the radial r and angular θ coordinates:

$$x = \frac{r - r_-}{r_+ - r_-}, \qquad y = \frac{1}{2}(1 - \cos\theta). \tag{8}$$

We point out here that the same transformation of coordinates was used in our earlier paper [9] to rewrite the corresponding equations for the wave function components in a standard form of Heun equation. However, the aforementioned paper only examined the $s = \pm 1/2$ cases. Now, we note that the radial equation (3) can be rewritten as follows:

$$\mathcal{X}_s'' + \left(-\tilde{\alpha}^2 + \frac{\tilde{\alpha}\tilde{\kappa} + \tilde{\lambda} + \frac{1}{2}\tilde{\kappa}^2}{x} + \frac{\frac{1}{4} - \tilde{\beta}^2}{x^2} + \frac{\tilde{\alpha}\tilde{\kappa} - \tilde{\lambda} - \frac{1}{2}\tilde{\kappa}^2}{x-1} + \frac{\frac{1}{4} - \tilde{\gamma}^2}{(x-1)^2}\right)\mathcal{X}_s = 0, \tag{9}$$

where

$$\tilde{\alpha} = i\omega\varkappa, \qquad \tilde{\beta} = \frac{s}{2} - \frac{i}{\varkappa}(\eta r_- + \xi), \qquad \tilde{\gamma} = \frac{s}{2} + \frac{i}{\varkappa}(\eta r_+ + \xi), \qquad \tilde{\kappa} = s - i\eta,$$
$$\tilde{\lambda} = \frac{1}{2} + \tilde{\alpha}(\tilde{\gamma} - \tilde{\beta}) - \frac{1}{2}(\tilde{\gamma} - \tilde{\beta})^2 + \lambda_{k,s} + \nu, \qquad \nu = s, \tag{10}$$

and here $\eta = 2m\omega(1 + s_1^2 + s_2^2)$, $\xi = 4\omega m^2 s_1^2 s_2^2 + ka$, $\varkappa = r_+ - r_-$. Here we have performed the transformation $\mathcal{X}_s = (x(x-1))^{1/2}\bar{\mathcal{R}}_s$ of the radial wave function. We point out that for the particular case $s_1 = s_2 = 0$ the relations (10) are reduced to corresponding parameters derived for standard Kerr solution [3]. For the scalar field ($s = 0$) the derived relations (10) are in agreement with coefficients, obtained in [4] for the particular case of pairwise equal charges.

For the angular part we perform the transformation $\mathcal{Y}_s = (y(1-y))^{1/2}\mathcal{S}_s$ and write

$$\mathcal{Y}_s'' + \left(-\alpha^2 + \frac{\alpha\kappa + \lambda + \frac{1}{2}\kappa^2}{y} + \frac{\frac{1}{4} - \beta^2}{y^2} + \frac{\alpha\kappa - \lambda - \frac{1}{2}\kappa^2}{y-1} + \frac{\frac{1}{4} - \gamma^2}{(y-1)^2}\right)\mathcal{Y}_s = 0, \tag{11}$$

and the coefficients in this equation are given by

$$\alpha = 2a\omega, \quad \beta = \frac{s+k}{2}, \quad \gamma = \frac{s-k}{2},$$
$$\kappa = s, \quad \lambda = \frac{1}{2} + \alpha(\gamma - \beta) - \frac{1}{2}(\gamma - \beta)^2 + \lambda_{k,s} + s. \tag{12}$$

Equations (9) and (11) are much more convenient for further computations.

Following Whiting's procedure [3] we introduce a new function $\bar{h}(x)$ defined by the integral transformation

$$\bar{h}_s(x) = e^{\hat{\alpha}x} x^{\hat{\beta}} (x-1)^{\hat{\gamma}} \int_1^{+\infty} e^{2\tilde{\alpha}xz} e^{-\tilde{\alpha}z} z^{-\frac{1}{2} - \tilde{\beta}} \times (z-1)^{-\frac{1}{2} - \tilde{\gamma}} \mathcal{X}_s(z) dz, \tag{13}$$

where

$$\hat{\alpha} = -\tilde{\alpha}, \quad \hat{\beta} = -\frac{1}{2}(\tilde{\beta} + \tilde{\gamma} + \tilde{\kappa}), \quad \hat{\gamma} = -\frac{1}{2}(\tilde{\beta} + \tilde{\gamma} - \tilde{\kappa}). \tag{14}$$

The transformed wave function $\bar{h}_s(x)$ solves the following equation:

$$x(x-1)\bar{h}_s'' + (2x-1)\bar{h}_s' + \left(4\omega(\eta m + \xi)x - \lambda_{k,s} - s + \omega^2\varkappa^2 x(x-1) - s^2\frac{x-1}{x} + \eta^2\frac{x}{x-1}\right)\bar{h}_s = 0. \tag{15}$$

To transform the angular equation (11) a differential transformation should be applied [3]. Let us briefly describe the key points of this transformation. It was shown that there is a relation between $\mathcal{Y}_s(y)$ and a new function $v_s(y)$ which satisfies an equation with the same structure as (11). Namely, if for chosen values of parameters $\varepsilon, \varepsilon', \varepsilon''$ ($\varepsilon, \varepsilon', \varepsilon'' = \pm 1$) the parameter n, defined as

$$n = \varepsilon''\gamma + \varepsilon'\beta + \varepsilon\kappa \tag{16}$$

takes integer values, then the function

$$v_s(y) = e^{\bar{\alpha}y} y^{\bar{\beta}} (1-y)^{\bar{\gamma}} \left(\frac{\partial}{\partial y}\right)^n e^{\varepsilon\alpha y} y^{\varepsilon'\beta} (1-y)^{\varepsilon''\gamma} \mathcal{Y}_s(y) \tag{17}$$

is a solution of an equation similar to (11), if barred parameters take the form

$$\bar{\alpha} = -\varepsilon\alpha, \quad \bar{\beta} = \frac{1}{2}(1+n) - \varepsilon'\beta, \quad \bar{\gamma} = \frac{1}{2}(1+n) - \varepsilon''\gamma. \tag{18}$$

In order to achieve our goal, only two options for the order of derivative n are possible, namely $n = |s-k|$ and $n = |s+k|$. Taking the latter one, we arrive at the equation

$$v_s''(y) + \left(-4a^2\omega^2 + \frac{\lambda_{k,s} + s + \frac{1}{2} - 4a\omega k}{y} - \frac{\lambda_{k,s} + s + \frac{1}{2}}{y-1} + \frac{1/4}{y^2} + \frac{1/4 - s^2}{(y-1)^2}\right)v_s(y) = 0. \tag{19}$$

Introducing new function $v_s(y) = \sqrt{y(1-y)}\bar{v}_s(y)$, we rewrite it in the form

$$\left(y(1-y)\partial_y^2 + (1-2y)\partial_y - 4a^2\omega^2 y(1-y) + 4a\omega k(y-1) + \lambda_{k,s} + s(s+1) - \frac{s^2}{1-y}\right)\bar{v}_s(y) = 0. \tag{20}$$

The key step toward proving the stability of Teukolsky equation solutions or the absence of unstable modes is the so-called "unseparation of variables." Since Eqs. (15) and (20) have the same separation constant $\lambda_{k,s}$, which comes from the original Teukolsky equation (7), it may be assumed that both Eqs. (15) and (20) are derived from a single equation through the same process we have used to obtain Eqs. (9) and (11) from Eq. (7), i.e., using separation of variables and the transformation of coordinates (8). This reasoning leads to the equation

$$\left(\frac{\partial}{\partial r}\left(X\frac{\partial}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - (P(r) + a^2\cos^2\theta)\frac{\partial^2}{\partial t^2} - 2a\left(\frac{r-m}{\varepsilon_0 m} - \cos\theta\right)\frac{\partial^2}{\partial t\partial\varphi} - s^2\left(\frac{r-r_+}{r-r_-} + \frac{1-\cos\theta}{1+\cos\theta}\right)\right)e^{i(\omega t + k\varphi)}\bar{h}_s(r)\bar{v}_s(\theta) = 0, \tag{21}$$

where $\varepsilon_0 = (r_+ - r_-)/(r_+ + r_-)$ and the function $P(r)$ is defined as follows:

$$P(r) = 4m^2(1 + s_1^2 + s_2^2)^2\frac{(r-r_-)}{(r-r_+)} + \frac{8m^2}{\varkappa}(1 + s_1^2 + s_2^2 + 2s_1^2 s_2^2)(r - r_-) + r^2 - 2mr. \tag{22}$$

Equation (21) conforms with the corresponding equations in the Kerr case [3], and with the recently derived equation for the scalar field ($s = 0$) in a more general background geometry with four distinct charges [4]. We also point out that the spin-dependent term in the Eq. (21) is of the same form as for the Kerr background, and supposedly similar dependence may take place for a more general background geometry. The only term in Eq. (21) which depends on the electric and magnetic charges is the function $P(r)$, similarly to what was observed in [4]. We also note that $P(r)$ is positive in the outer domain ($r > r_+$), which is very important for showing that there are no unstable modes.

Equation (21) allows us to derive the inverse auxiliary metric $\hat{g}^{\mu\nu}$ up to a conformal factor. Before we obtain this metric, we point out that Eq. (21) can be represented as a "massive" scalar field equation in terms of the auxiliary metric, namely: $\widehat{\nabla}_\mu\widehat{\nabla}^\mu\Phi - f(r,\theta)\Phi = 0$, where $\widehat{\nabla}$ denotes the covariant differentiation operator with respect to the metric $\hat{g}$. Now we easily write components of the inverse auxiliary metric

$$\hat{g}^{tt} = -\frac{1}{\Omega^2}(P(r) + a^2\cos^2\theta), \qquad \hat{g}^{rr} = \frac{X}{\Omega^2}, \qquad \hat{g}^{\theta\theta} = \frac{1}{\Omega^2}, \qquad \hat{g}^{t\varphi} = -\frac{a}{\Omega^2}\left(\frac{r-m}{\varepsilon_0 m} - \cos\theta\right). \tag{23}$$

Imposing that the equation is exactly of the form (21) we obtain explicitly the conformal factor Ω

$$\Omega^2 = a\sqrt{X}\left(\frac{r-m}{\varepsilon_0 m} - \cos\theta\right)\sin\theta. \tag{24}$$

Finally, we write the auxiliary metric

$$d\hat{s}^2 = \Omega^2\left(\frac{dr^2}{X} + d\theta^2\right) + \frac{X}{\Omega^2}(P + a^2\cos^2\theta)\sin^2\theta d\varphi^2 - 2\sqrt{X}\sin\theta dt d\varphi. \tag{25}$$

This metric is in agreement with the expression derived in [4] for the scalar case. The Killing vector of time translation $K^\mu = \partial/\partial t$ is everywhere null for the metric (25). We will show that for the auxiliary metric (25) the energy density current for the scalar field $J^\mu = K^\lambda T^\mu_\lambda$ is positive at least outside the black hole. As we have noted above, the Eq. (21) can be derived form a "massive" scalar field Lagrangian, which can be written in the form

$$\mathcal{L} = -\frac{1}{2}\widehat{\nabla}_\lambda\Phi^*\widehat{\nabla}^\lambda\Phi - \frac{f}{2}|\Phi|^2, \tag{26}$$

where $f = \frac{s^2}{\Omega^2}\left(\frac{r-r_+}{r-r_-} + \frac{1-\cos\theta}{1+\cos\theta}\right)$, that is positive in the outer region. The stress-energy tensor for the field Φ is

$$T_{\mu\nu} = -\frac{1}{2}\left(\widehat{\nabla}_\mu\Phi^*\widehat{\nabla}_\nu\Phi + \widehat{\nabla}_\mu\Phi\widehat{\nabla}_\nu\Phi^* - \hat{g}_{\mu\nu}\left(\widehat{\nabla}_\lambda\Phi^*\widehat{\nabla}^\lambda\Phi + f|\Phi|^2\right)\right). \tag{27}$$

Using the above relation for calculation of the energy current J^0 and taking into consideration the factor $\sqrt{-\hat{g}}$, we obtain

$$\sqrt{-\hat{g}}J^0 = \frac{1}{2}\left((P(r)+a^2\cos^2\theta)|\partial_t\Phi|^2 + X|\partial_r\Phi|^2 + |\partial_\theta\Phi|^2 + s^2\left(\frac{r-r_+}{r-r_-}+\frac{1-\cos\theta}{1+\cos\theta}\right)|\Phi|^2\right)\sin\theta. \tag{28}$$

This current is manifestly positive outside the black hole. The current in (28) generalizes the corresponding relation for the Kerr background [3]. It is also a higher-spin generalization for the STU black hole [4] with pairwise-equal charges. Conservation of energy $\mathcal{E} = \int d^3x\sqrt{-\hat{g}}J^0$ together with positive definiteness of the integrand give rise to the conclusion that unstable modes should be withdrawn from the solution.

IV. CONNECTION RELATIONS AND MODE STABILITY FOR MASSLESS FERMIONIC PERTURBATIONS

Fermionic perturbations should be considered separately, because the differential part of the transformations is not applicable directly for a half-integer spin. Indeed, if $s\in\mathbb{Z}/2$ the parameter $n = s+k$, which was assumed to be integer as an order of differentiation, turns to be a half-integer as well. Here we examine spin $s = 1/2$ fermion only and we use the Dirac equation instead of the general Teukolsky equation. Additionally there is no superradiance for spin $s = 1/2$ fermions, therefore the Whiting's transformation can be avoided. Recently it was shown that, to prove the absence of unstable modes in the Kerr case, some analytical properties of the differential equation and the so-called connection conditions, which relate the functions of opposite directions of spin, can be used [11].

Throughout this derivation, we shall assume that the angular frequency ω is real. For the Kerr black hole, one may show that this case is possible through recourse to the work of Hartle and Wilkins [13]. The basic reasoning goes as follows: (i) The Schwarzschild black hole, which corresponds to the $a\to 0$ limit of Kerr, is known to be stable, so that the ingoing modes $Y^{(\mathrm{in})}$ cannot have zeros. (ii) as a is continuously increased, a zero of $Y^{(\mathrm{in})}$ cannot move to the lower half of the complex plane without crossing the real axis. To rigorously prove the second assertion it is necessary to show that $Y^{(\mathrm{in})}$, viewed as a function of a and ω, is analytic in these variables, so that there are no branch points and zeros cannot reach the lower half-plane without crossing the real axis. This was carefully proved in the aforementioned Ref. [13]. For the purposes of this derivation, we shall temporarily assume that this reasoning is still valid for the background at hand, which amounts to the assumption that the addition of the U(1) charges in the metric (which may be seem as a smooth deformation of the Kerr background) does not create branch points or otherwise affect the reasoning above. A brief explanation of why the arguments given in [13] remain valid for the geometry considered in this paper will be given at the end of this section.

To derive the connection relations, we write two pairs of equations, for the radial and angular components of the Dirac spinor, which appear due to separation of variables [9]

$$\sqrt{X}\hat{\mathcal{D}}_-\bar{\mathcal{R}}_{\frac{1}{2}}(r) = \lambda\bar{\mathcal{R}}_{-\frac{1}{2}}(r), \tag{29}$$

$$\sqrt{X}\hat{\mathcal{D}}_+\bar{\mathcal{R}}_{-\frac{1}{2}}(r) = \lambda\bar{\mathcal{R}}_{\frac{1}{2}}(r), \tag{30}$$

where the differential operators $\hat{\mathcal{D}}_\pm = \partial_r \mp iK/X$, the functions $K(r)$ and $X(r)$ are defined above and λ is the separation constant, related to the parameter $\lambda_{k,s}$ of the radial Teukolsky equation (3). Radial components of Dirac spinor are denoted as $\bar{\mathcal{R}}_{\pm\frac{1}{2}}(r)$ to avoid confusion with the radial wave functions of the Teukolsky equation (3). The radial wave function for $s = -\frac{1}{2}$ in the system (29) and (30) is identical to the corresponding Teukolsky wave function (3), i.e., $\bar{\mathcal{R}}_{-\frac{1}{2}}(r) = \mathcal{R}_{-\frac{1}{2}}(r)$. To achieve complete agreement for $s = \frac{1}{2}$ instead of the Dirac wave function $\bar{\mathcal{R}}_{\frac{1}{2}}$ we define a new function $\mathcal{R}_{\frac{1}{2}}(r)$ as follows: $\mathcal{R}_{\frac{1}{2}}(r) = X^{-\frac{1}{2}}\bar{\mathcal{R}}_{\frac{1}{2}}(r)$. Now the system (29) and (30) can be rewritten as follows:

$$X\left(\hat{\mathcal{D}}_- + \frac{X'}{2X}\right)\mathcal{R}_{\frac{1}{2}}(r) = \lambda\mathcal{R}_{-\frac{1}{2}}(r), \tag{31}$$

$$\hat{\mathcal{D}}_+\mathcal{R}_{-\frac{1}{2}}(r) = \lambda\mathcal{R}_{\frac{1}{2}}(r). \tag{32}$$

The main advantage of the system (31) and (32) is the fact that both wave functions $\mathcal{R}_{\pm\frac{1}{2}}(r)$ coincide with the corresponding radial Teukolsky wave functions (3).

We also write the system for angular components of the spinor wave function

$$\hat{\mathcal{L}}_+\mathcal{S}_{\frac{1}{2}}(\theta) = \lambda\mathcal{S}_{-\frac{1}{2}}(\theta), \tag{33}$$

$$\hat{\mathcal{L}}_-\mathcal{S}_{-\frac{1}{2}}(\theta) = -\lambda\mathcal{S}_{\frac{1}{2}}(\theta), \tag{34}$$

where $\hat{\mathcal{L}}_\pm = \partial_\theta \pm (a\omega\sin\theta + \frac{k}{\sin\theta})$ and $\mathcal{S}_{\pm\frac{1}{2}}(\theta)$ denote corresponding angular components of the spinor wave function. If the frequency ω is real then the differential operators $\hat{\mathcal{L}}_\pm$ are also real. The angular components $\mathcal{S}_\pm(\theta)$ are real functions as natural generalizations of spin spherical harmonics. In this case the separation constant λ is real as well. These considerations are crucial for the proof of absence of unstable modes.

To obtain the connection relations it is necessary to have asymptotic relations for the radial wave functions $\mathcal{R}_{\pm\frac{1}{2}}(r)$ near the horizon and at infinity. To achieve this aim, we use the Teukolsky equation (3) and, to simplify analysis we introduce the tortoise coordinate r_*, defined as follows $dr_* = \frac{F(r)}{X(r)}dr$. Moreover, for convenience to introduce auxiliary function $\mathcal{X}_s = X^{\frac{s}{2}}F^{\frac{1}{2}}\mathcal{R}_s(r)$ instead of the Teukolsky wave function $\mathcal{R}_s$. At infinity $r\to\infty$ ($r_*\to+\infty$) from the Teukolsky equation (3) it follows that

$$\frac{\partial^2\mathcal{X}_s}{\partial r_*^2} + \left(\omega^2 + \frac{2is\omega}{r_*}\right)\mathcal{X}_s \approx 0. \tag{35}$$

This equation holds up to $\sim\mathcal{O}(1/r^2)$ terms, which definitely go to zero as $r\to\infty$. The asymptotic solutions of Eq. (35) enable us to write asymptotic relations for the Teukolsky wave function $\mathcal{R}_s$:

$$\mathcal{R}_s \simeq Y_s^{(\mathrm{in})}\frac{e^{-i\omega r_*}}{r} + Y_s^{(\mathrm{out})}\frac{e^{i\omega r_*}}{r^{2s+1}}, \tag{36}$$

where $Y_s^{(\mathrm{in})}$ and $Y_s^{(\mathrm{out})}$ are amplitudes for ingoing and outgoing waves respectively for the corresponding values of spin s.

Near the horizon ($r\to r_+$ or $r_*\to-\infty$) we obtain

$$\frac{\partial^2\mathcal{X}_s}{\partial r_*^2} + \left(\tilde{\omega} - is\frac{r_+ - m}{F(r_+)}\right)^2\mathcal{X}_s \approx 0, \tag{37}$$

where $\tilde{\omega} = \omega + \frac{ka}{F(r_+)}$. The solutions near the horizon should be ingoing waves (infalling particles are examined), therefore its solution and the corresponding Teukolsky wave function take the form

$$\mathcal{X}_s(r) \simeq Z_s^{(\mathrm{in})}e^{-i(\tilde{\omega} - is\frac{r_+ - m}{F(r_+)})r_*} \simeq Z_s^{(\mathrm{in})}X^{-\frac{s}{2}}e^{-i\tilde{\omega}r_*}$$
$$\Rightarrow\quad \mathcal{R}_s \simeq Z_s^{(\mathrm{in})}X^{-s}e^{-i\tilde{\omega}r_*}. \tag{38}$$

Here we point out that our asymptotic relations for the radial wave functions $\mathcal{R}_s(r)$ at the horizon and at the infinity conform with the corresponding relations obtained for the Teukolsky equation in Kerr case [2].

Now, using the asymptotic relations (36) and (38) and the Eqs. (31) and (32), we find the connection relations between the amplitudes $Y_s^{(\mathrm{in})}$, $Y_s^{(\mathrm{out})}$, and $Z_s^{(\mathrm{in})}$ for opposite values of spin. Taking the asymptotic relation (36) and the Eqs. (31) and (32) we derive respectively:

$$2i\omega Y_{\frac{1}{2}}^{(\mathrm{out})} = \lambda Y_{-\frac{1}{2}}^{(\mathrm{out})}, \qquad -2i\omega Y_{-\frac{1}{2}}^{(\mathrm{in})} = \lambda Y_{\frac{1}{2}}^{(\mathrm{in})}. \tag{39}$$

Finally, if we take the asymptotic relations (38) and the Eq. (32) we arrive at the following relation:

$$Z_{-\frac{1}{2}}^{(\mathrm{in})}(r_+ - m - 2i\tilde{\omega}F(r_+)) = \lambda Z_{\frac{1}{2}}^{(\mathrm{in})}. \tag{40}$$

Similar connection relations for the Kerr background were obtained in [11] and our relations are consistent with them.

The other type of connection relations associates the amplitudes $Y_s^{(\mathrm{in})}$, $Y_s^{(\mathrm{out}}$ and $Z_s^{(\mathrm{in})}$ at infinity and at the horizon for the same orientation of spin. This connection relation can be established by virtue of the equation for the auxiliary wave function $\mathcal{X}_s$ which takes the form

$$\frac{\partial^2\mathcal{X}_s}{\partial r_*^2} + U(r,s)\mathcal{X}_s = 0, \tag{41}$$

and we note that the explicit form of the function $U(r,s)$ is not important for the further consideration, even though it can be obtained easily. It is known that for any differential equation of the form (41), the Wronskian is a constant, namely for a pair of its two independent solutions $\mathcal{X}_s^{(1)}$ and $\mathcal{X}_s^{(2)}$ we can write

$$W(\mathcal{X}_s^{(1)}, \mathcal{X}_s^{(2)}) \equiv \mathcal{X}_{s,r_*}^{(1)}\mathcal{X}_s^{(2)} - \mathcal{X}_s^{(1)}\mathcal{X}_{s,r_*}^{(2)} = \mathrm{const}, \tag{42}$$

where the derivatives are taken with respect to r_*. The potential $U(r,s)$ satisfies $U^*(r,s) = U(r,-s)$ while all the other parameters of the potential $U(r,s)$ are held fixed, therefore the invariance of the Wronskian can be rewritten in the form

$$W\left(\mathcal{X}_{\frac{1}{2}}, \mathcal{X}_{-\frac{1}{2}}^*\right)_{r_+} = W\left(\mathcal{X}_{\frac{1}{2}}, \mathcal{X}_{-\frac{1}{2}}^*\right)_{\infty}. \tag{43}$$

Using the explicit relation for the function $\mathcal{X}_s$ and the asymptotic relations (36) and (38) we can calculate both sides of latter relation. Namely at the horizon we obtain

$$W\left(\mathcal{X}_{\frac{1}{2}}, \mathcal{X}_{-\frac{1}{2}}^*\right)_{r_+} = Z_{\frac{1}{2}}^{(\mathrm{in})}Z_{-\frac{1}{2}}^{(\mathrm{in})*}\left(-(r_+ - m) - 2i\tilde{\omega}F(r_+)\right), \tag{44}$$

and at infinity we arrive at the following form

$$W\left(\mathcal{X}_{\frac{1}{2}}, \mathcal{X}_{-\frac{1}{2}}^*\right)_{\infty} = 2i\omega\left(Y_{\frac{1}{2}}^{(\mathrm{out})}Y_{-\frac{1}{2}}^{(\mathrm{out})*} - Y_{\frac{1}{2}}^{(\mathrm{in})}Y_{-\frac{1}{2}}^{(\mathrm{in})*}\right). \tag{45}$$

Finally, substituting the connection relations (40) and (39) into the relations (44) and (45) respectively and equating both of them we obtain

$$|Z_{\frac{1}{2}}^{(\mathrm{in})}|^2 = |Y_{\frac{1}{2}}^{(\mathrm{in})}|^2 - \frac{4\omega^2}{\lambda^2}|Y_{\frac{1}{2}}^{(\mathrm{out})}|^2. \tag{46}$$

We also point out that to write the upper equation we have used our assumption that the frequency ω and the separation constant λ are real. Clearly, if $Y_{\frac{1}{2}}^{(\mathrm{in})} = 0$, the above equation does not have any solution corresponding to real ω. This means that a zero of $Y_{\frac{1}{2}}^{(\mathrm{in})}$ could never move to the lower half-plane by crossing the real axis. When $a = 0$, modal stability can be shown rather straightforwardly from an energy functional argument since, in this limit, the absence of an ergosphere makes it possible to define a nonnegative energy density directly from (4), without recourse to Whiting-type transformations, such as the one used in the previous section [14]. Modal stability implies that any nontrivial mode in the nonrotating case must correspond to a frequency such that $\mathrm{Im}(\omega) > 0$. It follows that there must exist a neighborhood of $a = 0$ in which the equation $Y_{\frac{1}{2}}^{(\mathrm{in})}(a, \omega) = 0$ corresponds to a frequency belonging to the upper half-plane. If an unstable mode exists for the rotating black hole, this zero must migrate to the lower half-plane as a is changed, crossing the real axis for some intermediate value of this parameter. The results above show that, if $Y_{\frac{1}{2}}^{(\mathrm{in})}$ is an analytic function of a and ω, it cannot develop a zero and cannot therefore give rise to an unstable mode. A similar connection relation can be written for $s = -\frac{1}{2}$ and the exact same argument may be conducted to rule out unstable perturbations.

As previously explained, the argument above rests on the premise that a zero of $Y_s^{(\mathrm{in})}$ can only move from the upper to the lower complex half-plane by crossing the real axis. This is true provided that $Y_s^{(\mathrm{in})}$, viewed as a function of a and ω, is analytic. To prove this statement, we must generalize the argument given in Ref. [13] to the geometry at hand. Fortunately, the formal analogy between the equations in the STU and Kerr backgrounds greatly simplifies what would otherwise be a strenuous task. The singularities of both the radial and the angular equation for Dirac fermions, as well as the behavior of the solutions in their neighborhood, have been investigated in detail in a previous work by two of the authors [9]. We thus know that, in the outer region, there are two singularities: a regular one located at the event horizon, and an irregular singularity at infinity. As shown in [13], it is possible to study the properties of the desired coefficient in terms of the behavior of the solution in a neighborhood of those two singularities. The behavior at infinity is governed by Eq. (35), which in exactly the same one found for Kerr. This is to be expected, since the difference between r_i and r becomes negligible at infinity. Thus, the demonstration given in [13] for the analyticity of the solution in this region still holds in our case, without need for any adjustments. Near the horizon, we have Eq. (37), which is of the form

$$\frac{\partial^2 \mathcal{X}_s}{\partial r_*^2} + \Omega^2 \mathcal{X}_s = 0, \tag{47}$$

where Ω^2 is a constant. Although Ω has a different value in relation to what is found in the Kerr background, the above equation has the same form in both geometries. The proof of analyticity given in [13], which is mainly conducted in terms of Ω, still works in our case, with some tedious but straightforward adjustments. The numeric value of Ω may be important for some applications, but it does not matter for our present purposes, since it does not change the signs in any of the bounds relevant to the demonstration of analyticity.[1] Thus, it may be directly verified, through the same steps conducted in the aforementioned reference, that $Y_s^{(\mathrm{in})}$ is still analytic in the appropriate domain. This, together with the fact that the behavior at infinity is the same as in the Kerr background, suffices to ensure that a zero cannot migrate from infinity or through branch cuts.

V. CONCLUSIONS

The mode stability for black holes has become a very active area of study in recent years among both mathematicians' and physicists' communities [1,4,5,15]. To examine mode stability for general relativistic backgrounds, the celebrated Teukolsky equation is often used [2,3], but for more general black hole backgrounds, such as those from string theory and/or supergravity, an equation of this kind is not yet known.

In a previous work from two of the authors [9], the separability of the Dirac equation on a rotating pair-wise equal charged STU black hole background has been analyzed. The results from this investigation, as well as earlier ones concerning scalar fields [4], have led us to conjecture the Teukolsky-like equation that has been used in this work to study minimally coupled probe fields of higher spins. Thus, we have used Eq. (4) to derive decoupled equations for the radial and angular parts, obtained though the method of separation of variables. Generalizing Whiting's procedure, we study mode stability for minimally coupled fields of higher integer spin. The obtained results are in agreement with the analysis conducted in Ref. [4] for scalar fields in a more general metric. There are some subtleties for the fermionic case, which preclude a straightforward application of Whiting's procedure. To examine mode stability for the torsion-modified massless Dirac equation, we derive connection relations relating the two components of this equation, following the approach considered in [11]. The connection relations are shown to be in direct contradiction with the existence of unstable modes in the theory, which allows us to rule them out. We note that spin $1/2$ fields were observed to cause

[1] Indeed, the charges only enter the calculations through $F(r_+)$, which generalizes the expression $r^2 + a^2 \approx 2mr_+$. Since $F(r_+)$ is also a positive constant, it plays the same role in the argument, regardless of the choice of δ_i. In particular, this generalization leads to $\mathrm{Im}(\omega) < C/F(r_+)$, where C is a positive constant, which suffices to cover the lower half-plane for any δ_i.

instability of the event horizon [11], but this issue needs careful study and will be examined elsewhere.

An important direction for further study is the application of the developed techniques for five dimensional STU black holes, where up to now only scalar field mode stability has been examined [5].

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