Lagrangian formulation for perfect fluid equations with the ℓ-conformal Galilei symmetry

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Lagrangian formulation for perfect fluid equations which hold invariant under the ℓ-conformal Galilei group with half-integer ℓ is proposed. It is based on a Clebsch-type parametrization and reproduces Lagrangian description of the Euler fluid equations for $\ell = \frac{1}{2}$. The transition from the Lagrangian formulation to the Hamiltonian one is analyzed in detail.

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I. INTRODUCTION

Fluid mechanics with conformal symmetries currently attracts considerable attention in connection with the AdS/CFT-correspondence [1] and the fluid/gravity duality [2]. In particular, the latter can be understood as a hydrodynamic limit of the former in which the formalism of fluid mechanics is applied with the aim to an effective description of a strongly coupled quantum field theory. At the same time successful efforts to extend holography to strongly coupled condensed matter systems [3–5] stimulate investigations of fluid dynamics with nonrelativistic conformal symmetries.

In contrast to the unique relativistic conformal algebra, there are several options available in the nonrelativistic case. A well known example is the Schrödinger algebra [6–8], which has been found to be relevant for a wide range of physical applications (see Ref. [9] and references therein). The Schrödinger group was originally discovered as the maximal kinematical invariance group of the Schrödinger equation for a free massive particle [7].[1] In addition to the Galilei transformations it contains dilatation and special conformal transformation. Surprisingly enough, the nonrelativistic contraction of the relativistic conformal algebra [12] does not result in the Schrödinger algebra. The latter fact stimulates interest in the study of other finite-dimensional conformal extensions of the Galilei algebra which are combined into a family known in the literature as the ℓ-conformal Galilei algebra [13,14].

The algebra is characterized by an arbitrary integer or half-integer parameter ℓ and, in addition to temporal translation, dilatation and special conformal transformation, it involves a set of vector generators $C_i^{(k)}$, where $i = 1, \ldots, d$ is a spatial index and $k = 0, \ldots, 2\ell$. $C_i^{(0)}$ and $C_i^{(1)}$ link to spatial translations and Galilei boosts while higher values of k correspond to the so called constant accelerations. The case $\ell = \frac{1}{2}$ reproduces the Schrödinger algebra, while $\ell = 1$ is recovered in the nonrelativistic limit of the relativistic conformal algebra. The latter is usually referred to as the conformal Galilei algebra [12].

As far as dynamical realizations of the ℓ-conformal Galilei group are concerned (see e.g., [15–23] and references therein), the physical meaning of the parameter ℓ may vary. In condensed matter physics, the reciprocal $z = 1/\ell$ is known as a critical dynamical exponent, which links to the fact that under dilatation temporal and spatial coordinates scale differently, $t' = \lambda t$, $x'_i = \lambda^\ell x_i$, i.e., ℓ characterizes the degree of scaling anisotropy. In mechanics, field theory, and fluid mechanics, ℓ determines the order of differential equations of motion. Because a number of functionally independent integrals of motion needed to integrate a differential equation correlates with its order, in order to accommodate symmetries generated by the tower of vector generators $C_i^{(k)}$, one has to make recourse to higher-derivative systems. In particular, in mechanics and field theory the order of differential equations of motion is $2\ell + 1$, while in fluid mechanics it is 2ℓ. To give a notable example, the celebrated Pais-Uhlenbeck oscillator [24] enjoys the ℓ-conformal Galilei symmetry for a special choice of its frequencies [22].

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[1] In fact, similar nonrelativistic conformal structure has been known since 19th century due to the work on classical mechanics [10] and the heat equation [11].

Previous studies of nonrelativistic conformal symmetries in the context of fluid mechanics revealed interesting results. A perfect fluid described by the Euler equations has the Schrödinger symmetry ($\ell = \frac{1}{2}$) provided a specific equation of state is chosen [25,26] which links pressure to density. For a viscous fluid the Schrödinger symmetries are partially broken leaving one with the dilatation and Galilei symmetries [27]. Attempts to discover the conformal Galilei symmetry ($\ell = 1$) for systems which derive from relativistic conformally invariant hydrodynamic equations did not lead to success [27–29]. In the nonrelativistic limit such systems proved to be of limited physical interest. Reasonable hydrodynamic equations can be obtained as a result of a more subtle nonrelativistic contraction, but they do not enjoy conformal Galilei symmetries. It was recently shown that one can construct generalized perfect fluid equations which accommodate the ℓ-conformal Galilei symmetries for an arbitrary ℓ [30,31]. In particular, these equations contain the generalized Euler equation with higher material derivatives, which reduces to the perfect fluid equations for $\ell = \frac{1}{2}$.

Given a set of equations of motion, it is always desirable to have a Lagrangian formulation. There are two approaches to describe the nonrelativistic perfect fluid equations [32–36] (for modern developments see also [37–41]). The first approach (the Lagrange picture) deals with coordinates of a fluid particle parametrized by a set of continuum labels (see the classic monograph [34]). In this case, the Lagrangian function has the same form as in Newtonian mechanics and the action is invariant under volume-preserving diffeomorphism. The second approach (the Euler picture) is akin to classical field theory and deals directly with physical variables which are interpreted as the fluid density and the velocity vector field. Here the fluid equations can be naturally put into the Hamiltonian form [42]. The peculiarity of the latter formulation is that the Poisson brackets among the physical fields are noncanonical. In order to identify canonical variables and go over to a Lagrangian description, the Clebsch parametrization [43] of the velocity vector field is usually used. The latter description as well as a relation between the two approaches to fluid mechanics are reviewed in detail in [35].

It was recently shown [44] that the approach involving noncanonical Poisson brackets [42] can be adapted to construct a Hamiltonian formulation of the generalized perfect fluid equations with the ℓ-conformal Galilei symmetries for an arbitrary half-integer ℓ. Aiming at a Lagrangian formulation, it is important to understand whether canonical variables exist in which the perfect fluid equations with the ℓ-conformal Galilei symmetries arise from the variational principle. The goal of this work is to elaborate on this issue.

The paper is organized as follows. In the next section, we briefly review the generalized perfect fluid equations invariant under the ℓ-conformal Galilei group [30] and their Hamiltonian formulation developed in [44]. In Sec. III, we construct the Lagrangian formulation based on a Clebsch-type parametrization in which the generalized perfect fluid equations arise from the variational principle. In Sec. IV, the Dirac method is used to analyze constraints which arise after transition to the Hamiltonian formalism. A relation to the Hamiltonian description in terms of noncanonical Poisson brackets [44] is studied in detail. In the concluding Sec. V, we summaries our results discuss possible further developments.

II. EQUATIONS OF MOTION AND HAMILTONIAN FORMULATION

Let us take briefly recall the structure of the ℓ-conformal Galilei algebra [14]. Its generators include temporal translation H, dilatation D, special conformal transformation K and a set of vector generators $C_i^{(k)}$, $k = 0, \ldots, 2\ell$. The latter correspond to spatial translations ($k = 0$), Galilei boosts ($k = 1$) and the so-called constant accelerations ($k > 1$). The structure relations of the algebra read,[2]

$$
\begin{aligned}
&[H, D] = H, \qquad [H, C_i^{(k)}] = k C_i^{(k-1)}, \\
&[H, K] = 2D, \qquad [D, C_i^{(k)}] = (k - \ell) C_i^{(k)}, \\
&[D, K] = K, \qquad [K, C_i^{(k)}] = (k - 2\ell) C_i^{(k+1)}.
\end{aligned} \tag{2.1}
$$

They can be realized in a nonrelativistic space-time parametrized by (t, x_i), $i = 1, \ldots, d$, by the following way [14][3]:

$$
\begin{aligned}
&H = \partial_0, \qquad D = t\partial_0 + \ell x_i \partial_i, \\
&K = t^2 \partial_0 + 2\ell t x_i \partial_i, \qquad C_i^{(k)} = t^k \partial_i.
\end{aligned}
$$

Generalized perfect fluid equations invariant under the ℓ-conformal Galilei group were formulated in a recent paper [30],

$$
\partial_0 \rho + \partial_i(\rho \upsilon_i) = 0, \quad \mathcal{D}^{2\ell} \upsilon_i = -\frac{1}{\rho} \partial_i p, \quad p = \nu \rho^{1+\frac{1}{\ell d}}, \tag{2.2}
$$

where $\rho(t, x)$, $\upsilon_i(t, x)$, and $p(t, x)$ are the density, the velocity vector field, and the pressure, respectively, and $\mathcal{D} = \partial_0 + \upsilon_i \partial_i$ is the material derivative. The first equation is the continuity equation, while the second and third equations describe the generalized Euler equation with higher derivatives and the equation of state which links the pressure to the density, ν being a constant. For $\ell = \frac{1}{2}$, Eqs. (2.2) reproduce the perfect fluid equations invariant under the action of the Schrodinger group [26]. In what

[2]The algebra also includes spatial rotation which in what follows will be disregarded.

[3]Throughout the text we use the notations $\partial_0 = \frac{\partial}{\partial t}$, $\partial_i = \frac{\partial}{\partial x_i}$. Summation over repeated indices is understood.

follows, we will refer to the model (2.2) as the ℓ-conformal perfect fluid.

For half-integer values $\ell = n + \frac{1}{2}, n = 0, 1, \ldots,$ Eqs. (2.2) admit a Hamiltonian formulation [44]. In order to construct it, auxiliary fields $v_i^{(k)},\ k = 0, 1, \ldots, 2n$ are introduced with $v_i^{(0)} = v_i$ and the second equation in (2.2) is rewritten as the equivalent first-order system,

$$\mathcal{D}v_i^{(k)} = v_i^{(k+1)}, \qquad \mathcal{D}v_i^{(2n)} = -\frac{1}{\rho}\partial_i p. \tag{2.3}$$

Then one can verify that the following Hamiltonian:

$$H = \int dx \left(\frac{1}{2}\rho \sum_{k=0}^{2n} (-1)^k v_i^{(k)} v_i^{(2n-k)} + V\right), \qquad V = \ell dp, \tag{2.4}$$

puts the original equations of motion into the Hamiltonian form $\partial_0\rho = \{\rho, H\}$, $\partial_0 v_i^{(k)} = \{v_i^{(k)}, H\}$ provided the noncanonical Poisson brackets,

$$\begin{aligned}\{\rho(x), v_i^{(k)}(y)\} &= -\delta_{(k)(2n)}\partial_i\delta(x-y),\\ \{v_i^{(k)}(x), v_j^{(m)}(y)\} &= \frac{1}{\rho}(\delta_{(k)(2n)}\partial_i v_j^{(m)} - \delta_{(m)(2n)}\partial_j v_i^{(k)}\\ &\quad + (-1)^{k+1}\delta_{(k+m)(2n-1)}\delta_{ij})\delta(x-y),\end{aligned} \tag{2.5}$$

are introduced. Here $\delta_{(k)(m)}$ is the Kronecker symbol.

Within the Hamiltonian formalism the ℓ-conformal Galilei algebra is realized as follows. The Hamiltonian H (the conserved energy) (2.4) links to temporal translation while conserved charges associated with the dilatation, special conformal transformation and vector generators read,

$$\begin{aligned} D &= tH - \frac{1}{2}\int dx\rho \sum_{k=0}^{2n} (-1)^k (k+1) v_i^{(k)} v_i^{(2n-k-1)},\\ K &= t^2 H - 2tD - \frac{1}{2}\int dx\rho \sum_{k=0}^{2n} (-1)^k ((n+1)(2n+1) - k(k+1)) v_i^{(k-1)} v_i^{(2n-k-1)},\\ C_i^{(k)} &= \sum_{s=0}^{k} (-1)^s \frac{k!}{(k-s)!} t^{k-s} \int dx\rho v_i^{(2n-s)}, \qquad k = 0, \ldots, 2n+1, \end{aligned} \tag{2.6}$$

where $v_i^{(-1)} = x_i$. Under the Poisson brackets (2.5), the conserved charges obey the algebra (2.1), which is extended by the central charge [45],

$$\{C_i^{(k)}, C_j^{(m)}\} = (-1)^k k! m! \delta_{(k+m)(2n+1)}\delta_{ij} M, \quad M = \int dx\rho. \tag{2.7}$$

For a perfect fluid ($\ell = \frac{1}{2},\ n = 0$) the Hamiltonian formulation involving noncanonical Poisson brackets was originally given in [42].

III. CLEBSCH PARAMETRIZATION AND LAGRANGIAN FORMULATION

In order to demonstrate how the Eqs. (2.2) can be obtained from the variational principle, let us first recall (for more details see e.g., [35]) how the Lagrangian for a perfect fluid is built which correctly reproduces the continuity equation and the Euler equation,

$$\partial_0\rho + \partial_i(\rho v_i) = 0, \tag{3.1}$$

$$\mathcal{D}v_i = -\frac{1}{\rho}\partial_i p. \tag{3.2}$$

In three spatial dimensions this is achieved by making recourse to the Clebsch parametrization for the velocity vector field,

$$v_i = \partial_i\theta + \alpha\partial_i\beta, \tag{3.3}$$

which involves three scalar functions θ, α, and β. Then the Lagrangian reads,

$$\begin{aligned} L &= -\int dx\rho(\partial_0\theta + \alpha\partial_0\beta) - H\\ &= -\int dx\rho(\partial_0\theta + \alpha\partial_0\beta) - \int dx\left(\frac{1}{2}\rho v_i v_i + V\right), \end{aligned} \tag{3.4}$$

where H is the Hamiltonian (the total energy) with v_i in (3.3). The variation under θ gives the continuity equation (3.1), while the variations with respect to α and β give

$$\mathcal{D}\alpha = 0, \qquad \mathcal{D}\beta = 0, \tag{3.5}$$

where (3.1) was taken into account.

Finally, varying with respect to ρ and using (3.5), one gets

$$\mathcal{D}\theta - \frac{1}{2}v_i v_i + V'_\rho = 0. \tag{3.6}$$

As a result, the Euler equation (3.2) is satisfied,

$$\mathcal{D}v_i = \mathcal{D}(\partial_i\theta + \alpha\partial_i\beta) = -\frac{1}{\rho}\frac{\partial p}{\partial x_i}, \quad p = \rho V'_\rho - V. \tag{3.7}$$

In order to generalize the construction above to the ℓ-conformal perfect fluid, we go over to the equivalent first-order system (2.3). In the case of half-integer $\ell = n + \frac{1}{2}$, the starting equations read,

$$\partial_0\rho + \partial_i(\rho v_i^{(0)}) = 0, \tag{3.8}$$

$$\mathcal{D}v_i^{(k)} = v_i^{(k+1)}, \quad k = 0, 1, \ldots, 2n-1, \tag{3.9}$$

$$\mathcal{D}v_i^{(2n)} = -\frac{1}{\rho}\partial_i p, \qquad p = \nu\rho^{1+\frac{1}{\ell d}}. \tag{3.10}$$

Note that these equations are completely equivalent to (2.2) and hence completely characterize the ℓ-conformal perfect fluid.

A key ingredient of the construction above was the Clebsch parametrization of the velocity vector variable. For the ℓ-conformal perfect fluid one has a set of vector variables $v_i^{(k)}$ and it seems natural to expect that a Clebsch-type decomposition will be needed for each of them. It turns out, however, that in order to obtain the equations (3.8)–(3.10) from the variational principle only the highest component $v_i^{(2n)}$ should be Clebsch-decomposed, while the remaining vector variables $v_i^{(k)}$ with $k < 2n$ may remain intact. Up to a field redefinition, a suitable Clebsch-type decomposition can be chosen in the form,

$$v_i^{(2n)} = \partial_i\theta + \alpha\partial_i\beta + \sum_{k=0}^{n-1}(-1)^{k+1}v_j^{(k)}\partial_i v_j^{(2n-k-1)}. \tag{3.11}$$

When $n = 0$, there is no sum on the right-hand side and the decomposition for the Euler fluid (3.3) is reproduced. The generalized Lagrangian reads,

$$L = -\int dx\rho\left(\partial_0\theta + \alpha\partial_0\beta + \sum_{k=0}^{n-1}(-1)^{k+1}v_i^{(k)}\partial_0 v_i^{(2n-k-1)}\right) - H, \tag{3.12}$$

where H is Hamiltonian (2.4) with $v_i^{(2n)}$ in (3.11). Thus, the basic variables for the Lagrangian (3.12) are the scalar fields ρ, θ, α, β, and a set of vector fields $v_i^{(k)}$ with $k < 2n$.

Let us demonstrate how the Eqs. (3.8)–(3.10) follow from the Lagrangian (3.12). By varying the Lagrangian with respect to θ, one obtains the continuity equation (3.8). Variations with respect to α and β give (3.5), as before. Varying with respect to $v_i^{(k)}$ and taking into account (3.8), Eqs. (3.9) are reproduced. Finally, varying with respect to ρ and using (3.5), one gets

$$\mathcal{D}\theta - v_i^{(0)}v_i^{(2n)} + \frac{(-1)^n}{2}v_i^{(n)}v_i^{(n)} + V'_\rho = 0. \tag{3.13}$$

As a result, the equation

$$\mathcal{D}v_i^{(2n)} = \mathcal{D}\left(\partial_i\theta + \alpha\partial_i\beta + \sum_{k=0}^{n-1}(-1)^{k+1}v_j^{(k)}\partial_i v_j^{(2n-k-1)}\right) = -\frac{1}{\rho}\partial_i p, \tag{3.14}$$

is satisfied as well, where $p = \rho V'_\rho - V$.

Because the Lagrangian (3.12) involves only the first temporal derivative, a transition to the Hamiltonian formalism will lead to constraints. In the next section, we use the Dirac method [46] to analyze such constraints and demonstrate how the noncanonical Poisson brackets (2.5) show up.

IV. DIRAC'S CONSTRAINT ANALYSIS

For simplicity of the presentation, let us focus on the $\ell = \frac{3}{2}$ case. The corresponding Lagrangian is given by (3.12) with $n = 1$,

$$\begin{aligned} L &= -\int dx\rho(\partial_0\theta + \alpha\partial_0\beta - v_i^{(0)}\partial_0 v_i^{(1)}) - H \\ &= -\int dx\rho(\partial_0\theta + \alpha\partial_0\beta - v_i^{(0)}\partial_0 v_i^{(1)}) \\ &\quad - \int dx\left(\rho v_i^{(0)}v_i^{(2)} - \frac{1}{2}\rho v_i^{(1)}v_i^{(1)} + V\right), \end{aligned} \tag{4.1}$$

where H is the Hamiltonian (2.4) with $v_i^{(2)}$ defined in (3.11)

$$v_i^{(2)} = \partial_i\theta + \alpha\partial_i\beta - v_j^{(0)}\partial_i v_j^{(1)}. \tag{4.2}$$

Further simplification occurs if one sets the scalar variables α and β to zero as particular solutions to Eqs. (3.5). This will not affect the final result but simplify the calculations. In this case, the phase space consists of basic variables $X^A = (\rho, \theta, v_i^{(0)}, v_i^{(1)})$ and their conjugate momenta $P^A = (p_\rho, p_\theta, p_i^{(0)}, p_i^{(1)})$, which obey the canonical Poisson brackets,

$$\begin{aligned} &\{\rho(x), p_\rho(y)\} = \delta(x-y), \quad \{v_i^{(0)}(x), p_j^{(0)}(y)\} = \delta_{ij}\delta(x-y), \\ &\{\theta(x), p_\theta(y)\} = \delta(x-y), \quad \{v_i^{(1)}(x), p_j^{(1)}(y)\} = \delta_{ij}\delta(x-y). \end{aligned} \tag{4.3}$$

From the conditions determining the canonical momenta $P^A = \frac{\partial L}{\partial(\partial_0 X^A)}$ the following primary constraints arise:

$$\Phi^A \equiv \begin{pmatrix} \phi_\rho \\ \phi_\theta \\ \phi_i^{(0)} \\ \phi_i^{(1)} \end{pmatrix} = \begin{pmatrix} p_\rho \\ p_\theta + \rho \\ p_i^{(0)} \\ p_i^{(1)} - \rho v_i^{(0)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.4}$$

Then according to Dirac's method [46] the total Hamiltonian reads,

$$H_T = H + \int dx \big(\lambda_\rho \phi_\rho + \lambda_\theta \phi_\theta + \lambda_i^{(0)} \phi_i^{(0)} + \lambda_i^{(1)} \phi_i^{(1)}\big), \tag{4.5}$$

where H is the canonical Hamiltonian in (4.1) and $(\lambda_\rho, \lambda_\theta, \lambda_i^{(0)}, \lambda_i^{(1)})$ are the Lagrangian multipliers. Requiring the constraints to be conserved over time, $\partial_0 \phi = \{\phi, H_T\} = 0$, one unambiguously specifies the Lagrangian multipliers,

$$\begin{aligned} \lambda_\rho &= -\partial_i(\rho v_i^{(0)}), \qquad \lambda_i^{(0)} = v_i^{(1)} - v_j^{(0)} \partial_j v_i^{(0)}, \\ \lambda_\theta &= \frac{1}{2} v_i^{(1)} v_i^{(1)} + v_j^{(0)} (v_j^{(2)} - \partial_j \theta) - V'_\rho, \\ \lambda_i^{(1)} &= v_i^{(2)} - v_j^{(0)} \partial_j v_i^{(1)}. \end{aligned} \tag{4.6}$$

The latter fact implies that all the constraints (4.4) are second-class. The same conclusion is reached by analyzing the Poisson brackets among the constraints $\Phi_A = (\phi_\rho, \phi_\theta, \phi_i^{(0)}, \phi_i^{(1)})$, which form the nondegenerate matrix,

$$\Lambda_{AB}(x, x') = \{\Phi_A(x), \Phi_B(x')\} = \begin{pmatrix} 0 & -1 & 0 & v_i^{(0)} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho\delta_{ij} \\ -v_i^{(0)} & 0 & -\rho\delta_{ij} & 0 \end{pmatrix}_x \delta(x - x'). \tag{4.7}$$

The inverse matrix reads,

$$\Lambda^{-1}_{AB}(x, x') = \{\Phi_A(x), \Phi_B(x')\}^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \frac{v_i^{(0)}}{\rho} & 0 \\ 0 & -\frac{v_i^{(0)}}{\rho} & 0 & -\frac{\delta_{ij}}{\rho} \\ 0 & 0 & \frac{\delta_{ij}}{\rho} & 0 \end{pmatrix}_x \delta(x - x'), \tag{4.8}$$

such that $\int dz \Lambda^{-1}_{AC}(x, z) \Lambda_{CB}(z, x') = \delta_{AB} \delta(x - x')$.

In order to make connection with the Hamiltonian formulation presented in Sec. II, one should resolve the constraints (4.4) and deal with the Dirac bracket,

$$\begin{aligned} \{A(x), B(y)\}_D = \{A(x), B(y)\} + \int dz \Big[&\{A(x), \phi_\theta(z)\}\{\phi_\rho(z), B(y)\} \\ &- \Big(\{A(x), \phi_\rho(z)\} - \frac{v_i^{(0)}(z)}{\rho(z)} \{A(x), \phi_i^{(0)}(z)\}\Big)\{\phi_\theta(z), B(y)\} \\ &- \Big(\frac{v_i^{(0)}(z)}{\rho(z)} \{A(x), \phi_\theta(z)\} + \frac{1}{\rho(z)} \{A(x), \phi_i^{(1)}(z)\}\Big)\{\phi_i^{(0)}(z), B(y)\} \\ &+ \frac{1}{\rho(z)} \{A(x), \phi_i^{(0)}(z)\}\{\phi_i^{(1)}(z), B(y)\}\Big], \end{aligned} \tag{4.9}$$

where $A(t, x)$ and $B(t, x)$ are two arbitrary field variables of the phase space.

By resolving the constraints, one eliminates the canonical momenta from the consideration reducing the set of fields to ρ, θ, $v_i^{(0)}$, and $v_i^{(1)}$. Substituting them in (4.9) and taking into account (4.3), one obtains the following nonzero Dirac brackets,

$$\begin{aligned} \{\rho(x), \theta(y)\}_D &= \delta(x - y), \\ \{\theta(x), v_i^{(0)}(y)\}_D &= \frac{v_i^{(0)}}{\rho} \delta(x - y), \\ \{v_i^{(0)}(x), v_j^{(1)}(y)\}_D &= -\frac{1}{\rho} \delta_{ij} \delta(x - y). \end{aligned} \tag{4.10}$$

When α and β are not zero, it suffices to add the following Dirac brackets:

$$\{\theta(x),\alpha(y)\}_D=\frac{\alpha}{\rho}\delta(x-y),\quad \{\alpha(x),\beta(y)\}_D=\frac{1}{\rho}\delta(x-y). \tag{4.11}$$

Using (4.10) and (4.11), one can verify that the noncanonical Poisson brackets (2.5) are reproduced for $n=1$ with $v_i^{(2)}$ defined in (4.2). Also one can easily identify the canonical pairs (ρ,θ), $(\rho\alpha,\beta)$, and $(\rho v_i^{(0)}, v_i^{(1)})$. The same pairs result from the Lagrangian (4.1).

The constraint analysis above can be readily generalized to the case of arbitrary half-integer ℓ. One can see from the Lagrangian (3.12) that the canonical pairs include (ρ,θ), $(\rho\alpha,\beta)$, and $(\rho v_i^{(k)}, v_i^{(2n-k-1)})$, where $k=0,1,\ldots,n-1$.

V. CONCLUSION

To summarize, in this work the Lagrangian formulation for the generalized higher-derivative perfect fluid equations, which hold invariant under the ℓ-conformal Galilei group with arbitrary half-integer parameter ℓ, was constructed. It is based on a suitably chosen Clebsch-type parametrization and correctly reproduces the Lagrangian description of a Euler fluid in [35] for $\ell=\frac{1}{2}$. The Dirac method was used in order to analyze constraints which arose after transition to the Hamiltonian formalism. It was demonstrated that all the constraints are second class. The corresponding Dirac brackets were computed, which reproduced the Hamiltonian description in [44] given in terms of noncanonical Poisson brackets.

The recent works on fluid mechanics with the ℓ-conformal Galilei symmetry was mostly focused on the development of the mathematical structure. It now calls for physical applications. Firstly, a clear-cut thermodynamic interpretation is needed. In this regard, the approach in [47] may pave the way. Possible link to statistical mechanics, in particular the universality classes of Hohenberg and Halperin [48], is interesting to explore. Because the generalized perfect fluid equations contain higher-derivative terms, they may find potential applications within the context of the hyperjerk theory [49].

Turning to other possible developments, it would be interesting to develop the Lagrange picture [34] for describing higher derivative fluid mechanics and relate it to the results presented in this paper. Supersymmetric extensions of the Lagrangian (3.12) in the spirit of [50,51] as well as possible applications within the context of the fluid/gravity correspondence are worth exploring.

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