Even-parity black hole perturbations in minimal theory of bigravity

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We study even-parity black hole perturbations in minimal theory of bigravity. We consider the Schwarzschild solution written in the spatially flat coordinates in the self-accelerating branch as the background solution. We clarify the gauge transformations for the $\ell = 0$, 1, and ≥ 2 modes with ℓ being the angular multipole moments under the joint foliation-preserving diffeomorphism transformation. Requiring that the asymptotic regions in the physical and fiducial sectors share the same Minkowski vacua, the solution to the $\ell = 0$ perturbations can be absorbed by a redefinition of the Schwarzschild background. In order to analyze the $\ell = 1$ and ≥ 2 modes, for simplicity we focus on the effectively massless case, where the constant parameter measuring the ratio of the proper times between the two sectors is set to unity and the effective mass terms in the equations of motion vanish. We also find that as a particular solution all the $\ell = 1$ perturbations vanish by imposing their regularity at spatial infinity. For each of the $\ell \geq 2$ modes, in the effectively massless case, we highlight the existence of the expected two propagating modes and four instantaneous modes.

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I. INTRODUCTION

Ghost-free massive gravity and bigravity theories are promising candidates to elucidate the origin of the present day's cosmic acceleration. The first model of ghost-free massive gravity was formulated within a linearized theory by Fierz and Pauli [1]. While naive nonlinear extensions of the Fierz-Pauli theory have not been successful because of the appearance of the Boulware-Deser (BD) ghost [2], the first model of massive gravity free from the BD ghost at the fully nonlinear level was provided by de Rham, Gabadadze, and Tolley [3]. de Rham, Gabadadze, and Tolley model was then extended to a bigravity theory by Hassan and Rosen (HR) [4], by promoting the second fiducial metric to be another dynamical field.[1]

HR bigravity was extended to the minimal theory of bigravity (MTBG) [8], where the four-dimensional spacetime diffeomorphism invariance is broken down to the three-dimensional spatial diffeomorphism invariance and time-reparametrization invariance. While the four-dimensional diffeomorphism invariance is explicitly broken, the absence of problematic scalar and vector degrees of freedom (d.o.f.) makes it easy for the theory to be consistent with experimental and observational tests. MTBG shares the same background cosmological dynamics with HR bigravity, but the number of propagating d.o.f. is down to four, two massless tensorial d.o.f. and the other two tensorial d.o.f. that are massive. The absence of the extra scalar and vector d.o.f. in MTBG implies the absence of ghost or gradient instabilities associated with them [9–11]. In the normal branch, deviations from GR in the dynamics of both background and the scalar sector could be already observed. The absence of extra d.o.f. also allows for a new production scenario of spin-2 dark matter based on the transition from an anisotropic fixed point solution to an isotropic one [12].

MTBG possesses constraints by which the unwanted modes can be removed nonlinearly from the theory. A consequence of the presence of these constraints is the appearance of instantaneous (or shadowy) modes [13,14], which are described by elliptic equations on a three-dimensional hypersurface. Such instantaneous modes appear not only in MTBG but also in other theories, for instance, higher-order scalar-tensor theories where the degeneracy conditions are met only in the unitary gauge, known as U-DHOST theories [15,16]. They satisfy elliptic equations for which some appropriate boundary conditions need to be imposed.

[1]HR bigravity still suffers from the BD ghost if matter is coupled to both the physical and fiducial metrics [5–7].

Reference [17] investigated static and spherically symmetric solutions in the self-accelerating and normal branches of MTBG. It was shown that a pair of Schwarzschild–de Sitter spacetimes with different cosmological constants and black hole (BH) masses written in the spatially flat (Gullstrand-Painlevé) coordinates is a solution in the self-accelerating branch of MTBG. On the other hand, in the normal branch, while the spatially flat coordinates of the paired Schwarzschild–de Sitter metrics cannot be solutions, those written in the coordinates with a constant trace of the extrinsic curvature tensors on the constant time hypersurfaces [15,18–20] could be solutions, provided that the two metrics are parallel.

Reference [14] investigated spherically symmetric gravitational collapse of pressureless dust in the self-accelerating branch of MTBG. While the interior region of a collapsing solution is described by a Friedmann-Lemaître-Robertson-Walker (FLRW) universe, the exterior region is described by a Schwarzschild spacetime with specific time slicings. The collapsing solution with the spatially flat slicings has been obtained under certain tuning of the initial conditions. In the spatially closed case corresponding to an extension of the Oppenheimer-Snyder model [21–23], gravitational collapse happens in the physical and fiducial sectors in the same manner as it would in two independent copies of GR under a certain tuning of the matter energy densities and Schwarzschild radii between the two sectors.

Reference [14] also studied odd-parity perturbations of the Schwarzschild–de Sitter solutions written in the spatially flat coordinates in the self-accelerating branch of MTBG. For the modes $\ell \geq 2$, where ℓ represents the angular multipole moment, there are four physical modes, where two of them are dynamical and the remaining two are instantaneous. Beside the case in which the ratio of the lapse functions in the physical and fiducial sectors are equal to unity, the two dynamical modes are coupled to each other and sourced by the two instantaneous modes. For the dipolar mode $\ell = 1$, the two copies of the slow-rotation limit of the Kerr–de Sitter metrics, in general, cannot be a solution in MTBG, indicating deviation from GR for rotating black holes.

In the present paper, we will study even-parity perturbations in the self-accelerating branch of MTBG. In order to simplify the analysis of the even-parity perturbations, we will focus on the Schwarzschild background solutions written in the spatially flat coordinates, where the effective cosmological constants are tuned to zero, instead of the Schwarzschild–de Sitter solutions. By construction, for $\ell \geq 2$ there should be two propagating d.o.f. We also expect the appearance of a number of instantaneous modes which obey a set of elliptic differential equations on each constant time hypersurface. To make the analysis of the $\ell \geq 1$ modes explicit, we will set the ratio of the lapse functions between the two sectors to be unity. In this case the effective graviton mass terms in the equations of motion of the perturbations vanish. We will also assume that the two gravitational radii of the Schwarzschild metrics in both the sectors coincide. We call this case the *effectively massless* case. Within these assumptions, we will be able to reduce the set of the perturbation equations, and finally identify the two propagating modes and four instantaneous modes.

For the $\ell = 0$ and 1 modes, we will show that there is no propagating d.o.f. as expected from the structure of MTBG, and will present the exact analytic solutions for the perturbations of the two spacetime metrics and the Lagrange multipliers. For these modes, we will show that under the suitable boundary conditions all the free functions of time can be fixed. For $\ell = 0$, the solution for the perturbations can be absorbed by the redefinition of the two gravitational radii of the Schwarzschild solution. We also find that all the components of the $\ell = 1$ perturbations vanish by imposing the regularity of the physical and fiducial metrics at spatial infinity.

The structure of this paper is as follows: In Sec. II, we briefly review MTBG. In Sec. III, we introduce the even-parity perturbations about the Schwarzschild solution written in the spatially flat coordinates in the self-accelerating branch of MTBG. In Sec. IV, we investigate the monopolar perturbations with $\ell = 0$ and relate the solution to the nonlinear analysis of the time-dependent spherically symmetric solutions in the spatially flat coordinates. In Sec. V, we investigate the dipolar perturbations with $\ell = 1$ within the effectively massless case and find the boundary conditions to fix all the free functions of time at the spatial infinities. In Sec. VI, we investigate the higher multipolar perturbations with $\ell \geq 2$ in the effectively massless case and identify the two propagating and four instantaneous modes. The last Sec. VII is devoted to giving a brief summary and conclusion.

II. MINIMAL THEORY OF BIGRAVITY AND MINKOWSKI VACUA

A. Theory

We start with the Arnowitt-Deser-Misner (ADM) decomposition of the physical and fiducial metrics, $g_{\mu\nu}$ and $f_{\mu\nu}$, respectively,

$$g_{\mu\nu}dx^\mu dx^\nu = -N^2 dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt),$$
$$f_{\mu\nu}dx^\mu dx^\nu = -M^2 dt^2 + \phi_{ij}(dx^i + M^i dt)(dx^j + M^j dt), \quad (1)$$

where $x^\mu = (t, x^i)$ represents the coordinates of the four-dimensional spacetime with t and x^i being the temporal coordinate and the coordinates of the three-dimensional spaces, respectively. N, N^i, and γ_{ij} represent the lapse function, shift vector, and three-dimensional spatial metric in the physical sector, and M, M^i, and ϕ_{ij} represent the corresponding quantities in the fiducial sector.

In the unitary gauge, the action of MTBG [8,14,17] is then given by

$$S = \frac{1}{2\kappa^2}\int d^4x\Big(\mathcal{L}_g[N, N^i, \gamma_{ij}; M, M^i, \phi_{ij}; \lambda, \bar{\lambda}, \lambda^i] + \mathcal{L}_m[N, N^i, \gamma_{ij}; M, M^i, \phi_{ij}; \Psi]\Big), \quad (2)$$

where $\kappa^2 = 8\pi G$ represents the gravitational constant in the physical sector, $\mathcal{L}_g$ and $\mathcal{L}_m$ represent the gravitational and matter parts of the Lagrangian, respectively, λ, $\bar{\lambda}$, and λ^i describe the two scalar and one spatial-vector Lagrange multipliers which are associated with the second-class constraints necessary to reduce the number of propagating d.o.f. to four in the original Hamiltonian formulation, and Ψ represents the matter fields. The gravitational Lagrangian $\mathcal{L}_g$ of MTBG is further decomposed into the "precursor" and "constraint" parts as

$$\mathcal{L}_g = \mathcal{L}_{\rm pre}[N, N^i, \gamma_{ij}; M, M^i, \phi_{ij}] + \mathcal{L}_{\rm con}[N, N^i, \gamma_{ij}; M, M^i, \phi_{ij}; \lambda, \bar{\lambda}, \lambda^i], \quad (3)$$

with

$$\mathcal{L}_{\rm pre} \coloneqq \sqrt{-g}R[g] + \alpha^2\sqrt{-f}R[f] - m^2(N\sqrt{\gamma}\mathcal{H}_0 + M\sqrt{\phi}\tilde{\mathcal{H}}_0), \quad (4)$$

$$\mathcal{L}_{\rm con} \coloneqq \sqrt{\gamma}\alpha_{1\gamma}(\lambda + \Delta_\gamma\bar{\lambda}) + \sqrt{\phi}\alpha_{1\phi}(\lambda - \Delta_\phi\bar{\lambda}) + \sqrt{\gamma}\alpha_{2\gamma}(\lambda + \Delta_\gamma\bar{\lambda})^2 + \sqrt{\phi}\alpha_{2\phi}(\lambda - \Delta_\phi\bar{\lambda})^2 - m^2[\sqrt{\gamma}U^i{}_k D_i\lambda^k - \beta\sqrt{\phi}\tilde{U}_k{}^i\tilde{D}_i\lambda^k], \quad (5)$$

and

$$\alpha_{1\gamma} \coloneqq -m^2 U^p{}_q K^q{}_p, \qquad \alpha_{1\phi} \coloneqq m^2\tilde{U}^p{}_q\Phi^q{}_p,$$
$$\alpha_{2\gamma} \coloneqq \frac{m^4}{4N}\left(U^p{}_q - \frac{1}{2}U^k{}_k\delta^p{}_q\right)U^q{}_p,$$
$$\alpha_{2\phi} \coloneqq \frac{m^4}{4M\alpha^2}\left(\tilde{U}_q{}^p - \frac{1}{2}\tilde{U}_k{}^k\delta_q{}^p\right)\tilde{U}_p{}^q, \quad (6)$$

where the constant α represents the ratio of the gravitational constants between the two sectors, m is a parameter with mass dimension one which can be regarded as the graviton mass, β is a constant, $\gamma \coloneqq \det(\gamma_{ij})$ and $\phi \coloneqq \det(\phi_{ij})$ are the determinants of the two three-dimensional spatial metrics γ_{ij} and ϕ_{ij}, respectively. We also note that $K^q{}_p = \gamma^{qr}K_{rp}$ and $\Phi^q{}_p = \phi^{qr}\Phi_{rp}$, where K_{ij} and Φ_{ij} represent the extrinsic curvature tensors on each constant time hypersurface in the physical and fiducial sectors, respectively. Furthermore, $\mathcal{H}_0$ and $\tilde{\mathcal{H}}_0$ are defined by $\mathcal{H}_0 \coloneqq \sum_{n=0}^3 c_{4-n}e_n(\mathcal{K})$ and $\tilde{\mathcal{H}}_0 \coloneqq \sum_{n=0}^3 c_n e_n(\tilde{\mathcal{K}})$ with

$$e_0(\mathcal{K}) = 1, \qquad e_1(\mathcal{K}) = [\mathcal{K}],$$
$$e_2(\mathcal{K}) = \frac{1}{2}([\mathcal{K}]^2 - [\mathcal{K}^2]), \qquad e_3(\mathcal{K}) = \det(\mathcal{K}),$$
$$e_0(\tilde{\mathcal{K}}) = 1, \qquad e_1(\tilde{\mathcal{K}}) = [\tilde{\mathcal{K}}],$$
$$e_2(\tilde{\mathcal{K}}) = \frac{1}{2}([\tilde{\mathcal{K}}]^2 - [\tilde{\mathcal{K}}^2]), \qquad e_3(\tilde{\mathcal{K}}) = \det(\tilde{\mathcal{K}}), \quad (7)$$

with $\mathcal{K}^i{}_k$ and $\tilde{\mathcal{K}}_k{}^i$ characterized by $\mathcal{K}^i{}_k\mathcal{K}^k{}_j = \gamma^{ik}\phi_{kj}$ and $\tilde{\mathcal{K}}_j{}^k\tilde{\mathcal{K}}_k{}^i = \gamma_{jk}\phi^{ki}$. $\Delta_\gamma \coloneqq \gamma^{ij}D_iD_j$ and $\Delta_\phi \coloneqq \phi^{ij}\tilde{D}_i\tilde{D}_j$ represent the Laplacian operators in the physical and fiducial sectors, respectively, where D_i and $\tilde{D}_i$ are the covariant derivatives associated with the spatial metrics γ_{ij} and ϕ_{ij}. The spatial tensors $U^i{}_j$ and $\tilde{U}_j{}^i$ are, respectively, defined by

$$U^i{}_j \coloneqq \frac{1}{2}\sum_{n=1}^3 c_{4-n}\Big(U_{(n)}{}^i{}_j + \gamma^{ik}\gamma_{j\ell}U_{(n)}{}^\ell{}_k\Big),$$
$$\tilde{U}_j{}^i \coloneqq \frac{1}{2}\sum_{n=1}^3 c_n\Big(\tilde{U}_{(n)j}{}^i + \phi^{ik}\phi_{j\ell}\tilde{U}_{(n)k}{}^\ell\Big), \quad (8)$$

with $U_{(n)}{}^i{}_k \coloneqq \frac{\partial e_n(\mathcal{K})}{\partial\mathcal{K}^k{}_i}$, $\tilde{U}_{(n)k}{}^i \coloneqq \frac{\partial e_n(\tilde{\mathcal{K}})}{\partial\tilde{\mathcal{K}}^k{}_i}$, and c_j ($j = 0, 1, 2, 3, 4$) being dimensionless coupling constants. In this paper, we will focus on the vacuum case and set $\mathcal{L}_m = 0$ in Eq. (2).

B. Minkowski vacua in the self-accelerating branch

Before considering the BH solutions, we briefly review de Sitter and Minkowski solutions in the spatially flat, homogeneous and isotropic FLRW metrics, which are, respectively, given by

$$g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + \left(dr - r\frac{\dot{a}(t)}{a(t)}dt\right)^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$
$$f_{\mu\nu}dx^\mu dx^\nu = C_0^2\left[-b^2C_m(t)^2dt^2 + \left(dr - r\frac{\dot{a}_f(t)}{a_f(t)}C_m(t)dt\right)^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right], \quad (9)$$

where $a(t)$ and $a_f(t)$ represent the scale factors in the physical and fiducial sectors, respectively, $C_m(t)$ is a function of the time t, and b is a constant that characterizes the relative light cone aperture in the fiducial sector. In the presence of a nontrivial form of $C_m(t)$, the time coordinate t cannot be the proper time in the fiducial sector. The general ansatz for the Lagrange multipliers is given by

$$\lambda = \lambda(t,r), \quad \bar{\lambda} = \bar{\lambda}(t,r), \quad \lambda^r = \lambda^r(t,r), \quad \lambda^\theta = \lambda^\varphi = 0. \quad (10)$$

In order for the equations of motion for λ, $\bar{\lambda}$, and λ^r to be automatically satisfied, we impose the condition for the self-accelerating branch [8,14,17] given by

$$c_3 + 2C_0c_2 + C_0^2c_1 = 0. \tag{11}$$

From the equations of motion for the lapse functions N and M, i.e., the Friedmann equations in both the sectors, we obtain the solutions for the scale factors in the physical and fiducial sectors given by

$$a(t) = a_0 \exp\left[\sqrt{\frac{\Lambda_g}{3}}t\right],$$
$$a_f(t) = a_{f,0} \exp\left[bC_0\sqrt{\frac{\Lambda_f}{3}}t\right], \tag{12}$$

where a_0 and $a_{0,f}$ are integration constants, and the effective cosmological constants in both the sectors are related to the parameters in the Lagrangian of MTBG (4) by

$$\Lambda_g = \frac{m^2(c_4 - 2C_0^3c_1 - 3C_0^2c_2)}{2},$$
$$\Lambda_f = \frac{(C_0^2c_0 + 2C_0c_1 + c_2)m^2}{2C_0^2\alpha^2}. \tag{13}$$

The remaining metric equations of motion provide the general solutions for the Lagrange multipliers

$$\lambda = C_\lambda(t) + \frac{1}{C_0^2}\frac{\sqrt{c_0C_0^2 + 2C_0c_1 + c_2} + C_0^2\sqrt{-2C_0^3c_1 - 3C_0^2c_2 + c_4}\alpha}{\sqrt{c_0C_0^2 + 2C_0c_1 + c_2} - \sqrt{-2C_0^3c_1 - 3C_0^2c_2 + c_4}\alpha}\left(\bar{\lambda}'' + \frac{2}{r}\bar{\lambda}'\right), \tag{14}$$

$$\lambda^r = -\frac{\sqrt{c_0C_0^2 + 2C_0c_1 + c_2} - \sqrt{-2C_0^3c_1 - 3C_0^2c_2 + c_4}\alpha}{\sqrt{6}\alpha(-1 + C_0\beta)}mrC_\lambda(t), \tag{15}$$

while $\bar{\lambda}$ remains undetermined. Imposing the regularity of λ^r as $r \to \infty$ yields

$$C_\lambda(t) = 0, \tag{16}$$

which leads to $\lambda^r = 0$ and, from Eq. (14),

$$\lambda = \frac{1}{C_0^2}\frac{\sqrt{c_0C_0^2 + 2C_0c_1 + c_2} + C_0^2\sqrt{-2C_0^3c_1 - 3C_0^2c_2 + c_4}\alpha}{\sqrt{c_0C_0^2 + 2C_0c_1 + c_2} - \sqrt{-2C_0^3c_1 - 3C_0^2c_2 + c_4}\alpha}\left(\bar{\lambda}'' + \frac{2}{r}\bar{\lambda}'\right). \tag{17}$$

Thus, λ and $\bar{\lambda}$ are not determined by the background equations of motion.[2]

In the limit of the vanishing effective cosmological constants $\Lambda_g = 0$ and $\Lambda_f = 0$, which from Eq. (13) are explicitly given by

$$c_4 - 2C_0^3c_1 - 3C_0^2c_2 = 0, \quad C_0^2c_0 + 2C_0c_1 + c_2 = 0, \tag{18}$$

the paired de Sitter solutions (12) can smoothly reduce to the paired Minkowski solutions

$$g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$
$$f_{\mu\nu}dx^\mu dx^\nu = C_0^2[-b^2C_m(t)^2dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)]. \tag{19}$$

We note that the nontrivial form of $C_m(t)$ represents a Minkowski vacuum in the fiducial sector which is different from that in the physical sector. In order that the physical and fiducial sectors share the same Minkowski vacua, we have to set $C_m(t) = 1$. In the next sections, we assume that in the two asymptotic regions of the paired Schwarzschild solution written in the spatially-flat coordinates, the physical and fiducial sectors share the same Minkowski vacua.

III. EVEN-PARITY PERTURBATIONS OF SCHWARZSCHILD SOLUTIONS

Under the ADM decomposition (1), the perturbed static and spherically symmetric spacetimes written in the spatially flat coordinates can be described as

[2] At higher-order the metric may depend on $\bar{\lambda}$ and/or $\bar{\bar{\lambda}}$. If this is the case then $\bar{\lambda}$ and/or $\bar{\bar{\lambda}}$ may be fixed by suitable boundary conditions for higher-order perturbations. However, this is beyond the scope of the present paper.

$$g_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + \gamma_{rr}(dr+N^r dt)^2 + 2\gamma_{ra}(dr+N^r dt)(d\theta^a + N^a dt) + \gamma_{ab}(d\theta^a+N^a dt)(d\theta^b + N^b dt),$$
$$f_{\mu\nu}dx^\mu dx^\nu = -b^2M^2dt^2 + \phi_{rr}(dr+M^r dt)^2 + 2\phi_{ra}(dr+M^r dt)(d\theta^a + M^a dt) + \phi_{ab}(d\theta^a+M^a dt)(d\theta^b + M^b dt), \quad (20)$$

where r is the radial coordinate, $\theta^a = (\theta, \varphi)$ represents the coordinates along the unit two-sphere, and

$$N = N_{(0)}(r)\left(1+\sum_{\ell,m} n_0(t,r)Y_{\ell m}(\theta^a)\right), \qquad N^r = N^r_{(0)}(r)\left(1+\sum_{\ell,m} n_1(t,r)Y_{\ell m}(\theta^a)\right),$$
$$\gamma_{rr} = 1+\sum_{\ell,m} n_2(t,r)Y_{\ell m}(\theta^a), \qquad N^a = \sum_{\ell,m} h_t(t,r)\theta^{ab}\tilde{\nabla}_b Y_{\ell m}(\theta^a), \qquad \gamma_{ra} = r\sum_{\ell,m} h_r(t,r)\tilde{\nabla}_a Y_{\ell m}(\theta^a),$$
$$\gamma_{ab} = r^2\left[\theta_{ab} + \sum_{\ell,m} h_1(t,r)\theta_{ab}Y_{\ell m}(\theta^a) + \sum_{\ell,m} h_2(t,r)\tilde{\nabla}_a\tilde{\nabla}_b Y_{\ell m}(\theta^a)\right],$$
$$M = C_0 M_{(0)}(r)\left(1+\sum_{\ell,m} m_0(t,r)Y_{\ell m}(\theta^a)\right), \qquad M^r = M^r_{(0)}(r)\left(1+\sum_{\ell,m} m_1(t,r)Y_{\ell m}(\theta^a)\right),$$
$$\phi_{rr} = C_0^2\left(1+\sum_{\ell,m} m_2(t,r)Y_{\ell m}(\theta^a)\right), \qquad M^a = \sum_{\ell,m} k_t(t,r)\theta^{ab}\tilde{\nabla}_b Y_{\ell m}(\theta^a), \qquad \phi_{ra} = C_0^2 r\sum_{\ell,m} k_r(t,r)\tilde{\nabla}_a Y_{\ell m}(\theta^a),$$
$$\phi_{ab} = C_0^2 r^2\left[\theta_{ab} + \sum_{\ell,m} k_1(t,r)\theta_{ab}Y_{\ell m}(\theta^a) + \sum_{\ell,m} k_2(t,r)\tilde{\nabla}_a\tilde{\nabla}_b Y_{\ell m}(\theta^a)\right]. \quad (21)$$

Here, $C_0 > 0$ is a constant fixed by solving the background equations of motion, $Y_{\ell m}(\theta^a)$ represents spherical harmonics with the multipole and magnetic moments (ℓ, m) with $-\ell \leq m \leq \ell$, and $\tilde{\nabla}_a$ represents the covariant derivative with respect to the metric of the unit two-sphere θ_{ab}, respectively. Because of the degeneracy between the different m modes for the same ℓ, without loss of generality, we may choose $m = 0$ so that $Y_{\ell m} = P_\ell(\cos\theta)$. The parameter $b(>0)$ measures the difference in the time passing in the physical and fiducial sectors, which acquires a nontrivial physical significance in MTBG where the two copies of the four-dimensional diffeomorphism invariance is broken down to the joint three-dimensional one. $N_{(0)}(r)$ and $N^r_{(0)}(r)$ represent the background lapse function and shift vector in the physical sector, while $M_{(0)}(r)$ and $M^r_{(0)}(r)$ represent the corresponding quantities in the fiducial sector, respectively. For each of the ℓ modes, n_0, n_1, n_2, h_t, h_r, h_1, and h_2 represent the (t,r) part of the metric perturbations in the physical sector, and m_0, m_1, m_2, k_t, k_r, k_1, and k_2 represent the (t,r) part of the metric perturbations in the fiducial sector.

In the self-accelerating branch of MTBG satisfying Eq. (11), the Schwarzschild–de Sitter solution written in the spatially flat coordinates is given by

$$g_{\mu\nu}dx^\mu dx^\nu = -N_{(0)}(r)^2dt^2 + (dr + N^r_{(0)}(r)dt)^2 + r^2\theta_{ab}d\theta^a d\theta^b,$$
$$f_{\mu\nu}dx^\mu dx^\nu = C_0^2\left[-b^2M_{(0)}(r)^2dt^2 + (dr + M^r_{(0)}(r)dt)^2 + r^2\theta_{ab}d\theta^a d\theta^b\right], \quad (22)$$

with

$$N_{(0)}(r) = M_{(0)}(r) = 1, \qquad N^r_{(0)}(r) = \sqrt{\frac{\Lambda_g r^2}{3} + \frac{r_g}{r}},$$
$$M^r_{(0)}(r) = \sqrt{\frac{r^2C_0^2b^2\Lambda_f}{3} + \frac{r_f}{r}}, \quad (23)$$

where Λ_g and Λ_f are the effective cosmological constants given by Eq. (13) [14,17].

In the rest, we focus on the Schwarzschild solution obtained in the limit of the vanishing effective cosmological constants $\Lambda_g = 0$ and $\Lambda_f = 0$, under the conditions explicitly given by Eq. (18). The background metric solution (23) in this limit reduces to

$$N_{(0)}(r) = M_{(0)}(r) = 1, \quad N^r_{(0)}(r) = \sqrt{\frac{r_g}{r}}, \quad M^r_{(0)}(r) = \sqrt{\frac{r_f}{r}}, \quad (24)$$

where r_g and r_f then correspond to the gravitational radii of the Schwarzschild spacetimes. We will assume that $C_0c_1 + c_2 \neq 0$. As mentioned previously, we also call the case of $b = 1$ (as well as $r_f = r_g$) the *effectively massless case*, where the effective graviton mass terms in the equations of motion of perturbations vanish in both the odd- and even-parity sectors [14].

Because of the spherical symmetry, the background part of the angular components of the vector Lagrange multiplier trivially vanish, $\lambda^\theta = \lambda^\varphi = 0$. We also choose the trivial solution for the remaining components of the background Lagrange multipliers

$$\lambda = 0, \qquad \bar{\lambda} = 0, \qquad \lambda^r = 0, \tag{25}$$

which is compatible with the background equations of motion. In general, the background solution for $\bar{\lambda}$ may be a solution for the Laplace equation in three-dimensional flat space. However, since in the Lagrangian (4) the $\bar{\lambda}$ dependence appears through the spatial Laplacian operators $\Delta_\gamma \bar{\lambda}$ and $\Delta_\phi \bar{\lambda}$, i.e., in our background case the Laplacian operator in the three-dimensional flat space acting on $\bar{\lambda}$, a solution of the Laplace equation in the three-dimensional flat space does not contribute to the background dynamics. Thus, without loss of generality, we may set $\bar{\lambda} = 0$. On top of the trivial solution (25), we consider the even-parity perturbations of the Lagrange multipliers given by

$$\lambda = \sum_{\ell,m} \lambda_0(t,r) Y_{\ell m}(\theta^a), \qquad \bar{\lambda} = \sum_{\ell,m} \lambda_1(t,r) Y_{\ell m}(\theta^a),$$
$$\lambda^r = \sum_{\ell,m} \lambda_2(t,r) Y_{\ell m}(\theta^a), \qquad \lambda^a = \sum_{\ell,m} \lambda_3(t,r) \theta^{ab} \tilde{\nabla}_b Y_{\ell m}(\theta^a). \tag{26}$$

A. The $\ell \geq 2$ modes

For the $\ell \geq 2$ modes, the perturbed physical and fiducial metrics in the even-parity sectors are, respectively, given by

$$\begin{aligned} g_{\mu\nu} dx^\mu dx^\nu = {}& -N_{(0)}(r)^2 dt^2 + (dr + N^r_{(0)}(r) dt)^2 + r^2 \theta_{ab} d\theta^a d\theta^b \\ & + \sum_{\ell \geq 2, m} \{ [-2 n_0(t,r) N_{(0)}(r)^2 + N^r_{(0)}(r)^2 (n_2(t,r) + 2 n_1(t,r))] Y_{\ell m}(\theta^a) dt^2 \\ & + 2 N^r_{(0)}(r)(n_1(t,r) + n_2(t,r)) Y_{\ell m}(\theta^a) dt dr + n_2(t,r) Y_{\ell m}(\theta^a) dr^2 \\ & + 2r(r h_t(t,r) + N^r_{(0)}(r) h_r(t,r)) \tilde{\nabla}_a Y_{\ell m} dt d\theta^a + 2r h_r(t,r) \tilde{\nabla}_a Y_{\ell m}(\theta^a) dr d\theta^a \\ & + r^2 (h_1(t,r) \theta_{ab} Y_{\ell m}(\theta^a) + h_2(t,r) \tilde{\nabla}_a \tilde{\nabla}_b Y_{\ell m}(\theta^a)) d\theta^a d\theta^b \}, \end{aligned} \tag{27}$$

$$\begin{aligned} f_{\mu\nu} dx^\mu dx^\nu = C_0^2 \Big[& -b^2 M_{(0)}(r)^2 dt^2 + (dr + M^r_{(0)}(r) dt)^2 + r^2 \theta_{ab} d\theta^a d\theta^b \\ & + \sum_{\ell \geq 2, m} \{ (-2 m_0 b^2 M_{(0)}(r)^2 + M^r_{(0)}(r)^2 ((m_2(t,r) + 2 m_1(t,r))) Y_{\ell m}(\theta^a) dt^2 \\ & + 2 M^r_{(0)}(r)(m_1(t,r) + m_2(t,r)) Y_{\ell m}(\theta^a) dt dr + m_2(t,r) Y_{\ell m}(\theta^a) dr^2 \\ & + 2r(r k_t(t,r) + M^r_{(0)}(r) k_r(t,r)) \tilde{\nabla}_a Y_{\ell m}(\theta^a) dt d\theta^a + 2r k_r(t,r) \tilde{\nabla}_a Y_{\ell m}(\theta^a) dr d\theta^a \\ & + r^2 (k_1(t,r) \theta_{ab} Y_{\ell m}(\theta^a) + k_2(t,r) \tilde{\nabla}_a \tilde{\nabla}_b Y_{\ell m}(\theta^a)) d\theta^a d\theta^b \} \Big]. \end{aligned} \tag{28}$$

The perturbed three-dimensional metrics in the even-parity sector are, respectively, given by

$$\begin{aligned} \gamma_{ij} dy^i dy^j = {}& dr^2 + r^2 \theta_{ab} d\theta^a d\theta^b + \sum_{\ell \geq 2, m} [n_2(t,r) Y_{\ell m}(\theta^a) dr^2 + 2r h_r(t,r) \tilde{\nabla}_a Y_{\ell m}(\theta^a) dr d\theta^a \\ & + r^2 (h_1(t,r) \theta_{ab} Y_{\ell m}(\theta^a) + h_2(t,r) \tilde{\nabla}_a \tilde{\nabla}_b Y_{\ell m}(\theta^a)) d\theta^a d\theta^b], \\ \phi_{ij} dy^i dy^j = {}& C_0^2 \Big\{ dr^2 + r^2 \theta_{ab} d\theta^a d\theta^b + \sum_{\ell \geq 2, m} [m_2(t,r) Y_{\ell m}(\theta^a) dr^2 + 2r k_r(t,r) \tilde{\nabla}_a Y_{\ell m}(\theta^a) dr d\theta^a \\ & + r^2 (k_1(t,r) \theta_{ab} Y_{\ell m}(\theta^a) + k_2(t,r) \tilde{\nabla}_a \tilde{\nabla}_b Y_{\ell m}(\theta^a)) d\theta^a d\theta^b] \Big\}. \end{aligned} \tag{29}$$

Under the spatial gauge transformation $t \to t$ and $x^i \to x^i + \xi^i(t, x^i)$ with

$$\xi^r = \sum_{\ell \geq 2, m} \Xi_r(t,r) Y_{\ell m}(\theta^a), \qquad \xi^a = \sum_{\ell \geq 2, m} \Xi_1(t,r) \theta^{ab} \tilde{\nabla}_b Y_{\ell m}(\theta^a), \tag{30}$$

the metric perturbations transform as

$$\bar{\delta}n_0(t,r) = -\frac{N'_{(0)}(r)}{N_{(0)}(r)}\Xi_r(t,r), \qquad \bar{\delta}n_1(t,r) = -\frac{N^r_{(0)}{}'(r)}{N^r_{(0)}(r)}\Xi_r(t,r) + \Xi'_r(t,r) - \frac{1}{N^r_{(0)}(r)}\dot{\Xi}_r(t,r),$$
$$\bar{\delta}n_2(t,r) = -2\Xi'_r(t,r), \qquad \bar{\delta}h_t(t,r) = N^r_{(0)}(r)\Xi'_1(t,r) - \dot{\Xi}_1(t,r), \qquad \bar{\delta}h_r(t,r) = -\frac{\Xi_r(t,r) + r^2\Xi'_1(t,r)}{r},$$
$$\bar{\delta}h_1(t,r) = -\frac{2\Xi_r(t,r)}{r}, \qquad \bar{\delta}h_2(t,r) = -2\Xi_1(t,r), \tag{31}$$

and

$$\bar{\delta}m_0(t,r) = -\frac{M'_{(0)}{}'(r)}{M_{(0)}(r)}\Xi_r(t,r), \qquad \bar{\delta}m_1(t,r) = -\frac{M^r_{(0)}{}'(r)}{M^r_{(0)}(r)}\Xi_r(t,r) + \Xi'_r(t,r) - \frac{1}{M^r_{(0)}(r)}\dot{\Xi}_r(t,r),$$
$$\bar{\delta}m_2(t,r) = -2\Xi'_r(t,r), \qquad \bar{\delta}k_t(t,r) = M^r_{(0)}(r)\Xi'_1(t,r) - \dot{\Xi}_1(t,r), \qquad \bar{\delta}k_r(t,r) = -\frac{\Xi_r(t,r) + r^2\Xi'_1(t,r)}{r},$$
$$\bar{\delta}k_1(t,r) = -\frac{2\Xi_r(t,r)}{r}, \qquad \bar{\delta}k_2(t,r) = -2\Xi_1(t,r), \tag{32}$$

where $\bar{\delta}$ represents the difference between the perturbed quantities before and after the gauge transformation. For each of the $\ell \geq 2$ modes, among 14 metric variables for the even parity perturbations, two of them can be eliminated by choosing Ξ_r and Ξ_1. Later, we move to the gauge

$$h_1(t,r) = h_2(t,r) = 0, \tag{33}$$

which will fix the two gauge functions Ξ_r and Ξ_1 completely.

B. The $\ell = 0$ mode

For the $\ell = 0$ mode, using $Y_{\ell m} = P_0(\cos\theta) = 1$, the metric perturbations can be written as

$$\begin{aligned} g_{\mu\nu}dx^\mu dx^\nu = {} & -N_{(0)}(r)^2dt^2 + (dr + N^r_{(0)}(r)dt)^2 + r^2\theta_{ab}d\theta^a d\theta^b \\ & + [-2n_0(t,r)N_{(0)}(r)^2 + N^r_{(0)}(r)^2(n_2(t,r) + 2n_1(t,r))]dt^2 + 2N^r_{(0)}(r)(n_1(r,r) + n_2(t,r))dtdr \\ & + n_2(t,r)dr^2 + r^2h_1(t,r)\theta_{ab}d\theta^a d\theta^b, \end{aligned} \tag{34}$$

$$\begin{aligned} f_{\mu\nu}dx^\mu dx^\nu = C_0^2\Big\{ & -b^2M_{(0)}(r)^2dt^2 + (dr + M^r_{(0)}(r)dt)^2 + r^2\theta_{ab}d\theta^a d\theta^b \\ & + [-2m_0(t,r)b^2M_{(0)}(r)^2 + M^r_{(0)}(r)^2(m_2(t,r) + 2m_1(t,r))]dt^2 + 2M^r_{(0)}(r)(m_1(t,r) + m_2(t,r))dtdr \\ & + m_2(t,r)dr^2 + r^2k_1(t,r)\theta_{ab}d\theta^a d\theta^b\Big\}. \end{aligned} \tag{35}$$

Under the spatial gauge transformation for $\ell = 0$, $t \to t$, and $x^i \to x^i + \xi^i(t, x^i)$ with $\xi^r = \Xi_r(t,r)$ and $\xi^a = 0$, the metric perturbations transform as

$$\bar{\delta}n_0(t,r) = -\frac{N'_{(0)}(r)}{N_{(0)}(r)}\Xi_r(t,r), \qquad \bar{\delta}n_1(t,r) = -\frac{N^r_{(0)}{}'(r)}{N^r_{(0)}(r)}\Xi_r(t,r) + \Xi'_r(t,r) - \frac{1}{N^r_{(0)}(r)}\dot{\Xi}_r(t,r),$$
$$\bar{\delta}n_2(t,r) = -2\Xi'_r(t,r), \qquad \bar{\delta}h_1(t,r) = -\frac{2\Xi_r(t,r)}{r}, \tag{36}$$

and

$$\bar{\delta}m_0(t,r) = -\frac{(M_{(0)})'}{M_{(0)}}\Xi_r(t,r), \qquad \bar{\delta}m_1(t,r) = -\frac{M^r_{(0)}{}'(r)}{M^r_{(0)}(r)}\Xi_r(t,r) + \Xi_r'(t,r) - \frac{1}{M^r_{(0)}(r)}\dot{\Xi}_r(t,r),$$
$$\bar{\delta}m_2(t,r) = -2\Xi_r'(t,r), \qquad \bar{\delta}k_1(t,r) = -\frac{2\Xi_r(t,r)}{r}. \tag{37}$$

We will move to the gauge

$$h_1(t,r) = 0, \tag{38}$$

which completely fixes Ξ_r. The even-parity perturbation of the Lagrange multipliers for $\ell = 0$ is given by

$$\lambda = \lambda_0(t,r), \qquad \bar{\lambda} = \lambda_1(t,r), \qquad \lambda^r = \lambda_2(t,r), \qquad \lambda^a = 0. \tag{39}$$

C. The $\ell = 1$ mode

For the $\ell = 1$ mode, using $Y_{\ell m} = P_1(\cos\theta) = \cos\theta$, the two perturbed metrics can be expanded as

$$\begin{aligned} g_{\mu\nu}dx^\mu dx^\nu = {}& -N_{(0)}(r)^2dt^2 + (dr + N^r_{(0)}(r)dt)^2 + r^2\theta_{ab}d\theta^a d\theta^b \\ & + \left[-2n_0(t,r)N_{(0)}(r)^2 + N^r_{(0)}(r)^2(n_2(t,r) + 2n_1(t,r))\right]\cos\theta dt^2 + 2N^r_{(0)}(r)(n_1(t,r) + n_2(t,r))\cos\theta dtdr \\ & + n_2(t,r)\cos\theta dr^2 - 2r(rh_t(t,r) + N^r_{(0)}(r)h_r(t,r))\sin\theta dtd\theta \\ & - 2rh_r(t,r)\sin\theta drd\theta + r^2(h_1(t,r) - h_2(t,r))\cos\theta\cdot\theta_{ab}d\theta^a d\theta^b, \end{aligned} \tag{40}$$

$$\begin{aligned} f_{\mu\nu}dx^\mu dx^\nu = {}& C_0^2\Big\{-b^2M_{(0)}(r)^2dt^2 + (dr + M^r_{(0)}(r)dt)^2 + r^2\theta_{ab}d\theta^a d\theta^b \\ & + \left[-2m_0(t,r)b^2M_{(0)}(r)^2 + M^r_{(0)}(r)^2(m_2 + 2m_1)\right]\cos\theta dt^2 + 2M^r_{(0)}(r)(m_1(t,r) + m_2(t,r))\cos\theta dtdr \\ & + m_2(t,r)\cos\theta dr^2 - 2r(rk_t(t,r) + M^r_{(0)}(r)k_r(t,r))\sin\theta dtd\theta \\ & - 2rk_r(t,r)\sin\theta drd\theta + r^2(k_1(t,r) - k_2(t,r))\cos\theta\cdot\theta_{ab}d\theta^a d\theta^b\Big\}. \end{aligned} \tag{41}$$

Under the spatial gauge transformation $t \to t$ and $x^i \to x^i + \xi^i(t,x^i)$ for $\ell = 1$ with $\xi^r = \Xi_r(t,r)\cos\theta$ and $\xi^\theta = -\Xi_1(t,r)\sin\theta$, the metric perturbations transform as

$$\bar{\delta}n_0(t,r) = -\frac{N'_{(0)}(r)}{N_{(0)}(r)}\Xi_r(t,r), \qquad \bar{\delta}n_1(t,r) = -\frac{N^r_{(0)}{}'(r)}{N^r_{(0)}(r)}\Xi_r(t,r) + \Xi_r'(t,r) - \frac{1}{N^r_{(0)}(r)}\dot{\Xi}_r(t,r),$$
$$\bar{\delta}n_2(t,r) = -2\Xi_r'(t,r), \qquad \bar{\delta}h_t(t,r) = N^r_{(0)}\Xi_1' - \dot{\Xi}_1,$$
$$\bar{\delta}h_r(t,r) = -\frac{\Xi_r(t,r) + r^2\Xi_1'(t,r)}{r}, \qquad \bar{\delta}(h_1(t,r) - h_2(t,r)) = 2\Xi_1(t,r) - \frac{2\Xi_r(t,r)}{r}, \tag{42}$$

and

$$\bar{\delta}m_0(t,r) = -\frac{M'_{(0)}(r)}{M_{(0)}}\Xi_r(t,r), \qquad \bar{\delta}m_1(t,r) = -\frac{M^r_{(0)}{}'(r)}{M^r_{(0)}(r)}\Xi_r(t,r) + \Xi_r'(t,r) - \frac{1}{M^r_{(0)}(r)}\dot{\Xi}_r(t,r),$$
$$\bar{\delta}m_2(t,r) = -2\Xi_r'(t,r), \qquad \bar{\delta}k_t(t,r) = M^r_{(0)}(r)\Xi_1'(t,r) - \dot{\Xi}_1(t,r),$$
$$\bar{\delta}k_r(t,r) = -\frac{\Xi_r(t,r) + r^2\Xi_1'(t,r)}{r}, \qquad \bar{\delta}(k_1(t,r) - k_2(t,r)) = 2\Xi_1(t,r) - \frac{2\Xi_r(t,r)}{r}. \tag{43}$$

Since $h_2(t,r)$ and $k_2(t,r)$ always appear as the combinations of $h_1(t,r)-h_2(t,r)$ and $k_1(t,r)-k_2(t,r)$ in Eqs. (40) and (41), respectively, we may set

$$h_2(t,r)=0, \qquad k_2(t,r)=0. \tag{44}$$

By fixing Ξ_r and Ξ_1 appropriately, for instance, we may choose the gauge

$$h_1(t,r)=0, \qquad h_r(t,r)=0. \tag{45}$$

The even-parity perturbation of the Lagrange multipliers for $\ell=1$ is given by

$$\lambda=\lambda_0(t,r)\cos\theta, \quad \bar{\lambda}=\lambda_1(t,r)\cos\theta, \quad \lambda^r=\lambda_2(t,r)\cos\theta,$$
$$\lambda^\theta=-\lambda_3(t,r)\sin\theta, \quad \lambda^\varphi=0. \tag{46}$$

IV. THE SOLUTIONS FOR THE MONOPOLAR PERTURBATIONS

In this section, we focus on the $\ell=0$ mode.

A. The case of the two copies of GR

First, we focus on the case of the two copies of GR. This will help illustrate the case of MTBG, whose Lagrangian can be understood as a particular coupling between the two copies of GR. In the case of the two copies of GR, after deriving all the equations of motion, with use of the two copies of the four-dimensional diffeomorphism invariance, for the perturbed metrics (34) and (35) we may choose the gauge

$$h_1=0, \qquad k_1=0, \qquad n_2=0, \qquad m_2=0. \tag{47}$$

For convenience, we introduce the new variables ψ and χ

$$n_1=n_0+\frac{1}{4\sqrt{r_g}}\psi, \qquad m_1=m_0+\frac{1}{4\sqrt{r_f}}\chi. \tag{48}$$

Integrating the equations of motion for n_0 and m_0, we find the solutions

$$\psi=C_\psi(t), \qquad \chi=C_\chi(t), \tag{49}$$

where $C_\psi(t)$ and $C_\chi(t)$ are functions of time. The equations of motion for n_2 and m_2 then, respectively, lead to

$$C_\psi(t)=C_{\psi,0}, \qquad C_\chi(t)=C_{\chi,0}, \tag{50}$$

where $C_{\psi,0}$ and $C_{\chi,0}$ are integration constants. The equations of motion for ψ and χ, respectively, give rise to

$$n_0=C_{n_0}(t), \qquad m_0=C_{m_0}(t), \tag{51}$$

where $C_{n_0}(t)$ and $C_{m_0}(t)$ are free functions of time. We note that the constants $C_{\psi,0}$ and $C_{\chi,0}$ can be absorbed into the redefinition of gravitational radii r_g and r_f, and the functions $C_{n_0}(t)$ and $C_{m_0}(t)$ can be set to zero by the redefinition of the time coordinates in each sector. Thus, the $\ell=0$ mode in the case of the two copies of GR can be absorbed by the redefinition of the Schwarzschild backgrounds.

B. The case of MTBG

We then focus on the $\ell=0$ mode in the self-accelerating branch of MTBG. For convenience, we also introduce the new variables ψ and χ

$$n_1=n_0+\frac{1}{4r_g}\Big[(6r-5r_g)h_1-2rn_2+2r(r-r_g)h_1'$$
$$+2\sqrt{r^3r_g}\dot{h}_1+\sqrt{r_g}\psi\Big], \tag{52}$$

$$m_1=m_0+\frac{1}{4r_f}\Big[(6b^2r-5r_f)k_1-2b^2rm_2+2b^2r^2k_1'$$
$$-2rr_fk_1'+2\sqrt{r^3r_f}\dot{k}_1+\sqrt{r_f}\chi\Big]. \tag{53}$$

Integrating the equations of motion for n_0 and m_0, respectively, we find the solutions

$$\psi=C_\psi(t)+\frac{2(r_g-2r)}{\sqrt{r_g}}h_1,$$
$$\chi=C_\chi(t)+\frac{2(r_f-2b^2r)}{\sqrt{r_f}}k_1, \tag{54}$$

where $C_\psi(t)$ and $C_\chi(t)$ are functions of time. The equations of motion for λ_0 and λ_1 yield

$$m_2=\frac{1}{2}(h_1-k_1)+n_2. \tag{55}$$

The equation of motion for λ_2 can then be integrated as

$$k_1=h_1+\frac{C_{hk}(t)}{r^{\frac{3}{2}}}, \tag{56}$$

where $C_{hk}(t)$ is a function of time. The equation of motion for m_2 leads to

$$\lambda_2=\frac{1}{2bC_0^2(C_0c_1+c_2)(-1+C_0\beta)m^2r^{\frac{3}{2}}}\Big[-(b-1)bC_0^3(C_0c_1+c_2)m^2rC_{hk}(t)+2C_0^2(C_0c_1+c_2)m^2r(\sqrt{r_f}-b\sqrt{r_g})\lambda_0$$
$$+C_0^3r\alpha^2C_\chi'(t)-4C_0c_1m^2\sqrt{r_f}\lambda_1'-4c_2m^2\sqrt{r_f}\lambda_1'-4bC_0^3c_1m^2\sqrt{r_g}\lambda_1'-4bC_0^2c_2m^2\sqrt{r_g}\lambda_1'$$
$$-2C_0c_1m^2r\sqrt{r_f}\lambda_1''-2c_2m^2r\sqrt{r_f}\lambda_1''-2bC_0^3c_1m^2r\sqrt{r_g}\lambda_1''-2bC_0^2c_2m^2r\sqrt{r_g}\lambda_1''\Big]. \tag{57}$$

Similarly, the equation of motion for n_2 leads to

$$C_\psi(t) = C_{\psi,0} - \frac{C_0^2\alpha^2}{b} C_\chi(t), \tag{58}$$

where $C_{\psi,0}$ is an integration constant.

The equation of motion for ψ reduces to the equation

$$2h_1' - 2n_0' - n_2' + rh_1'' + \sqrt{\frac{r}{r_g}}(-\dot{h}_1 + \dot{n}_2 - r\dot{h}_1') = 0, \tag{59}$$

and the equation of motion for χ reduces to the equation

$$2h_1' - 2m_0' - n_2' + rh_1'' + \sqrt{\frac{r}{r_f}}(-\dot{h}_1 + \dot{n}_2 - r\dot{h}_1') = 0. \tag{60}$$

The compatibility of the equations of motion for ψ and h_1 leads to the solution

$$\lambda_0 = C_{\lambda_0}(t) + \frac{\sqrt{r_f} + bC_0^2\sqrt{r_g}}{C_0^2(\sqrt{r_f} - b\sqrt{r_g})}\left(\lambda_1'' + \frac{2}{r}\lambda_1'\right), \tag{61}$$

where $C_{\lambda_0}(t)$ is a free function of time. The compatibility of the equations of motion for χ and k_1 leads to the same solution as Eq. (61). The combination of Eqs. (59) and (60) can be integrated as

$$m_0 = C_{m_0}(t) + \sqrt{\frac{r_g}{r_f}}n_0 + \frac{\sqrt{r_f} - \sqrt{r_g}}{2\sqrt{r_f}}(h_1 + rh_1' - n_2), \tag{62}$$

where $C_{m_0}(t)$ is a free function of time. Then, Eq. (60) reduces to Eq. (59).

So far, we have not fixed the gauge. From now on, we fix the gauge as Eq. (38) and then Eq. (59) reduces to

$$-2n_0' - n_2' + \sqrt{\frac{r}{r_g}}\dot{n}_2 = 0. \tag{63}$$

Under the condition (63), the general solution is given by

$$\begin{aligned}
n_1 &= \frac{C_{\psi_0}}{4\sqrt{r_g}} - \frac{C_0^2\alpha^2}{4b\sqrt{r_g}}C_\chi(t) + n_0 - \frac{r}{2r_g}n_2,\\
m_0 &= C_{m_0}(t) + \sqrt{\frac{r_g}{r_f}}n_0 - \frac{\sqrt{r_f} - \sqrt{r_g}}{2\sqrt{r_f}}n_2,\\
m_1 &= C_{m_0}(t) + \frac{1}{4\sqrt{r_f}}C_\chi(t) + \frac{1}{2\sqrt{r_f}}C_{hk}'(t)\\
&\quad + \frac{1}{2r_f}\left[2\sqrt{r_f r_g}n_0 + (-b^2 r - r_f + \sqrt{r_f r_g})n_2\right],\\
m_2 &= n_2 - \frac{1}{2r^{\frac{3}{2}}}C_{hk}(t), \qquad k_1 = \frac{1}{r^{\frac{3}{2}}}C_{hk}(t).
\end{aligned} \tag{64}$$

Imposing for simplicity the flatness of the spatial three-dimensional part of the $g_{\mu\nu}$ metric, we have $n_2 = 0$ and then the integration of Eq. (63) is given by $n_0 = C_{n_0}(t)$, where $C_{n_0}(t)$ is a free function of time. Imposing for simplicity the flatness of the spatial three-dimensional part of the $f_{\mu\nu}$ metric, we have $m_2 = k_1 = 0$, and hence $C_{hk}(t) = 0$. In general, imposing spatial flatness in the two sectors restricts the solution space for the $\ell = 0$ perturbations. However, we nonetheless choose to impose this condition to be compatible with the discussion in Sec. IV C. By the reparametrization,

$$C_{m_0}(t) = d_{m_0}(t) + \left(1 - \sqrt{\frac{r_g}{r_f}}\right)C_{n_0}(t), \tag{65}$$

where $d_{m_0}(t)$ is a function of time, the metric solution with the flatness of the three-dimensional spatial metrics is given by

$$\begin{aligned}
&n_0 = C_{n_0}(t), \qquad n_1 = C_{n_0}(t) + \frac{1}{4\sqrt{r_g}}\left(C_{\psi_0} - \frac{C_0^2\alpha^2}{b}C_\chi(t)\right), \qquad n_2 = 0,\\
&m_0 = C_{n_0}(t) + d_{m_0}(t), \qquad m_1 = C_{n_0}(t) + d_{m_0}(t) + \frac{1}{4\sqrt{r_f}}C_\chi(t), \qquad m_2 = k_1 = 0,
\end{aligned} \tag{66}$$

which satisfies the constraint

$$\sqrt{r_g}(n_1 - n_0) + \frac{C_0^2\alpha^2\sqrt{r_f}}{b}(m_1 - m_0) = \frac{C_{\psi_0}}{4}. \tag{67}$$

$C_{n_0}(t)$ corresponds to the d.o.f. of the redefinition of the time coordinate t. In other words, we may set $C_{n_0}(t) = 0$ by a suitable redefinition of the time coordinate. Although the metric solution satisfies the flatness of the spatial three-dimensional metrics and the asymptotic flatness of the two spacetimes, there are still the two functions of time, $C_\chi(t)$ and $d_{m_0}(t)$. We also impose that the two sectors share the same asymptotic Minkowski vacua, and set $d_{m_0}(t) = 0$.

The general solution for the Lagrange multipliers which do not affect the metric solutions is explicitly given by

TABLE I. The manipulations to reduce the equations of motion for the $\ell = 0$ mode. EOM means equation of motion.

Manipulation	Output	Remark	# of independent variables
Derive EOMs			11
EOM for n_0	Eliminate ψ	$C_\psi(t)$	10
EOM for m_0	Eliminate χ	$C_\chi(t)$	9
EOM for λ_0	Eliminate m_2	EOM for λ_1 is not independent	8
EOM for λ_2	Eliminate k_1	$C_{hk}(t)$	7
EOM for m_2	Eliminate λ_2		6
EOM for n_2		Eliminate $C_\psi(t)$	6
Combine EOM for ψ and EOM for h_1	Eliminate λ_0	C_{λ_0}	5
Combine EOM for ψ and EOM for χ	Eliminate m_0	C_{m_0}	4
Fix gauge	Eliminate h_1		3
Spatial flatness of $g_{\mu\nu}$ and $f_{\mu\nu}$	Set $n_2 = 0$	Eliminate $C_{hk}(t)$	2
EOM for ψ	Eliminate n_0	$C_{n_0}(t)$	1
Redefinition of time		Eliminate $C_{n_0}(t)$	1
Same Minowski vacua		Eliminate $C_{m_0}(t)$	1
Argument on collapse		Eliminate $C_\chi(t)$	0

$$\lambda_0 = C_{\lambda_0}(t) + \frac{\sqrt{r_f} + bC_0^2\sqrt{r_g}}{C_0^2(\sqrt{r_f} - b\sqrt{r_g})}\left(\lambda_1'' + \frac{2}{r}\lambda_1'\right), \tag{68}$$

$$\lambda_2 = \frac{2(C_0c_1 + c_2)m^2(\sqrt{r_f} - b\sqrt{r_g})C_{\lambda_0}(t) + C_0\alpha^2 C_\chi'(t)}{2b(C_0c_1 + c_2)m^2\sqrt{r}(-1 + C_0\beta)}, \tag{69}$$

while λ_1 is undetermined. If we consider the gravitational collapse of spherically symmetric stars in both sectors, then one should impose the regularity at the center and it is expected that the function $C_\chi(t)$ should be fixed to a constant value that results in a constant mass of matter in the interior region of each sector,

$$C_\chi(t) = C_{\chi,0} = \text{const.} \tag{70}$$

Thus, the solutions for the Lagrange multipliers reduce to

$$\lambda_0 = C_{\lambda_0}(t) + \frac{\sqrt{r_f} + bC_0^2\sqrt{r_g}}{C_0^2(\sqrt{r_f} - b\sqrt{r_g})}\left(\lambda_1'' + \frac{2}{r}\lambda_1'\right),$$
$$\lambda_2 = \frac{(\sqrt{r_f} - b\sqrt{r_g})C_{\lambda_0}(t)}{b\sqrt{r}(-1 + C_0\beta)}. \tag{71}$$

Because of the regularity of λ_2 as $r \to \infty$, no condition is imposed on $C_{\lambda_0}(t)$. On the other hand, λ_0 and λ_1 are not determined by the equations of motion for the $\ell = 0$ perturbations.[3]

[3]At higher-order the metric may depend on λ_0 and/or λ_1. If this is the case then λ_0 and/or λ_1 may be fixed by suitable boundary conditions for higher-order perturbations. However, this is beyond the scope of the present paper.

The manipulations to reduce the equations of motion for the $\ell = 0$ mode are summarized in Table I. In the next subsection, we confirm that the general solution for the $\ell = 0$ mode corresponds to the linearized limit of the time-dependent extension of the Schwarzschild solution in the spatially flat coordinates in the self-accelerating branch of MTBG.

C. Time-dependent extensions of the Schwarzschild solutions in the self-accelerating branch of MTBG

The most general spherically symmetric but time-dependent physical and fiducial metrics are given by, respectively,

$$g_{\mu\nu}dx^\mu dx^\nu = -A_0(t,r)dt^2 + A_1(t,r)(dr + N^r(t,r)dt)^2 + A_2(t,r)r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$
$$f_{\mu\nu}dx^\mu dx^\nu = -A_{0f}(t,r)dt^2 + A_{1f}(t,r)(dr^2 + N_f^r(t,r)dt)^2 + A_{2f}(t,r)r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{72}$$

Under the spherical symmetry, the most general ansatz for the Lagrange multipliers is given by Eq. (10).

Within the general ansatz of the metrics (72), we assume the ansatz for the time-dependent Schwarzschild metrics in the spatially flat coordinates

$$A_0 = C_n(t)^2, \qquad A_1 = A_2 = 1, \qquad N^r = C_g(t)\sqrt{\frac{1}{r}},$$
$$A_{0f} = b^2C_0^2C_m(t)^2, \quad A_{1f} = A_{2f} = C_0^2, \quad N_f^r = C_f(t)\sqrt{\frac{1}{r}}. \tag{73}$$

We also impose the condition for the self-accelerating branch Eq. (11) and the vanishing effective cosmological constants (18), under which the equations of motion for λ, $\bar{\lambda}$, and λ^r are automatically satisfied. The equations of motion for A_0, N^r, A_{0f}, and N^r_f are also automatically satisfied.

The equation of motion for A_1 relates λ^r with other variables as

$$\lambda^r = \frac{1}{\sqrt{r}}\frac{bC_0^2C_g(t)C_m(t)+C_f(t)C_n(t)}{bC_0^2C_m(t)C_n(t)(1-C_0\beta)}\left(\bar{\lambda}''+\frac{2}{r}\bar{\lambda}'\right) + \frac{1}{\sqrt{r}}\frac{-bC_g(t)C_m(t)+C_f(t)C_n(t)}{bC_m(t)C_n(t)(-1+C_0\beta)}\lambda - \frac{2}{\sqrt{r}}\frac{C_n(t)C_g'(t)-C_g(t)C_n'(t)}{C_0(C_0c_1+c_2)m^2(-1+C_0\beta)C_n(t)^2}. \tag{74}$$

The equation of motion for A_2 yields

$$\lambda = C_\lambda(t) - \frac{bC_0^2C_g(t)C_m(t)+C_f(t)C_n(t)}{C_0^2(bC_g(t)C_m(t)-C_f(t)C_n(t))}\left(\bar{\lambda}''+\frac{2}{r}\bar{\lambda}'\right). \tag{75}$$

The equations of motion for A_{1f} and A_{2f} provide a degenerate equation, which can be integrated as

$$C_g(t) = C_n(t)\left(C_{g,0} - \frac{C_0^2\alpha^2}{b}\frac{C_f(t)}{C_m(t)}\right), \tag{76}$$

where $C_{g,0}$ is an integration constant. Then, all the components of the equations of motion are satisfied.

By the redefinition of the functions of time, $C_m(t) = C_n(t)d_m(t)$ and $C_f(t) = C_n(t)d_f(t)d_m(t)$, where $d_m(t)$ and $d_f(t)$ are functions of time, from Eq. (76) we obtain $C_g(t) = C_n(t)(C_{g,0} - \frac{C_0^2\alpha^2}{b}d_f(t))$. The nontrivial metric solution is given by

$$A_0 = C_n(t)^2, \qquad N^r = \sqrt{\frac{1}{r}}\left(C_{g,0} - \frac{C_0^2\alpha^2}{b}d_f(t)\right)C_n(t),$$
$$A_{0f} = b^2C_0^2d_m(t)^2C_n(t)^2, \qquad N^r_f = \sqrt{\frac{1}{r}}d_f(t)d_m(t)C_n(t), \tag{77}$$

which satisfies

$$\sqrt{r}\left(\frac{N^r}{C_n(t)} + \frac{C_0^2\alpha^2}{b}\frac{M^r}{C_m(t)}\right) = C_{g,0}, \tag{78}$$

corresponding to the nonlinear extension of Eq. (67). While $\bar{\lambda}$ is undetermined by the equations of motion, the general solutions for the Lagrange multipliers λ and λ^r are given by

$$\lambda = C_\lambda(t) - \frac{bC_0^2C_{g,0}+d_f(t)-C_0^4\alpha^2d_f(t)}{C_0^2(bC_{g,0}-d_f(t)-C_0^2\alpha^2d_f(t))}\left(\bar{\lambda}''+\frac{2}{r}\bar{\lambda}'\right), \tag{79}$$

$$\lambda^r = \frac{1}{\sqrt{r}}\frac{(C_0c_1+c_2)m^2C_\lambda(t)(-bC_{g,0}+d_f(t)+C_0^2\alpha^2d_f(t))+2C_0\alpha^2d_f'(t)}{bm^2(C_0c_1+c_2)(-1+C_0\beta)}. \tag{80}$$

By the rescaling of the time coordinate, we may set $C_n(t) = 1$. In the absence of BHs, the solution reduces to the Minkowski solutions. In order for both the sectors to share the same asymptotic Minkowski vacua as $r \to \infty$, we impose $d_m(t) = 1$ and then obtain

$$A_0 = 1, \qquad N^r = \sqrt{\frac{1}{r}}\left(C_{g,0} - \frac{C_0^2\alpha^2}{b}d_f(t)\right),$$
$$A_{0f} = b^2C_0^2, \qquad N^r_f = \sqrt{\frac{1}{r}}d_f(t). \tag{81}$$

If we consider spherically symmetric stellar solutions in both the sectors, then Eq. (81) would describe the exterior solution outside the stars. Imposing the regularity at the center and integrating the corresponding equation of motion towards the exterior, the coefficient of the $1/\sqrt{r}$ terms in N^r and N^r_f should be determined by the total masses in the interior of the stars. Thus, $d_f(t)$ should be fixed to a constant value, because the total mass of matter forming the star in each sector has to be constant,

$$d_f(t) = d_{f,0}. \tag{82}$$

We note that $\bar{\lambda}$ is not fixed by the equations of motion, and the general solution for λ and λ^r is then given by

$$\lambda = C_\lambda(t) + \frac{bC_0^2C_{g,0}+d_{f,0}-C_0^4d_{f,0}\alpha^2}{C_0^2(-bC_{g,0}+d_{f,0}+C_0^2\alpha^2d_{f,0})}\left(\bar{\lambda}''+\frac{2}{r}\bar{\lambda}'\right), \tag{83}$$

$$\lambda^r = \frac{bC_{g,0}-d_{f,0}(1+C_0^2\alpha^2)}{b(1-C_0\beta)\sqrt{r}}C_\lambda(t). \tag{84}$$

Because of the regularity of λ_r as $r \to \infty$, no condition is imposed for $C_\lambda(t)$. λ and $\bar{\lambda}$ are not determined by the equations of motion in the spherically symmetric backgrounds.[4]

V. THE SOLUTIONS OF THE DIPOLAR PERTURBATIONS

In this section, we focus on the $\ell = 1$ mode.

A. The case of the two copies of GR

In this subsection, we review the case of the two copies of GR, to help illustrate the more complex case of MTBG. In the perturbed metrics for the $\ell = 1$ mode, Eqs. (40) and (41), we set $h_2 = 0$ and $k_2 = 0$, and under the two copies of the four-dimensional diffeomorphism invariance we may choose the gauge

$$h_t = 0, \qquad h_r = 0, \qquad h_1 = 0,$$
$$k_t = 0, \qquad k_r = 0, \qquad k_1 = 0. \tag{85}$$

Similar to the case of $\ell = 0$, we introduce the new variables ψ and χ to eliminate the perturbations of the radial components of the shift vectors n_1 and m_1

$$n_1 = \frac{1}{2rr_g}(\sqrt{rr_g}\psi + 2rr_g n_0 - r^2 n_2),$$
$$m_1 = \frac{1}{2rr_f}(\sqrt{rr_f}\chi + 2rr_f m_0 - b^2 r^2 m_2). \tag{86}$$

Using the equations of motion for n_0 and m_0, we can relate n_0 and m_0 with other variables as

$$n_0 = -\frac{1}{2r_g}(rn_2 + 2\sqrt{rr_g}\psi'),$$
$$m_0 = -\frac{1}{2r_f}(b^2 rm_2 + 2\sqrt{rr_f}\chi'). \tag{87}$$

The equations of motion for ψ and δh yield, respectively, the evolution equations of n_2. Their degeneracy leads to

$$n_2 = \frac{1}{3}\sqrt{\frac{r}{r_g}}(3\psi' + 2r\psi''). \tag{88}$$

Substituting it into the equation of motion for ψ or the equivalent equation of motion for δh,

$$2r(r - r_g)\psi' + 2\sqrt{r^3 r_g}\dot{\psi} + (r + r_g)\psi = 0, \tag{89}$$

with which all the other equations of motion in the $g_{\mu\nu}$ sector can be satisfied. Similarly, the equations of motion for χ and δk yield, respectively, the evolution equations of m_2, whose degeneracy leads to

$$m_2 = \frac{1}{3}\sqrt{\frac{r}{r_f}}(3\chi' + 2r\chi''). \tag{90}$$

Substituting it into the equation of motion for ψ or the equivalent equation of motion for δk,

$$2r(b^2 r - r_f)\chi' + 2\sqrt{r^3 r_f}\dot{\chi} + (b^2 r + r_f)\chi = 0, \tag{91}$$

with which all the other equations of motion in the $f_{\mu\nu}$ sector can be satisfied. Since the master equations (89) and (91) are of the first order, there is no propagating d.o.f. In order to satisfy the regularity boundary conditions at spatial infinity and at the horizon, we obtain the trivial solution

$$\psi = 0, \quad \chi = 0, \tag{92}$$

as the solutions of the $\ell = 1$ mode in the case of the two copies of GR.

B. The case of MTBG

We then focus on the self-accelerating branch of MTBG. In order to make the analysis of the $\ell = 1$ mode more explicit, we focus on the case of the effective massless case with $b = 1$ and $r_f = r_g$. Even with these conditions, the constraint part of the Lagrangian (5) does not vanish and hence the essential properties of MTBG are still retained.

After deriving the equations of motion, similar to the case of $\ell = 0$, we introduce the new variables ψ and χ to eliminate the perturbations of the radial components of the shift vectors, n_1 and m_1

$$n_1 = \frac{1}{4rr_g}\Big[2\sqrt{rr_g}\psi + r(6r - 5r_g)h_1 + 4r(r - r_g)h_r + 4rr_g n_0 - 2r^2 n_2 + 2r^2(r - r_g)h_1' + 2\sqrt{r^5 r_g}(\delta h + \dot{h}_1)\Big],$$
$$m_1 = \frac{1}{4rr_g}\Big[2\sqrt{rr_g}\chi + r(6r - 5r_g)k_1 + 4r(r - r_g)k_r + 4rr_g m_0 - 2r^2 m_2 + 2r^2(r - r_g)k_1' + 2\sqrt{r^5 r_g}(\delta k + \dot{k}_1)\Big]. \tag{93}$$

We also replace h_t and k_t by δh and δk, respectively, by

$$h_t = -\sqrt{\frac{r_g}{r^3}}h_r + \delta h, \qquad k_t = -\sqrt{\frac{r_g}{r^3}}k_r + \delta k. \tag{94}$$

In the case of the two copies of GR, with use of the temporal gauge d.o.f., we can set $\delta h = 0$ and $\delta k = 0$. In the case of MTBG, since there is no temporal gauge d.o.f., we cannot set $\delta h = 0$ and $\delta k = 0$ and instead treat δh and δk as independent variables. We then derive the equations of motion for the metric perturbations n_0, ψ, n_2, δh, m_0, χ, m_2, δk, k_r, and k_1, and also those for the perturbations of the Lagrange multipliers λ_0, λ_1, λ_2, and λ_3.

[4]Introducing small deviations from the spherically symmetry, the behavior of the metric may depend on λ and/or $\bar{\lambda}$. If this is the case then λ and/or $\bar{\lambda}$ may be fixed by some boundary conditions for the small deviations from the spherically symmetry. However, this is beyond the scope of the present paper.

Using the equations of motion for n_0 and m_0, we can relate n_0 and m_0 with the other variables as

$$\begin{aligned} n_0 = \frac{1}{4r_g}\Big\{&4r^{\frac{3}{2}}\sqrt{r_g}\delta h + (-18r + 5r_g)h_1 \\ &- 2[-2(r-2r_g)h_r + rn_2 + 5r^2 h_1' - 3rr_g h_1' \\ &+ 2\sqrt{rr_g}\psi' + r^{\frac{3}{2}}\sqrt{r_g}\dot{h}_1 - 2r^{\frac{3}{2}}\sqrt{r_g}\dot{h}_r]\Big\}, \\ m_0 = \frac{1}{4r_g}\Big\{&4r^{\frac{3}{2}}\sqrt{r_g}\delta k + (-18r + 5r_g)k_1 \\ &- 2[-2(r-2r_g)k_r + rm_2 + 5r^2 k_1' - 3rr_g k_1' \\ &+ 2\sqrt{rr_g}\chi' + r^{\frac{3}{2}}\sqrt{r_g}\dot{k}_1 - 2r^{\frac{3}{2}}\sqrt{r_g}\dot{k}_r]\Big\}. \end{aligned}$$

In the effectively massless case $b = 1$ and $r_f = r_g$, the equation of motion for λ_0 becomes trivial. The equation of motion for λ_1 yields

$$m_2 = n_2 + \frac{1}{2}(h_1 - k_1). \tag{95}$$

The equations for λ_2 and λ_3 can be integrated as

$$\begin{aligned} k_r &= h_r + \frac{1}{r^2}C_{k_1}(t) + \frac{1}{r^{\frac{5}{2}}}C_{k_r}(t), \\ k_1 &= h_1 + \frac{2}{r^2}C_{k_1}(t) + \frac{1}{r^{\frac{5}{2}}}C_{k_r}(t), \end{aligned} \tag{96}$$

where $C_{k_1}(t)$ and $C_{k_r}(t)$ are free functions of time. The equations of motion for ψ and δh yield, respectively, the evolution equations of n_2 as

$$\dot{n}_2 = F_{n_2,1}, \qquad \dot{n}_2 = F_{n_2,2}, \tag{97}$$

where

$$\begin{aligned} F_{n_2,1} \coloneqq{}& \frac{7}{2}\delta h - \frac{1}{2r^2}\psi + 3r\delta h' - \sqrt{\frac{r}{r_g}}n_2' + \sqrt{\frac{r_g}{r}}n_2' - 2\psi'' \\ &- \frac{6}{\sqrt{rr_g}}h_1 + \sqrt{\frac{r_g}{r^3}}h_r - 17\sqrt{\frac{r}{r_g}}h_1' + \frac{5}{2}\sqrt{\frac{r_g}{r}}h_1' + 2\sqrt{\frac{r}{r_g}}h_r' - 6\sqrt{\frac{r_g}{r}}h_r' - \frac{5r^{\frac{3}{2}}}{\sqrt{r_g}}h_1'' \\ &+ 2\sqrt{rr_g}h_1'' - \frac{1}{2}\dot{h}_1 + 3\dot{h}_r + 2r\dot{h}_r', \\ F_{n_2,2} \coloneqq{}& \frac{1}{2}\Bigg(-13\delta h - 6\sqrt{\frac{r_g}{r^3}}n_2 - \frac{\psi}{r^2} - 16r\delta h' - 2\sqrt{\frac{r}{r_g}}n_2' + 2\sqrt{\frac{r_g}{r}}n_2' + \frac{6}{r}\psi' - 4r^2\delta h'' \\ &+ \frac{(24r - 5r_g)}{\sqrt{r^3 r_g}}h_1 + 20\sqrt{\frac{r_g}{r^3}}h_r + 20\sqrt{\frac{r}{r_g}}h_1' - 5\sqrt{\frac{r_g}{r}}h_1' + 4\sqrt{\frac{r}{r_g}}h_r' + \frac{2r^{\frac{3}{2}}}{\sqrt{r_g}}h_1'' - \dot{h}_1 + 6\dot{h}_r + 4r\dot{h}_r'\Bigg). \end{aligned} \tag{98}$$

Similarly, the equations of motion for χ and δk yield, respectively, the other evolution equations of n_2 as

$$\dot{n}_2 = G_{n_2,1}, \qquad \dot{n}_2 = G_{n_2,2}, \tag{99}$$

where

$$\begin{aligned} G_{n_2,1} \coloneqq{}& \frac{1}{2r^{9/2}r_g}\Big[2r\sqrt{r_g}(-10r + 29r_g)C_{k_1}(t) + 3\sqrt{rr_g}(-9r + 19r_g)C_{k_r}(t) \\ &- r^2\sqrt{r_g}\Big(-7\sqrt{r^5 r_g}\delta k + \sqrt{rr_g}\chi + 2\sqrt{rr_g}C_{k_1}'(t) + 4\sqrt{r_g}C_{k_r}'(t) - 6\sqrt{r^7 r_g}\delta k' + 2r^3 n_2' - 2r^2 r_g n_2' + 4\sqrt{r^5 r_g}\chi'' \\ &+ 12r^2 h_1 - 2rr_g h_r + 34r^3 h_1' - 5r^2 r_g h_1' - 4r^3 h_r' + 12r^2 r_g h_r' + 10r^4 h_1'' \\ &- 4r^3 r_g h_1'' + \sqrt{r^5 r_g}\dot{h}_1 - 6\sqrt{r^5 r_g}\dot{h}_r - 4\sqrt{r^7 r_g}\dot{h}_r'\Big)\Big], \\ G_{n_2,2} \coloneqq{}& \frac{1}{2r^4\sqrt{r_g}}\Big[-20\sqrt{r}(r - 2r_g)C_{k_1}(t) + (-21r + 33r_g)C_{k_r}(t) - 13r^4\sqrt{r_g}\delta k - 6r^{\frac{5}{2}}r_g n_2 - r^2\sqrt{r_g}\chi \\ &- 2r^2\sqrt{r_g}C_{k_1}'(t) - 4\sqrt{r^3 r_g}C_{k_r}'(t) - 16r^5\sqrt{r_g}\delta k' - 2r^{\frac{9}{2}}n_2' + 2r^{\frac{7}{2}}r_g n_2' + 6r^3\sqrt{r_g}\chi' - 4r^6\sqrt{r_g}\delta k'' \\ &+ 24r^{\frac{7}{2}}h_1 - 5r^{\frac{5}{2}}r_g h_1 + 20r^{\frac{5}{2}}r_g h_r + 20r^{\frac{9}{2}}h_1' - 5r^{\frac{7}{2}}r_g h_1' + 4r^{\frac{9}{2}}h_r' + 2r^{\frac{11}{2}}h_1'' - r^4\sqrt{r_g}\dot{h}_1 \\ &+ 6r^4\sqrt{r_g}\dot{h}_r + 4r^5\sqrt{r_g}\dot{h}_r'\Big]. \end{aligned} \tag{100}$$

The consistency of Eq. (97), $F_{n_2,1} = F_{n_2,2}$, yields

$$n_2 = \frac{1}{3}\sqrt{\frac{r}{r_g}}(-10r\delta h - 11r^2\delta h' + 3\psi' - 2r^3\delta h'' + 2r\psi'') + \frac{1}{6r_g}\Big((36r - 5r_g)h_1 + 2(9r_g h_r + 27r^2 h_1' - 5rr_g h_1' + 6rr_g h_r' + 6r^3 h_1'' - 2r^2 r_g h_1'')\Big). \tag{101}$$

The consistency of Eq. (99), $G_{n_2,1} = G_{n_2,2}$, yields

$$\psi = \chi - 3\sqrt{\frac{r_g}{r^3}}C_{k_1}(t) + \frac{3r - 2r_g}{r^2\sqrt{r_g}}C_{k_r}(t) + r^2(\delta h - \delta k) + \frac{Q_1(t)}{\sqrt{r}} + Q_2(t), \tag{102}$$

where $Q_1(t)$ and $Q_2(t)$ are free functions of time. The compatibility of Eqs. (97) and (99) then yields

$$\delta h = \delta k + \frac{1}{6r^4}\Big[2(5r - r_g)\sqrt{\frac{r}{r_g}}C_{k_1}(t) - \sqrt{r_g}C_{k_r}(t) - 2r^{\frac{3}{2}}Q_1(t) - r^2 Q_2(t) + 6r^3 Q_3(t) + 6r^{\frac{7}{2}}Q_4(t) + 2r^2 C_{k_1}'(t) + 2r^{\frac{3}{2}}C_{k_r}'(t)\Big], \tag{103}$$

where $Q_3(t)$ and $Q_4(t)$ are free functions of time. After imposing Eqs. (101), (102), and (103), the equations of motion for ψ, δh, χ, and δk coincide, and hence we may focus on the equation of motion for ψ.

A combination of the equations of motion for n_2 and m_2 relates λ_3 to λ_1 and λ_2 as

$$\lambda_3 = \frac{2(1 + C_0^2)\sqrt{r_g}}{C_0^2(1 + C_0)\sqrt{r^7}}\left(\lambda_1 - r\lambda_1' - \frac{r^2}{2}\lambda_1''\right) + \frac{1}{r}\lambda_2 - \frac{C_0\alpha^2\Big[r\Big(Q_2(t) + 6r_g\Big(Q_3(t) + \sqrt{r}Q_4(t)\Big) - 12C_{k_1}'(t) + 2\sqrt{r_g}Q_1'(t)\Big) + 2\sqrt{r^3 r_g}Q_2'(t)\Big]}{2(1 + C_0)(C_0 c_1 + c_2)(1 + C_0^2\alpha^2)m^2\sqrt{r^7 r_g}}. \tag{104}$$

Substituting Eq. (104) back to the evolution equation for n_2, we obtain

$$\begin{aligned}\dot{\chi} = {}& r^2\delta\dot{k} + \sqrt{\frac{r^3}{r_g}}(r - r_g)\delta k' + \frac{5r - 9r_g}{2}\sqrt{\frac{r}{r_g}}\delta k - \frac{r + r_g}{2\sqrt{r^3 r_g}}\chi - \frac{-r + r_g}{\sqrt{rr_g}}\chi' \\ &+ \frac{-8r + 7r_g}{r^3}C_{k_1}(t) + \frac{2[r_g + C_0^2\alpha^2(-3r + r_g)]}{\sqrt{r^3 r_g}(1 + C_0^2\alpha^2)}C_{k_1}'(t) + \frac{3r^2 - 15rr_g + 11r_g^2}{2r_g r^{\frac{7}{2}}}C_{k_r}(t) + \frac{-3r + r_g}{r^2\sqrt{r_g}}C_{k_r}'(t) \\ &+ \left(\frac{11}{2} - \frac{6r}{r_g}\right)h_1 - 3\left(1 - \frac{r_g}{r}\right)h_r + \left(4r - \frac{3r^2}{r_g} - r_g\right)h_1' + \left(-\frac{3r^{\frac{3}{2}}}{\sqrt{r_g}} + \sqrt{rr_g}\right)\dot{h}_1, \\ &- \frac{Q_2(t) + 2(3r_g Q_3(t) + 3\sqrt{r}r_g Q_4(t) + \sqrt{r_g}Q_1'(t) + \sqrt{rr_g}Q_2'(t))}{2\sqrt{rr_g}(1 + C_0^2\alpha^2)}.\end{aligned} \tag{105}$$

Substituting Eq. (105) and its derivatives with respect to r into the equations of motion for h_1, k_1, h_r, and k_r, we find that the equations of motion for h_1 and k_1 coincide, and similarly the equations of motion for h_r and k_r coincide, respectively. This is because either h_1 or k_1 and either h_r or k_r can be eliminated by the gauge d.o.f. for the $\ell = 1$ mode. The consistency of the equations of motion for h_1 and h_r then requires

$$\lambda_1(t) = \frac{Q_5(t)}{r^2} + rQ_6(t) + r^3 Q_7(t) - \frac{4C_0^3\alpha^2 r^{\frac{3}{2}}\sqrt{r_g}(3\sqrt{r_g}Q_4(t) + Q_2'(t))}{21(1 + C_0^2)m^2 r_g(1 + C_0^2\alpha^2)(C_0 c_1 + c_2)}, \tag{106}$$

where $Q_5(t)$, $Q_6(t)$, and $Q_7(t)$ are free functions of time. We note that $Q_5(t)$ and $Q_6(t)$ describe the solutions of the Laplace equation for $\ell = 1$, and without loss of generality we can impose that $Q_5(t) = 0$ and $Q_6(t) = 0$.

At this stage, the equations of motion h_r, k_r, h_1, and k_1 coincide. Without loss of generality, we focus on the equation of motion for h_r, which yields

$$\lambda_2 = Q_8(t) - \frac{10(1+C_0^2)\sqrt{rr_g}}{C_0^2(1+C_0)} Q_7(t) + \frac{C_0\alpha^2(Q_2(t) + 6r_g Q_3(t) + 6\sqrt{r} r_g Q_4(t) + 2\sqrt{r_g} Q_1'(t) + 2\sqrt{rr_g} Q_2'(t) - 12 C_{k_1}'(t))}{6(1+C_0)(C_0c_1 + c_2)(1+C_0^2\alpha^2) m^2 r^{\frac{3}{2}} \sqrt{r_g}}. \tag{107}$$

Requiring that $\lambda_1 \to 0$ in the limit of $r \to \infty$, from Eq. (106), we impose that $Q_7(t) = 0$ and $Q_4(t) = -\frac{1}{3\sqrt{r_g}} Q_2'(t)$. Then, we obtain $\lambda_1 = 0$. Requiring that $\lambda_2 \to 0$ in the limit of $r \to \infty$, from Eq. (107), we impose that $Q_8(t) = 0$, and then obtain

$$\lambda_2 = \frac{C_0\alpha^2(Q_2(t) + 6r_g Q_3(t) - 12C_{k_1}'(t) + 2\sqrt{r_g} Q_1'(t))}{6(1+C_0)(C_0c_1+c_2)(1+C_0^2\alpha^2) m^2 r^{\frac{3}{2}}\sqrt{r_g}}, \qquad \lambda_3 = -\frac{2}{r}\lambda_2. \tag{108}$$

By introducing the new variable ϕ to eliminate δk by

$$\delta k = \frac{\chi}{r^2} + \phi, \tag{109}$$

and substituting Eq. (109) into Eq. (105), χ is related to ϕ as

$$\begin{aligned}\chi = {} & \frac{r^{\frac{7}{2}}}{3\sqrt{r_g}}\dot{\phi} + \frac{r^3(r-r_g)}{3r_g}\phi' + \frac{r^2(5r-9r_g)}{6r_g}\phi + \frac{-8r+7r_g}{3\sqrt{r^3 r_g}} C_{k_1}(t) + \frac{3r^2 - 15rr_g + 11r_g^2}{6r^2 r_g^{\frac{3}{2}}} C_{k_r}(t) + \frac{-3r+r_g}{3\sqrt{rr_g}} C_{k_r}'(t) \\ & - \frac{rQ_2(t) + 6rr_gQ_3(t) - 4r_gC_{k_1}'(t) + 12C_0^2 r\alpha^2 C_{k_1}'(t) - 4C_0^2 r_g\alpha^2 C_{k_1}'(t) + 2r\sqrt{r_g}Q_1'(t)}{6r_g(1+C_0^2\alpha^2)} \\ & + \frac{1}{6r^{\frac{11}{2}} r_g^{\frac{3}{2}}}\left(r^7(-12r+11r_g)h_1 + 6r^6 r_g(-r+r_g)h_r - 6r^9 h_1' + 8r^8 r_g h_1' - 2r^7 r_g^2 h_1' + 2r^{\frac{15}{2}} r_g^{\frac{3}{2}}\dot{h}_1 - 6\sqrt{r^{17} r_g}\dot{h}_1\right).\end{aligned} \tag{110}$$

Since we have already employed all the equations of motion, in general ϕ is undetermined. However, since n_2 and m_2 are given by

$$n_2 = -\frac{3C_{k_1}(t)}{r^2} + \frac{(r-4r_g)C_{k_r}(t)}{r^{\frac{5}{2}} r_g} - \frac{r^{\frac{3}{2}}}{3\sqrt{r_g}}(10\phi + 11r\phi' + 2r^2\phi'') + \frac{1}{6r_g}\Big((36r-5r_g)h_1 + 18r_g h_r + 2r((27r-5r_g)h_1' + 6r_g h_r' + 2r(3r-r_g)h_1'')\Big), \tag{111}$$

$$m_2 = -\frac{4C_{k_1}(t)}{r^2} + \frac{(2r-9r_g)C_{k_r}(t)}{2r^{\frac{5}{2}} r_g} - \frac{r^{\frac{3}{2}}}{3\sqrt{r_g}}(10\phi + 11r\phi' + 2r^2\phi'') + \frac{1}{6r_g}\Big((36r-5r_g)h_1 + 18r_g h_r + 2r((27r-5r_g)h_1' + 6r_g h_r' + 2r(3r-r_g)h_1'')\Big). \tag{112}$$

we find that the gauge invariant [see Eqs. (31) and (32)] combination $n_2 - m_2$ does not depend on ϕ, h_r, and h_1. Because of the presence of the gauge degrees of freedom, Ξ_r and Ξ_1 we may choose the gauge $h_1 = 0$ and $h_r = 0$ as Eq. (45). Furthermore, as a particular solution we may choose

$$\phi = 0, \tag{113}$$

which avoids the growing terms in n_2 and m_2 as $r \to \infty$.

Below, we focus on the leading behavior of the perturbed metric components in the large distance limit $r \to \infty$ and impose their regularity. The imposition of a sufficient number of boundary conditions in the limit of $r \to \infty$ will fix the remaining free functions of time, $Q_1(t)$, $Q_2(t)$, $Q_3(t)$, $C_{k_r}(t)$, and $C_{k_1}(t)$. We will clarify the boundary conditions which are necessary to eliminate these free functions of time.

The leading terms in the (t,t) components of the metric perturbations, which are obtained from linear combinations of n_0, $\frac{r_g}{r}n_1$, $\frac{r_g}{r}n_2$, m_0, $\frac{r_g}{r}m_1$, and $\frac{r_g}{r}m_2$ are proportional to $Q_2'(t)r$. In order to satisfy the regularity of both the sectors in the large distance $r \to \infty$, we require that the $\mathcal{O}(r)$ terms in the (t,t) components of the perturbed metrics vanish, and hence impose

$$Q_2(t) = q_2, \tag{114}$$

where q_2 is an integration constant.

The leading terms in the (t,r) components of the metric perturbations which are obtained from the linear combinations of $\sqrt{\frac{r_g}{r}}n_1$, $\sqrt{\frac{r_g}{r}}n_2$, $\sqrt{\frac{r_g}{r}}m_1$, and $\sqrt{\frac{r_g}{r}}m_2$ are given by the $\mathcal{O}(r^0)$ terms, which lead to the divergence of the ADM masses in both the sectors in the large distance limit $r \to \infty$. The regularity of the ADM masses requires

$$Q_3(t) = 0,$$
$$Q_1(t) = q_1 - \frac{1}{2\sqrt{r_g}}\left(q_2 t + 12C_0^2\alpha^2 C_{k_1}(t)\right), \tag{115}$$

where q_1 is an integration constant. The leading terms in the (r,r) components of the metric perturbations which are obtained from n_2 and m_2 are given by the $\mathcal{O}(\frac{1}{r^{\frac{3}{2}}})$, where the coefficients are proportional to $C_{k_r}(t)$.

The leading and subleading terms in the (t,θ) components of the metric perturbations which are obtained from the linear combinations of r^2h_t, $r^2\sqrt{\frac{r_g}{r}}h_r$, r^2k_t, and $r^2\sqrt{\frac{r_g}{r}}k_r$ are given by the $\mathcal{O}(r^{\frac{1}{2}})$ and $\mathcal{O}(r^0)$ terms, respectively. For the asymptotic flatness of the spacetimes in both the sectors, to eliminate $\mathcal{O}(r^{\frac{1}{2}})$ term requires

$$C_{k_r}(t) = c_{k_r}, \tag{116}$$

where c_{k_r} is an integration constant. Similarly, eliminating the $\mathcal{O}(r^0)$ terms in the (t,θ) components of the metric perturbations requires that

$$C_{k_1}(t) = \frac{q_2}{2}t + c_{k_1}, \qquad c_{k_r} = -\frac{2}{3}q_2 r_g^{\frac{3}{2}}, \tag{117}$$

where c_{k_1} is an integration constant. The (r,θ) components of the metric perturbations obtained from rh_r and rk_r automatically vanish and need not to be considered in the rest. The leading terms in the angular components of the perturbed metric are proportional to $\mathcal{O}(r^0)$, which are suppressed by the factor $\frac{1}{r^2}$ compared to the background metrics.

After imposing Eq. (116), the $\mathcal{O}(r^0)$ terms of the (t,t) components of the metric perturbations automatically vanish. Similarly, after imposing Eq. (116), the $\mathcal{O}(\frac{1}{r^{\frac{1}{2}}})$ terms of the (t,r) components of the metric perturbations automatically vanish. Then, the $\mathcal{O}(\frac{1}{r^{\frac{1}{2}}})$ terms in the (t,t) components of the metric perturbations are proportional to $\frac{q_2}{\sqrt{r}}$. Requiring that these terms vanish as well imposes

$$q_2 = 0, \tag{118}$$

and from Eq. (117) we obtain that $c_{k_r} = 0$. The $\mathcal{O}(\frac{1}{r})$ terms in the (t,r) components of the metric perturbations vanish. With Eq. (118), the $\mathcal{O}(\frac{1}{r})$ and $\mathcal{O}(\frac{1}{r^{\frac{3}{2}}})$ terms in the (t,t) components of the metric perturbations automatically vanish.

The next-order terms in the (t,θ) components of the metric perturbations are given by the $\mathcal{O}(\frac{1}{r^{\frac{1}{2}}})$ terms. Requiring that these terms vanish imposes

$$q_1 = 0, \qquad c_{k_1} = 0. \tag{119}$$

After imposing Eq. (119), the $\mathcal{O}(\frac{1}{r^{3/2}})$ terms in the (t,r) components of the metric perturbations automatically vanish. As a consequence, all the components of the $\ell = 1$ perturbations vanish.

The manipulations to reduce the equations of motion for the $\ell = 1$ mode are summarized in Table II.

VI. THE SOLUTIONS OF THE HIGHER MULTIPOLAR PERTURBATIONS

In this section, we focus on the $\ell \geq 2$ modes.

A. The case of the two copies of GR

Again, as a point of comparison, we first review the case of the two copies of GR, i.e., the case of $m = 0$. For the modes $\ell \geq 2$, after deriving the equations of motion for $m = 0$, under the two copies of the four-dimensional diffeomorphism invariance, for the perturbed metrics (27) and (28) we fix the gauge as

$$h_t = -\sqrt{\frac{r_g}{r^3}}h_r, \qquad h_1 = 0, \qquad h_2 = 0,$$
$$k_t = -\sqrt{\frac{r_f}{r^3}}k_r, \qquad k_1 = 0, \qquad k_2 = 0. \tag{120}$$

We then introduce the master variables ψ and χ as

TABLE II. The manipulations to reduce the equations of motion for the $\ell = 1$ mode. EOM means equation of motion.

Manipulation	Output	Remark	# of independent variables
Derive EOMs			18
Symmetry	$h_2 = 0$ and $k_2 = 0$		16
Gauge fixing	$h_1 = 0$ and $h_r = 0$		14
EOM for n_0	Eliminate n_0		13
EOM for m_0	Eliminate m_0		12
EOM for λ_1	Eliminate m_2		11
EOM for λ_2 and λ_3	Eliminate k_r and k_1	$C_{k_1}(t)$, $C_{k_r}(t)$	9
Combine EOM for ψ and EOM for δh	Eliminate n_2		8
Combine EOM for χ and EOM for δk	Eliminate ψ	$Q_1(t)$, $Q_2(t)$	7
Combine EOM for ψ and EOM for χ	Eliminate δh	$Q_3(t)$, $Q_4(t)$	6
Combine EOM for n_2 and EOM for m_2	Eliminate λ_3		5
Combine EOM for h_1 and EOM for h_r	Eliminate λ_1	$Q_7(t)$	4
EOM for h_r	Eliminate λ_2	$Q_8(\mathrm{t})$	3
Regularity of λ_1 at $r \to \infty$	Set $\lambda_1 = 0$	$Q_7(t) = 0$, fix $Q_4(t)$	3
Regularity of λ_2 at $r \to \infty$		$Q_8(t) = 0$	3
EOM for n_2			2
Introduce ϕ [Eq. (109)]	Eliminate χ		1
Gauge invariance of $n_2 - m_2$	$\phi = 0$		1
Set λ_0	$\lambda_0 = 0$		0
Eliminating $\mathcal{O}(r)$ in the (t,t) components	Eq. (114)	Fix $Q_2(t)$	0
Eliminating $\mathcal{O}(r^0)$ in the (t,r) components	Eq. (115)	Fix $Q_3(t)$ and $Q_1(t)$	0
Eliminating $\mathcal{O}(\sqrt{r})$ and $\mathcal{O}(r^0)$ in the (t,θ) components	Eqs. (116) and (117)	Fix $C_{k_1}(t)$ and $C_{k_1}(t)$	0
Eliminating $\mathcal{O}(\frac{1}{\sqrt{r}})$ in the (t,t) components	Eq. (118)	Fix q_2	0
Eliminating $\mathcal{O}(\frac{1}{\sqrt{r}})$ in the (t,θ) components	Eq. (119)	Fix q_1 and c_{k_1}	0

$$\begin{aligned} n_1 &= \frac{1}{2r_g}\left[\ell(\ell+1)(r-r_g)h_r + 2r_g n_0 - r n_2 + \kappa^2\sqrt{\ell(\ell+1)}\frac{(\ell^2+\ell-2)r+3r_g}{r}\psi\right], \\ m_1 &= \frac{1}{2r_f}\left[\ell(\ell+1)(b^2 r-r_f)k_r + 2r_f m_0 - b^2 r m_2 + \frac{\kappa^2}{\alpha^2}\sqrt{\ell(\ell+1)}\frac{(\ell^2+\ell-2)b^2 r+3r_f}{r}\chi\right]. \end{aligned} \tag{121}$$

The equations of motion for n_0 and m_0 relate n_0 and m_0 to other variables, respectively. The combination of the equations of motion for h_1 and h_2 is used to eliminate n_2. Similarly, the combination of the equations of motion for k_1 and k_2 is used to eliminate m_2. Then, the combination of the equations of motion for ψ and h_1 yields

$$\begin{aligned} &\ddot{\psi} - 2\sqrt{\frac{r_g}{r}}\dot{\psi}' - \left(1-\frac{r_g}{r}\right)\psi'' + \frac{1}{2}\sqrt{\frac{r_g}{r^3}}\dot{\psi} - \frac{r_g}{r^2}\psi' \\ &\quad + \frac{\ell(\ell+1)(\ell^2+\ell-2)^2 r^3 + 3(\ell^2+\ell-2)^2 r^2 r_g + 9(\ell^2+\ell-2) r r_g^2 + 9r_g^3}{r^3((\ell^2+\ell-2)r+3r_g)^2}\psi = 0. \end{aligned} \tag{122}$$

Similarly, the combination of the equations of motion for χ and k_1 yields

$$\begin{aligned} &\ddot{\chi} - 2\sqrt{\frac{r_f}{r}}\dot{\chi}' - \left(b^2-\frac{r_f}{r}\right)\chi'' + \frac{1}{2}\sqrt{\frac{r_f}{r^3}}\dot{\chi} - \frac{r_f}{r^2}\chi' \\ &\quad - \frac{\ell(\ell+1)(\ell^2+\ell-2)^2 b^6 r^3 + 3b^4(\ell^2+\ell-2)^2 r^2 r_f + 9b^2(\ell^2+\ell-2) r r_f^2 + 9r_f^3}{r^3(b^2(\ell^2+\ell-2)r+3r_f)^2}\chi = 0. \end{aligned} \tag{123}$$

With Eqs. (122) or (123), we confirm that the rest of the equations of motion in the physical and fiducial sectors are satisfied, respectively. Thus, ψ and χ play the role of the master variables in the physical and fiducial sectors, respectively.

B. The case of MTBG

We then focus on the $\ell \geq 2$ modes in the self-accelerating branch of MTBG. In order to eliminate the dependence on h_t and k_t, we introduce the new variables δh and δk as Eq. (94). We also introduce the new variables ψ and χ to replace n_1 and m_1 by

$$\begin{aligned} n_1 = \frac{1}{8r_g}\Big[&2\ell(\ell+1)\sqrt{r^3 r_g}\delta h + 2(6r-5r_g)h_1 - \ell(\ell+1)(6r-5r_g)h_2 + 4\ell(\ell+1)(r-r_g)h_r \\ &+ 4r(r-r_g)h_1' - 2\ell(\ell+1)r(r-r_g)h_2' + 4\sqrt{r^3 r_g}\dot{h}_1 - 2\ell(\ell+1)\sqrt{r^3 r_g}\dot{h}_2 \\ &+ 8r_g n_0 - 4rn_2 + \frac{4\kappa^2}{r}\sqrt{\ell(\ell+1)}\big((\ell^2+\ell-2)r + 3r_g\big)\psi\Big], \\ m_1 = \frac{1}{8r_f}\Big[&2\ell(\ell+1)\sqrt{r^3 r_f}\delta k + 2(6b^2 r-5r_f)k_1 - \ell(\ell+1)(6b^2 r-5r_f)k_2 + 4\ell(\ell+1)(b^2 r-r_f)k_r \\ &+ 4r(b^2 r-r_f)k_1' - 2\ell(\ell+1)r(b^2 r-r_f)k_2' + 4\sqrt{r^3 r_f}\dot{k}_1 - 2\ell(\ell+1)\sqrt{r^3 r_f}\dot{k}_2 \\ &+ 8r_f m_0 - 4b^2 r m_2 + \frac{4\kappa^2}{\alpha^2 r}\sqrt{\ell(\ell+1)}\big((\ell^2+\ell-2)b^2 r + 3r_f\big)\chi\Big]. \end{aligned} \tag{124}$$

After deriving the totally 18 components of the equations of motion for the 18 variables, we fix h_1 and h_2 to 0 by the gauge conditions (33). The equation of motion for λ_0 fixes m_2 as

$$m_2 = -\frac{1}{2}k_1 + \frac{\ell(\ell+1)}{4}k_2 + n_2. \tag{125}$$

The equation of motion for λ_1 is not independent of that for λ_0. The equation of motion for λ_2 fixes k_r as

$$k_r = h_r + \frac{1}{2\ell(\ell+1)}[-6k_1 - 4rk_1' + \ell(\ell+1)(3k_2 + 2rk_2')]. \tag{126}$$

The equations of motion for n_0 and m_0 fix n_0 and m_0, respectively. The compatibility of the equations of motion for h_1 and h_2 fixes n_2. Similarly, the compatibility of the equations of motion for k_1 and k_2 fixes λ_3. The equation of motion for λ_3 provides the constraint relation

$$k_2'' = \frac{2(\ell^2+\ell+18)k_1 + 44rk_1' + \ell(\ell+1)[(\ell^2+\ell-22)k_2 - 22rk_2'] + 8r^2 k_1''}{4\ell(\ell+1)r^2}. \tag{127}$$

The combination of the equations of motion for h_r and k_r provides the evolution equation for k_2

$$\ddot{k}_2 = G_{k_2}, \tag{128}$$

where G_{k_2} represents at most first-order time derivative terms of the perturbation variables. The compatibility of the equations of motion for δh and δk provides the constraint relation

$$\delta k'' = H_{\delta k}, \tag{129}$$

where $H_{\delta k}$ includes at most first-order time derivative terms of the perturbation variables. The compatibility of the equations of motion for ψ and χ provides the constraint relation

$$\chi'' = H_\chi, \tag{130}$$

where H_χ includes at most first-order time and radial derivative terms of the perturbation variables. The equation of motion for n_2 fixes λ_2. The compatibility of the equations of motion for m_2 and n_2 provides a constraint relation denoted by

$$\mathcal{C}_1 = 0, \tag{131}$$

which includes at most first-order time derivative terms of the perturbation variables. The compatibility of the equations of motion for h_1 and k_1 provides

$$k_1'' = H_{k_1}, \tag{132}$$

TABLE III. The manipulations to reduce the equations of motion for the $\ell \geq 2$ modes. EOM means equation of motion.

Manipulation	Output	Remark	# of independent variables
Derive EOMs			18
Fix the gauge	$h_1 = 0$ and $h_2 = 0$		16
EOM for λ_0	Eliminate m_2	EOM for λ_1 is not independent	15
EOM for λ_2	Eliminate k_r		14
EOM for n_0	Eliminate n_0		13
EOM for m_0	Eliminate m_0		12
Combine EOM for h_1 and EOM for h_2	Eliminate n_2		11
Combine EOM for k_1 and EOM for k_2	Eliminate λ_3		10
EOM for λ_3	Eq. (127)	First constraint	9
Combine EOM for h_r and EOM for k_r	Eq. (128)	First evolution equation	8
Combine EOM for δh and EOM for δk	Eq. (129)	Second constraint	7
Combine EOM for ψ and EOM for χ	Eq. (130)	Third constraint	6
Combine EOM for n_2 and EOM for m_2	$\mathcal{C}_1 = 0$	Fourth constraint	5
EOM for n_2 (or EOM for m_2)	Eliminate λ_2		4
Combine EOM for h_1 and EOM for k_1	Eq. (132)	Fifth constraint	3
EOM for h_1 (or EOM for k_1)	$\mathcal{E}_1 = 0$	Second evolution equation	2
EOM for h_r (or EOM for k_r)	$\mathcal{C}_2 = 0$	Sixth constraint	1
EOM for δh (or EOM for δk)	$\mathcal{E}_2 = 0$	Degenerate to $\mathcal{E}_1 = 0$	1
EOM for ψ (or EOM for χ)	$\mathcal{E}_3 = 0$	Degenerate to $\mathcal{E}_1 = 0$ and $\mathcal{E}_2 = 0$	1
Take the difference between $\mathcal{E}_1 = 0$ and $\mathcal{E}_2 = 0$	Eq. (136)	Seventh constraint	0
Take the difference between $\mathcal{E}_2 = 0$ and $\mathcal{E}_3 = 0$	$\mathcal{C}_1 = 0$	Not an independent constraint	0

where H_{k_1} contains at most first-order time derivative terms including $\dot{\psi}''$, $\dot{\chi}''$, $\dot{\delta h}''$, $\dot{\delta k}''$, $\delta h''$, $\delta k''$, ψ'', h_r'', λ_1''', and λ_1''.

With use of the constraint (131), the equations of motion for h_1, δh, and ψ lead to the evolution equations

$$\mathcal{E}_1 = 0, \qquad \mathcal{E}_2 = 0, \qquad \mathcal{E}_3 = 0, \tag{133}$$

which include at most second-order time derivative terms. We find that the evolution equations in Eq. (133) are degenerate with respect to the second-order derivatives of t and can be rewritten as the evolution equation of

$$\begin{aligned}\Upsilon &:= 2\kappa^2\sqrt{\frac{r}{r_g}}\big((\ell^2+\ell-2)r+3r_g\big)(\psi+C_0^2\chi) \\ &\quad - \sqrt{\ell(\ell+1)}r^4(\delta h + C_0^2\alpha^2\delta k) \\ &\quad + 2\sqrt{\frac{1}{\ell(\ell+1)}}C_0^2\alpha^2 r^2\sqrt{\frac{r}{r_g}}\left(\frac{\ell^2+\ell+4}{2}r - r_g\right)k_1.\end{aligned} \tag{134}$$

The equation of motion for h_r leads to the constraint relation

$$\mathcal{C}_2 = 0, \tag{135}$$

which includes at most first-order time derivative terms of the perturbation variables. In the case of two copies of GR with $m = 0$, ψ and χ play the role of the master variables in each sector. Instead, in the case of MTBG with $m \neq 0$, the evolution of the perturbations in the two sectors are coupled to each other.

The degeneracy between $\mathcal{E}_1 = 0$ and $\mathcal{E}_2 = 0$ in Eq. (133) leads to the constraint relation

$$\psi'' = H_\psi, \tag{136}$$

where H_ψ represents at most the first-order time derivative terms of the perturbation variables. Similarly, the degeneracy between $\mathcal{E}_2 = 0$ and $\mathcal{E}_3 = 0$ in Eq. (133) reduces to the constraint relation Eq. (131) and does not produce any more constraint. The manipulations to reduce the equations of motion for the $\ell \geq 2$ modes are summarized in Table III. In the effectively massless case with $b = 1$ and $r_f = r_g$, λ_0 does not appear and hence without loss of generality we may set[5]

$$\lambda_0 = 0. \tag{137}$$

[5] λ_0 may appear in higher-order perturbations even in the effectively massless case. In such a case, imposing the regularity of higher-order perturbations may fix λ_0 uniquely, although this is beyond the scope of the present paper.

Each of the degenerate evolution equations $\mathcal{E}_1 = 0$ ($\mathcal{E}_2 = 0$ or $\mathcal{E}_3 = 0$) still contains the terms with two time and one radial derivatives as $\ddot{\psi}'$. However, by combining $\mathcal{E}_3 = 0$ with $\dot{\mathcal{C}}_1' = 0$, all the two time and one radial derivative terms cancel and the resultant evolution equation

$$\tilde{\mathcal{E}}_3 = 0, \tag{138}$$

is purely a second-order derivative equation with respect to time.

We then replace δh with Υ using Eq. (134). We also relate h_r to other variables through Eq. (131). Then, Eq. (138) reduces to

$$\bar{\mathcal{E}}_3 = 0, \tag{139}$$

which has the structure

$$\ddot{k}_2 + \left(-1 + \frac{r_g}{r}\right) k_2'' + \frac{2r_g^2}{C_0^2\alpha^2(5r_g - 6r)}\left(\frac{r}{r_g}\right)^{\frac{3}{2}}\left[\ddot{\Upsilon} + \left(-1 + \frac{r_g}{r}\right)\Upsilon''\right] = R_{\bar{\mathcal{E}}_3}, \tag{140}$$

where the remaining terms $R_{\bar{\mathcal{E}}_3}$ do not contain second-order derivative terms with respect to time. We note that the evolution equation (128) is not independent of Eq. (139). After eliminating h_r with Eq. (131), the constraint Eq. (135) now turns into an evolution equation, because the $\dot{h}_r$ term in the original (135) turns into second-order derivative terms with respect to time.

After eliminating h_r with Eq. (131), the Eqs. (129), (130), and (136) finally reduce to the elliptic equations

$$\psi'' = K_\psi, \tag{141}$$

$$\chi'' = K_\chi, \tag{142}$$

$$k_1'' = -\frac{2(\ell^2 + \ell + 18)k_1 + 44rk_1' + \ell(\ell+1)[(\ell^2 + \ell - 22)k_2 - 22rk_2'] - 4\ell(\ell+1)r^2k_2''}{8r^2}, \tag{143}$$

where the last equation coincides with Eq. (127), and K_ψ and K_χ in Eqs. (141) and (142) do not contain second-order derivative terms with respect to r besides k_2'' and Υ''. The constraint relation (135) now turns to the evolution equation

$$\bar{\mathcal{C}}_2 = 0, \tag{144}$$

which has the structure

$$\ddot{k}_2 + \left(-1 + \frac{r_g}{r}\right)k_2'' + \frac{r_g^2}{C_0^2\alpha^2(r_g - \frac{\ell^2+\ell+4}{2}r)}\left(\frac{r}{r_g}\right)^{\frac{3}{2}} \times \left(\ddot{\Upsilon} + \left(-1 + \frac{r_g}{r}\right)\Upsilon''\right) = R_{\bar{\mathcal{C}}_2}, \tag{145}$$

where the remaining terms $R_{\bar{\mathcal{C}}_2}$ do not contain second-order derivative terms with respect to time. The combination of (140) and (145) leads to the individual evolution equations for Υ and k_2, which are schematically given by

$$\ddot{\Upsilon} + \left(-1 + \frac{r_g}{r}\right)\Upsilon'' = R_\Upsilon, \tag{146}$$

$$\ddot{k}_2 + \left(-1 + \frac{r_g}{r}\right)k_2'' = R_{k_2}, \tag{147}$$

where R_{k_2} and R_Υ are given by the linear combinations of $R_{\bar{\mathcal{E}}_3}$ and $R_{\bar{\mathcal{C}}_2}$. These equations tell that in the case of the effectively massless case with $b = 1$ and $r_f = r_g$ the two modes Υ and k_2 propagate with the speed of light in the radial directions.

Equation (132) reduces to the constraint

$$\bar{\mathcal{C}}_3 = 0, \tag{148}$$

which includes λ_1'''' and $\ddot{k}_2$. After eliminating $\ddot{k}_2$ by Eq. (147), Eq. (148) can be written solely as the fourth-order differential equation with respect to r for λ_1

$$2r^4\lambda_1'''' + 5r^3\lambda_1''' - \frac{12 + 5\ell(\ell+1)}{2}r^2\lambda_1'' + 6(\ell^2 + \ell + 1)r\lambda_1' + \frac{1}{2}(\ell - 4)\ell(\ell+1)(\ell+5)\lambda_1 = 0, \tag{149}$$

whose general solution is given by

$$\lambda_1 = C_{\lambda_1,1}(t)r^{-\frac{\ell-4}{2}} + C_{\lambda_1,2}(t)r^{\frac{\ell+5}{2}} + C_{\lambda_1,3}(t)r^{-\ell-1} + C_{\lambda_1,4}(t)r^{\ell}, \tag{150}$$

where $C_{\lambda_1,1(t)}$, $C_{\lambda_1,2}(t)$, $C_{\lambda_1,3}(t)$, and $C_{\lambda_1,4}(t)$ are functions of t. Depending on the boundary conditions, these free functions should be chosen appropriately. The fact that in the even-parity sector the equation for the instantaneous mode from the Lagrange multipliers is independent of the other modes is reminiscent of the case of the odd-parity perturbations in the effectively massless case where the equation for the instantaneous mode Λ associated with the Lagrange multiplier is independent of the other variables and can be solved analytically [see Eq. (117) in Ref. [14]]. We note that the solutions of $C_{\lambda_1,3}(t)$ and $C_{\lambda_1,4}(t)$ are those of the Laplace equation in the three-dimensional Euclid space for the ℓ mode. Plugging Eq. (150) into the other independent equations, $C_{\lambda_1,3}(t)$ and $C_{\lambda_1,4}(t)$ do not contribute to the remaining equation of motion. Thus, λ_1 contains only one physical instantaneous mode, in spite of the fact that it follows the fourth-order differential equation (149).

In Eqs. (127), (141), and (142), the right-hand sides still contain Υ'', k_2'', and $\delta k''$. We introduce the new variables $\bar{\psi}$, $\bar{\chi}$, and $\bar{k}_1$ by

$$\psi = \bar{\psi} + \frac{\sqrt{2+3(\ell-1)+(\ell-1)^2}}{2(1+C_0^2\alpha^2)[3r_g+(\ell^2+\ell-2)r]\kappa^2} r^{\frac{7}{2}} r_g^{\frac{1}{2}}[(1+C_0^2\alpha^2)\delta k + \Upsilon] + \frac{C_0^2\alpha^2}{8(1+C_0^2\alpha^2)} \frac{\sqrt{2+3(\ell-1)+(\ell-1)^2}\{4r_g - 2[6+3(\ell-1)+(\ell-1)^2]r\}}{[3r_g+(\ell^2+\ell-2)r]\kappa^2} r^2 k_2, \tag{151}$$

$$\chi = \bar{\chi} + \alpha^2 \frac{\sqrt{2+3(\ell-1)+(\ell-1)^2}}{2(1+C_0^2\alpha^2)[3r_g+(\ell^2+\ell-2)r]\kappa^2} r^{\frac{7}{2}} r_g^{\frac{1}{2}}[(1+C_0^2\alpha^2)\delta k + \Upsilon] + \frac{C_0^2\alpha^4}{8(1+C_0^2\alpha^2)} \frac{\sqrt{2+3(\ell-1)+(\ell-1)^2}\{4r_g - 2[6+3(\ell-1)+(\ell-1)^2]r\}}{[3r_g+(\ell^2+\ell-2)r]\kappa^2} r^2 k_2, \tag{152}$$

$$k_1 = \bar{k}_1 + \frac{\ell(\ell+1)}{2} k_2, \tag{153}$$

and eliminate ψ, χ, and k_1, so that Eqs. (127), (141), and (142) become purely the second-order spatial equations for $\bar{\psi}, \bar{\chi}$, and $\bar{k}_1$. We note that after introducing $\bar{\psi}, \bar{\chi}$, and $\bar{k}_1$, the dependence on δk never shows up in the equations of motion, and turns out to be a gauge mode.

By introducing $\bar{\psi}, \bar{\chi}$, and $\bar{k}_1$, the master equations for the two dynamical modes (146) and (147) turn to be

$$\ddot{\Upsilon} - 2\sqrt{\frac{r_g}{r}}\dot{\Upsilon}' + \left(-1 + \frac{r_g}{r}\right)\Upsilon'' = \bar{R}_\Upsilon. \tag{154}$$

$$\ddot{k}_2 - 2\sqrt{\frac{r_g}{r}}\dot{k}_2' + \left(-1 + \frac{r_g}{r}\right)k_2'' = \bar{R}_{k_2}, \tag{155}$$

where $\bar{R}_\Upsilon$ and $\bar{R}_{k_2}$ contain at most first-order derivatives with respect to the time and radial coordinates. We note that the left-hand sides of (154) and (155) correspond to the GR operators in the even-parity perturbations. Equations (127), (141), (142), and (150) now reduce to three elliptic equations and the physical solution for λ_1,

$$\bar{k}_1'' = -\frac{2(\ell^2+\ell+18)\bar{k}_1 + 44r\bar{k}_1' + 2(\ell+2)(\ell+1)\ell(\ell-1)k_2}{8r^2} \tag{156}$$

$$\bar{\psi}'' = \bar{K}_\psi, \tag{157}$$

$$\bar{\chi}'' = \bar{K}_\chi, \tag{158}$$

$$\lambda_1 = C_{\lambda_1,1}(t) r^{-\frac{\ell-4}{2}} + C_{\lambda_1,2}(t) r^{\frac{\ell+5}{2}}, \tag{159}$$

where $\bar{K}_\psi$ and $\bar{K}_\chi$ include the at most the first-order radial derivatives. Thus, in the effectively massless case, there are two dynamical modes Υ and k_2, while there are four instantaneous modes from $\bar{k}_1, \bar{\psi}, \bar{\chi}$, and λ_1. As in the case of the odd-parity sector [14], in the effectively massless case the two modes in the even-parity sector propagate with the speeds of light at least in the radial direction. Combined with the analysis of the odd-parity sector [14], in the effectively massless case there are four propagating d.o.f. and six instantaneous d.o.f. for each of the $\ell \geq 2$ modes.

If the effectively massless condition is relaxed, there may be more instantaneous modes, while the number of propagating modes should remain the same because of the structure of MTBG. Also as in the case of the odd-parity sector [14], we expect that if the condition is relaxed the speeds of the two propagating modes would differ from each other. We also confirm that the squared angular propagation speeds of the two evolution modes Υ and k_2 read from Eqs. (146) and (147) are positive, indicating that at least in the effectively massless case there is no instability in the angular directions in the large distance limits.

VII. CONCLUSIONS

As a continuation of the previous works [14,17], we have studied even-parity perturbations about static and spherically symmetric Schwarzschild solutions in the self-accelerating branch of MTBG. Before performing the analysis of even-parity perturbations, in Sec. II we have reviewed the Minkowski solutions in the self-accelerating branch of MTBG as the limit of the vanishing effective cosmological constants of the de Sitter solutions written in the spatially-flat FLRW coordinates. We have shown that in general the physical and fiducial sectors do not share the same Minkowski vacuum because under the joint foliation-preserving diffeomorphism invariance the time coordinate cannot describe the proper time in both the physical and fiducial sectors. In order to share the same Minkowski vacuum in both the sectors, the free function which measures the difference in the proper times between the two sectors has to be constant. In Sec. III, we have reviewed the even-parity perturbations on the Schwarzschild solutions in the spatially flat coordinates in the self-accelerating branch of MTBG. For each of the $\ell = 0$, 1, and ≥ 2 modes, we have clarified the gauge transformations under the joint three-dimensional diffeomorphism transformation.

In Sec. IV, we have investigated the solution for the $\ell = 0$ mode. As expected from the structure of MTBG, we confirmed that there is no propagating d.o.f. for $\ell = 0$. The general solution to the $\ell = 0$ mode allows that the mass of a BH in each sector varies with time, while the summation of masses in both the sectors remains constant. However, by requiring that the total mass of matter in each sector inside a star is constant before gravitational collapse, the mass of the BH in each sector has to be constant, which fixes one of the free functions of time to be a constant. Moreover, the requirement that the asymptotic regions of the spacetimes in both the sectors share the same Minkowski vacuum completely fixed the free functions of time. As a consequence, as in the two copies of GR, the general solution to the $\ell = 0$ mode can be absorbed by a redefinition of the parameters in the Schwarzschild background, namely the two gravitational radii. We also confirmed that the consequences of the linearized analysis for the $\ell = 0$ mode can be naturally extended to the nonlinear case as the spherically symmetric but time-dependent vacuum solutions written in the spatially flat coordinates in the self-accelerating branch of MTBG.

In Secs. V and VI, we have analyzed the even-parity perturbations for the $\ell = 1$ and $\ell \geq 2$ modes, respectively. Since the equations of motion for these modes have become quite involved, we have focused on the effectively massless case. First, we have set the constant parameter b which measures the ratio of the proper times between the two sectors to be unity. By imposing that $b = 1$, as in the case of the odd-parity perturbations [14], the effective mass term in the equations of motion for the perturbations vanishes. However, even in the case of $b = 1$, the terms depending on the Lagrange multipliers associated with the second-class constraints still remain nontrivial, and the essential structure of MTBG is maintained. Second, we have also assumed that the background gravitational radii in the physical and fiducial sectors coincide. With these assumptions, the system of the perturbed equations of motion in the even-parity sector has become somewhat tractable, while the essential features of the even-parity perturbations remain nontrivial.

In Sec. V, we have analyzed the even-parity perturbations for the $\ell = 1$ mode in the effectively massless case. In the effectively massless case we have exactly solved the set of equations of motion for the $\ell = 1$ mode. However, the general solution for the $\ell = 1$ mode contains several free functions of time. Under the choice of the gauge (45), as a particular solution obtained by setting $\phi = 0$ where the mode ϕ does not appear in the gauge-invariant combination $n_2 - m_2$, these functions of time could be fixed by imposing the suitable boundary conditions at the spatial infinity, requiring that the leading order corrections to each component of the metrics are suppressed by a factor of r^{-2} compared to the background quantities. The gauge-invariant parts of the resultant solutions to the $\ell = 1$ mode vanish, leaving no observable effect.

In Sec. VI, we have analyzed the even-parity perturbations for the $\ell \geq 2$ modes. In the effectively massless case, in contrast to the cases of the $\ell = 0$ and $\ell = 1$ modes, we have found that there are two dynamical modes and four instantaneous modes. We have also found the equations for the two dynamical modes Υ and k_2, and the elliptic equations for the four instantaneous modes, $\bar{\psi}$, $\bar{\chi}$, $\bar{k}_1$, and λ_1. The equation for λ_1 is independent of other modes, and can be solved analytically. We have shown that among the four solutions of λ_1, two of them are the solutions of the Laplace equation in the three-dimensional Euclid space and do not physically contribute to the dynamics of the other modes. Since the number of the physically independent solutions of λ_1 is two, there is one instantaneous mode arising from λ_1. Combined with the analysis of the odd-parity sector, in the effectively massless case there are four propagating d.o.f. and six instantaneous modes for each of the $\ell \geq 2$ modes in the self-accelerating branch of MTBG.

There are still many remaining issues. The analysis of the even-parity perturbations should be extended by relaxing the assumptions of the effectively massless case. The odd- and even-parity perturbations should be analyzed in the normal branch. As an application of the even-party and odd-parity perturbations, BH quasinormal modes should be analyzed to distinguish BHs in MTBG from the case of the two copies of GR. Finally, numerical simulation techniques for gravitational collapse in the self-accelerating and normal branch of MTBG should be developed. These subjects would be definitively interesting but are left for future studies.

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[1] M. Fierz and W. Pauli, On relativistic wave equations for particles of arbitrary spin in an electromagnetic field, Proc. R. Soc. A **173**, 211 (1939).
[2] D. G. Boulware and S. Deser, Can gravitation have a finite range?, Phys. Rev. D **6**, 3368 (1972).
[3] C. de Rham, G. Gabadadze, and A. J. Tolley, Resummation of massive gravity, Phys. Rev. Lett. **106**, 231101 (2011).
[4] S. Hassan and R. Rosen, Bimetric gravity from ghost-free massive gravity, J. High Energy Phys. 02 (2012) 126.
[5] Y. Yamashita, A. De Felice, and T. Tanaka, Appearance of Boulware–Deser ghost in bigravity with doubly coupled matter, Int. J. Mod. Phys. D **23**, 1443003 (2014).
[6] C. de Rham, L. Heisenberg, and R. H. Ribeiro, Ghosts and matter couplings in massive gravity, bigravity and multi-gravity, Phys. Rev. D **90**, 124042 (2014).
[7] A. E. Gumrukcuoglu, L. Heisenberg, S. Mukohyama, and N. Tanahashi, Cosmology in bimetric theory with an effective composite coupling to matter, J. Cosmol. Astropart. Phys. 04 (2015) 008.
[8] A. De Felice, F. Larrouturou, S. Mukohyama, and M. Oliosi, Minimal theory of bigravity: Construction and cosmology, J. Cosmol. Astropart. Phys. 04 (2021) 015.
[9] S. Garcia-Saenz, A. Held, and J. Zhang, Destabilization of black holes and stars by generalized Proca fields, Phys. Rev. Lett. **127**, 131104 (2021).
[10] H. O. Silva, A. Coates, F. M. Ramazanoğlu, and T. P. Sotiriou, Ghost of vector fields in compact stars, Phys. Rev. D **105**, 024046 (2022).
[11] E. S. Demirboğa, A. Coates, and F. M. Ramazanoğlu, Instability of vectorized stars, Phys. Rev. D **105**, 024057 (2022).
[12] Y. Manita, K. Aoki, T. Fujita, and S. Mukohyama, Spin-2 dark matter from anisotropic Universe in bigravity, Phys. Rev. D **107**, 104007 (2023).
[13] A. De Felice, D. Langlois, S. Mukohyama, K. Noui, and A. Wang, Generalized instantaneous modes in higher-order scalar-tensor theories, Phys. Rev. D **98**, 084024 (2018).
[14] M. Minamitsuji, A. De Felice, S. Mukohyama, and M. Oliosi, Gravitational collapse and odd-parity black hole perturbations in minimal theory of bigravity, Phys. Rev. D **107**, 064070 (2023).
[15] A. De Felice, A. Doll, and S. Mukohyama, A theory of Type-II minimally modified gravity, J. Cosmol. Astropart. Phys. 09 (2020) 034.
[16] A. De Felice, S. Mukohyama, and K. Takahashi, Nonlinear definition of the shadowy mode in higher-order scalar-tensor theories, J. Cosmol. Astropart. Phys. 12 (2021) 020.
[17] M. Minamitsuji, A. De Felice, S. Mukohyama, and M. Oliosi, Static and spherically symmetric general relativity solutions in minimal theory of bigravity, Phys. Rev. D **105**, 123026 (2022).
[18] A. De Felice, S. Mukohyama, and M. C. Pookkillath, Addressing H_0 tension by means of VCDM, Phys. Lett. B **816**, 136201 (2021).
[19] A. De Felice, A. Doll, F. Larrouturou, and S. Mukohyama, Black holes in a type-II minimally modified gravity, J. Cosmol. Astropart. Phys. 03 (**2021**) 004.
[20] A. De Felice, K.-i. Maeda, S. Mukohyama, and M. C. Pookkillath, Comparison of two theories of Type-IIa minimally modified gravity, Phys. Rev. D **106**, 024028 (2022).
[21] J. R. Oppenheimer and H. Snyder, On Continued gravitational contraction, Phys. Rev. **56**, 455 (1939).
[22] Y. Kanai, M. Siino, and A. Hosoya, Gravitational collapse in Painleve-Gullstrand coordinates, Prog. Theor. Phys. **125**, 1053 (2011).
[23] M. Blau, *Lecture notes on general relativity* (2017).