Charging Kerr-Schild spacetimes in higher dimensions

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We study higher dimensional charged Kerr-Schild (KS) spacetimes that can be constructed by a KS transformation of a vacuum solution with an arbitrary cosmological constant, and for which the vector potential is aligned with the KS vector $\boldsymbol{k}$. Focusing on the case of an *expanding* $\boldsymbol{k}$, we first characterize the presence of shear as an obstruction to non-null fields (thereby extending an early no-go result of Myers and Perry). We next obtain the complete family of shearfree solutions. In the twistfree case, they coincide with charged Schwarzschild-Tangherlini-like black holes. Solutions with a twisting $\boldsymbol{k}$ consist of a four-parameter family of higher dimensional charged Taub-NUT metrics with a base space of constant holomorphic sectional curvature. In passing, we identify the configurations for which the test-field limit gives rise to instances of the KS double copy. Finally, it is shown that null fields define a branch of twistfree but shearing solutions, exemplified by the product of a Vaidya-like radiating spacetime with an extra dimension.

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I. INTRODUCTION

A. Background

The celebrated vacuum black hole metric of Kerr [1] can be written as

$$\boldsymbol{g} = \boldsymbol{\eta} - 2H\boldsymbol{k} \otimes \boldsymbol{k}, \tag{1}$$

where the "background" metric,

$$\boldsymbol{\eta} = -\mathrm{d}u^2 + 2\mathrm{d}r(\mathrm{d}u + a\sin^2\theta\mathrm{d}\phi) + (r^2 + a^2\cos^2\theta)\mathrm{d}\theta^2 + (r^2 + a^2)\sin^2\theta\mathrm{d}\phi^2, \tag{2}$$

is flat, the covector field,

$$\boldsymbol{k} = \mathrm{d}u + a\sin^2\theta\mathrm{d}\phi, \tag{3}$$

is null (with respect to both metrics $\boldsymbol{\eta}$ and $\boldsymbol{g}$), and the scalar function H is given by

$$2H = 2H_{\mathrm{Kerr}} \equiv -\frac{2mr}{r^2 + a^2\cos^2\theta}. \tag{4}$$

It is remarkable that the charged Kerr-Newman solution [2] can be described by exactly the same metric (1)–(3) provided one takes a vector potential of the form

$$\boldsymbol{A} = -\frac{er}{r^2 + a^2\cos^2\theta}\boldsymbol{k}, \tag{5}$$

and simply replaces (4) in (1) by the new function[1]

$$2H = 2H_{\mathrm{KN}} \equiv -\frac{2mr - e^2}{r^2 + a^2\cos^2\theta}. \tag{6}$$

One can also easily obtain the Kerr-Newman-(A)dS solution from the Kerr-(A)dS metric with a similar method [where $\boldsymbol{\eta}$ is now the (anti) de–Sitter [(A)dS)] metric, cf. appendix A and references therein].

More generally, line element (1) with $\boldsymbol{\eta}$ flat and $\boldsymbol{k}$ null (but not necessarily of the form (2), (3), and with H *a priori* unspecified) defines the Kerr-Schild (KS) class of spacetimes [5] (see also the earlier [6]), which clearly includes both the Kerr and the Kerr-Newman metrics as particular members. It readily follows from the results of [7] (see also [3]) that, similarly as in the Kerr case, all diverging vacuum KS metrics can be charged by simply taking

$$\boldsymbol{A} = \alpha\boldsymbol{k}, \tag{7}$$

for a suitable spacetime function α, and accordingly redefining H

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[1] In order to match the standard four-dimensional conventions [3,4], both in (6) and in Appendix A we set the gravitational constant $\kappa = 2$ [cf. (9)]. However, in the rest of the paper we find it more convenient to leave it unspecified.

$$H = H_{\rm vac} \to H_{\rm elec}, \tag{8}$$

such as to keep into account the backreaction of the electromagnetic field in the Einstein equation. The form of the metric (1) (in particular $\boldsymbol{k}$) is otherwise unchanged in the charging process.[2]

In view of such a simple relation connecting vacuum KS solutions to their charged counterparts, it is desirable to clarify whether and to what extent a similar method can be used to produce electrovac solutions also in spacetime dimensions other than four. This is particularly relevant in the context of black holes, given that the higher-dimensional analogues of the Kerr black hole, i.e., the Myers-Perry spacetimes, are precisely of the KS form [11] (see also [12] in eight dimensions),[3] and that a similar property holds also in the presence of a cosmological constant Λ [16,17].[4]

An early attempt at obtaining charged rotating black holes within the KS class was already performed in [11].[5] It was concluded there that for metrics with precisely one nonzero spin, the Maxwell and the Einstein equations become incompatible if the KS ansatz is assumed. However, one might hope that such a no-go result could be circumvented in a more general context, e.g., by adding spins or a cosmological constant to the seed black hole. There are indeed several reasons why an ansatz as (1) with (7), (8) seems a promising one in order to construct a charged solution from a vacuum one, and which motivate its further exploration in an arbitrary number of dimensions n. The most important ones, which will be useful throughout the paper, are summarized below.

(1) If $\boldsymbol{g}$ in (1) is an Einstein spacetime, then the null congruence defined by $\boldsymbol{k}$ must be *geodesic*. More generally, geodesicity is equivalent to the much milder condition $R_{ab}k^ak^b = 0$ [15,23]. The geodesic property holds simultaneously in any geometry (1) (i.e., regardless of the choice of H, including the background with $H = 0$). It plays a crucial role in what follows and will be understood from now on. Similarly, also the optical matrix of $\boldsymbol{k}$ (cf. Sec. II B) does not depend on the function H, and must obey the *optical constraint* [15,23] [this is defined in (28) below, cf. [15,23–26], and becomes trivial if $\boldsymbol{k}$ is non-expanding].

(2) For any H, the Riemann tensor of the geometry $\boldsymbol{g}$ is of type II (or more special), aligned with $\boldsymbol{k}$ [15,23] in the classification of [27] (cf. also [26]), which thus also defines a multiple Weyl aligned null direction (mWAND) [28]. Imposing the Einstein equation, this implies, in particular, that the energy-momentum tensor of any geometry ds^2 must satisfy $T_{ab}k^ak^b = 0 = k^aT_{a[b}k_{c]}$ (i.e., it is also of type II or more special). This condition is compatible, in particular, with a Maxwell field of the form $\boldsymbol{F} = \mathrm{d}\boldsymbol{A}$ with (7), which is automatically aligned with $\boldsymbol{k}$ (i.e., $F^a{}_bk^b \propto k^a$ or, equivalently, $k^aF_{a[b}k_{c]} = 0$; cf. also [29]).

(3) The mixed Ricci components $R^a{}_b$ are linear in the function H and its derivatives [30,31]. The same is true for the frame Ricci components [15,23] (cf. Appendix B).

(4) One can easily see that $\sqrt{-g} = \sqrt{-\eta}$ and [with (7)] $g^{ac}g^{bd}F_{cd} = \eta^{ac}\eta^{bd}F_{cd}$, therefore the Maxwell equation for $\boldsymbol{F}$, i.e., $(\sqrt{-g}F^{ab})_{,b} = 0$, is independent of H—this means that one can solve it in the background $\boldsymbol{\eta}$, thereby obtaining a test-field solution $\boldsymbol{F}$ valid for any H. The specific form of H will be subsequently determined by solving the Einstein equation (9), (11), thus arriving at a full Einstein-Maxwell solution which keeps into account the backreaction. In the case $\Lambda = 0$ this was already observed in [11].

The above results were already known for the special case $n = 4$ and are scattered in various papers [5,7–9,30,32–35] (see also [3]).

It is also worth mentioning that the ansatz (1) with (7) is further motivated from the viewpoint of the KS double copy [36] (further comments will be given in Secs. III B 3 and IV A).

B. Summary of results

The present contribution aims at classifying all higher-dimensional electrovac KS spacetimes that can be obtained by charging a vacuum KS spacetime via the ansatz (1), (7) with the redefinition (8), where $\boldsymbol{k}$ is null and *expanding* (cf. Sec. II A), and $\boldsymbol{\eta}$ of constant curvature. After preliminarily defining certain key quantities and setting up the field equations for the KS ansatz in Sec. II, our main results can be summarized as follows (see also Table I).

(i) If $\boldsymbol{k}$ is twisting (Sec. III), then it must be *shearfree* and $\boldsymbol{F} = \mathrm{d}\boldsymbol{A}$ is necessarily *non-null*. This is possible only in *even* dimensions and, as it turns out, the only such solution is given by the special charged Taub-NUT solution (57)–(60). This is specified by four

[2]This can be also understood as a particular case of a KS transformation [8–10]. However, there also exist electrovac KS solutions for which $\boldsymbol{A}$ is not of the form (7) [3,7].

[3]That this is not the case for black rings [13] has been proven in [14,15].

[4]In the latter case, it is understood that in (1) $\boldsymbol{\eta}$ is a spacetime of constant nonzero curvature, such that (1) defines the KS-(A)dS class. For the sake of brevity, however, throughout this paper we shall simply call any $\boldsymbol{g}$ of the form (1) a KS spacetime, provided $\boldsymbol{\eta}$ is of constant (positive, negative or zero) curvature. In order to disregard the trivial (vacuum) case $\boldsymbol{g} = \boldsymbol{\eta}$, it will also be understood that $H \neq 0$.

[5]When the angular momentum vanishes, a static charged black hole (with an arbitrary Λ) has been known for some time [18], and it is easy to see it indeed admits a KS representation (e.g. using Robinson-Trautman coordinates [19], cf. also Sec. IV A). Five-dimensional charged black holes with nonzero angular momentum exist [20–22] which do *not* belong to the KS class [22].

TABLE I. Summary of the $n > 4$ charged KS solutions obtained with the ansatz (1), (7), (8), where $\boldsymbol{k}$ is assumed to be expanding ($\theta \neq 0$). The optical scalars are defined in (18). Note that the solution (75), (76) only represents a particular five-dimensional example for the null field branch $\omega = 0 \neq \sigma$ (which will deserve a separate study elsewhere).

ω	σ	$\boldsymbol{F}$	Solution	n
$\neq 0$	0	Non-null	(57)–(60)	Even
0	0	Non-null	(72), (73)	Any
0	$\neq 0$	Null	(75), (76)	Any

independent parameters μ, ℓ, α_0 and β_0 (roughly corresponding to mass, NUT, electric charge and asymptotic magnetic field strength), in addition to an arbitrary cosmological constant.[6] The base space metric $\boldsymbol{h}$ must be Kähler-Einstein and of constant holomorphic sectional curvature [cf. (C32)]. We observe that the no-go result of [11] was obtained for a KS ansatz based on a shearing $\boldsymbol{k}$, which explains why there is no contradiction with the KS solutions found here.

(ii) If $\boldsymbol{k}$ is twistfree (Sec. IV) there are two possibilities. (i) For zero shear, one is left with a subclass of the Robinson-Trautman electrovac solutions [42], for which $\boldsymbol{F}$ is again *non-null*. It eventually reduces to the solution (72), (73), which describes electrically charged Schwarzschild-Tangherlini black holes [18,43] (see also [44]), characterized by mass and charge parameters μ and α_0, and with a base space of constant curvature. (ii) For nonzero shear, $\boldsymbol{F}$ is instead *null*. Solutions in this branch, which include products of a Vaidya-like radiating spacetime with flat extra dimensions, will be studied in more detail elsewhere.

The appendices contain several auxiliary results. Appendix A reviews the KS form of the four-dimensional Kerr-Newman-(A)dS solution and its "topological" counterparts [45,46]. Appendix B presents the connection (Ricci rotation coefficients) and the Riemann and Ricci tensors [15,23] in an arbitrary adapted frame for KS spacetimes with a geodesic $\boldsymbol{k}$ in n dimensions. Appendix C summarizes basic properties of the higher-dimensional Taub-NUT metrics of [47–49], and reviews the integration of the corresponding Einstein equation in vacuum. A few new observations are also added, in particular concerning the overlap with the KS class. Finally, Appendix D contains the details of the integration of the Einstein-Maxwell equations for the case of a shearfree twisting KS vector field $\boldsymbol{k}$, relevant to the branch of solutions of Sec. III B 3.

II. PRELIMINARIES AND FIELD EQUATIONS

A. Notation

We will consider the Einstein-Maxwell equations in n spacetime dimensions in the form

$$G_{ab} + \Lambda g_{ab} = \kappa T_{ab}, \tag{9}$$

$$\nabla_b F^{ab} = 0, \tag{10}$$

where G_{ab} is the Einstein tensor and

$$T_{ab} = F_{ac} F_b{}^c - \frac{1}{4} g_{ab} F_{cd} F^{cd}. \tag{11}$$

It will be also convenient to define a rescaled cosmological constant as

$$\lambda \equiv \frac{2\Lambda}{(n-1)(n-2)}. \tag{12}$$

We will use a frame adapted to the KS ansatz (1), i.e., a set of n vectors $\boldsymbol{m}_{(a)}$ which consists of two null vectors $\boldsymbol{k} \equiv \boldsymbol{m}_{(0)}$, $\boldsymbol{n} \equiv \boldsymbol{m}_{(1)}$ and $n-2$ orthonormal spacelike vectors $\boldsymbol{m}_{(i)}$, with $a, b \ldots = 0, \ldots, n-1$ while $i, j \ldots = 2, \ldots, n-1$ [26,28]).[7] The Ricci rotation coefficients L_{ab}, N_{ab} and $\overset{i}{M}_{ab}$ are defined by (the following quantities are meant as projections onto the basis vectors) [50]

$$L_{ab} = k_{a;b}, \qquad N_{ab} = n_{a;b}, \qquad \overset{i}{M}_{ab} = m^{(i)}_{a;b}, \tag{13}$$

and satisfy the identities $L_{0a} = N_{1a} = N_{0a} + L_{1a} = \overset{i}{M}_{0a} + L_{ia} = \overset{i}{M}_{1a} + N_{ia} = \overset{i}{M}_{ja} + \overset{j}{M}_{ia} = 0$.

Since by construction $\boldsymbol{k}$ is geodesic (cf. point 1.1 in Sec. I A), with no loss of generality we will also assume it is affinely parametrized, and define an affine parameter r such that

$$k^a \partial_a = \partial_r. \tag{14}$$

By also using a frame parallelly transported along $\boldsymbol{k}$ [41], one thus has

$$L_{i0} = L_{10} = \overset{i}{M}_{j0} = N_{i0} = 0. \tag{15}$$

[6]Vacuum spacetimes with multiple NUT parameters [37] cannot be charged with the procedure employed in this paper precisely because they are (twisting and) shearing [25,38–40] (although some of them belong to the KS class [38]). For the same reason, the charged version of the odd-dimensional solutions of [37] is also ruled out (cf. [41]).

[7]With a slight abuse of notation, we will use the symbol $\boldsymbol{k}$ to denote both the vector field $k^a \partial_a$ and the corresponding covector $k_a \mathrm{d}x^a$ (where $k_a = g_{ab} k^b$). In all cases, it will be clear from the context what is the object under consideration.

Furthermore, it also follows that the rank of the *optical matrix*

$$L_{ij} = k_{a;b} m^a_{(i)} m^b_{(j)}, \tag{16}$$

is a frame-independent property (cf. Sec. 2.2 of [41]).

The symmetric and antisymmetric parts of the optical matrix

$$S_{ij} \equiv L_{(ij)} = \sigma_{ij} + \theta\delta_{ij}, \qquad A_{ij} \equiv L_{[ij]}, \tag{17}$$

where $\sigma_{ii} = 0$, can be used to define the expansion, shear and twist scalars, namely [41,50]

$$\theta \equiv \frac{1}{n-2} S_{ii}, \qquad \sigma^2 \equiv \sigma_{ij}\sigma_{ij}, \qquad \omega^2 \equiv A_{ij}A_{ij}. \tag{18}$$

Covariant derivatives along the frame vectors are denoted as

$$D \equiv k^a \nabla_a, \qquad \Delta \equiv n^a \nabla_a, \qquad \delta_i \equiv m^a_{(i)} \nabla_a. \tag{19}$$

Throughout the paper we will consider only solutions for which $\boldsymbol{k}$ is *expanding*, i.e., from now on we assume $\theta \neq 0$. In addition, $\boldsymbol{k}$ will define a null direction doubly aligned with the Riemann tensor (see point 1.1 in Sec. I A), i.e.,

$$R_{0i0j} = 0, \qquad R_{0ijk} = 0 = R_{010i}. \tag{20}$$

Upon using (15) and (20), a subset of the components of the Ricci identity [41] take the form

$$DL_{1i} = -L_{1j}L_{ji}, \qquad DL_{i1} = -L_{ij}L_{j1}, \tag{21}$$

$$DL_{ij} = -L_{ik}L_{kj}, \tag{22}$$

$$\delta_{[j|}L_{i|k]} = L_{1[j|}L_{i|k]} + L_{i1}L_{[jk]} + L_{il}\overset{l}{M}_{[jk]} + L_{l[j|}\overset{l}{M}_{i|k]}, \tag{23}$$

while the commutators of the derivative operators [51] reduce to

$$\Delta D - D\Delta = L_{11}D + L_{i1}\delta_i, \tag{24}$$

$$\delta_i D - D\delta_i = L_{1i}D + L_{ji}\delta_j, \tag{25}$$

$$\delta_i \Delta - \Delta\delta_i = N_{i1}D + (L_{i1} - L_{1i})\Delta + (N_{ji} + \overset{j}{M}_{i1})\delta_j, \tag{26}$$

$$\delta_{[i}\delta_{j]} = N_{[ij]}D + L_{[ij]}\Delta + \overset{k}{M}_{[ij]}\delta_k. \tag{27}$$

B. Optical matrix

Throughout the paper a frame as described in Sec. II A will be employed. The optical matrix (16) of a vacuum KS seed must obey the optical constraint [15,23]

$$L_{ik}L_{jk} = \frac{L_{lk}L_{lk}}{(n-2)\theta} S_{ij}. \tag{28}$$

Together with the Sachs equation (22), this implies that one can always choose a parallelly transported frame such that L_{ij} takes the block diagonal form [15,23–26]

$$L_{ij} = \begin{pmatrix} \boxed{\mathcal{L}_{(1)}} & & & \\ & \ddots & & \\ & & \boxed{\mathcal{L}_{(p)}} & \\ & & & \boxed{\tilde{\mathcal{L}}} \end{pmatrix}, \tag{29}$$

where the first p blocks are 2×2 and the last block $\tilde{\mathcal{L}}$ is an $(n-2-2p) \times (n-2-2p)$-dimensional diagonal matrix. They read

$$\mathcal{L}_{(\mu)} = \begin{pmatrix} s_{(2\mu)} & A_{2\mu,2\mu+1} \\ -A_{2\mu,2\mu+1} & s_{(2\mu)} \end{pmatrix} \quad (\mu = 1, \ldots, p),$$

$$s_{(2\mu)} = \frac{r}{r^2 + (a^0_{(2\mu)})^2}, \qquad A_{2\mu,2\mu+1} = \frac{a^0_{(2\mu)}}{r^2 + (a^0_{(2\mu)})^2}, \tag{30}$$

$$\tilde{\mathcal{L}} = \frac{1}{r}\,\mathrm{diag}(\underbrace{1, \ldots, 1}_{(m-2p)}, \underbrace{0, \ldots, 0}_{(n-2-m)}), \tag{31}$$

where $0 \leq 2p \leq m \leq n-2$, r is defined in (14), and the functions $a^0_{(2\mu)}$ are independent of r. The integer m denotes the rank of L_{ij} (which must be nonzero since $\boldsymbol{k}$ is expanding), so that L_{ij} is nondegenerate when $m = n-2$.

Using the above expression for L_{ij}, one obtains for the optical scalars (18)

$$(n-2)\theta = 2\sum_{\mu=1}^{p} \frac{r}{r^2 + \left(a^0_{(2\mu)}\right)^2} + \frac{m-2p}{r}, \tag{32}$$

$$\omega^2 = 2\sum_{\mu=1}^{p} \left(\frac{a^0_{(2\mu)}}{r^2 + \left(a^0_{(2\mu)}\right)^2}\right)^2, \tag{33}$$

$$\sigma^2 = 2\sum_{\mu=1}^{p} \left(\frac{r}{r^2 + \left(a^0_{(2\mu)}\right)^2} - \theta\right)^2 + (m-2p)\left(\frac{1}{r} - \theta\right)^2 + (n-2-m)\theta^2. \tag{34}$$

Clearly $\boldsymbol{k}$ is twistfree ($\omega = 0 \Leftrightarrow A_{ij} = 0$) if and only if $p = 0$, so that nonzero twist requires $p \geq 1$ and thus $m \geq 2$, $n \geq 4$ (cf. also [52]). It is shearfree ($\sigma = 0 \Leftrightarrow \sigma_{ij} = 0$) when $m = n - 2$ and additionally either: (i) $p = 0$ (i.e., for Robinson-Trautman spacetimes [19]) or (ii) $(a^0_{(2)})^2 = (a^0_{(4)})^2 = \ldots = (a^0_{(2p)})^2$ and $2p = m = n - 2$, which is clearly possible only when n is even (cf. [15,24] for more details).

C. Maxwell equation

Thanks to (7), the nonzero components of $\boldsymbol{F}$ read

$$F_{01} = D\alpha, \quad F_{ij} = -2\alpha A_{ij}, \quad F_{1i} = -2\alpha L_{[1i]} - \delta_i\alpha. \tag{35}$$

We observe that $\boldsymbol{F}$ is (nonzero and) *null* (cf., e.g., [51,53,54]) if and only if $D\alpha = 0 = A_{ij}$.

The Maxwell equation (10) can thus be written as (cf., e.g., [55,56])

$$D^2\alpha + (n-2)\theta D\alpha + 2\alpha\omega^2 = 0, \tag{36}$$

$$\begin{aligned}&[D\delta_i + (3A_{ji} - \sigma_{ij})\delta_j + (n-3)\theta\delta_i + (L_{1i} - 2L_{i1})D]\alpha \\ &\quad + 2\alpha\Big[\delta_j A_{ji} + (n-4)\theta L_{[1i]} - 2\sigma_{ij}L_{[1j]} - 2A_{ji}L_{j1} \\ &\quad + \overset{k}{M}_{jj}A_{ki} + \overset{k}{M}_{ij}A_{jk}\Big] = 0,\end{aligned} \tag{37}$$

$$\begin{aligned}&\Big(\Delta D + \delta_j\delta_j + 4L_{[1j]}\delta_j + N_{jj}D + \overset{k}{M}_{jj}\delta_k\Big)\alpha \\ &\quad + 2\alpha\Big(2L_{[1j]}L_{[1j]} + \delta_j L_{[1j]} + \overset{k}{M}_{jj}L_{[1k]} + N_{kj}A_{jk}\Big) = 0,\end{aligned} \tag{38}$$

where in (37) we have used (21) to get rid of a term proportional to $DL_{[1i]}$. Using (B1)–(B4) and (B5), it is easy to see that the components (36)–(38) of the Maxwell equation do not contain H (although some of the individual terms appearing above do) and thus reduce to equations in the background space $\boldsymbol{\eta}$, as expected from the comments in point 1.1 in Sec. I A.

For later use, let us note that (36) can be alternatively rewritten as $D^2 \ln\alpha + (D\ln\alpha)[D\ln\alpha + (n-2)\theta] + 2\omega^2 = 0$, from which it follows

$$\begin{aligned}&(D\ln\alpha)^2[(D\ln\alpha)^2 + 2\omega^2]^2 \\ &\quad = \left\{\frac{1}{2}D[(D\ln\alpha)^2] + (n-2)\theta(D\ln\alpha)^2\right\}^2.\end{aligned} \tag{39}$$

1. Twisting case ($p \geq 1$)

Equation (36) can be used to fix the r dependence of α. After some manipulations one arrives at

$$\alpha = \frac{\beta}{r^{m-2p-1}}\prod_{\mu=1}^{p}\frac{1}{r^2 + (a^0_{2\mu})^2}, \tag{40}$$

where the auxiliary function β is defined for even and odd m as, respectively,

$$\beta = \alpha_0 + \beta_0\sum_{\mu=0}^{p}\frac{\mathcal{A}^0_\mu}{m-1-2\mu}r^{m-1-2\mu} \quad (m \geq 2 \text{ even}), \tag{41}$$

$$\beta = \alpha_0 + \beta_0\left(\mathcal{A}^0_{\frac{m-1}{2}}\ln r + \sum_{\substack{\mu=0\\(2\mu\neq m-1)}}^{p}\frac{\mathcal{A}^0_\mu}{m-1-2\mu}r^{m-1-2\mu}\right) \quad (m \geq 3 \text{ odd}), \tag{42}$$

with

$$\begin{aligned}&\mathcal{A}^0_0 = 1, \\ &\mathcal{A}^0_\mu = \sum_{\nu_1<\nu_2<\ldots<\nu_\mu}(a^0_{(2\nu_1)})^2(a^0_{(2\nu_2)})^2\ldots(a^0_{(2\nu_\mu)})^2 \ (\mu = 1, \ldots, p),\end{aligned} \tag{43}$$

and α_0 and β_0 are two r-independent integration functions.

2. Twistfree case ($p = 0$)

When there is no twist ($p = 0$) a solution to (36) can be written simply as

$$\alpha = \alpha_0 r^{1-m} + \frac{\beta_0}{m-1} \quad (m \neq 1), \tag{44}$$

$$\alpha = \alpha_0 + \beta_0\ln r \quad (m = 1). \tag{45}$$

D. Einstein equation

Using (35), one finds $F_{cd}F^{cd} = -2(D\alpha)^2 + 4\alpha^2\omega^2$ and thus the nonzero components of the energy-momentum tensor (11) take the form

$$\begin{aligned}&T_{01} = -\frac{1}{2}(D\alpha)^2 - \alpha^2\omega^2, \\ &T_{ij} = 4\alpha^2 A_{ik}A_{jk} - \frac{1}{2}[-(D\alpha)^2 + 2\alpha^2\omega^2]\delta_{ij},\end{aligned} \tag{46}$$

$$T_{1i} = -(2\alpha L_{[1i]} + \delta_i\alpha)D\alpha + 2\alpha A_{ij}(2\alpha L_{[1j]} + \delta_j\alpha), \tag{47}$$

$$T_{11} = (2\alpha L_{[1i]} + \delta_i\alpha)(2\alpha L_{[1i]} + \delta_i\alpha). \tag{48}$$

Using (B14) with (28) and (46), the (spatial) trace and the tracefree part of the (ij) component of the Einstein equation (9) read, respectively,

$$2DH + 2H(n-2)\theta - 2H\frac{L_{mn}L_{mn}}{(n-2)\theta} = -\frac{\kappa}{(n-2)\theta}[2\alpha^2\omega^2 + (D\alpha)^2], \quad (49)$$

$$S_{ij}\frac{(D\alpha)^2 + 2\alpha^2\omega^2}{(n-2)\theta} = 4\alpha^2 A_{ik}A_{jk} + \frac{\delta_{ij}}{n-2}[(D\alpha)^2 - 2\alpha^2\omega^2], \quad (50)$$

while the (01) component is satisfied identically as a consequence of the Bianchi identity, and can thus from now on be omitted.

Owing to the block structure (29)–(31) of the matrix L_{ij}, Eq. (50) contains only diagonal terms, which give rise to different conditions depending on what component one is looking at. If $p \geq 1$, from the block $\mathcal{L}_{(1)}$ one obtains

$$\frac{r}{r^2 + (a^0_{(2)})^2}\frac{(D\alpha)^2 + 2\alpha^2\omega^2}{(n-2)\theta} = 4\alpha^2\left(\frac{a^0_{(2)}}{r^2 + (a^0_{(2)})^2}\right)^2 + \frac{1}{n-2}[(D\alpha)^2 - 2\alpha^2\omega^2] \quad (p \geq 1), \quad (51)$$

with (32) and (33), and similarly for the remaining blocks $\mathcal{L}_{(\mu)}$, up to $\mathcal{L}_{(p)}$.

Nonzero and zero entries of $\tilde{\mathcal{L}}$ are present, respectively, when $m > 2p$ and $m < n-2$, and give rise to

$$(D\alpha)^2 - 2\alpha^2\omega^2 = \frac{1}{r\theta}[(D\alpha)^2 + 2\alpha^2\omega^2] \quad (m > 2p), \quad (52)$$

$$(D\alpha)^2 - 2\alpha^2\omega^2 = 0 \quad (m < n-2). \quad (53)$$

For the time being, we do not need to display the remaining components [i.e., the $(1i)$ and (11) ones] of the Einstein equation.

III. TWISTING SOLUTIONS ($n \geq 4$, $p \geq 1$)

As mentioned in Sec. II B, twist is nonzero if and only if $p \geq 1$, which will thus be assumed throughout the present section. Equations (52) and (53) are clearly incompatible when $\alpha^2\omega^2 \neq 0$, therefore we have only to consider here the following possible cases [cf. (31)]: (i) $m = n-2 > 2p$, i.e., $\tilde{\mathcal{L}} = r^{-1}\mathrm{diag}(1, \ldots, 1)$; (ii) $m = n-2 = 2p$, i.e., the block $\tilde{\mathcal{L}}$ is absent in (29) (n even); (iii) $m = 2p < n-2$, i.e., $\tilde{\mathcal{L}} = \mathrm{diag}(0, \ldots, 0)$ (m even). These are analyzed in what follows. Note that necessarily $n > 4$ in cases (i) and (iii), and recall that L_{ij} is nondegenerate if and only if $m = n-2$, i.e., in cases (i) and (ii). The fact that the standard case $n = 4$ [7] is possible only in branch (ii) is a consequence of the Goldberg-Sachs theorem [3,57–59].

A. Nondegenerate L_{ij} with $2p < n-2$

Combining (51) and (52) one obtains

$$(D\alpha)^2 + 2\alpha^2\omega^2 = -4\alpha^2\frac{(n-2)\theta r}{r^2 + (a^0_{(2)})^2}, \quad (54)$$

which is clearly inconsistent since the lhs and the rhs have opposite signs [notice that here (32) gives $\theta r > 0$]. Therefore this case cannot occur. We observe that we arrived at this conclusion without using the Maxwell equation.

B. Nondegenerate L_{ij} with $2p = n-2$ (n even)

Here one has $p = 1 \Leftrightarrow m = 2 \Leftrightarrow n = 4$, and since the four-dimensional case has been already elucidated [7],[8] we can focus hereafter on the case $p \geq 2$. Along with (51) we thus have a similar equation with $a^0_{(2)}$ replaced by $a^0_{(4)}$. If $a^0_{(2)} \neq a^0_{(4)}$, a linear combination of those equations gives

$$(D\ln\alpha)^2 - 2\omega^2 = -\frac{4(n-2)r^2}{(r^2 + (a^0_{(2)})^2)(r^2 + (a^0_{(4)})^2)} = 0 \quad (\text{if } a^0_{(2)} \neq a^0_{(4)}). \quad (55)$$

Further analysis requires us to consider three possible subcases separately, depending on the multiplicity of the functions $(a^0_{(2\mu)})^2$.

1. Case with at least three distinct $(a^0_{(2\mu)})^2$ ($n \geq 8$)

Let us assume $p \geq 3$ and that there exist at least three distinct $(a^0_{(2\mu)})^2$, say (up to a relabeling of the frame vectors) $a^0_{(2)}$, $a^0_{(4)}$ and $a^0_{(6)}$. Since the lhs of (55) is independent of the choice of the block, considering the remaining two equations obtained by performing the substitutions of indices $(24) \to (46) \to (62)$ in (55), one concludes that $(r^2 + (a^0_{(2)})^2)(r^2 + (a^0_{(4)})^2) = (r^2 + (a^0_{(4)})^2)(r^2 + (a^0_{(6)})^2) = (r^2 + (a^0_{(6)})^2)(r^2 + (a^0_{(2)})^2)$. This is possible only if $(a^0_{(2)})^2 = (a^0_{(4)})^2 = (a^0_{(6)})^2$, thus contradicting our assumption. We have thus proven that this case cannot occur, i.e., there can be at most two distinct $(a^0_{(2\mu)})^2$.

2. Case with two distinct $(a^0_{(2\mu)})^2$ ($n \geq 6$)

Let us assume $p \geq 2$ and that there exist precisely two distinct $(a^0_{(2\mu)})^2$, say $a^0_{(2)} \neq a^0_{(4)}$, with respective multiplicities p_1 and p_2 (such that $2(p_1 + p_2) = 2p = n-2$). Equations (32) and (33) thus give

[8] To be precise, only the case $\Lambda = 0$ was studied in [7].

$$(n-2)\theta = 2r\left(\frac{p_1}{r^2+(a^0_{(2)})^2}+\frac{p_2}{r^2+(a^0_{(4)})^2}\right),$$
$$\omega^2 = 2p_1\left(\frac{a^0_{(2)}}{r^2+(a^0_{(2)})^2}\right)^2 + 2p_2\left(\frac{a^0_{(4)}}{r^2+(a^0_{(4)})^2}\right)^2. \tag{56}$$

Substituting (55) with (56) into (39) leads to an inconsistency, therefore this case can also be ruled out.

3. Shearfree case [all the $(a^0_{(2\mu)})^2$ coincide]

The last nondegenerate case to consider arises when $2p = m = n-2$ and all the functions $(a^0_{(2\mu)})^2$ coincide. As mentioned in Sec. II B, the null vector field $\boldsymbol{k}$ becomes then shearfree, while still expanding and twisting. In this case, no inconsistencies as the one found above arise, which means one can proceed with the full integration of the Einstein-Maxwell equations. According to the charging ansatz (1), (7), (8), one thus needs to start from a vacuum KS solution with a shearfree twisting $\boldsymbol{k}$, which is necessarily an mWAND (recall points 1.1 and 1.1 in Sec. I A).

For $n > 4$, the first example of a Ricci-flat spacetime admitting a shearfree twisting mWAND was identified for $n = 6$ in [25] (cf. also [26]) among the Taub-NUT vacua [47–49] (see also [37,60–63]). Subsequently, all $\Lambda = 0$ vacua possessing a twisting shearfree mWAND were obtained in six dimensions in [38], and in $n \geq 6$ (even) dimensions (including Λ) in [39] (see also [40] for related results).[9] As it turns out [39], they coincide with the Taub-NUT metrics of [47–49]. These are reviewed in Appendix C, from which it follows that the only KS vacua possessing a shearfree, twisting KS vector field $\boldsymbol{k}$ are given by the special Taub-NUT metrics (C33) with (C32). Then, using those metrics as vacuum seeds, the integration of the Einstein-Maxwell equations resulting from (1), (7), (8) is straightforward but lengthy, and we relegate technicalities to Appendix D.

As it turns out, the only $n > 4$ charged KS solution admitting an expanding, twisting and shearfree KS vector field is given by [cf. (D1), (D2), (D11), (D16), (D17), (D21)]

$$\boldsymbol{g} = \mathrm{d}r\otimes\boldsymbol{k} + \boldsymbol{k}\otimes\mathrm{d}r + (r^2+\ell^2)(\boldsymbol{h}+\lambda\boldsymbol{k}\otimes\boldsymbol{k}) + r\frac{\mu_0-\kappa f(r)}{(r^2+\ell^2)^{\frac{n-2}{2}}}\boldsymbol{k}\otimes\boldsymbol{k}, \qquad \boldsymbol{k} = \mathrm{d}u - 2\boldsymbol{Z}, \tag{57}$$

$$\boldsymbol{F} = \alpha'\mathrm{d}r\wedge\boldsymbol{k} - 2\alpha(r)\boldsymbol{\mathcal{F}}, \qquad \boldsymbol{\mathcal{F}} \equiv \mathrm{d}\boldsymbol{Z}, \tag{58}$$

where $\boldsymbol{h} = h_{\alpha\beta}(x)\mathrm{d}x^\alpha\mathrm{d}x^\beta$ is a Kähler-Einstein metric of constant holomorphic sectional curvature [64–66], cf. (C32) (parametrized by the coordinates x^α with $\alpha = 1,\ldots,n-2$, denoted collectively as x), $\boldsymbol{Z} = Z_\alpha(x)\mathrm{d}x^\alpha$ is a 1-form which lives in the base space, with Kähler 2-form $\boldsymbol{\mathcal{F}}$, and $\alpha' \equiv \mathrm{d}\alpha/\mathrm{d}r$ [while λ was defined in (12)]. The functions α and f are given by

$$\alpha = \frac{r}{(r^2+\ell^2)^{\frac{n-2}{2}}}\left[\alpha_0 + \beta_0\sum_{\mu=0}^{\frac{n-2}{2}}\binom{\frac{n-2}{2}}{\mu}\frac{\ell^{2\mu}}{n-3-2\mu}r^{n-3-2\mu}\right], \tag{59}$$

$$f' = -r^{-2}(r^2+\ell^2)^{\frac{n-4}{2}}\left[2\alpha^2\ell^2 + \frac{1}{n-2}(r^2+\ell^2)^2(\alpha')^2\right], \tag{60}$$

and μ_0, α_0, β_0 are integration constants [see also (D22), (D23)]. The above spacetime is of Weyl type D. From (B6)–(B12) it follows that for $r\to\infty$ all the components of the Weyl tensor fall off as r^{1-n} or faster, which implies that these metrics are locally asymptotically (A)dS [67] (or locally asymptotically flat if $\lambda = 0$).

Starting form a stationary Taub-NUT ansatz, for $n > 4$ such kind of solutions (but generically not in the KS class) were constructed in [68] in the special case $\beta_0 = 0$, and in full generality in [69] (see also [70]). In the limit $n = 4$ one recovers solutions obtained in [71–73]. While we refer to [68–70,74] for a thorough discussion of the above solutions as well as their Euclidean counterparts, let us just mention that, for $r\to\infty$, the asymptotic behavior of $\boldsymbol{g}$ and $\boldsymbol{F}$ is determined by

$$2H = -\kappa\frac{2\beta_0^2\ell^2}{(n-3)^2(n-5)}\frac{1}{r^2}+\ldots,$$
$$\alpha = \frac{\beta_0}{n-3}+\frac{n-2}{(n-3)(n-5)}\frac{\beta_0\ell^2}{r^2}+\ldots \quad (n>4), \tag{61}$$

$$2H = -\frac{\mu_0}{r}+\ldots, \quad \alpha = \beta_0+\frac{\alpha_0}{r}+\ldots \quad (n=4). \tag{62}$$

Elementary dimensional analysis reveals that μ_0 and α_0^2 enter H at the orders r^{3-n} and $r^{2(3-n)}$, respectively [while α_0 enters α at the "Coulombian" order r^{3-n}, cf. (59)], and thus both give rise to subleading terms when $n > 4$ (and $\beta_0 \neq 0$). From (58) and (61) one finds that the quadratic electromagnetic invariant behaves as

$$F_{ab}F^{ab} = \frac{4(n-2)\beta_0^2\ell^2}{(n-3)^2}\frac{1}{r^4}+\ldots \quad (n>4), \tag{63}$$

[9] To be precise, a further condition on the asymptotic behavior of the Weyl tensor was imposed in [38], which however is irrelevant from the viewpoint of the present discussion, for it is obeyed by all shearfree KS vacua [15]. Such a condition was not assumed in [39], where the assumptions are, in fact, slightly milder than the existence of an mWAND. It is also worth noticing that the solutions of [49] contain also some (non-KS) vacua for which the shearfree, twisting congruence is not an mWAND (Appendix C 2 a) or not even a WAND (Appendix C 2 b).

while for $n = 4$ both β_0^2 and α_0^2 contribute at order r^{-4}. The constants μ_0, α_0 and β_0 can thus be related to mass, electric charge and asymptotic magnetic field strength (cf. [74] for related comments), respectively.

Let us further note that, in the weak-field limit, the spacetime metric is given by (57) with $\kappa = 0$ [which corresponds to the vacuum geometry (C33)], and (7) with (59) thus describes a test electromagnetic field solving the Maxwell equation in that background. For $\lambda = 0 = \beta_0$, the same test-field solution can be obtained by using the Killing vector field ∂_u as a vector potential [75,76]. In this case H and α have the same functional form, which provides an example of the KS double copy [36], whereas this does not hold when $\beta_0 \neq 0$.

Solution for $n = 6$. For definiteness, let us present the explicit form of the above solution in the case $n = 6$. As discussed in Appendix C 2 a, if $n = 6$ and if we assume $\lambda > 0$, then the base space must be $\mathbb{C}P^2$ with the Fubini-Study metric [64–66], namely (cf. also the real coordinates used, e.g., in [77])

$$\boldsymbol{h} = P^{-2}\left\{\mathrm{d}\rho^2 + \frac{\rho^2}{4}[(\mathrm{d}\psi + \cos\theta\mathrm{d}\phi)^2 + P(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)]\right\}, \qquad P = 1 + \frac{a^2}{6}\rho^2, \tag{64}$$

where a is a real constant related to the four-dimensional Ricci scalar by $\tilde{R} = 4a^2$. Indeed metric (64) satisfies (C32) with $\lambda > 0$ if and only if

$$a^2 = 6\lambda\ell^2, \tag{65}$$

which corresponds to the second of (C31).

The 1-form $\boldsymbol{Z}$ can be taken to be

$$\boldsymbol{Z} = \ell P^{-1}\frac{\rho^2}{4}(\mathrm{d}\psi + \cos\theta\mathrm{d}\phi), \tag{66}$$

and the full six-dimensional KS solution is then given by (7), (57) with [from (59), (60)]

$$\alpha = \frac{r^4}{(r^2+\ell^2)^2}\left[\frac{\alpha_0}{r^3} + \beta_0\left(\frac{1}{3} + \frac{2\ell^2}{r^2} - \frac{\ell^4}{r^4}\right)\right], \tag{67}$$

$$f = \frac{9\alpha_0^2(3r^2+\ell^2) + 96\alpha_0\beta_0\ell^2r^3 + 8\beta_0^2\ell^2(-r^6 + 15\ell^2r^4 + 9\ell^4r^2 + 9\ell^6)}{36r(r^2+\ell^2)^2}. \tag{68}$$

(A further additive constant in f has been omitted since it simply amounts to a redefinition of μ_0.) In the limit $\lambda = 0$ the base space becomes flat [thanks to (65)] and the above spacetime becomes a KS metric with a Minkowski background. The negative curvature version of the above solution (i.e., with base space D^2 and $\lambda < 0$) is obtained by replacing $a \mapsto ia$. A similar example with $\mathbb{C}P^2$ base and $\lambda < 0$ (thus not a KS metric) was presented in [69] (see also [68,70]).

C. Degenerate L_{ij} ($2p = m$)

First, by plugging (53) into (51) and into the corresponding equations for $\mu = 2, \ldots, p$ (i.e., with $a^0_{(2)}$ replaced by $a^0_{(2\mu)}$) one concludes that $a^0_{(2)} = a^0_{(4)} = \ldots = a^0_{2p}$. Then (53) with (33) gives

$$(D\ln\alpha)^2 = 2\omega^2 = 4p\left(\frac{a^0_{(2)}}{r^2 + (a^0_{(2)})^2}\right)^2, \tag{69}$$

which is incompatible with (39) [to see this one should also note that, thanks to (32), here $(n-2)\theta = 2pr/(r^2 + (a^0_{(2)})^2)$]. Therefore this case cannot occur.

IV. TWISTFREE SOLUTIONS ($p = 0$)

We now move to solutions for which $\boldsymbol{k}$ is twistfree. In this case Eq. (51) is absent, while (52) and (53) (with $\omega = 0$) imply that either $m = n-2$ (i.e., $L_{ij} = S_{ij} = \theta\delta_{ij}$ is nondegenerate, with $\theta = 1/r$) or $D\alpha = 0$. In the former case $\boldsymbol{k}$ is shearfree and the spacetime thus belongs to the Robinson-Trautman class, while in the latter case we can assume, without loss of generality, that $\boldsymbol{k}$ is shearing (or else we would be again in the Robinson-Trautman branch). Let us study these two possibilities separately.

A. Robinson-Trautman solutions ($m = n-2$)

Since $m = n-2$, eqs. (44), (45) give

$$\alpha = \alpha_0 r^{3-n} + \frac{\beta_0}{n-3} \quad (n \geq 4), \tag{70}$$

$$\alpha = \alpha_0 + \beta_0 \ln r \quad (n = 3). \tag{71}$$

The case $n = 3$ has been fully explored in [78] (cf. also [79] and references therein), while for $n = 4$ we refer again the reader to [7]. For $n \geq 5$, the complete family of Robinson-Trautman electrovac solutions has been obtained in [42] under the assumption that $\boldsymbol{F}$ is aligned with the

Robinson-Trautman null vector field. In the case considered here, the Robinson-Trautman vector field coincides with the KS one, i.e. $\boldsymbol{k}$, which is indeed aligned with $\boldsymbol{F}$ by construction [Eq. (35)]. It follows that charged KS solutions of the Robinson-Trautman class must be a subset of the solutions of [42]. Among those, the only ones that are also KS reduce to the electrically charged Schwarzschild-Tangherlini metric [18] and its extensions with hyperbolic or planar symmetry [43] (this was noted in [15] in the vacuum case).[10] Without the need of any further calculations we thus arrive at

$$\boldsymbol{g} = 2\mathrm{d}u\mathrm{d}r + r^2\boldsymbol{h} - \left(K - \lambda r^2 - \frac{\mu_0}{r^{n-3}} + \kappa\frac{(n-3)\alpha_0^2}{n-2}\frac{1}{r^{2(n-3)}}\right)\mathrm{d}u^2, \tag{72}$$

$$\boldsymbol{F} = -\frac{(n-3)\alpha_0}{r^{n-2}}\mathrm{d}r \wedge \mathrm{d}u, \tag{73}$$

where λ, μ_0 and α_0 are constants proportional, respectively, to Λ [cf. (12)], mass and electric charge, and $\boldsymbol{h} = h_{\alpha\beta}(x)\mathrm{d}x^\alpha\mathrm{d}x^\beta$ is an $(n-2)$-dimensional Riemannian metric of constant curvature, whose Ricci scalar is normalized as $\mathcal{R} = K(D-2)(D-3)$ with $K = \pm 1, 0$ [β_0 in (70) becomes also a constant in this case and thus a purely gauge term]. Here $\boldsymbol{k} = \mathrm{d}u$, $2H = -\mu_0 r^{3-n} + \kappa\frac{(n-3)\alpha_0^2}{n-2}r^{2(3-n)}$, and the constant curvature background metric $\boldsymbol{\eta}$ is obtained by setting $\mu_0 = 0 = \alpha_0$. For an appropriate parameter range, these solutions describe electrically charged black holes [44]. These spacetimes are of Weyl type D.

Similarly as in Sec. III B 3, the weak-field limit corresponds to setting $\kappa = 0$ in (72), in which case (73) (i.e., $\boldsymbol{A} = \alpha_0 r^{3-n}\boldsymbol{k}$) represents a test electric field living in an Einstein spacetime. For $\lambda = 0$, it can be produced with the method of [75,76] owing to the presence of a Killing vector field ∂_u. It was discussed in the context of the KS double copy in [36] (see also [81,82] for the case $\lambda \neq 0$, and [83] for related comments).

Let us further note that the Robinson-Trautman solutions of [15] contain an additional magnetic branch (in even dimensions), which does not belong to the charged KS class because it has $F_{ij} \neq 0$ and yet $A_{ij} = 0$, which is clearly incompatible with (35) (in other words, in that case $\boldsymbol{A}$ must contain also a spatial component and cannot thus be simply proportional to $\boldsymbol{k}$, as we have assumed). Indeed $F_{ij} = 0$ in (73).

[10]This follows from the fact that for Robinson-Trautman charged metrics which are also KS, the frame spatial components C_{ijkl} of the Weyl tensor fall as r^{1-n} or faster as $r \to \infty$ [as follows from (B9) with (49)], thus forcing the base space to be of constant curvature [19,26,80].

B. Shearing solutions with $D\alpha = 0$ ($n \geq 5$)

For $n = 3$ all null geodesic congruences are shearfree [52], while for $n = 4$ the same is true for KS congruences, as a consequence of the Goldberg-Sachs theorem [3,57–59] (cf. also [7]). Therefore here we can assume $n \geq 5$.

For this class of solutions one simply has

$$\alpha = \frac{\beta_0}{m-1}, \tag{74}$$

which is compatible with (44), (45). Since this is the only case corresponding to a *null* Maxwell field (cf. Sec. II C), this branch will be studied in detail elsewhere. Here we only point out that it is not empty by presenting an explicit five-dimensional example in what follows.

An example for $n = 5$ ($\lambda = 0$). A simple solution with $\lambda = 0$ in five-dimension (and $m = 2$) can be constructed by taking a direct product of a Vaidya-like spacetime obtained in [84] (see also related comments in [85]) with a flat extra dimension, which gives rise to

$$\boldsymbol{g} = 2\mathrm{d}u\mathrm{d}r + r^2(\mathrm{d}x^2 + \mathrm{d}y^2) + \mathrm{d}z^2 + \frac{\mu_0 + h(u)}{r}\mathrm{d}u^2,$$
$$\dot{h} = \kappa(a_1^2 + a_2^2), \tag{75}$$

$$\boldsymbol{F} = [a_1(u)\mathrm{d}x + a_2(u)\mathrm{d}y] \wedge \mathrm{d}u, \tag{76}$$

where μ_0 is a constant, a_1, a_2 are arbitrary functions of the advanced time u, and $\dot{h} \equiv \mathrm{d}h/\mathrm{d}u$. The radiative (null) field (76) [or (7) with $\alpha = a_1(u)x + a_2(u)y$] is responsible for the time dependence of line element and corresponds to a pure radiation energy-momentum tensor given by

$$\boldsymbol{T} = \frac{a_1^2 + a_2^2}{r^2}\mathrm{d}u^2. \tag{77}$$

This can be interpreted as an energy flux along $k^a\partial_a = \partial_r$, which produces (loosely speaking) a mass gain due to incoming radiation as u evolves.[11] We have checked that the above spacetime does not admit any mWAND distinct from $\boldsymbol{k}$ and is therefore of Weyl type II. However, in regions where $r(h + \mu_0) < 0$ there exists a unique Weyl aligned null direction (WAND) of multiplicity 1 [26,28] defined by $\partial_u - \frac{1}{2}(\zeta^2 + \frac{\mu_0 + h}{r})\partial_r + \zeta\partial_z$ with $\zeta^2 = -\frac{r\dot{h}}{3(h+\mu_0)}$. Metric (75) is flat iff $a_1 = 0 = a_2$ and $\mu_0 = 0$.

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[11]Alternatively, after the replacement $u \mapsto -u$, one can interpret the solution as describing emission of radiation (and mass loss) as the retarded time u evolves (cf., e.g., [85]).

APPENDIX A: THE KERR-NEWMAN-(A)DS SOLUTION IN KS FORM ($n = 4$)

1. Spherical solution

Similarly as described in Sec. I in the Kerr-Newman case, the four-dimensional Kerr-Newman-(A)dS solution in KS coordinates can be obtained with the same method by "charging" the Kerr-(A)dS metric. Starting from the vacuum line element in KS coordinates given in [17], one easily arrives at the Einstein-Maxwell solution

$$\boldsymbol{g} = \boldsymbol{\eta} + \frac{2mr - e^2}{\rho^2}\boldsymbol{k} \otimes \boldsymbol{k}, \tag{A1}$$

where[12]

$$\boldsymbol{\eta} = -\frac{\Delta_\theta}{\Xi}(1 - \lambda r^2)\mathrm{d}t^2 + \frac{\rho^2}{(1 - \lambda r^2)(r^2 + a^2)}\mathrm{d}r^2 + \frac{\rho^2}{\Delta_\theta}\mathrm{d}\theta^2 + \frac{r^2 + a^2}{\Xi}\sin^2\theta \mathrm{d}\phi^2, \tag{A2}$$

$$\Delta_\theta = 1 + \lambda a^2\cos^2\theta, \quad \rho^2 = r^2 + a^2\cos^2\theta, \quad \Xi = 1 + \lambda a^2, \quad \lambda = \frac{\Lambda}{3}, \tag{A3}$$

is a suitable form of the (A)dS metric, and

$$\boldsymbol{k} = \frac{\Delta_\theta}{\Xi}\mathrm{d}t + \frac{\rho^2}{(1 - \lambda r^2)(r^2 + a^2)}\mathrm{d}r - \frac{a\sin^2\theta}{\Xi}\mathrm{d}\phi, \tag{A4}$$

represents an affinely parametrized geodesic null vector field. The vector potential corresponding to metric (A1) is given by[13]

$$\boldsymbol{A} = -\frac{er}{\rho^2}\boldsymbol{k}. \tag{A5}$$

For $\lambda = 0$ the above solution reduces to (1)–(5) upon redefining the coordinates as $\mathrm{d}t \mapsto \mathrm{d}u - \mathrm{d}r$, $\mathrm{d}\phi \mapsto -\mathrm{d}\phi - a\mathrm{d}r/(r^2 + a^2)$.

The Kerr-Newman-(A)dS solution (A1)–(A5) is usually presented in Boyer-Lindquist-type coordinates. These are defined by (cf. [17] in the vacuum case)

$$\mathrm{d}t = \frac{\mathrm{d}\tau}{\Xi} + \frac{2mr - e^2}{(1 - \lambda r^2)\Delta_r}\mathrm{d}r, \quad \mathrm{d}\phi = \mathrm{d}\varphi - a\lambda\frac{\mathrm{d}\tau}{\Xi} + a\frac{2mr - e^2}{(r^2 + a^2)\Delta_r}\mathrm{d}r, \tag{A6}$$

$$\Delta_r = (a^2 + r^2)(1 - \lambda r^2) - 2mr + e^2, \tag{A7}$$

resulting in

$$\boldsymbol{g} = -\frac{\Delta_r}{\Xi^2\rho^2}(\mathrm{d}\tau - a\sin^2\theta\mathrm{d}\varphi)^2 + \frac{\Delta_\theta\sin^2\theta}{\Xi^2\rho^2}[a\mathrm{d}\tau - (r^2 + a^2)\mathrm{d}\varphi]^2 + \frac{\rho^2}{\Delta_r}\mathrm{d}r^2 + \frac{\rho^2}{\Delta_\theta}\mathrm{d}\theta^2, \tag{A8}$$

with (A3), (A5), and (A4) taking the form

$$\boldsymbol{k} = \frac{\mathrm{d}\tau}{\Xi} + \frac{\rho^2}{\Delta_r}\mathrm{d}r - \frac{a\sin^2\theta}{\Xi}\mathrm{d}\varphi. \tag{A9}$$

After observing that the term proportional to $\mathrm{d}r$ in (A9) gives rise to a removable gauge term in (A5), this is the form of the solution given in [72,86] (cf. also [4]).

2. Hyperbolic solution

In addition to the Kerr-Newman-(A)dS solutions considered above, the family of metrics obtained by Carter [72,87] and Plebański [88] (see also [86,89]) also contains certain "topological" extensions, as described in [45,46,90]. The hyperbolic counterpart of (A1)–(A5) can be obtained straightforwardly by the following analytic continuation of coordinates and parameters [45,46]:

$$t \mapsto it, \quad r \mapsto ir, \quad \theta \mapsto i\theta, \quad m \mapsto -im, \quad a \mapsto ia. \tag{A10}$$

The resulting solution is given again by (A1) and (A5), but now with (after redefining $\rho \mapsto i\rho$, $\boldsymbol{k} \mapsto -i\boldsymbol{k}$)

$$\boldsymbol{\eta} = -\frac{\Delta_\theta}{\Xi}(-1 - \lambda r^2)\mathrm{d}t^2 + \frac{\rho^2}{(-1 - \lambda r^2)(r^2 + a^2)}\mathrm{d}r^2 + \frac{\rho^2}{\Delta_\theta}\mathrm{d}\theta^2 + \frac{r^2 + a^2}{\Xi}\sinh^2\theta \mathrm{d}\phi^2, \tag{A11}$$

$$\Delta_\theta = 1 - \lambda a^2\cosh^2\theta, \quad \rho^2 = r^2 + a^2\cosh^2\theta, \quad \Xi = 1 - \lambda a^2, \tag{A12}$$

$$\boldsymbol{k} = \frac{\Delta_\theta}{\Xi}\mathrm{d}t + \frac{\rho^2}{(1 + \lambda r^2)(r^2 + a^2)}\mathrm{d}r + \frac{a\sinh^2\theta}{\Xi}\mathrm{d}\phi. \tag{A13}$$

The solutions of [45,46] were not given in the above KS form, but in Boyer-Lindquist-type coordinates—these can be obtained by an obvious modification of (A6).

[12] Cf. [16] for the special case $\Xi = 0$ (for $\lambda < 0$).

[13] We observe that, in the test-field limit, the KS form of the solution provides a natural rephrasing of the Killing 1-form "background subtraction" of [83].

3. Planar solution

The KS form of the planar solution of [45,46] is given by (A1) and (A5) with

$$\boldsymbol{\eta} = \frac{N}{\rho^2}\mathrm{d}t^2 + \frac{\rho^2}{a^2-\lambda r^4}\mathrm{d}r^2 + \frac{\rho^2}{\Delta_P}\mathrm{d}P^2 - \frac{2aS}{\rho^2}\mathrm{d}t\mathrm{d}\phi + (r^2 - a^2P^2)\mathrm{d}\phi^2, \tag{A14}$$

$$\boldsymbol{k} = (1-\lambda a^2P^2)\mathrm{d}t + \frac{\rho^2}{a^2-\lambda r^4}\mathrm{d}r + aP^2\mathrm{d}\phi, \tag{A15}$$

where

$$\Delta_P = 1-\lambda a^2P^4, \qquad \rho^2 = r^2 + a^2P^2,$$
$$N = a^2(1+\lambda r^2)^2\Delta_P - (a^2-\lambda r^4)(1-\lambda a^2P^2)^2,$$
$$S = P^2(a^2-\lambda r^4)(1-\lambda a^2P^2) + r^2(1+\lambda r^2)\Delta_P. \tag{A16}$$

As in the hyperbolic case, the solutions of [45,46] were presented in Boyer-Lindquist-type coordinates, which can be obtained by defining

$$\mathrm{d}t = \mathrm{d}\tau + \frac{(2mr-e^2)r^2}{(a^2-\lambda r^4)\Delta_r}\mathrm{d}r,$$
$$\mathrm{d}\phi = \mathrm{d}\varphi + a\lambda\mathrm{d}t + a\frac{2mr-e^2}{(a^2-\lambda r^4)\Delta_r}\mathrm{d}r, \tag{A17}$$

$$\Delta_r = a^2 - \lambda r^4 - 2mr + e^2. \tag{A18}$$

APPENDIX B: CONNECTION AND CURVATURE OF KS SPACETIMES ($\boldsymbol{k}$ GEODESIC)

The Ricci rotation coefficients (13) for the geometry (1) in a null frame $\{\boldsymbol{k},\boldsymbol{n},\boldsymbol{m}_{(i)}\}$ adapted to $\boldsymbol{k}$ (as defined in section II A) read

$$L_{i0} = L_{i0}|_0, \quad L_{10} = L_{10}|_0, \quad L_{ij} = L_{ij}|_0,$$
$$\overset{i}{M}_{j0} = \overset{i}{M}_{j0}|_0, \quad \overset{i}{M}_{jk} = \overset{i}{M}_{jk}|_0 \tag{B1}$$

$$N_{i0} = N_{i0}|_0, \quad L_{i1} = L_{i1}|_0, \quad L_{1i} = L_{1i}|_0 - HL_{i0}|_0,$$
$$N_{ij} = N_{ij}|_0 + HL_{ji}|_0, \tag{B2}$$

$$\overset{i}{M}_{j1} = \overset{i}{M}_{j1}|_0 + H\Big(\overset{i}{M}_{j0}|_0 + 2L_{[ij]}|_0\Big),$$
$$L_{11} = L_{11}|_0 - HL_{10}|_0 - D|_0H, \tag{B3}$$

$$N_{i1} = N_{i1}|_0 + H(N_{i0}|_0 + 2L_{1i}|_0 - HL_{i0}|_0 - L_{i1}|_0) + \delta_i|_0H, \tag{B4}$$

where a subscript $|_0$ denotes quantities pertaining to the background geometry $\boldsymbol{\eta}$ (defined by $H=0$). We note that the frames of $\boldsymbol{g}$ and $\boldsymbol{\eta}$ are related by $k^a = k^a|_0$, $m^a_{(i)} = m^a_{(i)}|_0$ and $n^a = n^a|_0 + Hk^a|_0$ (and thus $n_a = n_a|_0 - Hk_a|_0$), so that for the derivative operators (19), when acting on scalar functions, we have

$$D = D|_0, \qquad \delta_i = \delta_i|_0, \qquad \Delta = \Delta|_0 + HD|_0. \tag{B5}$$

When $\boldsymbol{k}$ is *geodesic* ($\Leftrightarrow L_{i0}=0$, which is necessarily the case, e.g., in vacuum or with aligned matter [15,23]) and an *affine parametrization* is employed ($\Leftrightarrow L_{10}=0$), and $\boldsymbol{\eta}$ is a spacetime of constant curvature, the curvature tensor of metric (1) takes the form (with (12) [15,23][14]

$$R_{0i0j} = 0, \quad R_{010i} = 0, \quad R_{0ijk} = 0, \tag{B6}$$

$$R_{0101} = D^2H - \lambda, \quad R_{01ij} = -2A_{ij}DH + 4HS_{k[j}A_{i]k}, \tag{B7}$$

$$R_{0i1j} = -L_{ij}DH + 2HA_{ik}L_{kj} + \lambda\delta_{ij}, \tag{B8}$$

$$R_{ijkl} = 4H(A_{ij}A_{kl} + A_{l[i}A_{j]k} + S_{l[i}S_{j]k}) + 2\lambda\delta_{i[k}\delta_{l]j}, \tag{B9}$$

$$R_{011i} = (-\delta_iD + 2L_{[i1]}D + L_{ji}\delta_j)H + 2H(L_{1j}L_{ji} - L_{j1}S_{ij}), \tag{B10}$$

$$R_{1ijk} = 2(L_{[j|i}\delta_{|k]} + A_{jk}\delta_i)H - 2H\Big(\delta_iA_{kj} + 2L_{1[j}L_{k]i} - 2L_{[j|1}A_{|k]i} + 2L_{[1i]}A_{kj} + 2A_{l[j}\overset{l}{M}_{k]i}\Big), \tag{B11}$$

$$R_{1i1j} = \Big[\delta_{(i}\delta_{j)} + \overset{k}{M}_{(ij)}\delta_k + 2(2L_{1(j|} - L_{(j|1})\delta_{|i)} + N_{(ij)}D - S_{ij}\Delta\Big]H + 2H\Big(\delta_{(i|}L_{1|j)} - \Delta S_{ij} - 2L_{1(i}L_{j)1} + 2L_{1i}L_{1j} - L_{k(i|}N_{k|j)} + L_{1k}\overset{k}{M}_{(ij)} - 2S_{k(i}\overset{k}{M}_{j)1} - 2HS_{k(i}A_{j)k}\Big). \tag{B12}$$

It follows that the nonvanishing components of the Ricci tensor are [15,23]

$$R_{01} = -[D^2H + (n-2)\theta DH + 2H\omega^2] + (n-1)\lambda, \tag{B13}$$

$$R_{ij} = 2HL_{ik}L_{jk} - 2[DH + (n-2)\theta H]S_{ij} + (n-1)\lambda\delta_{ij}, \tag{B14}$$

$$R_{1i} = (-\delta_iD + 2L_{[i1]}D + 2L_{ij}\delta_j - L_{jj}\delta_i)H + 2H(\delta_jA_{ij} + A_{ij}\overset{j}{M}_{kk} - A_{kj}\overset{i}{M}_{kj} - L_{jj}L_{1i} + 3L_{ij}L_{[1j]} + L_{ji}L_{(1j)}), \tag{B15}$$

[14]We emphasize that the following expressions hold in any null frame adapted to $\boldsymbol{k}$, i.e., not necessarily a parallelly transported one.

$$R_{11} = \Big[\delta_i\delta_i + (N_{ii} - 2HL_{ii})D + \Big(4L_{1j} - 2L_{j1} - \overset{i}{M}_{ji}\Big)\delta_j - L_{ii}\Delta\Big]H + 2H\Big[2\delta_i L_{[1i]} + 4L_{1i}L_{[1i]} + L_{i1}L_{i1} - L_{11}L_{ii} + 2L_{[1j]}\overset{j}{M}_{ii} - 2A_{ij}N_{ij} - 2H\omega^2 + (n-2)\lambda\Big]. \tag{B16}$$

[The Ricci identity [41] has also been used in (B16).] We observe that the Ricci tensor is linear in H [this is manifest for the components (B13)–(B15), while in (B16) one has to use (B2) and (B5)]. A similar property of the mixed coordinate components R^a_b was pointed out in [30,31].

For later purposes, it is useful to observe that, when $\boldsymbol{k}$ is shearfree, the second component in (B7) reduces to

$$R_{01ij} = 2A_{ij}(-DH + 2H\theta) \quad (\sigma = 0). \tag{B17}$$

This implies that if $A_{ij} \neq 0$, then necessarily $R_{01ij} \neq 0$ for vacuum solutions [as can be seen easily using (32) and the form of H in vacuum obtained in [15,23]], except in the trivial case $H = 0$. This observation will be used in Sec. C 2.

For $n = 4$, equivalent results were obtained in [8,33].

APPENDIX C: TAUB-NUT SPACETIMES IN HIGHER DIMENSIONS

The Taub-NUT spacetime is a well-known vacuum solution of Einstein gravity in four dimensions [73,91,92] (cf. also [3,4] and references therein). Extensions to arbitrary even dimensions have been known for some time [37,47–49,60–63]. More recently, those have been characterized by the presence of two twisting shearfree congruences of null geodesics and have thus attracted attention in the context of higher-dimensional formulations of the Goldberg-Sachs theorem [25,26,38,39]. In this appendix we review the basic properties of such spacetimes, first "off shell" and subsequently in the Einstein (vacuum) case (relevant to Sec. III B 3). We also present a few new observations regarding the Weyl type and the overlap with the KS class, and further point out that Taub-NUT Einstein spacetimes (and thus shearfree congruences of null geodesics) also exist in odd dimensions in the special branch of Sec. C 2 b. In this appendix, we will denote by $\mathrm{d}s^2$ those line elements that are not necessarily KS (as opposed to the symbol $\boldsymbol{g}$ used for KS metrics throughout the paper).

1. Metric ansatz and off-shell properties

a. Line element in stationary (NUT) coordinates

We consider the n-dimensional stationary line element [48,49]

$$\mathrm{d}s^2 = -A^2(r)(\mathrm{d}t - 2\boldsymbol{Z})^2 + B^2(r)\mathrm{d}r^2 + C^2(r)\boldsymbol{h}, \tag{C1}$$

where the coordinates x^α ($\alpha = 1, \ldots, n-2$, also denoted collectively simply as x) parametrize a Riemannian base space of dimension $n-2$ which carries a metric $\boldsymbol{h} = h_{\alpha\beta}(x)\mathrm{d}x^\alpha\mathrm{d}x^\beta$, $\boldsymbol{Z} = Z_\alpha(x)\mathrm{d}x^\alpha$ is a 1-form which lives in the base space, and A, B and C are functions of r.

For later purposes, it is useful to define the (purely spatial) 2-form

$$\mathcal{F} \equiv \mathrm{d}\boldsymbol{Z}, \tag{C2}$$

which we assume to be nonzero, as well as its positive definite quadratic invariant

$$\mathcal{F}^2 \equiv h^{\alpha\gamma}h^{\beta\delta}\mathcal{F}_{\alpha\beta}\mathcal{F}_{\gamma\delta}. \tag{C3}$$

Both the above quantities are r independent.

b. Line element in null coordinates and twisting, shearfree null congruences

The two 1-forms

$$\boldsymbol{\ell}_\pm \equiv \mathrm{d}t - 2\boldsymbol{Z} \pm BA^{-1}\mathrm{d}r, \tag{C4}$$

define two congruences of *shearfree* null geodesics [25,38–40] with expansion and twist given by, respectively [cf. also (C11) below; hereafter a prime denotes differentiation with respect to r]

$$\theta = \pm C'(ABC)^{-1}, \quad \omega^2 = C^{-4}\mathcal{F}^2. \tag{C5}$$

Note that the "reflection" $t \mapsto -t$, $\boldsymbol{Z} \mapsto -\boldsymbol{Z}$ leaves the metric invariant while sending $\boldsymbol{\ell}_+ \mapsto -\boldsymbol{\ell}_-$ and $\boldsymbol{\ell}_- \mapsto -\boldsymbol{\ell}_+$. This means these two null directions have identical geometric properties (up to certain signs) [93].

The change of coordinates

$$\mathrm{d}t = \mathrm{d}u - \frac{B}{A}\mathrm{d}r, \tag{C6}$$

puts metric (C1) in the form $\mathrm{d}s^2 = -A^2(\mathrm{d}u - 2\boldsymbol{Z})^2 + 2AB(\mathrm{d}u - 2\boldsymbol{Z})\mathrm{d}r + C^2(r)\boldsymbol{h}$, and gives $\boldsymbol{\ell}_+ = \mathrm{d}u - 2\boldsymbol{Z}$, $\boldsymbol{\ell}_- = \boldsymbol{\ell}_+ - 2BA^{-1}\mathrm{d}r$. Without losing generality, a redefinition of r enables one to set

$$AB = 1, \tag{C7}$$

which we shall assume hereafter. After relabeling

$$2\mathcal{H} = A^2, \tag{C8}$$

we thus arrive at

$$\mathrm{d}s^2 = \mathrm{d}r \otimes \boldsymbol{\ell}_+ + \boldsymbol{\ell}_+ \otimes \mathrm{d}r + C^2(r)\boldsymbol{h} - 2\mathcal{H}(r)\boldsymbol{\ell}_+ \otimes \boldsymbol{\ell}_+, \quad \boldsymbol{\ell}_+ = \mathrm{d}u - 2\boldsymbol{Z}, \tag{C9}$$

where the function $\mathcal{H}$ can now have any sign or vanish (thus describing also possible time-dependent Taub regions).

Let us introduce the following null coframe:

$$\boldsymbol{\omega}^0 = -\mathcal{H}\boldsymbol{\ell}_-, \quad \boldsymbol{\omega}^1 = \boldsymbol{\ell}_+, \quad \boldsymbol{\omega}^i = C\tilde{\omega}^i \quad (i = 2, \ldots, n-1), \tag{C10}$$

where $\{\tilde{\omega}^i\}$ is an orthonormal coframe of the metric $\boldsymbol{h}$, such that $\boldsymbol{h} = \delta_{\tilde{i}\tilde{j}}\boldsymbol{\omega}^{\tilde{i}}\boldsymbol{\omega}^{\tilde{j}}$ and $\mathrm{d}s^2 = \boldsymbol{\omega}^0\boldsymbol{\omega}^1 + \boldsymbol{\omega}^1\boldsymbol{\omega}^0 + \delta_{ij}\boldsymbol{\omega}^i\boldsymbol{\omega}^j$. In the frame dual to (C10) (for which, in particular, $\boldsymbol{e}_0 = \ell_+^a\partial_a = \partial_r$), the nonzero Ricci rotation coefficients (13) read

$$L_{ij} = C'C^{-1}\delta_{ij} + C^{-2}\mathcal{F}_{\tilde{i}\tilde{j}}, \qquad \overset{i}{M}_{j0} = -C^{-2}\mathcal{F}_{\tilde{i}\tilde{j}}$$
$$\overset{i}{M}_{jk} = -C^{-1}\Gamma^{\tilde{i}}_{\tilde{j}\tilde{k}}, \tag{C11}$$

$$N_{ij} = \mathcal{H}L_{ji}, \quad \overset{i}{M}_{j1} = \mathcal{H}C^{-2}\mathcal{F}_{\tilde{i}\tilde{j}}, \quad L_{11} = -\mathcal{H}', \tag{C12}$$

where hereafter indices with a tilde denote components in the tilded frame and $\Gamma^{\tilde{i}}_{\tilde{j}\tilde{k}}$ are the connection coefficients of the geometry $\boldsymbol{h}$. In particular, the twist matrix of $\boldsymbol{\ell}_+$ is thus given by

$$A_{ij} = C^{-2}\mathcal{F}_{\tilde{i}\tilde{j}}. \tag{C13}$$

We note that, since $\overset{i}{M}_{j0} \neq 0$, the above frame is *not* parallelly transported along the null congruence generated by $\boldsymbol{\ell}_+$.

c. Curvature

The nonzero frame components of the Riemann tensor can be computed straightforwardly and take the form (cf. also [39,49])

$$R_{0i0j} = C^{-4}(-C^3C''\delta_{ij} + \mathcal{F}_{\tilde{i}\tilde{k}}\mathcal{F}_{\tilde{j}\tilde{k}}), \quad R_{0ijk} = -2C^{-3}\tilde{\nabla}_{[\tilde{k}}\mathcal{F}_{\tilde{j}]\tilde{i}}, \tag{C14}$$

$$R_{01ij} = -(2\mathcal{H}C^{-2})'\mathcal{F}_{\tilde{i}\tilde{j}}, \qquad R_{0101} = \mathcal{H}'', \tag{C15}$$

$$R_{0i1j} = -(\mathcal{H}C^{-2})'\mathcal{F}_{\tilde{i}\tilde{j}} - C^{-1}(\mathcal{H}C')'\delta_{ij} - C^{-4}\mathcal{H}\mathcal{F}_{\tilde{i}\tilde{k}}\mathcal{F}_{\tilde{j}\tilde{k}}, \tag{C16}$$

$$R_{ijkl} = C^{-2}\tilde{R}_{\tilde{i}\tilde{j}\tilde{k}\tilde{l}} + 4\mathcal{H}C^{-4}[-(CC')^2\delta_{i[k}\delta_{l]j} + \mathcal{F}_{\tilde{i}\tilde{j}}\mathcal{F}_{\tilde{k}\tilde{l}} - \mathcal{F}_{\tilde{i}[\tilde{k}}\mathcal{F}_{\tilde{l}]\tilde{j}}], \tag{C17}$$

$$R_{1i1j} = \mathcal{H}^2R_{0i0j}, \quad R_{1ijk} = -\mathcal{H}R_{0ijk}, \tag{C18}$$

where $\tilde{\nabla}$ and $\tilde{R}_{\tilde{i}\tilde{j}\tilde{k}\tilde{l}}$ are, respectively, the covariant derivative and the Riemann tensor associated to $\boldsymbol{h}$. In passing, let us observe that, by the first of (C14), $\boldsymbol{\ell}_\pm$ are WANDs if and only if $(n-2)\mathcal{F}_{\tilde{i}\tilde{k}}\mathcal{F}_{\tilde{j}\tilde{k}} = \mathcal{F}^2\delta_{\tilde{i}\tilde{j}}$ (which is an identity for $n = 4$).

The Ricci components then follow (cf. also [39,47–49])

$$R_{00} = C^{-4}[-(n-2)C^3C'' + \mathcal{F}^2], \quad R_{0i} = C^{-3}\tilde{\nabla}_{\tilde{k}}\mathcal{F}_{\tilde{k}\tilde{i}}, \tag{C19}$$

$$R_{01} = -\mathcal{H}'' - (n-2)C^{-1}(\mathcal{H}C')' - C^{-4}\mathcal{H}\mathcal{F}_{\tilde{i}\tilde{k}}\mathcal{F}_{\tilde{i}\tilde{k}}, \tag{C20}$$

$$R_{ij} = C^{-2}(\tilde{R}_{\tilde{i}\tilde{j}} + 4\mathcal{H}C^{-2}\mathcal{F}_{\tilde{i}\tilde{k}}\mathcal{F}_{\tilde{j}\tilde{k}}) - 2\delta_{ij}C^{-2}[C(\mathcal{H}C')' + (n-3)\mathcal{H}(C')^2], \tag{C21}$$

$$R_{11} = \mathcal{H}^2R_{00}, \quad R_{1i} = -\mathcal{H}R_{0i}. \tag{C22}$$

2. Vacuum solutions

For the purposes of this paper, we are interested in Taub-NUT metrics that solve the Λ-vacuum Einstein equation

$$R_{ab} = \frac{R}{n}g_{ab}, \tag{C23}$$

which gives [cf. (9) with $T_{ab} = 0$ and (12)]

$$R = n(n-1)\lambda. \tag{C24}$$

Imposing $R_{00} = 0$ gives $(n-2)C^3C'' = \mathcal{F}^2$. Since C depends only on r, this means that $\mathcal{F}^2 = \text{const}$. The solution of this ODE can be written (up to a linear redefinition of r and suitable rescalings of u and $\boldsymbol{Z}$, such as to preserve (C7) as [48,49]

$$C^2 = r^2 + \ell^2, \qquad \ell^2 \equiv \frac{\mathcal{F}^2}{n-2}(= \text{const}). \tag{C25}$$

From $R_{0i} = 0$ one obtains

$$\tilde{\nabla}_{\tilde{k}}\mathcal{F}_{\tilde{k}\tilde{i}} = 0, \tag{C26}$$

so that $\boldsymbol{F}$ is also co-closed [in addition to being closed, cf. (C2)].

In order to solve $R_{ij} = \frac{R}{n}\delta_{ij}$ one needs to consider two possibilities separately, depending on whether or not $\mathcal{H}C^{-2}$ is a constant.[15]

a. Generic case $\mathcal{H}C^{-2} \neq$ const (n even)

In this case, requiring $R_{ij} = \frac{R}{n}\delta_{ij}$ gives rise to three separate equations. Two of those are tensorial and read

$$\mathcal{F}_{\tilde{i}\tilde{k}}\mathcal{F}_{\tilde{j}\tilde{k}} = \ell^2\delta_{ij}, \tag{C27}$$

$$\tilde{R}_{\tilde{i}\tilde{j}} = \frac{\tilde{R}}{n-2}\delta_{ij}, \tag{C28}$$

[15]This was noticed already in [49], but the special case $\mathcal{H}C^{-2} = \text{const}$ (Sec. C 2 b) was not studied there.

where $\tilde{R}_{\tilde{i}\tilde{j}} = \tilde{R}_{\tilde{i}\tilde{k}\tilde{j}\tilde{k}}$ and $\tilde{R} = \tilde{R}_{\tilde{k}\tilde{k}}$. This means that the spatial geometry must be (almost-)Kähler-Einstein [39,49] and n is necessarily even. We further observe that (C27) ensures that the two null directions defined by (C4) are WANDs (cf. Sec. C 1 c). They are mWANDs (i.e., the Weyl type is D) if, and only if, $\tilde{\nabla}_{\tilde{k}}\mathcal{F}_{\tilde{i}\tilde{j}} = 0$, i.e., if the base space is Kähler-Einstein (cf. also [39]).

The remaining (scalar) equation determines $\mathcal{H}(r)$ up to an arbitrary integration constant, and can be conveniently written as [49]

$$\frac{r^2}{(r^2+\ell^2)^{n/2}}[r^{-1}(r^2+\ell^2)^{(n-2)/2}2\mathcal{H}]' + \frac{1}{n-2}\left(2\Lambda - \frac{\tilde{R}}{r^2+\ell^2}\right) = 0. \qquad \text{(C29)}$$

Note that $2\mathcal{H}(r^2+\ell^2)^{(n-2)/2}$ is a polynomial of degree n in r [40,49,61]. For large values of r this gives the asymptotic behavior

$$2\mathcal{H} = -\lambda r^2 + \frac{-\lambda\ell^2(2n-3) + \frac{\tilde{R}}{n-2}}{n-3} + \ldots, \qquad \text{(C30)}$$

with an integration constant μ_0 appearing at the order r^{3-n}.

Upon using the Bianchi identity, one can then verify that all components of the Einstein equation are now satisfied. The resulting vacuum metric is thus given by (C9) with (C2) and (C25)–(C29). In the limit $\ell \to 0$, the 1-form $\boldsymbol{Z}$ can be gauged away in (C9) and one obtains static Schwarzschild-Tangherlini-like metrics [18,43,94] of the Robinson-Trautman class [19], for which the base space can carry any Einstein metric.

Intersection with KS metrics. Let us now discuss under what conditions the Einstein spacetimes determined above belong to the KS class, with $\boldsymbol{\ell}_+$ being a (shearfree, twisting) KS vector field. First, since $\boldsymbol{\ell}_+$ is now necessarily an mWAND [15,23], the base space must be Kähler-Einstein (as remarked above). Next, by comparing the Riemann component R_{01ij} of (C15) with the corresponding result for KS spacetimes (B17) and using (C29) one obtains

$$-2\mathcal{H} = \lambda(r^2+\ell^2) + \frac{\mu_0 r}{(r^2+\ell^2)^{\frac{n-2}{2}}}, \qquad \tilde{R} = n\lambda\mathcal{F}^2, \qquad \text{(C31)}$$

where μ_0 is an integration constant. Furthermore, by comparing the component R_{ijkl} of (C17) with the KS one (B9) one arrives at a constraint on the curvature of the base space (in addition to the already mentioned Kähler-Einstein condition), namely

$$\tilde{R}_{\tilde{i}\tilde{j}\tilde{k}\tilde{l}} = 2\lambda(\ell^2\delta_{i[k}\delta_{l]j} + \mathcal{F}_{\tilde{i}\tilde{j}}\mathcal{F}_{\tilde{k}\tilde{l}} - \mathcal{F}_{\tilde{i}[\tilde{k}}\mathcal{F}_{\tilde{l}]\tilde{j}}). \qquad \text{(C32)}$$

This means that the base manifold is a space of *constant holomorphic sectional curvature* [64–66], and therefore it is uniquely given by (assuming it to be simply connected and complete; cf. Theorems 7.8 and 7.9 of [66]): (i) the complex projective space $\mathbb{C}P^{\frac{n-2}{2}}$ if $\lambda > 0$; (ii) the open unit ball $D^{\frac{n-2}{2}}$ in $\mathbb{C}^{\frac{n-2}{2}}$ if $\lambda < 0$; (iii) $\mathbb{C}^{\frac{n-2}{2}}$ (flat space) if $\lambda = 0$. This also implies that for $\lambda \neq 0$ and $n > 4$ it is *not* a space of constant curvature.

To summarize, after relabeling $\boldsymbol{\ell}_+ = \boldsymbol{k}$, the only $n > 4$ Taub-NUT vacuum metrics which are also KS are given by the line element[16]

$$\boldsymbol{g} = \mathrm{d}r \otimes \boldsymbol{k} + \boldsymbol{k} \otimes \mathrm{d}r + (r^2+\ell^2)(\boldsymbol{h} + \lambda\boldsymbol{k}\otimes\boldsymbol{k}) + \frac{\mu_0 r}{(r^2+\ell^2)^{\frac{n-2}{2}}}\boldsymbol{k}\otimes\boldsymbol{k}, \qquad \boldsymbol{k} = \mathrm{d}u - 2\boldsymbol{Z}, \qquad \text{(C33)}$$

where the base space metric $\boldsymbol{h}$ must be Kähler-Einstein and of constant holomorphic sectional curvature, with Kähler 2-form $\boldsymbol{\mathcal{F}} = \mathrm{d}\boldsymbol{Z}$ [recall (C3), (C13), (C25), (C27), (C28), (C32)]. Metric (C33) is indeed of the KS form (1) with $2H = -\mu_0 r(r^2+\ell^2)^{\frac{2-n}{2}}$. For $\mu_0 \neq 0$ the Weyl type is D (cf. [25,38,39]), while $\mu_0 = 0$ corresponds to a spacetime of constant curvature. For example, for the $n = 6$ KS-Taub-NUT metric with $\lambda > 0$, the base space is given by $\mathbb{C}P^2$ with the Fubini-Study metric, cf. (64), (65).

The case $n = 4$ is special in that (C32) becomes equivalent to the second of (C31), which means that the (two-dimensional) base space has constant curvature for any value of λ. Four-dimensional Taub-NUT metrics of the KS class can thus be written as

$$\begin{aligned} \boldsymbol{g} &= \mathrm{d}r \otimes \boldsymbol{\ell}_+ + \boldsymbol{\ell}_+ \otimes \mathrm{d}r + (r^2+\ell^2)(2P^{-2}\mathrm{d}\zeta\mathrm{d}\bar{\zeta} \\ &\quad + \lambda\boldsymbol{\ell}_+\otimes\boldsymbol{\ell}_+) + \frac{\mu_0 r}{r^2+\ell^2}\boldsymbol{\ell}_+\otimes\boldsymbol{\ell}_+, \\ \boldsymbol{\ell}_+ &= \mathrm{d}u - i\ell P^{-1}(\bar{\zeta}\mathrm{d}\zeta - \zeta\mathrm{d}\bar{\zeta}), \\ P &= 1 + 2\lambda\ell^2\zeta\bar{\zeta} \quad (n=4). \end{aligned} \qquad \text{(C34)}$$

Upon using (C6), metric (C34) corresponds to a fine tuned version of (12.19,[4]). For $\mu_0 = 0$ it reduces to a spacetime of constant curvature.

b. Special case $\mathcal{H}C^{-2} = \text{const}$

Let us set here $2\mathcal{H} = -c_0 C^2$, where c_0 is a constant.[17] The condition $R_{01} = \frac{R}{n}$ with (C25) reveals that c_0 is fixed by the cosmological constant [recall (12)]

[16]To be precise, at this stage we have proven this only in the case $\mathcal{H}C^{-2} \neq$ const. However, in Sec. C 2 b we will prove that the case $\mathcal{H}C^{-2} =$ const does not contain any KS-Taub-NUT vacua (except for spacetimes of constant curvature), therefore the general statement about the uniqueness of metric (C33) is indeed true.

[17]This special case can also be characterized by $C^2(r)\boldsymbol{\ell}_\pm$ becoming conformal Killing vector fields [39] [this is true also off-shell, i.e., for any metric (C1) or (C9) with $A^2(r) = 2\mathcal{H}(r) = -c_0 C^2(r)$].

$$c_0 = \lambda. \tag{C35}$$

Next, imposing $R_{ij} = \frac{R}{n}\delta_{ij}$ results in

$$\tilde{R}_{\tilde{i}\tilde{j}} = \lambda(2\mathcal{F}_{\tilde{i}\tilde{k}}\mathcal{F}_{\tilde{j}\tilde{k}} + \mathcal{F}^2\delta_{\tilde{i}\tilde{j}}), \tag{C36}$$

such that $\tilde{R} = n\lambda\mathcal{F}^2$. All components of the Einstein equation are now satisfied.

The final form of the metric is thus given by

$$\mathrm{d}s^2 = \mathrm{d}r \otimes \boldsymbol{\ell}_+ + \boldsymbol{\ell}_+ \otimes \mathrm{d}r + (r^2 + \ell^2)(\boldsymbol{h} + \lambda\boldsymbol{\ell}_+ \otimes \boldsymbol{\ell}_+), \tag{C37}$$

where $\boldsymbol{\ell}_+$ and ℓ^2 are defined in (C9) and (C25), respectively, and the base space metric $\boldsymbol{h}$ must obey (C36). It is easy to see (e.g. using (C6) backwards) that metric (C37) belongs to the class of Brinkmann warps [95] (cf. also, e.g., [96,97] and references therein)[18]—except when $\lambda = 0$. Let us emphasize that, by (C36), the base metric $\boldsymbol{h}$ is not restricted to be Einstein when $\lambda \neq 0$, since, generically, $\mathcal{F}_{\tilde{i}\tilde{k}}\mathcal{F}_{\tilde{j}\tilde{k}}$ is not proportional to $\delta_{\tilde{i}\tilde{j}}$ (in particular, it cannot be so for an odd n). For the same reason, the two null directions defined by (C4) are not WANDs, in general (cf. section C 1 c). They are iff n is even and $\boldsymbol{h}$ is (almost-) Kähler-Einstein, which is the case of the "Fefferman-Einstein" metrics studied in [39]. As in Sec. C 2 a, they are mWANDs if and only if, in addition, $\tilde{\nabla}_{\tilde{k}}\mathcal{F}_{\tilde{i}\tilde{j}} = 0$. The limit $\ell \to 0$ gives rise to "massless" Schwarzschild-Tangherlini-like metrics with a Ricci-flat base space [18,19,43,94].

Noticing the similarity of (C36) to the Einstein equation sourced by a Maxwell field $\boldsymbol{\mathcal{F}} = \mathrm{d}\boldsymbol{Z}$ (in $n-2$ dimensions), it is not difficult to find suitable spatial geometries $\boldsymbol{h}$ and thus construct explicit examples of vacuum solutions (C37). For instance, in seven spacetime dimensions with a positive cosmological constant one can take (C37) with

$$\boldsymbol{h} = \frac{1}{10\lambda\ell^2}[4[\mathrm{d}\theta^2 + \sin^2\theta(\mathrm{d}\phi^2 + \sin^2\phi\mathrm{d}\xi^2)] + \mathrm{d}\chi^2 + \sin^2\chi\mathrm{d}\psi^2] \quad (n = 7, \lambda > 0), \tag{C38}$$

$$\boldsymbol{Z} = \frac{1}{2\sqrt{10}\lambda\ell}\cos\chi\mathrm{d}\psi, \tag{C39}$$

which is clearly a (non-Einstein) direct product $S^3 \times S^2$. Similar solutions can be constructed in other dimensions.

Intersection with KS metrics. The first of (C15) gives

$$R_{01ij} = 0. \tag{C40}$$

Since this contradicts the result (B17) for KS spacetimes in vacuum, it follows that the two shearfree null directions (C4) cannot be KS vector fields of the spacetime (C37) (except in the trivial case $H = 0$, i.e., for spacetimes of constant curvature, which is equivalent to (C33) with $\mu_0 = 0$).

APPENDIX D: INTEGRATION OF THE EINSTEIN-MAXWELL EQUATIONS FOR SHEARFREE TWISTING KS SOLUTIONS

In this appendix we present the details of the integration of the Einstein-Maxwell equations for the case of a shearfree twisting KS vector field $\boldsymbol{k}$, corresponding to the branch of solutions described in Sec. III B 3. As proven in Appendix C, it follows from the results of [39] that the only $n > 4$ KS vacua possessing a shearfree, twisting KS vector field $\boldsymbol{k}$ are given by the Taub-NUT metrics (C33) with (C32). In order to charge those spacetimes according to (1), (7), (8), we can thus focus on KS line elements of the form

$$\boldsymbol{g} = \mathrm{d}r \otimes \boldsymbol{k} + \boldsymbol{k} \otimes \mathrm{d}r + (r^2 + \ell^2)(\boldsymbol{h} + \lambda\boldsymbol{k} \otimes \boldsymbol{k}) - 2H(u, r, x)\boldsymbol{k} \otimes \boldsymbol{k}, \qquad \boldsymbol{k} = \mathrm{d}u - 2\boldsymbol{Z}, \tag{D1}$$

where ℓ is a constant. The coordinates x^α ($\alpha = 1, \ldots, n-2$, also denoted collectively simply as x) parametrize a Riemannian base space of dimension $n-2$ which carries a Kähler-Einstein metric $\boldsymbol{h} = h_{\alpha\beta}(x)\mathrm{d}x^\alpha\mathrm{d}x^\beta$ of constant holomorphic sectional curvature [cf. (C32)], and $\boldsymbol{Z} = Z_\alpha(x)\mathrm{d}x^\alpha$ is a 1-form which lives in the base space, with Kähler 2-form given by [cf. (C2), (C3), (C25), (C27), (C28)]

$$\boldsymbol{\mathcal{F}} \equiv \mathrm{d}\boldsymbol{Z}. \tag{D2}$$

The latter is related to the twist matrix of $\boldsymbol{k}$ and to the parameter ℓ by (C13), (C27).

An adapted parallelly transported null frame for metric (D1) is given by

$$\boldsymbol{k} = \partial_r, \qquad \boldsymbol{n} = \partial_u + \left[-\frac{\lambda}{2}(r^2 + \ell^2) + H\right]\boldsymbol{k}, \qquad \boldsymbol{m}_{(i)} = (r^2 + \ell^2)^{-\frac{1}{2}}X_i{}^j(\tilde{\boldsymbol{m}}_{(j)} + 2Z_{\tilde{j}}\partial_u), \tag{D3}$$

where the vectors $\tilde{\boldsymbol{m}}_{(i)}$ define an orthonormal frame of the base space metric $\boldsymbol{h}$, the orthogonal matrix $X_i{}^j$ reads (cf., e.g., Appendix A.5 of [38])

$$X_i^j = \ell(r^2 + \ell^2)^{-\frac{1}{2}}(\delta_i^j + r\ell^{-2}\mathcal{F}_{\tilde{i}}{}^{\tilde{j}}), \tag{D4}$$

and indices with a tilde denote components of base space quantities in the tilded frame. In the frame (D3), one finds that the Ricci rotation coefficients needed in the following

[18]These are the unique Einstein spaces which are properly conformal to other Einstein spaces. The case $n = 4$ is special, for such Einstein spacetimes are necessarily of constant curvature [95–97].

[along with (15)] are given by[19]

$$L_{ij} = (r^2+\ell^2)^{-1}(r\delta_{ij} + \mathcal{F}_{\tilde{i}\tilde{j}}), \quad L_{i1} = 0 = L_{1i}, \tag{D5}$$

$$N_{ij} = \left[-\frac{\lambda}{2}(r^2+\ell^2) + H\right] L_{ji}, \quad L_{11} = \lambda r - DH. \tag{D6}$$

We observe that (D5) gives

$$\theta = r(r^2+\ell^2)^{-1}, \quad A_{ij} = (r^2+\ell^2)^{-1}\mathcal{F}_{\tilde{i}\tilde{j}}, \tag{D7}$$

which, with (33), reveals that the $(a^0_{(2\mu)})^2$ become constants, i.e.,

$$(a^0_{(2)})^2 = (a^0_{(4)})^2 = \ldots = (a^0_{(n-2)})^2 = \ell^2. \tag{D8}$$

Further, Eq. (23) gives

$$\delta_j A_{ji} + \overset{k}{M}_{jj} A_{ki} + \overset{k}{M}_{ij} A_{jk} = 0, \tag{D9}$$

while (25) becomes $\delta_i D - D\delta_i = \theta\delta_i + A_{ji}\delta_j$. The Maxwell component (37) thus reduces to

$$[\delta_i D + 2A_{ji}\delta_j + (n-4)\theta\delta_i]\alpha = 0. \tag{D10}$$

Thanks to (D8), Eq. (40) with (41), (43) takes the form

$$\alpha = \frac{r}{(r^2+\ell^2)^{\frac{n-2}{2}}}\left[\alpha_0 + \beta_0 \sum_{\mu=0}^{\frac{n-2}{2}} \binom{\frac{n-2}{2}}{\mu} \frac{\ell^{2\mu}}{n-3-2\mu} r^{n-3-2\mu}\right]. \tag{D11}$$

Using (D7) and (D11), Eq. (D10) can be written as

$$(\ell^2 - r^2)\delta_i\alpha_0 + \delta_i\beta_0 \sum_{\mu=0}^{\frac{n-2}{2}} \binom{\frac{n-2}{2}}{\mu} \times \frac{(n-4-2\mu)r^2 + (n-2-2\mu)\ell^2}{n-3-2\mu}\ell^{2\mu} r^{n-3-2\mu} + 2\mathcal{F}_{\tilde{k}\tilde{i}}\delta_k\left[r\alpha_0 + \beta_0 \sum_{\mu=0}^{\frac{n-2}{2}} \binom{\frac{n-2}{2}}{\mu} \frac{\ell^{2\mu}}{n-3-2\mu} r^{n-2-2\mu}\right] = 0. \tag{D12}$$

It is easy to see that the highest power of r in the above equation gives $(n-4)\delta_i\beta_0 = 0$, and then (hereafter we assume $n > 4$) the subleading terms imply also $\delta_i\alpha_0 = 0$, so that from now on

$$\delta_i\beta_0 = 0, \qquad \delta_i\alpha_0 = 0. \tag{D13}$$

Next, using (D3), (D5), (D6), (D9) and (36), the Maxwell component (38) becomes simply

$$\alpha_{,ru} = 0, \tag{D14}$$

which means [with (D11), (D13) and (D3)] that α_0 and β_0 are constants, i.e., α is only a function of r.

Let us now consider the Einstein equation. Equation (50) is already satisfied thanks to (D5), while (49) can be written as

$$\frac{r^2}{(r^2+\ell^2)^{\frac{n-4}{2}}} D[r^{-1}(r^2+\ell^2)^{\frac{n-2}{2}} 2H] = -\kappa\left[2\alpha^2\ell^2 + \frac{1}{n-2}(r^2+\ell^2)^2(D\alpha)^2\right]. \tag{D15}$$

Since the rhs is a function of r only, this means that

$$2H = -r\frac{\mu_0 - \kappa f(r)}{(r^2+\ell^2)^{\frac{n-2}{2}}}, \tag{D16}$$

with $D\mu_0 = 0$,

$$Df = -r^{-2}(r^2+\ell^2)^{\frac{n-4}{2}} \times \left[2\alpha^2\ell^2 + \frac{1}{n-2}(r^2+\ell^2)^2(D\alpha)^2\right]. \tag{D17}$$

Now, thanks to (D5) and (D13) we have [cf. (47), (48)]

$$T_{1i} = 0, \qquad T_{11} = 0. \tag{D18}$$

With (B15) and (D16), the component $(1i)$ of the Einstein equation thus reduces to $[(r^2-\ell^2)\delta_i + 2r\mathcal{F}_{\tilde{i}\tilde{j}}\delta_j]\mu_0 = 0$, which clearly implies

$$\delta_i\mu_0 = 0. \tag{D19}$$

Finally, with (B16) and (D19), the component (11) of the Einstein equation becomes simply

$$\mu_{0,u} = 0, \tag{D20}$$

i.e., μ_0 is a constant.

To summarize, the only charged KS solution admitting an expanding, twisting and shearfree KS vector field is given by metric (D1) with the vector potential (7), where α and f are functions of r determined by (D11) and (D17), μ_0, α_0, β_0 are integration constants, and λ is the

[19]This can be seen by first obtaining the Ricci rotation coefficients in a simpler but nonparallelly transported frame given by $\boldsymbol{k} = \partial_r$, $\boldsymbol{n} = \partial_u + [-\frac{\lambda}{2}(r^2+\ell^2) + H]\boldsymbol{k}$, $\boldsymbol{m}_{(i)} = (r^2+\ell^2)^{-\frac{1}{2}}(\tilde{\boldsymbol{m}}_{(i)} + 2Z_{\tilde{i}}\partial_u)$, which follow readily from (B1)–(B3), where the background quantities now refer to spacetime (C33) with $\mu_0 = 0$ [cf. (C11), (C12) with $-2\mathcal{H} = \lambda(r^2+\ell^2)$ and (C25)]. Using the known transformation rules under spins [41] one then arrives at (D5), (D6).

cosmological constant [cf. (12)]. Using (35), it follows that the electromagnetic field strength reads

$$\boldsymbol{F} = (D\alpha)\mathrm{d}r \wedge \boldsymbol{k} - 2\alpha\boldsymbol{\mathcal{F}}, \tag{D21}$$

with (D2) and (D11). Recall that the Kähler-Einstein base space metric $\boldsymbol{h}$ must also obey the additional constraint (C32), i.e., it is of constant holomorphic sectional curvature.

If desired, one can obtain the explicit form of the function $f(r)$ in (D16) by integrating (D17), upon noticing that [using (D11) and the binomial theorem]

$$D\alpha = \frac{-(n-3)r^2 + \ell^2}{r(r^2+\ell^2)}\alpha + \frac{\beta_0}{r}, \tag{D22}$$

and therefore

$$Df = -\frac{(r^2+\ell^2)^{\frac{n-4}{2}}}{(n-2)r^4}\left\{\alpha^2[(n-3)^2r^4 + 2\ell^2r^2 + \ell^4] - \beta_0(r^2+\ell^2)^2\left[2\alpha\frac{(n-3)r^2-\ell^2}{r^2+\ell^2} - \beta_0\right]\right\}, \tag{D23}$$

with α as in (D11).

[1] R. P. Kerr, Gravitational field of a spinning mass as an example of algebraically special metrics, Phys. Rev. Lett. **11**, 237 (1963).

[2] E. T. Newman, R. Couch, K. Chinnapared, A. Exton, A. Prakash, and R. Torrence, Metric of a rotating, charged mass, J. Math. Phys. (N.Y.) **6**, 918 (1965).

[3] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein's Field Equations*, 2nd ed. (Cambridge University Press, Cambridge, England, 2003).

[4] J. B. Griffiths and J. Podolský, *Exact Space-Times in Einstein's General Relativity* (Cambridge University Press, Cambridge, England, 2009).

[5] R. P. Kerr and A. Schild, Some algebraically degenerate solutions of Einstein's gravitational field equations, Proc. Symp. Appl. Math. **17**, 199 (1965).

[6] A. Trautman, On the propagation of information by waves, in *Recent Developments in General Relativity* (Pergamon Press and PWN, New York and Warszawa, 1962), pp. 459–463.

[7] G. C. Debney, R. P. Kerr, and A. Schild, Solutions of the Einstein and Einstein-Maxwell equations, J. Math. Phys. (N.Y.) **10**, 1842 (1969).

[8] D. G. B. Edelen, The null-bundle of an Einstein-Riemann space I: General theory, J. Math. Mech. **16**, 351 (1966).

[9] B. C. Xanthopoulos, The optical scalars in Kerr–Schild-type spacetimes, Ann. Phys. (N.Y.) **149**, 286 (1983).

[10] B. Coll, S. R. Hildebrandt, and J. M. M. Senovilla, Kerr-Schild symmetries, Gen. Relativ. Gravit. **33**, 649 (2001).

[11] R. C. Myers and M. J. Perry, Black holes in higher dimensional space-times, Ann. Phys. (N.Y.) **172**, 304 (1986).

[12] A. Chakrabarti, Kerr metric in eight dimensions, Phys. Lett. B **172**, 175 (1986).

[13] R. Emparan and H. S. Reall, A rotating black ring solution in five dimensions, Phys. Rev. Lett. **88**, 101101 (2002).

[14] V. Pravda and A. Pravdová, WANDs of the black ring, Gen. Relativ. Gravit. **37**, 1277 (2005).

[15] M. Ortaggio, V. Pravda, and A. Pravdová, Higher dimensional Kerr-Schild spacetimes, Classical Quantum Gravity **26**, 025008 (2009).

[16] S. W. Hawking, C. J. Hunter, and M. M. Taylor-Robinson, Rotation and the AdS/CFT correspondence, Phys. Rev. D **59**, 064005 (1999).

[17] G. W. Gibbons, H. Lü, D. N. Page, and C. N. Pope, The general Kerr-de Sitter metrics in all dimensions, J. Geom. Phys. **53**, 49 (2005).

[18] F. R. Tangherlini, Schwarzschild field in n dimensions and the dimensionality of space problem, Nuovo Cimento (1955–1965) **27**, 636 (1963).

[19] J. Podolský and M. Ortaggio, Robinson-Trautman spacetimes in higher dimensions, Classical Quantum Gravity **23**, 5785 (2006).

[20] J. Kunz, F. Navarro-Lérida, and A. K. Petersen, Five-dimensional charged rotating black holes, Phys. Lett. B **614**, 104 (2005).

[21] J. Kunz, F. Navarro-Lérida, and J. Viebahn, Charged rotating black holes in odd dimensions, Phys. Lett. B **639**, 362 (2006).

[22] M. B. Fröb, I. Khavkine, T. Málek, and V. Pravda, On well-posedness and algebraic type of the five-dimensional charged rotating black hole with two equal-magnitude angular momenta, Eur. Phys. J. C **82**, 215 (2022).

[23] T. Málek and V. Pravda, Kerr-Schild spacetimes with (A)dS background, Classical Quantum Gravity **28**, 125011 (2011).

[24] M. Ortaggio, V. Pravda, and A. Pravdová, Type III and N Einstein spacetimes in higher dimensions: General properties, Phys. Rev. D **82**, 064043 (2010).

[25] M. Ortaggio, V. Pravda, and A. Pravdová, On the Goldberg-Sachs theorem in higher dimensions in the non-twisting case, Classical Quantum Gravity **30**, 075016 (2013).

[26] M. Ortaggio, V. Pravda, and A. Pravdová, Algebraic classification of higher dimensional spacetimes based on null alignment, Classical Quantum Gravity **30**, 013001 (2013).

[27] R. Milson, A. Coley, V. Pravda, and A. Pravdová, Alignment and algebraically special tensors in Lorentzian geometry, Int. J. Geom. Methods Mod. Phys. **02**, 41 (2005).

[28] A. Coley, R. Milson, V. Pravda, and A. Pravdová, Classification of the Weyl tensor in higher dimensions, Classical Quantum Gravity **21**, L35 (2004).

[29] S. Hervik, M. Ortaggio, and L. Wylleman, Minimal tensors and purely electric or magnetic spacetimes of arbitrary dimension, Classical Quantum Gravity **30**, 165014 (2013).
[30] M. Gürses and F. Gürsey, Lorentz covariant treatment of the Kerr-Schild geometry, J. Math. Phys. (N.Y.) **16**, 2385 (1975).
[31] T. Dereli and M. Gürses, The generalized Kerr-Schild transform in eleven-dimensional supergravity, Phys. Lett. B **171**, 209 (1986).
[32] A. H. Thompson, A class of related space-times, Tensor **17**, 92 (1966).
[33] I. M. Dozmorov, Solutions of the Einstein equations with zero coupling, Sov. Phys. J. **13**, 1284 (1970).
[34] B. C. Xanthopoulos, Exact vacuum solutions of Einstein's equation from linearized solutions, J. Math. Phys. (N.Y.) **19**, 1607 (1978).
[35] A. H. Taub, Generalized Kerr–Schild space-times, Ann. Phys. (N.Y.) **134**, 326 (1981).
[36] R. Monteiro, D. O'Connell, and C. D. White, Black holes and the double copy, J. High Energy Phys. 12 (2014) 056.
[37] R. Mann and C. Stelea, Nuttier (A)dS black holes in higher dimensions, Classical Quantum Gravity **21**, 2937 (2004).
[38] M. Ortaggio, On the uniqueness of the Myers-Perry spacetime as a type II(D) solution in six dimensions, J. High Energy Phys. 06 (2017) 042.
[39] A. Taghavi-Chabert, Twisting non-shearing congruences of null geodesics, almost CR structures and Einstein metrics in even dimensions, Ann. Mat. Pura Appl. **201**, 655 (2022).
[40] D. V. Alekseevsky, M. Ganji, G. Schmalz, and A. Spiro, Lorentzian manifolds with shearfree congruences and Kähler-Sasaki geometry, Differential Geometry and its Applications **75**, 101724 (2021).
[41] M. Ortaggio, V. Pravda, and A. Pravdová, Ricci identities in higher dimensions, Classical Quantum Gravity **24**, 1657 (2007).
[42] M. Ortaggio, J. Podolský, and M. Žofka, Robinson-Trautman spacetimes with an electromagnetic field in higher dimensions, Classical Quantum Gravity **25**, 025006 (2008).
[43] G. W. Gibbons and D. L. Wiltshire, Space-time as a membrane in higher dimensions, Nucl. Phys. **B287**, 717 (1987).
[44] H. Kodama and A. Ishibashi, Master equations for perturbations of generalized static black holes with charge in higher dimensions, Prog. Theor. Phys. **111**, 29 (2004).
[45] D. Klemm, V. Moretti, and L. Vanzo, Rotating topological black holes, Phys. Rev. D **57**, 6127 (1998); **60**, 109902(E) (1999).
[46] M. M. Caldarelli and D. Klemm, Supersymmetry of anti-de Sitter black holes, Nucl. Phys. **B545**, 434 (1999).
[47] L. Bérard Bergery, Sur de nouvelles variétés riemanniennes d'Einstein, in *Institut Elie Cartan*, Equipe de recherche associée au CNRS d'Analyse Globale n° 839 (Université de Nancy I, Nancy, 1982), Vol. 6, pp. 1–60.
[48] F. A. Bais and P. Batenburg, A new class of higher dimensional Kaluza-Klein monopole and instanton solutions, Nucl. Phys. **B253**, 162 (1985).
[49] D. N. Page and C. N. Pope, Inhomogeneous Einstein metrics on complex line bundles, Classical Quantum Gravity **4**, 213 (1987).
[50] V. Pravda, A. Pravdová, A. Coley, and R. Milson, Bianchi identities in higher dimensions, Classical Quantum Gravity **21**, 2873 (2004); **24**, 1691(C) (2007).
[51] A. Coley, R. Milson, V. Pravda, and A. Pravdová, Vanishing scalar invariant spacetimes in higher dimensions, Classical Quantum Gravity **21**, 5519 (2004).
[52] D. D. K. Chow, C. N. Pope, and E. Sezgin, Kundt spacetimes as solutions of topologically massive gravity, Classical Quantum Gravity **27**, 105002 (2010).
[53] L. M. Sokolowski, F. Occhionero, M. Litterio, and L. Amendola, Classical electromagnetic radiation in multidimensional space-times, Ann. Phys. (N.Y.) **225**, 1 (1993).
[54] M. Ortaggio and V. Pravda, Electromagnetic fields with vanishing scalar invariants, Classical Quantum Gravity **33**, 115010 (2016).
[55] M. Durkee, V. Pravda, A. Pravdová, and H. S. Reall, Generalization of the Geroch-Held-Penrose formalism to higher dimensions, Classical Quantum Gravity **27**, 215010 (2010).
[56] M. Ortaggio, Asymptotic behaviour of Maxwell fields in higher dimensions, Phys. Rev. D **90**, 124020 (2014).
[57] J. N. Goldberg and R. K. Sachs, A theorem on Petrov types, Acta Phys. Pol. Suppl. **22**, 13 (1962).
[58] W. Kundt and A. Thompson, Le tenseur de Weyl et une congruence associée de géodésiques isotropes sans distorsion, C.R. Hebd. Seances Acad. Sci. **254**, 4257 (1962).
[59] I. Robinson and A. Schild, Generalization of a theorem by Goldberg and Sachs, J. Math. Phys. (N.Y.) **4**, 484 (1963).
[60] D. Lorenz-Petzold, Higher-dimensional Taub-NUT-de Sitter solutions, Prog. Theor. Phys. **78**, 11 (1987).
[61] A. Chamblin and G. W. Gibbons, Topology and time reversal, in *String Gravity and Physics at the Planck Energy Scale*, edited by N. Sánchez and A. Zichichi (Kluwer Academic Publishers, Dordrecht/Boston/London, 1996), Vol. 476, pp. 233–253 [arXiv:gr-qc/9510006].
[62] M. Taylor-Robinson, Higher dimensional Taub-Bolt solutions and the entropy of noncompact manifolds, arXiv:hep-th/9809041.
[63] A. M. Awad and A. Chamblin, A bestiary of higher-dimensional Taub-NUT-AdS spacetimes, Classical Quantum Gravity **19**, 2051 (2002).
[64] S. Bochner, Curvature in Hermitian metric, Bull. Am. Math. Soc. **53**, 179 (1947).
[65] K. Yano, *Differential Geometry on Complex and Almost Complex Spaces* (Pergamon Press, Oxford, 1965).
[66] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry* (Interscience, New York, 1969), Vol. 2.
[67] A. Ashtekar and S. Das, Asymptotically anti-de Sitter spacetimes: Conserved quantities, Classical Quantum Gravity **17**, L17 (2000).
[68] R. Mann and C. Stelea, New Taub-NUT-Reissner-Nordström spaces in higher dimensions, Phys. Lett. B **632**, 537 (2006).
[69] A. M. Awad, Higher dimensional Taub-nuts and Taub-bolts in Einstein-Maxwell gravity, Classical Quantum Gravity **23**, 2849 (2006).
[70] M. H. Dehghani and A. Khodam-Mohammadi, Thermodynamics of Taub-NUT/bolt black holes in Einstein-Maxwell gravity, Phys. Rev. D **73**, 124039 (2006).

[71] D. R. Brill, Electromagnetic fields in a homogeneous, nonisotropic universe, Phys. Rev. **133**, B845 (1964).
[72] B. Carter, Hamilton-Jacobi and Schrodinger separable solutions of Einstein's equations, Commun. Math. Phys. **10**, 280 (1968).
[73] V. A. Ruban, Non-singular metrics of Taub-Newman-Unti-Tamburino type with an electromagnetic field, Dokl. Akad. Nauk SSSR **204**, 1086 (1972).
[74] D. Flores-Alfonso and H. Quevedo, Topological characterization of higher-dimensional charged Taub-NUT instantons, Int. J. Geom. Methods Mod. Phys. **16**, 1950154 (2019).
[75] A. Papapetrou, Champs gravitationnels stationnaires à symétrie axiale, Ann. Inst. Henri Poincaré A **4**, 83 (1966).
[76] R. M. Wald, Black hole in a uniform magnetic field, Phys. Rev. D **10**, 1680 (1974).
[77] G. W. Gibbons and C. N. Pope, $\mathbb{C}P^2$ as a gravitational instanton, Commun. Math. Phys. **61**, 239 (1978).
[78] J. Podolský and M. Papajčík, All solutions of Einstein-Maxwell equations with a cosmological constant in $2+1$ dimensions, Phys. Rev. D **105**, 064004 (2022).
[79] A. A. García Díaz, *Exact Solutions in Three-Dimensional Gravity* (Cambridge University Press, Cambridge, England, 2017).
[80] M. Ortaggio, J. Podolský, and M. Žofka, Static and radiating p-form black holes in the higher dimensional Robinson-Trautman class, J. High Energy Phys. 02 (2015) 045.
[81] N. Bahjat-Abbas, A. Luna, and C. D. White, The Kerr-Schild double copy in curved spacetime, J. High Energy Phys. 12 (2017) 004.
[82] M. Carrillo-González, R. Penco, and M. Trodden, The classical double copy in maximally symmetric spacetimes, J. High Energy Phys. 04 (2018) 028.
[83] A. Aliev, Electromagnetic properties of Kerr-anti-de Sitter black holes, Phys. Rev. D **75**, 084041 (2007).
[84] I. Robinson and A. Trautman, Some spherical gravitational waves in general relativity, Proc. R. Soc. A **265**, 463 (1962).
[85] J. M. M. Senovilla, Black hole formation by incoming electromagnetic radiation, Classical Quantum Gravity **32**, 017001 (2015).
[86] B. Carter, Black hole equilibrium states, in *Black Holes*, edited by C. De Witt and B. S. De Witt (Gordon and Breach, New York, 1973), pp. 57–214.
[87] B. Carter, A new family of Einstein spaces, Phys. Lett. **26A**, 399 (1968).
[88] J. F. Plebański, A class of solutions of Einstein-Maxwell equations, Ann. Phys. (N.Y.) **90**, 196 (1975).
[89] J. F. Plebański and M. Demiański, Rotating, charged, and uniformly accelerating mass in general relativity, Ann. Phys. (N.Y.) **98**, 98 (1976).
[90] D. Klemm, Rotating black branes wrapped on Einstein spaces, J. High Energy Phys. 11 (1998) 019.
[91] A. H. Taub, Empty space-times admitting a three parameter group of motions, Ann. Math. **53**, 472 (1951).
[92] E. Newman, L. Tamburino, and T. Unti, Empty-space generalization of the Schwarzschild metric, J. Math. Phys. (N.Y.) **4**, 915 (1963).
[93] V. Pravda, A. Pravdová, and M. Ortaggio, Type D Einstein spacetimes in higher dimensions, Classical Quantum Gravity **24**, 4407 (2007).
[94] D. Birmingham, Topological black holes in anti-de Sitter space, Classical Quantum Gravity **16**, 1197 (1999).
[95] H. W. Brinkmann, Einstein spaces which are mapped conformally on each other, Math. Ann. **94**, 119 (1925).
[96] A. Z. Petrov, *Einstein Spaces* (Pergamon Press, Oxford, 1969), translation of the 1961 Russian ed.
[97] M. Ortaggio, V. Pravda, and A. Pravdová, On higher dimensional Einstein spacetimes with a warped extra dimension, Classical Quantum Gravity **28**, 105006 (2011).