Conformal Killing gravity in static spherically symmetric spacetimes

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We identify an anisotropic divergence-free conformal Killing tensor K_{jl} for static spherically symmetric space-times, and write the conformal Killing gravity equations as Einstein equations augmented by this tensor. The field equations are of second order; this fact allows for analytic solutions and considerably simplifies the derivation of results of previous studies based on the original Harada equations. In particular, we prove the equivalence of the known third-order field equations, with the second-order ones obtained by us in the conformal Killing parametrization. The structure of the Ricci tensor and of the conformal Killing tensor are compatible with both anisotropic fluid sources and (non)linear electrodynamics. We reobtain covariantly and in simple steps the general static spherical solutions for vacuum and linear electrodynamics. Moreover we recover the purely magnetic Lagrangian functions that induce metrics of interest for black holes.

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I. CONFORMAL KILLING GRAVITY

Recently J. Harada [1,2] introduced a new theory of gravity, with field equations

$$
\begin{aligned}
H_{jkl} &= 8\pi T_{jkl} \\
H_{jkl} &= \nabla_j R_{kl} + \nabla_k R_{lj} + \nabla_l R_{jk} \\
&\quad - \frac{1}{3}(g_{kl}\nabla_j R + g_{lj}\nabla_k R + g_{jk}\nabla_l R) \\
T_{jkl} &= \nabla_j T_{kl} + \nabla_k T_{lj} + \nabla_l T_{jk} \\
&\quad - \frac{1}{6}(g_{kl}\nabla_j T + g_{lj}\nabla_k T + g_{jk}\nabla_l T). \qquad (1)
\end{aligned}
$$

R_{jk} is the Ricci tensor with trace R, and T_{kl} is the stress-energy tensor with trace T. The Bianchi identity $\nabla_j R^j{}_k = \frac{1}{2}\nabla_k R$ implies $\nabla_j T^j{}_k = 0$. Solutions of the Einstein equations are solutions of the new theory.

Shortly after, we found a parametrization showing that Harada's equations are equivalent to the Einstein equations modified by a supplemental conformal Killing tensor (CKT) that is also divergence free [3]:

$$R_{kl} - \frac{1}{2}R g_{kl} = T_{kl} + K_{kl} \qquad (2)$$

$$
\begin{aligned}
&\nabla_j K_{kl} + \nabla_k K_{jl} + \nabla_l K_{jk} \\
&\quad = \frac{1}{6}(g_{kl}\nabla_j K + g_{jl}\nabla_k K + g_{jk}\nabla_l K). \qquad (3)
\end{aligned}
$$

For this reason the theory was named conformal Killing gravity (CKG). The reformulation makes Harada's extension of general relativity (GR) explicit through the conformal Killing term, that satisfies $\nabla^k K_{kl} = 0$ and enters as a new source term in the equations.

Feng and Chen [4] proposed an action principle for CKG. Some references on geometrical and physical applications of conformal Killing tensors are [5–8].

As R_{kl} contains second order derivatives of the metric tensor, higher orders in the field equations (2) may arise with the tensor K_{jk}. This does not occur in the present work, as well as in [3], where we obtained a realization of the conformal Killing parametrization in the Friedmann-Robertson-Walker (FRW) background. In [3] the CKT has the perfect fluid form, is a candidate for representing the dark sector, and contains the scale factor with no derivatives. The Friedmann equations are thus second order in the metric, and reproduced the same forecasting obtained by Harada [2] with Eqs. (1). Vacuum cosmological solutions, and wormhole and black hole solutions were obtained by Clément and Nouicer [9]. In the FRW background, CKG is embraced by a general parametrization of Codazzi tensors [10].

In the next work [11] we deepened the geometrical aspects and the cosmological consequences. In particular we showed that the density contrast in the matter era behaved as in ΛCDM and provided a fit of the Hubble parameter versus redshift with cosmic chronometers (CC) and baryonic acoustic oscillations (BAO), with a forecast of future singularities.

Most papers on CKG dealt with a static background. Barnes [12] found the general spherically symmetric vacuum solution and [13,14] the general solution with a

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Maxwell source. Junior *et al.* [15] investigated regular black hole solutions of CKG coupled to nonlinear electrodynamics and scalar fields. Further in [16] they explored black bounce solutions in CKG coupled to nonlinear electrodynamics and scalar fields. pp-wave solutions were studied by Barnes [17].

In this paper we show a realization of the conformal Killing parametrization in a static spherically symmetric background

$$ds^2 = -b^2(r)dt^2 + f_1^2(r)dr^2 + f_2^2(r)d\Omega_2^2.$$

There are clear advantages in the parametrization approach (2): the results based on the equations (1) are here reobtained with simplicity and covariantly.

In Sec. II we recollect the covariant description of the Ricci tensor of a spherically symmetric static space-time found in [18]. Then we show that an anisotropic conformal Killing tensor is naturally hosted in such spaces. It extends the perfect fluid form recently shown by Barnes in [17] in spherical symmetry.

In Sec. III we write the field equations of CKG for an anisotropic fluid source: they are second order differential equations in the metric functions, as in GR.

An interesting result for the metric coefficients is obtained: if $f_2 = r$ then $p_r + \mu = 0$ if and only if $bf_1 = (\kappa_3 r^2 + \kappa_4)^{-1/2}$. It extends the result obtained by Barnes [13] and Clément and Noucier [9] for vacuum space-times and linear electrodynamics.

In Sec. IV the vacuum case is analyzed. We prove the equivalence of the third order differential equation (26) for $b^2(r)$ by Barnes with the second-order one obtained by us, Eq. (25). The first one descends from Harada's equations (1), and the latter descends from (2).

We show that the (unique) Schwarzschild-like solution is the vacuum by Harada and remark that it cannot originate from a perfect fluid CKT.

In Sec. V we write the CKG equations for nonlinear electrodynamics. The same result for the metric functions in Sec. III is proven in the nonlinear case.

In the linear case (electric and magnetic monopoles) the general solution by Barnes [13] and Clément and Noucier [9] is here obtained as a solution of a second-order equation.

For the purely magnetic case, we write the second-order equation (44) for b^2 with generic magnetic Lagrangian. It straightforwardly reproduces the Lagrangians obtained for black-hole solutions by Junior *et al.* [15], and a generalization of the Hayward metric.

II. ANISOTROPIC CKT

A covariant characterization [18–20] of static space-times is the existence of a timelike unit vector field that is shear, expansion, and vorticity free, with closed acceleration $\dot{u}_j = u^k\nabla_k u_j$:

$$\nabla_k u_j = -u_k\dot{u}_j, \qquad \nabla_j\dot{u}_k = \nabla_k\dot{u}_j. \tag{4}$$

With $\eta = \dot{u}^k\dot{u}_k$ the normalized acceleration vector is

$$\chi_k = \frac{\dot{u}_k}{\sqrt{\eta}}. \tag{5}$$

We focus on static spherically symmetric space-times

$$ds^2 = -b^2(r)dt^2 + f_1^2(r)dr^2 + f_2^2(r)(d\theta^2 + \sin^2\theta d\phi^2). \tag{6}$$

In this (comoving) frame, $u_0 = -b$, $u_\mu = 0$, $\dot{u}_0 = 0$, $\dot{u}_r = b'/b$, $\dot{u}_\theta = \dot{u}_\phi = 0$. A prime denotes a derivative in the variable r. Then,

$$\sqrt{\eta} = \frac{b'}{f_1 b} \tag{7}$$

and $\chi_0 = 0$, $\chi_r = f_1$, $\chi_\theta = \chi_\phi = 0$.

The covariant expression of the Ricci tensor was obtained in [18], Eq. (85):

$$R_{kl} = \frac{R + 4\nabla_p\dot{u}^p}{3}u_k u_l + \frac{R + \nabla_p\dot{u}^p}{3}g_{kl} + \Sigma(r)\left[\chi_k\chi_l - \frac{u_k u_l + g_{kl}}{3}\right], \tag{8}$$

where the coefficients are [Eqs. (87) and (89) of [18]]

$$\nabla_p\dot{u}^p = \frac{1}{bf_1^2}\left[b'' - b'\left(\frac{f_1'}{f_1} - 2\frac{f_2'}{f_2}\right)\right], \tag{9}$$

$$\Sigma(r) = -\frac{1}{bf_1^2}\left[b'' - b'\left(\frac{f_1{}'}{f_1} + \frac{f_2{}'}{f_2}\right)\right] - \frac{1}{f_1^2}\left[\frac{f_1^2}{f_2^2} + \frac{f_2{}''}{f_2} - \left(\frac{f_2{}'}{f_2}\right)^2 - \frac{f_1{}'f_2{}'}{f_1 f_2}\right]. \tag{10}$$

$R = R^k{}_k$, and the curvature $R^\star$ of the spacelike submanifold [Eqs. (90) and (91) of [18]] are:

$$R = R^\star - 2\nabla_p\dot{u}^p, \tag{11}$$

$$R^\star = \frac{2}{f_2^2} - \frac{2}{f_1^2}\left[2\frac{f_2''}{f_2} - 2\frac{f_1' f_2'}{f_1 f_2} + \left(\frac{f_2'}{f_2}\right)^2\right]. \tag{12}$$

In the background (6) we consider a symmetric tensor with the anisotropic structure of the Ricci tensor:

$$K_{kl} = \mathsf{A}(r)u_k u_l + \mathsf{B}(r)g_{kl} + \mathsf{C}(r)\chi_k\chi_l. \tag{13}$$

The following result shows that a static spherically symmetric space-time always hosts an anisotropic conformal Killing tensor (the proof is in Appendix B):

Theorem 1. In the static spherically symmetric spacetime (6) the tensor (13) is a divergence-free conformal Killing tensor if and only if

$$\mathsf{A} = \kappa_2 f_2^2 - 2\kappa_3 b^2,$$
$$\mathsf{B} = \kappa_1 + 2\kappa_2 f_2^2 + \kappa_3 b^2$$
$$\mathsf{C} = -\kappa_2 f_2^2,$$

where κ_1, κ_2 and κ_3 are arbitrary constants.

A, B, and C are combinations of the metric functions b, f_1, f_2 with no derivatives. Therefore, the CKG equations (2) and (3) contain derivatives of the metric not higher than second order.

Example 1: Perfect fluid CKT. $Z_j = bu_j$ is a Killing vector, $\nabla_j Z_k + \nabla_k Z_j = 0$. Following the strategy proposed in [7], $K_{jk} = \alpha Z_j Z_k + \beta g_{jk}$ with constant α and scalar function β, is a CKT. The divergence-free condition is $\beta = -\frac{1}{2}\alpha b^2 + \gamma$. The resulting tensor is:

$$K_{jk} = \alpha b^2\left(u_j u_k - \frac{1}{2} g_{jk}\right) + \gamma g_{jk}$$

is the divergence-free CKT recently shown by Barnes in [17]. It is a particular case of Theorem 1 with $\kappa_2 = 0$.

III. CKG FOR THE ANISOTROPIC FLUID

The structures of the Ricci tensor (8) and of the CKT (13) are compatible with the stress-energy tensor of an anisotropic fluid without heat flow:

$$T_{kl} = (\mu + p_\perp)u_k u_l + p_\perp g_{kl} + (p_r - p_\perp)\chi_k\chi_l. \tag{14}$$

μ is the energy density, p_r and $p_\perp$ are the radial and transversal pressures in the comoving frame, and $\chi_k = \dot{u}_k/\sqrt{\eta}$. The tensor is rewritten as

$$T_{kl} = (\mu + P)u_k u_l + P g_{kl} + (p_r - p_\perp)\left[\chi_k\chi_l - \frac{g_{kl} + u_k u_l}{3}\right],$$

where $P = \frac{1}{3}(p_r + 2p_\perp)$ is the total pressure. The CKG equations with the fluid source are

$$\begin{aligned}&\frac{R + 4\nabla_p \dot{u}^p}{3} u_k u_l + \frac{2\nabla_p \dot{u}^p - R}{6} g_{kl} + \Sigma\left[\chi_k\chi_l - \frac{u_k u_l + g_{kl}}{3}\right]\\ &= (\mu + P)u_k u_l + P g_{kl} + (p_r - p_\perp)\left[\chi_k\chi_l - \frac{u_k u_l + g_{kl}}{3}\right]\\ &\quad + \left[\mathsf{A} + \frac{\mathsf{C}}{3}\right] u_k u_l + \left[\mathsf{B} + \frac{\mathsf{C}}{3}\right] g_{kl} + \mathsf{C}\left[\chi_k\chi_l - \frac{u_k u_l + g_{kl}}{3}\right].\end{aligned}$$

They give three scalar equations:

$$\begin{aligned}\frac{1}{3}(R + 4\nabla_p \dot{u}^p) &= (\mu + P) + \mathsf{A} + \frac{1}{3}\mathsf{C}\\ \frac{1}{6}(2\nabla_p \dot{u}^p - R) &= P + \mathsf{B} + \frac{1}{3}\mathsf{C}\\ \Sigma &= (p_r - p_\perp) + \mathsf{C}.\end{aligned}$$

Rearranging terms and using $R = R^\star - 2\nabla_p \dot{u}^p$ and Theorem 1 we obtain the following.

Proposition 1. The field equations of CKG in the static spherically symmetric metric (6) with the anisotropic CKT Eq. (13) are

$$\begin{aligned}\frac{R^\star}{2} &= \mu - 3\kappa_3 b^2 - \kappa_2 f_2^2 - \kappa_1\\ \nabla_p \dot{u}^p &= \frac{3}{2}P + \frac{1}{2}\mu + 2\kappa_2 f_2^2 + \kappa_1\\ \Sigma &= (p_r - p_\perp) + \mathsf{C}.\end{aligned} \tag{15}$$

Remark 1. Let $f_2 = r$ and consider Eqs. (15) for a perfect fluid ($p_r = p_\perp$). In GR they are $\mu = \frac{1}{2}R^\star$, $P = \frac{2}{3}\nabla_p \dot{u}^p - \frac{1}{6}R^\star$ and $\Sigma = 0$: three equations for the unknowns μ, p, b, and f_1. A further condition, such as an equation of state, is needed. The same occurs for the conformal Killing equations.

A. Properties of metric functions

Hereafter we consider the static spherical metric with $f_2 = r$, and set $f_1 = h/b$.

$$ds^2 = -b^2(r)dt^2 + \frac{h^2(r)}{b^2(r)}dr^2 + r^2 d\Omega^2. \tag{16}$$

We prove a remarkable property of the metric function $h(r)$. First we assert the following geometric result.

Lemma 1.

$$\frac{R^\star}{2} + \nabla_p \dot{u}^p + \Sigma = \frac{3b^2}{r}\frac{h'}{h^3} \tag{17}$$

Proof. Using (9), (10), and (12) the following relations are straightforwardly obtained:

$$R^\star = \frac{2}{r^2} + \frac{4b^2}{r}\frac{h'}{h^3} - \frac{4bb'}{rh^2} - \frac{2}{r^2}\frac{b^2}{h^2} \tag{18}$$

$$\nabla_p \dot{u}^p = \frac{1}{h^2}(b'^2 + bb'') - \frac{bb'}{h^2}\frac{h'}{h} + \frac{2bb'}{rh^2} \tag{19}$$

$$\Sigma = -\frac{1}{h^2}(b'^2 + bb'') + bb'\frac{h'}{h^3} + \frac{1}{r^2}\frac{b^2}{h^2} - \frac{1}{r^2} + \frac{b^2}{r}\frac{h'}{h^3}. \tag{20}$$

The result (17) follows. ∎

We are ready to prove the following.

Proposition 2. Consider CKG in the metric (16) with an anisotropic fluid source. Then

$$\frac{1}{h^2} = \kappa_3 r^2 + \kappa_4 \Leftrightarrow p_r = -\mu. \tag{21}$$

Proof. The addition of Eqs. (15) with $\mathsf{C} = -\kappa_2 r^2$ and $P = \frac{1}{3}(p_r + 2p_\perp)$ gives

$$\frac{R^\star}{2} + \nabla_p \dot{u}^p + \Sigma = \frac{3}{2}(p_r + \mu) - 3\kappa_3 b^2.$$

With (17) we obtain

$$\frac{2b^2}{r}\frac{h'}{h^3} + 2\kappa_3 b^2 = p_r + \mu.$$

If $p_r + \mu = 0$ it is $\frac{h'}{h^3} = -\kappa_3 r$ that integrates to (21) with κ_4 a constant. Conversely if $h^{-2} = \kappa_3 r^2 + \kappa_4$ then $p_r + \mu = 0$. ∎

If $p_r + \mu = 0$ it is

$$f_1(r) = \frac{1}{b(r)\sqrt{\kappa_3 r^2 + \kappa_4}}. \tag{22}$$

Equation (22) was obtained by Barnes [12] while investigating vacuum solutions and also in solving CKG field equations of linear electrodynamics [13].

IV. VACUUM SOLUTIONS

In the absence of matter Eqs. (15) become

$$\begin{aligned} R^\star &= -6\kappa_3 b^2 - 2\kappa_2 r^2 - 2\kappa_1 \\ \nabla_p \dot{u}^p &= 2\kappa_2 r^2 + \kappa_1 \\ \Sigma &= \mathsf{C} = -\kappa_2 r^2. \end{aligned} \tag{23}$$

It is also $h^{-2} = \kappa_3 r^2 + \kappa_4$. The remaining metric function b^2 is determined by $\Sigma = \mathsf{C}$. With the substitution $y = b^2$ the anisotropic term (20) now is

$$\Sigma = -(\kappa_3 r^2 + \kappa_4)\frac{y''}{2} - \kappa_3 r\frac{y'}{2} + \kappa_4\frac{y}{r^2} - \frac{1}{r^2}, \tag{24}$$

and the equation $\Sigma = -\kappa_2 r^2$ for $y(r)$ is

$$(\kappa_3 r^2 + \kappa_4)y'' + \kappa_3 r y' - 2\frac{\kappa_4}{r^2}y + \frac{2}{r^2} = 2\kappa_2 r^2. \tag{25}$$

Proposition 3. The second-order equation (25) is equivalent to the third-order equation obtained by Barnes in [12], Eq. (16), with computer algebra:

$$\begin{aligned} &(\kappa_3 r^2 + \kappa_4)r^3 y''' + (\kappa_3 r^2 - 2\kappa_4)r^2 y'' \\ &\quad - (\kappa_3 r^2 + 2\kappa_4)r y' + 8\kappa_4 y = 8. \end{aligned} \tag{26}$$

Proof. The equation $\Sigma = \mathsf{C}$ gives $\Sigma' = \mathsf{C}'$. Being $\mathsf{C} = -\kappa_2 r^2$ it is $\mathsf{C}' = \frac{2}{r}\mathsf{C}$. Thus the anisotropic term satisfies $\Sigma' = \frac{2}{r}\Sigma$, and the integral is $\Sigma = \mathsf{C}$:

$$\Sigma' - \frac{2}{r}\Sigma = 0 \Leftrightarrow \Sigma = \mathsf{C}. \tag{27}$$

The left-hand side of (27) is evaluated,

$$\begin{aligned} \Sigma' - \frac{2}{r}\Sigma = &-\frac{y'''}{2}(\kappa_3 r^2 + \kappa_4) - y''\left(\frac{\kappa_3}{2}r - \frac{\kappa_4}{r}\right) \\ &+ y'\left(\frac{\kappa_3}{2} + \frac{\kappa_4}{r^2}\right) - y\frac{4\kappa_4}{r^3} + \frac{4}{r^3}, \end{aligned}$$

and entails the equation by Barnes. ∎

The second-order equation (25) is solved ($x = r\sqrt{\kappa_3}$, $\kappa_4 = 1$): the sum of the homogeneous (with coefficients $c_{1,2}$) and the inhomogeneous solutions,

$$\begin{aligned} b^2(r) = &c_1\frac{\sqrt{1+x^2}}{x} + c_2\left[\frac{\sqrt{1+x^2}}{x}\mathrm{arcsh}x - 1\right] \\ &+ 1 - \frac{3\kappa_2}{2\kappa_3^2}\left[\frac{\sqrt{1+x^2}}{x}\mathrm{arcsh}x - 1 - \frac{x^2}{3}\right]. \end{aligned} \tag{28}$$

Example 2: Case $\mathbf{h = 1}$. The vacuum static spherical solution of CKG with $h = 1$ (i.e., $\kappa_3 = 0$ and $\kappa_4 = 1$) was found by Harada [1], and extends the Schwarzschild-de Sitter solution of GR:

$$b^2 = 1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2 - \frac{\lambda}{5}r^4. \tag{29}$$

The following result holds.

Proposition 4. Equation (29) is the unique solution with $h = 1$ of the vacuum equations (23) with $\kappa_1 = -\Lambda$, $\kappa_2 = -\lambda$.

Proof. With $\kappa_3 = 0$ and $\kappa_4 = 1$ in (25), the equation $\Sigma = \mathsf{C}$ is the Euler equation $r^2 y'' - 2y = -2 + 2\kappa_2 r^4$ with solution

$$y = b^2 = 1 + \frac{c_1}{r} + c_2 r^2 + \frac{1}{5}\kappa_2 r^4. \tag{30}$$

Thus $c_1 = -2M$, $c_2 = -\frac{1}{3}\Lambda$, and $\lambda = -\kappa_2$.

The solution must also solve the other equations in (23): $R^\star = -2\kappa_2 r^2 - 2\kappa_1$ and $\nabla_p \dot{u}^p = 2\kappa_2 r^2 + \kappa_1$.

With $h = 1$ and $b^2 = y$ Eqs. (18) and (19) become

$$\begin{aligned} R^\star &= -\frac{2}{r}y' - \frac{2}{r^2}y + \frac{2}{r^2} = 2\lambda r^2 + 2\Lambda \\ \nabla_p \dot{u}^p &= \frac{1}{2}y'' + \frac{1}{r}y' = -2\lambda r^2 - \Lambda. \end{aligned}$$

The equations are satisfied with $\kappa_1 = -\Lambda$. ∎

The Schwarzschild de-Sitter metric occurs if and only if $\kappa_2 = 0$, i.e., $\mathsf{C} = 0$, giving the perfect fluid CKT. Then, a perfect fluid CKT in the field equations cannot originate Harada's vacuum solution (29).

V. (NON)LINEAR ELECTRODYNAMICS IN CKG

In [18] we specified the covariant form of the stress-energy tensor of nonlinear electrodynamics in static spherically symmetric space-times:

$$T^{nlin}_{jk} = 2(\mathbb{E}^2 + \mathbb{B}^2)[u_j u_k - \chi_j \chi_k]\mathcal{L}_F(F) + 2g_{jk}[\mathbb{B}^2\mathcal{L}_F(F) - \mathcal{L}(F)]. \tag{31}$$

$F = \frac{1}{4}F_{jk}F^{jk}$ is the Faraday scalar, $\mathcal{L}_F = d\mathcal{L}/dF$. In the static setting it is $F = \frac{1}{2}(\mathbb{B}^2 - \mathbb{E}^2)$, where we have the following.

Proposition 5.

$$\mathbb{E}(r) = \frac{q_e}{r^2\mathcal{L}_F(F)}, \qquad \mathbb{B} = \frac{q_m}{r^2}. \tag{32}$$

Proof. Equation (28) in [18] is $\nabla_j[\chi^j\mathbb{E}\mathcal{L}_F] = \sqrt{\eta}\mathbb{E}\mathcal{L}_F$. Explicitly, using (7) and the formulas in Appendix A,

$$\left(\frac{2}{rf_1} + \frac{b'}{f_1 b}\right)(\mathbb{E}\mathcal{L}_F) + \frac{1}{f_1}\frac{d}{dr}(\mathbb{E}\mathcal{L}_F) = \frac{b'}{f_1 b}(\mathbb{E}\mathcal{L}_F).$$

Terms cancel, and the linear equation gives the first result, where q_e is an integration constant.

Equation (18) in [18] is $\sqrt{\eta}\chi^p\nabla_p\mathbb{B} = \mathbb{B}[\nabla_p\dot{u}^p - \eta - \frac{\chi^s\nabla_s\eta}{2\sqrt{\eta}}]$. Using (A4) the equation is integrated and yields the monopole solution. (Different deductions of the proposition are in [21,22].) ∎

The tensor structure of the Ricci and of the stress-energy tensors is matched by the conformal Killing tensor. Thus the field equations of CKG are

$$\begin{aligned}
&\frac{R + 4\nabla_p\dot{u}^p}{3}u_k u_l + \frac{2\nabla_p\dot{u}^p - R}{6}g_{kl} + \Sigma(r)\left[\chi_k\chi_l - \frac{u_k u_l + g_{kl}}{3}\right] \\
&= \frac{4}{3}(\mathbb{E}^2 + \mathbb{B}^2)\mathcal{L}_F(F)u_k u_l \\
&\quad + 2\left[\frac{1}{3}(2\mathbb{B}^2 - \mathbb{E}^2)\mathcal{L}_F(F) - \mathcal{L}(F)\right]g_{kl} \\
&\quad - 2(\mathbb{E}^2 + \mathbb{B}^2)\mathcal{L}_F(F)\left[\chi_k\chi_l - \frac{u_k u_l + g_{kl}}{3}\right] \\
&\quad + \left(\mathsf{A} + \frac{\mathsf{C}}{3}\right)u_k u_l + \left(\mathsf{B} + \frac{\mathsf{C}}{3}\right)g_{kl} + \mathsf{C}\left[\chi_k\chi_l - \frac{u_k u_l + g_{kl}}{3}\right].
\end{aligned}$$

By equating the coefficients one obtains three scalar equations. A rearrangement of terms and use of $R = R^\star - 2\nabla_p\dot{u}^p$ give

$$\begin{aligned}
\frac{1}{2}R^\star &= 2\mathcal{L}_F(F)\mathbb{E}^2 + 2\mathcal{L}(F) + \mathsf{A} - \mathsf{B} \\
\nabla_p\dot{u}^p &= 2\mathcal{L}_F(F)\mathbb{B}^2 - 2\mathcal{L}(F) + \frac{\mathsf{A}}{2} + \mathsf{B} + \frac{\mathsf{C}}{2} \\
\Sigma &= -2(\mathbb{E}^2 + \mathbb{B}^2)\mathcal{L}_F(F) + \mathsf{C}.
\end{aligned} \tag{33}$$

Also in this case we are able to show the validity of (22). The following result holds.

Proposition 6. Consider CKG coupled with nonlinear electrodynamics in the metric (16). Then

$$\frac{1}{h^2} = \kappa_3 r^2 + \kappa_4.$$

Proof. The sum of Eqs. (33) is $\frac{1}{2}R^\star + \nabla_p\dot{u}^p + \Sigma = \frac{3}{2}(\mathsf{A} + \mathsf{C})$. Now use Lemma 1 and $\mathsf{A} + \mathsf{C} = -2\kappa_3 b^2$. ∎

A. Linear electrodynamics

In linear electrodynamics $\mathcal{L}(F) = F = \frac{1}{2}(\mathbb{B}^2 - \mathbb{E}^2)$. The electric field is Coulomb $\mathbb{E}(r) = q_e/r^2$, so that

$$\mathbb{E}^2 + \mathbb{B}^2 = \frac{q^2}{r^4},$$

with $q^2 = q_e^2 + q_m^2$. Equations (33) take the form

$$\frac{1}{2}R^\star = \frac{q^2}{r^4} - 3\kappa_3 b^2 - \kappa_2 r^2 - \kappa_1 \tag{34}$$

$$\nabla_p\dot{u}^p = \frac{q^2}{r^4} + 2\kappa_2 r^2 + \kappa_1 \tag{35}$$

$$\Sigma = -2\frac{q^2}{r^4} + \mathsf{C}. \tag{36}$$

Σ is given by (24) and $\mathsf{C} = -\kappa_2 r^2$. Equation (36) is the second-order differential equation for $y = b^2$:

$$(\kappa_3 r^2 + \kappa_4)y'' + \kappa_3 r y' - 2\kappa_4\frac{y}{r^2} + \frac{2}{r^2} = 2\kappa_2 r^2 + \frac{4q^2}{r^4}. \tag{37}$$

Proposition 7. In CKG coupled with linear electrodynamics the second-order equation for $b^2(r)$ in a static spherically symmetric background is equivalent to the following third-order equation:

$$(\kappa_3 r^2 + \kappa_4)r^3 y''' + (\kappa_3 r^2 - 2\kappa_4)r^2 y'' - (\kappa_3 r^2 + 2\kappa_4)r y' + 8\kappa_4 y = 8 - \frac{24q^2}{r^2}. \tag{38}$$

This is Eq. (14) worked by Barnes in [13] and the "master equation" by Clément and Noucier [9].

Proof. As in the vacuum case, $\Sigma = -2\frac{q^2}{r^4} + \mathsf{C}$ gives $\Sigma' = \frac{8q^2}{r^5} + \mathsf{C}'$. Since $\mathsf{C} = -\kappa_2 r^2$ it is $\mathsf{C}' = \frac{2}{r}\mathsf{C} = \frac{2}{r}[\Sigma + 2\frac{q^2}{r^4}]$.

Thus the anisotropic term satisfies $\Sigma' - \frac{2}{r}\Sigma = \frac{12q^2}{r^5}$; on the other hand the integral is $\Sigma = -2\frac{q^2}{r^4} + \mathsf{C}$. The following equivalence holds:

$$\Sigma' - \frac{2}{r}\Sigma = \frac{12q^2}{r^5} \Leftrightarrow \Sigma = -2\frac{q^2}{r^4} + \mathsf{C}. \tag{39}$$

Using (24) the left-hand side of (39) is evaluated and gives the third-order equation. ∎

As a check, the second-order equation (37) is solved for $\kappa_4 = 1$, $x = r\sqrt{\kappa_3}$. The solution, evaluated with *Mathematica*, coincides with that by Clément and Noucier [9]:

$$y(x) = c_1\frac{\sqrt{1+x}}{x} + c_2\left[\frac{\sqrt{1+x^2}}{x}\mathrm{arcsh}(x) - 1\right] + 1$$
$$-\frac{3\kappa_2}{2\kappa_3^2}\left[\frac{\sqrt{1+x^2}}{x}\mathrm{arcsh}(x) - 1 - \frac{x^2}{3}\right] + \kappa_3 q^2\left(2 + \frac{1}{x^2}\right).$$

Example 3: Case $\mathbf{h = 1}$. The metric function

$$b^2(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2 + \frac{q^2}{r^2} - \frac{\lambda}{5}r^4 \tag{40}$$

was obtained in [13,15] with different strategies. In [15] Junior, Lobo, and Rodriguez posed a functional form of $b(r)$ including the term $-\frac{\lambda}{5}r^4$ that characterizes CKG, and then engineered a nonlinear electrodynamics Lagrangian and its derivative. On the other hand, in [13] Barnes used computer algebra to obtain the same and more general solutions.

Here the solution (40) is achieved in a much simpler way.

Proposition 8. The unique solution of (34)–(36) with $h = 1$ (i.e., $\kappa_3 = 0$, $\kappa_4 = 1$) is the metric function (40), with $\kappa_1 = -\Lambda$, $\kappa_2 = -\lambda$.

Proof. The equation $\Sigma = -2\frac{q^2}{r^4} + \mathsf{C}$ with Σ in (24) and $y = b^2$ is Euler's equation $r^2 y'' - 2y + 2 = 4\frac{q^2}{r^2} + 2\kappa_2 r^4$ with the solution

$$y = b^2 = 1 + \frac{c_1}{r} + c_2 r^2 + \frac{q^2}{r^2} + \frac{\kappa_2}{5}r^4. \tag{41}$$

This is (40) with $c_1 = -2M$, $c_2 = -\frac{\Lambda}{3}$, and $\lambda = -\kappa_2$.

Let us show that it also solves (34) and (35):

$$\frac{1}{2}R^\star = q^2/r^4 - \kappa_2 r^2 - \kappa_1$$
$$\nabla_p \dot{u}^p = q^2/r^4 + 2\kappa_2 r^2 + \kappa_1.$$

With (40), $h = 1$, and $b^2 = y$, Eqs. (18) and (19) give

$$\frac{1}{2}R^\star = \frac{1}{r^2} - \frac{1}{r}y' - \frac{1}{r^2}y = q^2/r^4 - \kappa_2 r^2 - 3c_2$$
$$\nabla_p \dot{u}^p = \frac{1}{2}y'' + \frac{1}{r}y' = q^2/r^4 + 2\kappa_2 r^2 + 3c_2.$$

Equality is achieved with $\kappa_1 = 3c_2$. ∎

Example 4. For $\kappa_3 = 1$ and $\kappa_4 = 0$ Eq. (37) is solved by

$$y = \lambda - \frac{\Lambda}{3}r^2 + m\ln r - \frac{1}{2r^2} + \frac{q^2}{4r^4}. \tag{42}$$

This is Eq. (19) in Barnes or (2.21) in Clément and Nouicer.

B. Nonlinear electrodynamics: Purely magnetic solutions

In this case $\mathbb{E}(r) = 0$ and $\mathbb{B} = q_m/r^2$ (magnetic monopole). Then $F = \frac{\mathbb{B}^2}{2} = \frac{q_m^2}{2r^4}$. The third equation of (33) is

$$\Sigma = -2\frac{q_m^2}{r^4}\mathcal{L}_F(F) + \mathsf{C}.$$

Now $\mathcal{L}_F = \frac{d\mathcal{L}}{dF} = \frac{d\mathcal{L}}{dr}\frac{dr}{dF} = -\mathcal{L}'\frac{r^5}{2q_m^2}$, so that the previous equation rewrites as

$$\Sigma = r\mathcal{L}' + \mathsf{C}. \tag{43}$$

The knowledge of $\mathcal{L}(r)$ in (43) gives a second-order differential equation for the metric function $b^2 = y$:

$$(\kappa_3 r^2 + \kappa_4)y'' + \kappa_3 r y' - \frac{2\kappa_4}{r^2}y + \frac{2}{r^2} = -2(r\mathcal{L}' - \kappa_2 r^2). \tag{44}$$

Example 5: Case $\mathbf{h = 1}$. For $\kappa_3 = 0$ and $\kappa_4 = 1$ the solution by the method of variation of parameters is

$$y(r) = 1 + c_1 r^2 + \frac{c_2}{r} - \frac{\kappa_2}{5}r^4 - \frac{2}{r}\int^r dr' r'^2 \mathcal{L}(r'). \tag{45}$$

Given $\mathcal{L}(F)$, $F = q_m^2/(2r^4)$, one evaluates $b(r)$.

In [15] a different strategy is pursued: an input metric with faithful properties (such as regularity in the origin) is chosen, and the corresponding Lagrangian is reconstructed.

Here this procedure is greatly facilitated by Eq. (43). It is illustrated to reproduce two interesting examples:

(i) Metric Eq. (34) in [15]:

$$y = 1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2 + \frac{q_m^2}{r^2} - \frac{\lambda}{5}r^4 + \frac{k_0}{r^4}. \tag{46}$$

With $\Sigma = \frac{y-1}{r^2} - \frac{1}{2}y''$ we get $\Sigma = \lambda r^2 - \frac{2q_m^2}{r^4} - 9\frac{k_0}{r^6}$. Equation (43) is easily solved by

$$\mathcal{L} = \phi_0 + \frac{q_m^2}{2r^4} + \phi_1 r^2 + 3\frac{k_0}{2r^6} = \phi_0 + F + \phi_1 \frac{q_m}{\sqrt{2F}} + 3\frac{k_0(2F)^{3/2}}{2q_m^3} \tag{47}$$

with constants ϕ_0, ϕ_1. This is Eq. (41) in [15].

(ii) Bardeen-type metric Eq. (72) in [15]):

$$y = 1 - \frac{2Mr^2}{(q_m^2 + r^2)^{3/2}} - \frac{\Lambda}{3}r^2 - \frac{\lambda}{5}r^4. \tag{48}$$

Now $\Sigma = \lambda r^2 - \frac{15Mr^2}{(q_m^2+r^2)^{7/2}}$. Equation (43) is easily solved:

$$\mathcal{L} = \frac{3Mq_m^2}{(q_m^2 + r^2)^{5/2}} + \phi_1 r^2 + \phi_0, \tag{49}$$

with ϕ_0 and ϕ_1 constants. This is Eq. (73) in [15]. With $F = q_m^2/(2r^4)$, $\mathcal{L}$ can be rewritten as

$$\mathcal{L}(F) = \frac{3Mq_m^2}{\left(q_m^2 + \frac{q_m}{\sqrt{2F}}\right)^{5/2}} + \phi_1 \frac{q_m}{\sqrt{2F}} + \phi_0. \tag{50}$$

This is Eq. (77) presented in [15].

This is a new example:

(iii) Hayward-like solution. The metric

$$b^2 = 1 - \frac{2Mr^2}{q_m^3 + r^3} - \frac{\Lambda}{3}r^2 - \frac{\lambda}{5}r^4 \tag{51}$$

is the Hayward black-hole when $\lambda, \Lambda = 0$ [23]. It is inferred that $\Sigma = \lambda r^2 - \frac{18Mr^3 q_m^3}{(q_m^3+r^3)^3}$. Then (43) is solved by

$$\mathcal{L} = \frac{3Mq_m^3}{(q_m^3 + r^3)^2} + \phi_1 r^2 + \phi_0 = \frac{3M(2F)^{3/2}}{(1 + (2Fq_m^2)^{3/4})^2} + \phi_1 \frac{q_m}{\sqrt{2F}} + \phi_0. \tag{52}$$

VI. CONCLUSIONS

We have shown that static spherically symmetric spacetimes naturally host an anisotropic divergence-free conformal Killing tensor (Theorem 1). This makes the parametrization (2) of the Harada equations as modified Einstein equations effective for such a background.

The CKG equations can support an anisotropic fluid source as well as (non)linear electrodynamics. In both cases the equations are second order. This is a great advantage with respect to the solution by components of the third-order Harada equations found in the existing literature. We prove the equivalence of our second-order equations with the third-order ones. Our approach recovers several results obtained so far in a simple and covariant way, and gives some new ones.

APPENDIX A: USEFUL FORMULAS

We collect useful formulas for static spherical spacetimes, taken from [18].

A scalar function $F(r)$ has gradient in the radial direction, given by the unit spacelike vector χ:

$$\nabla_j F = \chi_j \chi^k \nabla_k F = \chi_j \frac{F'}{f_1}$$

(the prime is d/dr). In particular,

$$\frac{\chi^s \nabla_s \eta}{2\sqrt{\eta}} = \frac{1}{2\sqrt{\eta}} \frac{1}{f_1} \frac{d}{dr}\left(\frac{b'^2}{b^2 f_1^2}\right) = \frac{1}{f_1^2}\left[\frac{b''}{b} - \frac{b'^2}{b^2} - \frac{b' f_1'}{b f_1}\right]. \tag{A1}$$

The vector $\chi_k = \dot{u}_k/\sqrt{\eta}$ is normalized, $\chi^k \chi_k = 1$.

Its covariant derivative is

$$\nabla_j \chi_k = \frac{\nabla_j \dot{u}_k}{\sqrt{\eta}} - \frac{\chi^s \nabla_s \eta}{2\eta} \chi_j \chi_k. \tag{A2}$$

Since $\dot{u}_k$ is closed, χ_j is also closed: $\nabla_j \chi_k = \nabla_k \chi_j$. A consequence is $\chi^j \nabla_j \chi_k = \chi^j \nabla_k \chi_j = 0$.

The gradient of the acceleration is [Eq. (16) in [19]]

$$\nabla_j \dot{u}_k = -\eta u_k u_j + \frac{\chi^s \nabla_s \eta}{2\sqrt{\eta}} \chi_j \chi_k + \frac{1}{2} N_{jk}\left[\nabla_p \dot{u}^p - \eta - \frac{\chi^s \nabla_s \eta}{2\sqrt{\eta}}\right], \tag{A3}$$

where $N_{jk} = g_{jk} + u_j u_k - \chi_j \chi_k$ is a projection.

Equation (87) in [18], and Eqs. (7) and (A1) give

$$\nabla_p \dot{u}^p - \eta - \frac{\chi^s \nabla_s \eta}{2\sqrt{\eta}} = \frac{2}{f_1^2}\left[\frac{b'}{b} \frac{f_2'}{f_2}\right]. \tag{A4}$$

One then obtains

$$\nabla_j \chi_k = \frac{1}{f_1}\left[\frac{f_2'}{f_2} - \frac{b'}{b}\right] u_k u_j + \frac{f_2'}{f_1 f_2}(g_{jk} - \chi_j \chi_k) \tag{A5}$$

$$\nabla_p \chi^p = 2\frac{f_2'}{f_1 f_2} + \frac{b'}{f_1 b}. \tag{A6}$$

APPENDIX B

Let $K_{kl} = \mathsf{A} u_k u_l + \mathsf{B} g_{kl} + \mathsf{C} \chi_k \chi_l$.

The conformal Killing condition (3) with the static equation (4) for u_i and the closedness of χ_i is

$$0=\nabla_i\left(\mathsf{B}-\frac{1}{6}K\right)g_{jk}+\left(\nabla_j\mathsf{B}-\frac{1}{6}K\right)g_{ki}+\left(\nabla_k\mathsf{B}-\frac{1}{6}K\right)g_{ij}$$
$$+(\nabla_i\mathsf{A}-2\mathsf{A}\dot{u}_i)u_ju_k+(\nabla_j\mathsf{A}-2\mathsf{A}\dot{u}_j)u_ku_i$$
$$+(\nabla_k\mathsf{A}-2\mathsf{A}\dot{u}_k)u_iu_j+(\nabla_i\mathsf{C})\chi_j\chi_k+(\nabla_j\mathsf{C})\chi_k\chi_i$$
$$+(\nabla_k\mathsf{C})\chi_i\chi_j+2\mathsf{C}(\chi_i\nabla_j\chi_k+\chi_j\nabla_k\chi_i+\chi_k\nabla_i\chi_j).$$

Since A only depends on r, it is $\nabla_i\mathsf{A}=\chi_i\chi^k\nabla_k\mathsf{A}=\chi_i\mathsf{A}'/f_1$, and similarly for B and C and K. The equation, multiplied by f_1 becomes

$$0=\left(\mathsf{B}'-\frac{1}{6}K'\right)(\chi_ig_{jk}+\chi_jg_{ki}+\chi_kg_{ij})$$
$$+\left(\mathsf{A}'-2\frac{b'}{b}\mathsf{A}\right)(\chi_iu_ju_k+\chi_ju_ku_i+\chi_ku_iu_j)+3\mathsf{C}'\chi_i\chi_j\chi_k$$
$$+2f_1\mathsf{C}(\chi_i\nabla_j\chi_k+\chi_j\nabla_k\chi_i+\chi_k\nabla_i\chi_j). \quad \text{(B1)}$$

Contraction with $\chi^i\chi^j\chi^k$:

$$0=3\mathsf{B}'+3\mathsf{C}'-\frac{1}{2}K'=\frac{1}{2}(\mathsf{A}'+2\mathsf{B}'+5\mathsf{C}'). \quad \text{(B2)}$$

The identity is used to simplify K' and A' from the equation

$$0=\left(\mathsf{A}'-2\frac{b'}{b}\mathsf{A}\right)(\chi_iu_ju_k+\chi_ju_ku_i+\chi_ku_iu_j)$$
$$+\mathsf{C}'[3\chi_i\chi_j\chi_k-(\chi_ig_{jk}+\chi_jg_{ki}+\chi_kg_{ij})]$$
$$+2f_1\mathsf{C}(\chi_i\nabla_j\chi_k+\chi_j\nabla_k\chi_i+\chi_k\nabla_i\chi_j).$$

Contraction with $u^iu^j\chi^k$: $0=(\mathsf{A}'-2\frac{b'}{b}\mathsf{A})+\mathsf{C}'+2f_1\mathsf{C}u^j\dot{\chi}_j$. It is $u^j\dot{\chi}_j=-\dot{u}^j\chi_j=-\sqrt{\eta}=-b'/(bf_1)$. Then, $\mathsf{A}'+\mathsf{C}'=2\frac{b'}{b}(\mathsf{A}+\mathsf{C})$, with solution $\mathsf{A}+\mathsf{C}=-2\kappa_3b^2(r)$. Contraction with $g^{ij}\chi^k$:

$$0=-\mathsf{A}'+2\frac{b'}{b}\mathsf{A}-3\mathsf{C}'+2f_1\mathsf{C}\nabla^j\chi_j.$$

Use (A6) and obtain

$$0=-(\mathsf{A}'+\mathsf{C}')+2\frac{b'}{b}(\mathsf{A}+\mathsf{C})-2\mathsf{C}'+4\frac{f_2'}{f_2}\mathsf{C}.$$

It follows that $\mathsf{C}=-\kappa_2f_2^2$. Then $\mathsf{A}=-2\kappa_3b^2+\kappa_2f_2^2$. Equation (B2) gives $\mathsf{B}=\kappa_3b^2+2\kappa_2f_2^2+\kappa_1$ and $K=6\kappa_3b^2+6\kappa_2f_2^2+4\kappa_1$.

The found parameters $\mathsf{A},\mathsf{B},\mathsf{C}$ are inserted in (B1). Up to a factor of $2\kappa_2f_1f_2^2$ it is

$$0=\frac{f_2{}'}{f_1f_2}(\chi_ig_{jk}+\chi_jg_{ki}+\chi_kg_{ij})$$
$$+\left[\frac{f_2{}'}{f_1f_2}-\frac{b'}{bf_1}\right](\chi_iu_ju_k+\chi_ju_ku_i+\chi_ku_iu_j)$$
$$-3\frac{f_2{}'}{f_2f_1}\chi_i\chi_j\chi_k-(\chi_i\nabla_j\chi_k+\chi_j\nabla_k\chi_i+\chi_k\nabla_i\chi_j). \quad \text{(B3)}$$

With the expression (A5) the conformal Killing equation is satisfied for any choice of the constants.

Now we prove the opposite statement: in the metric (6), if $\mathsf{A},\mathsf{B},\mathsf{C}$ are those in Theorem 1, then the tensor (13) is conformal Killing. This is expressed by condition (B1), that becomes (B3) after substitutions. The latter equation is identically satisfied. ∎

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