Revisiting linear stability of black hole odd-parity perturbations in Einstein-Aether gravity

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In Einstein-Aether gravity, we revisit the issue of linear stabilities of black holes against odd-parity perturbations on a static and spherically symmetric background. In this theory, superluminal propagation is allowed and there is a preferred timelike direction along the unit Aether vector field. If we choose the usual spherically symmetric background coordinates with respect to the Killing time t and the areal radius r, it may not be appropriate for unambiguously determining the black hole stability because the constant t hypersurfaces are not necessarily always spacelike. Unlike past related works of black hole perturbations, we choose an Aether-orthogonal frame in which the timelike Aether field is orthogonal to spacelike hypersurfaces over the whole background spacetime. In the short wavelength limit, we show that no-ghost conditions as well as radial and angular propagation speeds coincide with those of vector and tensor perturbations on the Minkowski background. Thus, the odd-parity linear stability of black holes for large radial and angular momentum modes is solely determined by constant coefficients of the Aether derivative couplings.

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I. INTRODUCTION

General relativity (GR) is a fundamental pillar of modern physics for describing the gravitational interaction. GR enjoys the invariance under transformations in the Lorentz group. From the perspective of quantum gravity and high-energy theories, however, there are some indications that Lorentz invariance may not be an exact symmetry at all energies [1–5]. Lorentz violation at high energies may allow for the possibility of regularizing field theories, while recovering Lorentz symmetry at low energies [6]. Although broken Lorentz invariance for the standard model matter fields is highly constrained from numerous experiments, the bounds on Lorentz violation in the gravitational sector are not so stringent yet [7–9].

To accommodate broken Lorentz invariance for the gravitational fields without losing the covariant property of GR, there is a way of introducing a unit timelike vector field u^μ satisfying the relation $u^\mu u_\mu = -1$. This is known as Einstein-Aether theory [10], in which a preferred threading with respect to the Aether field is present. To maintain general covariance of Einstein gravity, we require that the preferred threading is dynamical. Since the timelike Aether field is nonvanishing at any spacetime points, it always breaks local Lorentz invariance. In this sense, Einstein-Aether theory is distinguished from other Lorentz-violating theories restoring Lorentz invariance at some particular energy scales.

The covariant action of Einstein-Aether theory, which was introduced by Jacobson and Mattingly [10], contains four derivative couplings of the Aether field with dimensionless coupling constants $c_{1,2,3,4}$ besides the Ricci scalar R. The unit vector constraint on the timelike Aether field can be incorporated into the action as a Lagrange multiplier of the form $\lambda(u^\mu u_\mu + 1)$. We should mention that there was also an equivalent approach based on a tetrad formalism advocated by Gasperini [11]. The Einstein-Aether framework can encompass several classes of vector-tensor theories such as the spontaneous breaking of Lorentz invariance in string theory [8] and cuscuton theories with a quadratic scalar potential [12,13]. There are also extended versions of Einstein-Aether theory in which a symmetry-breaking potential for the vector is introduced [14] or the Aether coupling functions are generalized [15,16]. The generalized Einstein-Aether theory of Ref. [15] is subject to severe constraints on the coupling functions, if $c_1 + c_3 \neq 0$ [17].

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The perturbative analysis of Einstein-Aether theory on the Minkowski background (with all nonvanishing coupling constants $c_{1,2,3,4}$) shows that there are one scalar, two vector, and two tensor propagating degrees of freedom [18]. As we will review in Sec. II, their squared propagation speeds are given, respectively, by Eqs. (2.10), (2.11), and (2.12), all of which are different from that of light. The gravitational-wave event GW170817 of a black hole (BH) neutron star binary, along with the gamma-ray burst 170817A, put a stringent limit $|c_T - 1| \lesssim 10^{-15}$ on the tensor propagation speed c_T [19], thereby translating to the bound $|c_1 + c_3| \lesssim 10^{-15}$ [20,21]. The coupling constants $c_{1,2,3,4}$ have been constrained from other experiments and observations such as gravitational Cerenkov radiation [22], big bang nucleosynthesis [23], solar-system tests of gravity [24], binary pulsars [25–28], and gravitational waveforms [29–31]. Despite those numerous observational data, there are still wide regions of parameter space that are compatible with all these constraints.

If we apply Einstein-Aether theory to the physics on a static and spherically symmetric (SSS) background, it is known that there are some nontrivial BH solutions endowed with the Aether hair [32–50]. Besides the usual metric horizon at which the time translation Killing vector ζ^μ becomes null ($\zeta^\mu \zeta_\mu = 0$), broken Lorentz invariance can give rise to the existence of a universal horizon at which the Aether field u_μ is orthogonal to ζ^μ, i.e., $u_\mu \zeta^\mu = 0$ [51]. This universal horizon, which lies inside the metric horizon, can be interpreted as a causal boundary of any speeds of propagation. In other words, once a wave signal is trapped inside the universal horizon, it does not escape from BHs toward spatial infinity. These distinguished features in Einstein-Aether theory may manifest themselves for inspiral gravitational waveforms emitted from BH binaries and BH quasinormal modes in the ringdown phase. Thus, the upcoming high-precision observations of gravitational waves will offer the possibility of probing the signature of BHs with the Aether hair.

The BH perturbation theory, which was originally developed by Regge-Wheeler [52] and Zerilli [53], plays a crucial role in computing the quasinormal modes of BHs. Moreover, the linear stability of BHs is known by studying conditions for the absence of ghosts and Laplacian instabilities in the small-scale limit. In scalar-tensor Horndeski theories [54], for example, the second-order actions of odd- and even-parity perturbations on the SSS background were derived in Refs. [55–57] for exploring the linear stability of hairy BHs. In Refs. [58–60], it was found that the angular propagation speeds of even-parity perturbations, besides other stability conditions, are important to exclude a large class of hairy BHs due to Laplacian instabilities. As a result, the presence of a Gauss-Bonnet term coupled to the scalar field plays a prominent role in the realization of linearly stable BH solutions in Horndeski theories [61].

In Einstein-Aether theory, the second-order action of odd-parity perturbations was derived in Ref. [62] by using a standard SSS coordinate introduced later in Eq. (3.23). The odd-parity sector contains two propagating degrees of freedom: (1) one tensor mode arising from the gravitational perturbation χ, and (2) one vector mode arising from the Aether perturbation δu. The no-ghost conditions and propagation speeds for χ and δu were obtained by dealing with the t coordinate as a time clock [62]. In Einstein-Aether theory, however, there is a preferred timelike direction along the unit Aether field. Since the timelike property of t coordinate is not always ensured in this setup, the choice of t and r coordinates should not be necessarily appropriate for discussing the linear stability of BHs.

On the SSS spacetime where the background Aether field does not have vorticity, it is possible to locally choose a timelike coordinate ϕ in the form $u_\mu|_{\rm background} = -\eta \partial_\mu \phi$, where η is a nonvanishing function. This scalar field ϕ, which was named "khronon" in Ref. [63], defines the timelike direction in the foliation structure of spacetime. On the SSS background, one can introduce an Aether-orthogonal frame in which the Aether field is orthogonal to spacelike hypersurfaces. Indeed, it is known that [64] Einstein-Aether theory in such a configuration is equivalent to the infrared limit of the nonprojectable version of Hořava gravity [65].[1]

Since the Aether-orthogonal frame is a proper choice of the timelike coordinate orthogonal to spacelike hypersurfaces, we will revisit the linear stability analysis of BHs in the odd-parity sector for this coordinate system. In Sec. II, we briefly review current constraints on the coupling constants $c_{1,2,3,4}$ of derivative couplings of the Aether field. In Sec. III, we will see how a naive choice of the usual SSS coordinate (3.23) can cause apparent instabilities and introduce the Aether-orthogonal frame as well as relations between two different frames. In Sec. IV, we transform the second-order action of odd-parity perturbations derived in Ref. [62] to that in the Aether-orthogonal frame and show that, for large radial and angular momentum modes, the no-ghost conditions and speeds of propagation are identical to those of vector and tensor perturbations on the Minkowski background. Thus, unlike the results in Ref. [62], the linear stability of BHs against odd-parity perturbations does not add new conditions to those known in the literature. Sec. V is devoted to conclusions.

[1]It should be noted that the equivalence between Einstein-Aether gravity and khronometric theory (or the infrared limit of nonprojectable Hořava gravity [66]) holds only when the Aether field has zero vorticity. In particular, their Hamiltonian structures are different. In fact, while in Einstein-Aether gravity there are five propagating local physical degrees of freedom, in khronometric theory the number of propagating local physical degrees of freedom is three [51,67], so is in Hořava gravity [68].

Throughout the paper, we will use the natural unit in which the speed of light c and the reduced Planck constant $\hbar$ are unity. We also adopt the metric signature $(-,+,+,+)$.

II. EINSTEIN-AETHER THEORY AND CURRENT CONSTRAINTS

Einstein-Aether theory is given by the action [10]

$$\mathcal{S} = \frac{1}{16\pi G_{æ}} \int d^4x\sqrt{-g}[R + \mathcal{L}_{æ} + \lambda(g_{\mu\nu}u^\mu u^\nu + 1)], \quad (2.1)$$

where $G_{æ}$ is a constant corresponding to the gravitational coupling, R is the Ricci scalar, g is the determinant of metric tensor $g_{\mu\nu}$, λ is a Lagrange multiplier, u^μ is the Aether vector field, and

$$\mathcal{L}_{æ} = -M^{\alpha\beta}{}_{\mu\nu}\nabla_\alpha u^\mu \nabla_\beta u^\nu, \quad (2.2)$$

with

$$M^{\alpha\beta}{}_{\mu\nu} \coloneqq c_1 g^{\alpha\beta}g_{\mu\nu} + c_2\delta^\alpha_\mu\delta^\beta_\nu + c_3\delta^\alpha_\nu\delta^\beta_\mu - c_4u^\alpha u^\beta g_{\mu\nu}. \quad (2.3)$$

The Greek indices run from 0 to 3, ∇_α is a covariant derivative operator with respect to $g_{\mu\nu}$, and $c_{1,2,3,4}$ are four dimensionless coupling constants.

Varying the action (2.1) with respect to λ, it follows that

$$g_{\mu\nu}u^\mu u^\nu = -1. \quad (2.4)$$

This constraint ensures the existence of a timelike unit vector field at any spacetime points, so that there is a preferred threading responsible for the breaking of Lorentz invariance. Varying Eq. (2.1) with respect to u^μ, we obtain

$$\nabla_\mu J^\mu{}_\alpha + \lambda u_\alpha + c_4 u^\beta\nabla_\beta u^\mu \nabla_\alpha u_\mu = 0, \quad (2.5)$$

where

$$J^\mu{}_\alpha \coloneqq M^{\mu\nu}{}_{\alpha\beta}\nabla_\nu u^\beta. \quad (2.6)$$

Multiplying Eq. (2.5) by u^α and using Eq. (2.4), the Lagrange multiplier can be expressed as

$$\lambda = u^\alpha\nabla_\mu J^\mu{}_\alpha + c_4(u^\beta\nabla_\beta u^\mu)(u^\rho\nabla_\rho u_\mu). \quad (2.7)$$

The gravitational field equations derived by the variation of (2.1) with respect to $g_{\mu\nu}$ are

$$\begin{aligned} G_{\alpha\beta} = {} & \nabla_\mu[u_{(\alpha}J^\mu{}_{\beta)} + u^\mu J_{(\alpha\beta)} - u_{(\alpha}J_{\beta)}{}^\mu] \\ & + c_1(\nabla_\alpha u^\nu\nabla_\beta u_\nu - \nabla^\nu u_\alpha\nabla_\nu u_\beta) \\ & + c_4(u^\rho\nabla_\rho u_\alpha)(u^\nu\nabla_\nu u_\beta) \\ & + \frac{1}{2}g_{\alpha\beta}\mathcal{L}_{æ} + \lambda u_\alpha u_\beta, \end{aligned} \quad (2.8)$$

where $G_{\alpha\beta}$ is the Einstein tensor.

In general, the theory contains three different species of propagating degrees of freedoms, i.e., spin-0 (scalar), spin-1 (vector), and spin-2 (tensor) modes. On the Minkowski background with the line element

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = -dt^2 + \delta_{ij}dx^i dx^j, \quad (2.9)$$

the Aether field is aligned along the t direction, as $u^\mu = \delta^\mu_0$. According to the perturbative analysis on the background (2.9), the squared propagation speeds of spin-0, spin-1, and spin-2 modes are given, respectively, by [18,21]

$$c_S^2 = \frac{c_{123}(2 - c_{14})}{c_{14}(1 - c_{13})(2 + c_{13} + 3c_2)}, \quad (2.10)$$

$$c_V^2 = \frac{2c_1 - c_{13}(2c_1 - c_{13})}{2c_{14}(1 - c_{13})}, \quad (2.11)$$

$$c_T^2 = \frac{1}{1 - c_{13}}, \quad (2.12)$$

where $c_{ij} \coloneqq c_i + c_j$ and $c_{ijk} \coloneqq c_i + c_j + c_k$. The coefficients of the kinetic terms for each mode are

$$q_S = \frac{(1 - c_{13})(2 + c_{13} + 3c_2)}{c_{123}}, \quad (2.13)$$

$$q_V = c_{14}, \quad (2.14)$$

$$q_T = 1 - c_{13}. \quad (2.15)$$

So long as the denominators in Eqs. (2.10)–(2.13) do not vanish, there are one scalar, two vector, and two tensor propagating degrees of freedom in general.

If we require that the theory: (i) be self-consistent, such as free of ghosts and Laplacian instabilities; and (ii) be compatible with all the experimental and observational constraints obtained so far, it was found that the coupling constants must satisfy the following conditions [21]:

$$|c_{13}| \lesssim 10^{-15}, \quad (2.16)$$

$$0 < c_{14} \leq 2.5 \times 10^{-5}, \quad (2.17)$$

$$c_{14} \leq c_2 \leq 0.095, \quad (2.18)$$

$$c_4 \le 0. \tag{2.19}$$

It should be noted that the recent studies of the neutron star binary systems showed that one of the parametrized post-Newtonian parameters, $\alpha_1 = -4c_{14}$, is further restricted to $|\alpha_1| < 10^{-5}$ [28]. This translates to the limit

$$0 \lesssim c_{14} \lesssim 2.5 \times 10^{-6}, \tag{2.20}$$

which is stronger than the bound derived from lunar laser ranging experiments by one order of magnitude [21].

III. DISFORMAL TRANSFORMATION AND AETHER-ORTHOGONAL FRAME

A. Disformal transformation

Under a redefinition of the metric accompanied with the Aether field in the form $\tilde{g}_{\mu\nu} = g_{\mu\nu} + Bu_\mu u_\nu$, where B is a constant, the structure of the action (2.1) is preserved with a change of the coupling constants $\tilde{c}_{1,2,3,4}$ in the transformed frame [69]. This redefinition stretches the metric tensor in the Aether direction by a factor $1-B$. On choosing $B = 1 - c_I^2$ for the Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu}$, where the subscript I is either S, V, T with the squared propagation speeds c_I^2 given by Eqs. (2.10)–(2.12), it is possible to transform to a metric frame $\tilde{g}_{\mu\nu}$ in which one of the speeds is equivalent to 1 [32,70].

One can perform a more general disformal transformation [71] of the form

$$\bar{g}_{\mu\nu} = \Omega^2 (g_{\mu\nu} + Bu_\mu u_\nu), \tag{3.1}$$

where the conformal factor Ω and the disformal factor B are constants, and the Aether field u_μ satisfies the unit-vector constraint (2.4). The corresponding inverse metric and determinant are

$$\bar{g}^{\mu\nu} = \frac{1}{\Omega^2}\left(g^{\mu\nu} - \frac{B}{1-B}u^\mu u^\nu\right),$$
$$\sqrt{-\bar{g}} = \Omega^4 \sqrt{-(1-B)g}, \tag{3.2}$$

where we are assuming that $1 - B > 0$. The Aether field has a different (but constant) norm with respect to $\bar{g}^{\mu\nu}$ as

$$\bar{g}^{\mu\nu} u_\mu u_\nu = -\frac{1}{\Omega^2(1-B)}. \tag{3.3}$$

Hence, it makes sense to define

$$\bar{u}_\mu = \Omega\sqrt{1-B}\,u_\mu, \qquad \bar{u}^\mu = \bar{g}^{\mu\nu}\bar{u}_\nu. \tag{3.4}$$

The first covariant derivative of the Aether field with respect to $\bar{g}_{\mu\nu}$ is given by [72]

$$\bar{\nabla}_\mu u_\nu = \nabla_\mu u_\nu - B\delta\Gamma^\rho_{\mu\nu} u_\rho, \tag{3.5}$$

where

$$\delta\Gamma^\rho_{\mu\nu} = u_{(\mu}F_{\nu)}{}^\rho + \frac{u^\rho}{1-B}[\nabla_{(\mu}u_{\nu)} + Bu_\lambda u_{(\mu}\nabla^\lambda u_{\nu)}],$$
$$F_{\mu\nu} = \nabla_\mu u_\nu - \nabla_\nu u_\mu. \tag{3.6}$$

The Einstein-Hilbert action transforms as [72]

$$\int d^4x\sqrt{-\bar{g}}\bar{R} = \int d^4x\sqrt{-g}\left[\Omega^2\sqrt{1-B}R - \frac{\Omega^2 B}{\sqrt{1-B}}\{(\nabla_\mu u^\mu)^2 - \nabla_\rho u^\sigma \nabla_\sigma u^\rho\} + \frac{\Omega^2 B^2}{2\sqrt{1-B}}\left(u^\mu u^\nu F_{\mu\rho}F_\nu{}^\rho + \frac{1}{2}F_{\mu\nu}F^{\mu\nu}\right)\right]. \tag{3.7}$$

On using these relations, in the absence of matter fields, Einstein-Aether theory for the combination $(g_{\mu\nu}, u_\mu)$ with the constant parameters $(G_æ, c_1, c_2, c_3, c_4)$ is equivalent to Einstein-Aether theory for the combination $(\bar{g}_{\mu\nu}, \bar{u}_\mu)$ with a different set of parameters $(\bar{G}_æ, \bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4)$, where [69]

$$\bar{G}_æ = \Omega^2\sqrt{1-B}\,G_æ, \qquad \bar{c}_{14} = c_{14},$$
$$\bar{c}_{123} = (1-B)c_{123}, \qquad \bar{c}_{13} - 1 = (1-B)(c_{13}-1),$$
$$\bar{c}_1 - \bar{c}_3 - 1 = \frac{1}{1-B}(c_1 - c_3 - 1). \tag{3.8}$$

More explicitly, the coefficients $\bar{c}_i$ are related to c_i, as

$$\bar{c}_1 = \frac{2c_1 - 2(c_1+c_3)B + (c_1+c_3-1)B^2}{2(1-B)}, \tag{3.9}$$

$$\bar{c}_2 = c_2(1-B) - B, \tag{3.10}$$

$$\bar{c}_3 = \frac{2c_3 - (c_1+c_3-1)B(2-B)}{2(1-B)}, \tag{3.11}$$

$$\bar{c}_4 = \frac{2c_4 + 2(c_3-c_4)B - (c_1+c_3-1)B^2}{2(1-B)}. \tag{3.12}$$

We consider the Minkowski background characterized by the metric tensor $g_{\mu\nu} = \eta_{\mu\nu}$ and perform the disformal transformation (3.1). Upon choosing

$$B = 1 - c_I^2, \quad \text{where } I = S, V, T, \tag{3.13}$$

and using Eqs. (2.10)–(2.15) and Eqs. (3.9)–(3.12), the squared propagation speeds $\bar{c}_I^2$ in the frame $(\bar{g}_{\mu\nu}, \bar{u}_\mu)$ yield

$$\bar{c}_I^2 = 1, \tag{3.14}$$

for each subscript $I = S,\ V,\ T$. Furthermore, the coefficients of the time kinetic terms yield

$$\bar{Q}_I = c_I^2 Q_I, \tag{3.15}$$

where[2]

$$Q_S \equiv q_S, \qquad Q_V \equiv \frac{q_V}{c_V^2}, \qquad Q_T \equiv q_T. \tag{3.16}$$

Here, we have assumed

$$c_I^2 > 0, \tag{3.17}$$

in order to avoid the gradient instability of perturbations. Under this assumption, the inequality $1 - B > 0$ holds and thus the disformal transformation does not change the Lorentzian signature of the metric.

B. Aether-orthogonal frame

If the background Aether field has zero vorticity, which is the case for any spherically symmetric configurations, one can locally choose the time coordinate ϕ such that

$$u_\mu|_{\text{background}} = -\eta \partial_\mu \phi = -\eta \delta_\mu^\phi, \tag{3.18}$$

where η is a nonvanishing function. This choice of the time coordinate ϕ for the metric frame $g_{\mu\nu}$ is called the Aether-orthogonal frame. The unit-vector constraint (2.4) gives $g^{\phi\phi}|_{\text{background}} = -\eta^{-2}$ and $u^\phi|_{\text{background}} = \eta^{-1}$.

At each point of physical interest, one can choose a local Lorentz frame and then perform a local Lorentz transformation so that the metric and Aether field are of the form

$$g_{\mu\nu}|_{\text{local}} = -\eta^2 \mathrm{d}\phi^2 + \delta_{ij}\mathrm{d}x^i \mathrm{d}x^j, \qquad u_\mu|_{\text{local}} = -\eta \delta_\mu^\phi. \tag{3.19}$$

At leading order in the geometrical optics approximation, i.e., for modes whose wavelengths are much shorter than the time and length scales of the background, the background in the vicinity of the point of interest can be approximated by Eq. (3.19). In particular, in the vicinity of the point of interest, one can decompose the perturbations into spin-0 (scalar, $I = S$), spin-1 (vector, $I = V$) and spin-2 (tensor, $I = T$) modes. Let us consider the perturbations $\delta\chi_I$ corresponding to the I-excitation ($I = S, V, T$). Then, by definition of Q_I and c_I^2, the kinetic and gradient terms of $\delta\chi_I$ should locally have the following structure:

$$L_{\text{kin},I} = \frac{1}{2} C_I Q_I [(\eta^{-1}\partial_\phi \delta\chi_I)^2 - c_I^2 \delta^{ij} \partial_i \delta\chi_I \partial_j \delta\chi_I] + \cdots, \tag{3.20}$$

where C_I are positive definite coefficients, which should not be confused with $C_1, C_2, \cdots$ introduced later in Sec. IV, and $\cdots$ represents higher-order terms in the geometrical optics approximation. By undoing the local Lorentz transformation and going back to the original coordinate system before choosing the local Lorentz frame, the leading kinetic and gradient terms (3.20) can be written in a general coordinate system as

$$L_{\text{kin},I} = -\frac{1}{2} C_I \bar{Q}_I \bar{g}_I^{\mu\nu} \partial_\mu \delta\chi_I \partial_\nu \delta\chi_I + \cdots, \tag{3.21}$$

at leading order in the geometrical optics approximation, where $\cdots$ again represents higher-order terms in the geometrical optics approximation. We have assumed that the time and spatial scales involved in the coordinate transformation between the original coordinate system and the local Lorentz frame in the vicinity of the point of interest are sufficiently longer than the wavelengths of the modes of interest.

The local structure (3.21) clearly states that the perturbations $\delta\chi_I$ in the Aether-orthogonal frame do not behave as ghosts so long as $Q_I > 0$. Indeed, if we choose the time coordinate ϕ in the Aether-orthogonal frame, then the leading time kinetic term is

$$\begin{aligned} L_{\text{kin},I} &\ni -\frac{1}{2} C_I \bar{Q}_I \bar{g}_I^{\phi\phi} (\partial_\phi \delta\chi_I)^2 + \cdots \\ &= \frac{C_I c_I^2 Q_I}{2\Omega_I^2 \eta^2} (\partial_\phi \delta\chi_I)^2 + \cdots, \end{aligned} \tag{3.22}$$

which is positive.

Rigorously speaking, we have only given a heuristic argument for the local structure (3.21) without a proof, which requires analysis similar to that in Ref. [75]. In the rest of the present paper, we shall show that the kinetic and gradient terms for odd-parity perturbations (including $I = V$ and $I = T$ modes) around spherically symmetric BHs indeed have the local structure (3.21). The same analysis for even-parity perturbations (including $I = S, V, T$ modes) will be left for a future work.

[2]For the vector perturbation, the no-ghost condition changes from $q_V > 0$ to $Q_V > 0$ if one swaps the roles of the dynamical variable and its canonical momentum by a canonical transformation. See e.g., Appendix B of [73] or/and Sec. IV of [74] for a technique to perform canonical transformations at the level of the Lagrangian.

C. Spherically symmetric background and apparent instabilities

Let us consider the SSS background given by the line element

$$ds^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{h(r)} + r^2\Omega_{pq}\mathrm{d}\vartheta^p \mathrm{d}\vartheta^q, \tag{3.23}$$

together with the Aether-field configuration

$$u^\mu \partial_\mu = a(r)\partial_t + b(r)\partial_r, \tag{3.24}$$

where f, h, a, b are functions of r. In Eq. (3.23) the angular part contains two angles θ and φ, such that $\Omega_{pq}\mathrm{d}\vartheta^p\mathrm{d}\vartheta^q = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2$.

The unit-vector constraint (2.4) gives the following relation:

$$b = \epsilon\sqrt{(a^2 f - 1)h}, \tag{3.25}$$

where $\epsilon = \pm 1$. The existence of the Aether-field profile (3.25) with $b \neq 0$ requires that $(a^2 f - 1)h > 0$.

A metric (Killing) horizon is defined at the radius $r = r_{\rm g}$ at which the time translation Killing vector $\zeta^\mu = \delta^\mu_t$ is null, i.e., $\zeta^\mu \zeta_\mu = 0$. This translates to the condition

$$f(r_{\rm g}) = 0. \tag{3.26}$$

So long as h also vanishes on the metric horizon, the product $a^2 f h$ is at least a positive constant at $r = r_{\rm g}$ to satisfy the condition $(a^2 f - 1)h > 0$. This means that, as $r \to r_{\rm g}^+$, the temporal Aether-field component diverges as $a^2 \propto (fh)^{-1}$ for the coordinate system (3.23).

The leading-order kinetic and gradient terms of perturbations $\delta\chi_I$ (where $I = S, V, T$) for the effective metric $\bar{g}^I_{\mu\nu} = \Omega_I^2[g_{\mu\nu} + (1 - c_I^2)u_\mu u_\nu]$ (under the geometric optics approximation) are

$$L_{\mathrm{kin},I} = -\frac{1}{2}C_I\bar{Q}_I\bar{g}_I^{\mu\nu}\partial_\mu\delta\chi_I\partial_\nu\delta\chi_I = \frac{C_I\bar{Q}_I}{2\Omega_I^2}\left[\frac{1}{f}(\partial_t\delta\chi_I)^2 - h(\partial_r\delta\chi_I)^2 - \frac{1}{r^2}\Omega^{pq}\partial_p\delta\chi_I\partial_q\delta\chi_I + \frac{1-c_I^2}{c_I^2}(a\partial_t\delta\chi_I + b\partial_r\delta\chi_I)^2\right]$$
$$\ni \frac{1}{2}fC_I Q_I^{\rm apparent}\left[\frac{1}{f}(\partial_t\delta\chi_I)^2 - (c_{I,\Omega}^{\rm apparent})^2\frac{1}{r^2}\Omega^{pq}\partial_p\delta\chi_I\partial_q\delta\chi_I\right], \tag{3.27}$$

where

$$Q_I^{\rm apparent} = \frac{\bar{Q}_I}{\Omega_I^2}\left(\frac{1}{f} + \frac{1-c_I^2}{c_I^2}a^2\right), \tag{3.28}$$

$$(c_{I,\Omega}^{\rm apparent})^2 = \left(1 + \frac{1-c_I^2}{c_I^2}a^2 f\right)^{-1}. \tag{3.29}$$

Therefore, the coefficient $Q_I^{\rm apparent}$ of the kinetic term with respect to the Killing time t may become negative, despite the fact that the kinetic term with respect to the time coordinate in the Aether-orthogonal frame is always positive as in Eq. (3.22). Also, the apparent angular sound speed squared $(c_{I,\Omega}^{\rm apparent})^2$ would have nontrivial position dependence, although the propagation speed squared c_I^2 relative to the Aether-orthogonal frame is a constant given by the theory.

These behaviors are due to the deviation of the Killing time slicing from the Aether-orthogonal frame. The sound cones are not only narrowed or widened but also tilted relative to the Killing time slicing.[3] On the other hand, the sound cones are not tilted relative to the Aether-orthogonal frame.

Near the metric horizon $r = r_{\rm g}$, we have the following expansions:

$$f = f_1(r - r_{\rm g}) + \cdots, \qquad a = -\frac{1}{2f_1\alpha_0}(r - r_{\rm g})^{-1} + \cdots, \tag{3.30}$$

where f_1 and α_0 are constants [62]. Note that α_0 is the value of α at $r = r_{\rm g}$, where α is defined later in Eq. (3.36). In this regime, the quantities (3.28) and (3.29) can be estimated as[4]

$$Q_I^{\rm apparent} = \frac{\bar{Q}_I}{4f_1^2\alpha_0^2\Omega_I^2}\frac{1-c_I^2}{c_I^2}(r - r_{\rm g})^{-2} + \cdots, \tag{3.31}$$

$$(c_{I,\Omega}^{\rm apparent})^2 = 4f_1\alpha_0^2\frac{c_I^2}{1-c_I^2}(r - r_{\rm g}) + \cdots. \tag{3.32}$$

Therefore, if $c_I^2 > 1$, the apparent no-ghost condition is violated near the horizon. Moreover, the apparent angular

[3] See Ref. [76] for similar behaviors of sound cones for open string modes in the context of inhomogeneous tachyon condensation in string theory.

[4] For $I = V$ and $c_{13} = 0$, the result is to be compared with (5.32) and (5.33) of Ref. [62] up to an arbitrary positive overall factor for $q_V^{\rm apparent}$.

sound speed squared vanishes at the horizon. Let us stress again that the kinetic term with respect to the time coordinate in the Aether-orthogonal frame is always positive for $Q_I > 0$ and that the propagation speed squared c_I^2 relative to the Aether-orthogonal frame is constant given by the theory. In Sec. IV, we will address the linear stability of BHs in Einstein-Aether theory by choosing the Aether-orthogonal frame as a timelike coordinate.

D. Aether-orthogonal frame on the spherically symmetric background

The SSS background is described by the line element (3.23) with the Aether-field profile (3.24). Alternatively, we can choose the following Eddington-Finkelstein coordinate [48]:

$$\mathrm{d}s^2 = -f(r)\mathrm{d}v^2 + 2B(r)\mathrm{d}v\mathrm{d}r + r^2\Omega_{pq}\mathrm{d}\vartheta^p\mathrm{d}\vartheta^q, \tag{3.33}$$

which is related to the coordinate (3.23) as

$$\mathrm{d}v = \mathrm{d}t + \frac{\mathrm{d}r}{\sqrt{fh}}, \qquad B(r) = \sqrt{\frac{f}{h}}. \tag{3.34}$$

The transformation to the coordinate (3.33) is valid in the regime $f/h > 0$, i.e., for the same signs of f and h. On this background, we take the Aether-field configuration

$$u^\mu\partial_\mu = -\alpha(r)\partial_v - \beta(r)\partial_r. \tag{3.35}$$

The r-dependent functions α and β are related to a and b in Eq. (3.24), as

$$a + \frac{b}{\sqrt{fh}} = -\alpha, \qquad b = -\beta. \tag{3.36}$$

From Eq. (3.25), we obtain the following relations:

$$a = -\frac{1+f\alpha^2}{2f\alpha}, \qquad b = \sqrt{fh}\frac{1-f\alpha^2}{2f\alpha}. \tag{3.37}$$

The sign of b depends on that of $(1-f\alpha^2)/(f\alpha)$. Then, the Aether field u^μ has nonvanishing components,

$$u^v = -\alpha, \qquad u^r = \sqrt{fh}\frac{1-f\alpha^2}{2f\alpha}. \tag{3.38}$$

Even though a is divergent on the metric horizon ($r = r_{\rm g}$), the quantity α can take a finite value α_0 due to the relation $fa = -1/(2\alpha_0)$ at $r = r_{\rm g}$, see Eq. (3.30). On using the metric components $g_{vv} = -f$, $g_{vr} = g_{rv} = B$, and $g_{rr} = 0$, the nonvanishing components of u_μ are $u_v = f\alpha - B\beta$ and $u_r = -B\alpha$, so that

$$u_v = \frac{1+f\alpha^2}{2\alpha}, \qquad u_r = -\alpha\sqrt{\frac{f}{h}}. \tag{3.39}$$

A vector field s_μ that is orthogonal to u^μ obeys the relation $s_\mu u^\mu = 0$. This has the following nonvanishing components

$$s_v = \frac{1-f\alpha^2}{2\alpha}, \qquad s_r = \alpha\sqrt{\frac{f}{h}}. \tag{3.40}$$

Moreover, it satisfies the relation $s_\mu s^\mu = 1$.

Now, we introduce the two coordinates ϕ and ψ, as

$$\mathrm{d}\phi = \mathrm{d}v + \frac{u_r}{u_v}\mathrm{d}r = \mathrm{d}v - \frac{2\alpha^2}{1+f\alpha^2}\sqrt{\frac{f}{h}}\mathrm{d}r, \tag{3.41}$$

$$\mathrm{d}\psi = -\mathrm{d}v - \frac{s_r}{s_v}\mathrm{d}r = -\mathrm{d}v - \frac{2\alpha^2}{1-f\alpha^2}\sqrt{\frac{f}{h}}\mathrm{d}r, \tag{3.42}$$

which mean that ϕ is constant on a hypersurface orthogonal to u_μ and that ψ is constant on a hypersurface orthogonal to s_μ. Note that s_v goes to 0 as $r \to \infty$ and hence Eq. (3.42) is valid for a finite distance r. Since we already know the linear stability conditions in the asymptotically flat regime [18,21], it is sufficient to focus on the stability in the region with finite r. If we introduce the coordinate $\tilde{\psi}$ as $\mathrm{d}\tilde{\psi} = -s_v\mathrm{d}v - s_r\mathrm{d}r$ instead of ψ to avoid the divergence of s_r/s_v at spatial infinity, then $\tilde{\psi}$ is not integrable due to the r dependence in s_v. In this sense we choose the coordinate system (ϕ, ψ), which satisfies the integrability condition.

Since $\partial_\mu\phi = \delta^v_\mu - 2\alpha^2/(1+f\alpha^2)\sqrt{f/h}\delta^r_u$ from Eq. (3.41), the background Aether field can be expressed in the form

$$u_\mu = \frac{1+f\alpha^2}{2\alpha}\partial_\mu\phi, \tag{3.43}$$

and hence $\eta = -(1+f\alpha^2)/(2\alpha) = fa$ in Eq. (3.18). Thus, the coordinate system (ϕ, ψ) corresponds to the Aether-orthogonal frame in which ϕ is the time measured by observers comoving with the Aether field.

Substituting the first of Eq. (3.34) into Eqs. (3.41) and (3.42), the relation between the Aether-orthogonal frame and the coordinate (3.23) is given by [38]

$$\mathrm{d}\phi = \mathrm{d}t + \frac{1}{\sqrt{fh}}\frac{1-f\alpha^2}{1+f\alpha^2}\mathrm{d}r, \tag{3.44}$$

$$\mathrm{d}\psi = -\mathrm{d}t - \frac{1}{\sqrt{fh}}\frac{1+f\alpha^2}{1-f\alpha^2}\mathrm{d}r. \tag{3.45}$$

Solving these equations for $\mathrm{d}t$ and $\mathrm{d}r$ and substituting them into Eq. (3.23), the line element is expressed as

$$ds^2 = -\frac{(1+f\alpha^2)^2}{4\alpha^2}d\phi^2 + \frac{(1-f\alpha^2)^2}{4\alpha^2}d\psi^2 + r^2\Omega_{pq}d\vartheta^p d\vartheta^q. \tag{3.46}$$

Since $g_{\phi\phi}$ is always nonpositive, ϕ is a timelike coordinate. Similarly, ψ is a spacelike coordinate due to the positivity of $g_{\psi\psi}$. Note that a universal horizon is defined as the radius $r_{\rm UH}$ at which the time translation Killing vector ζ^μ is orthogonal to u_μ, i.e., $\zeta^\mu u_\mu = 0$. This corresponds to the radius satisfying

$$(1+f\alpha^2)|_{r=r_{\rm UH}} = 0, \tag{3.47}$$

at which the metric component $g_{\phi\phi}$ in Eq. (3.46) is vanishing. It is clear that Eq. (3.47) has solutions only when $f<0$, that is, the universal horizon always exists inside the metric horizon. For more details, see Ref. [38].

From Eqs. (3.44) and (3.45), we have

$$\frac{\partial\phi}{\partial t} = 1, \qquad \frac{\partial\phi}{\partial r} = \frac{1}{\sqrt{fh}}\frac{1-f\alpha^2}{1+f\alpha^2}, \tag{3.48}$$

$$\frac{\partial\psi}{\partial t} = -1, \qquad \frac{\partial\psi}{\partial r} = -\frac{1}{\sqrt{fh}}\frac{1+f\alpha^2}{1-f\alpha^2}. \tag{3.49}$$

Then, for a given function $\mathcal{F}$ of t and r, the t and r derivatives of $\mathcal{F}$ are given, respectively, by

$$\frac{\partial\mathcal{F}}{\partial t} = \mathcal{F}_{,\phi} - \mathcal{F}_{,\psi}, \tag{3.50}$$

$$\frac{\partial\mathcal{F}}{\partial r} = \frac{1}{\sqrt{fh}}\left(\frac{1-f\alpha^2}{1+f\alpha^2}\mathcal{F}_{,\phi} - \frac{1+f\alpha^2}{1-f\alpha^2}\mathcal{F}_{,\psi}\right), \tag{3.51}$$

where $\mathcal{F}_{,\phi} \coloneqq \partial\mathcal{F}/\partial\phi$ and $\mathcal{F}_{,\psi} \coloneqq \partial\mathcal{F}/\partial\psi$. The relations (3.50) and (3.51) will be used in Sec. IV.

IV. ODD-PARITY STABILITY IN THE AETHER-ORTHOGONAL FRAME

In this section, we study the linear stability of SSS BHs against odd-parity perturbations in the Aether-orthogonal frame. The second-order action in the odd-parity sector was derived in Ref. [62] for the line element (3.23). Since the constant t hypersurfaces are not always spacelike, the coordinate choice (3.23) is not suitable for studying the linear stability of BHs. We will express the second-order action of odd-parity perturbations by using the derivatives with respect to ϕ and ψ. In the following, we will discuss the two cases: (A) $l \geq 2$, and (B) $l = 1$, in turn, where l's are spherical multipoles.

A. $l \geq 2$

On the SSS background (3.23) with $\Omega_{pq}d\vartheta^p d\vartheta^q = d\theta^2 + \sin^2\theta d\varphi^2$, metric perturbations $h_{\mu\nu}$ can be separated into odd-parity (axial) and even-parity (polar) sectors [52,53]. We express $h_{\mu\nu}$ in terms of the spherical harmonics $Y_{lm}(\theta,\varphi)$. We will focus on axial perturbations with the parity $(-1)^{l+1}$. We choose the gauge in which the components h_{ab} vanish, where the subscripts a and b denote either θ or φ. Then, the nonvanishing components of odd-parity metric perturbations are given by

$$h_{ta} = \sum_{l,m} Q_{lm}(t,r)E_{ab}\nabla^b Y_{lm}(\theta,\varphi), \tag{4.1}$$

$$h_{ra} = \sum_{l,m} W_{lm}(t,r)E_{ab}\nabla^b Y_{lm}(\theta,\varphi), \tag{4.2}$$

where Q_{lm} and W_{lm} are functions of t and r. The tensor E_{ab} is antisymmetric with nonvanishing components $E_{\theta\varphi} = -E_{\varphi\theta} = \sin\theta$.

In the odd-parity sector, the Aether field has the following components:

$$u_t = -a(r)f(r), \qquad u_r = \frac{b(r)}{h(r)},$$
$$u_a = \sum_{l,m}\delta u_{lm}(t,r)E_{ab}\nabla^b Y_{lm}(\theta,\varphi), \tag{4.3}$$

where δu_{lm} is a function of t and r.

In the following, we will set $m=0$ without loss of generality. We also omit the subscripts l and m from the perturbations Q_{lm}, W_{lm}, and δu_{lm}. Expanding the action (2.1) up to quadratic order and integrating it with respect to θ and φ, we obtain the second-order action containing the fields Q, W, δu and their t, r derivatives. The dynamical field associated with the gravitational (tensor) perturbation is given by [62]

$$\chi \coloneqq \dot{W} - Q' + \frac{2}{r}Q + \frac{C_2\dot{\delta u} + C_3\delta u' + C_4\delta u}{C_1}, \tag{4.4}$$

where a dot and prime represent the derivatives with respect to t and r, respectively, and

$$C_1 = \frac{(1-c_{13})h}{2r^2 f}, \qquad C_2 = -\frac{c_{13}b}{2r^2 f}, \qquad C_3 = -\frac{c_{13}ah}{2r^2},$$
$$C_4 = \frac{[(2c_{14}-c_{13})(fa'+af')r + 2c_{13}af]h}{2r^3 f}. \tag{4.5}$$

Taking into account the field χ as a form of the Lagrange multiplier and varying the corresponding second-order action with respect to W and Q, we can eliminate W, Q, and their derivatives from the action. Then, the resulting quadratic-order action can be expressed in the form [62]

$$\mathcal{S}_{\rm odd} = \sum_l L \int {\rm d}t{\rm d}r \mathcal{L}_{\rm odd}, \tag{4.6}$$

where

$$L = l(l+1), \tag{4.7}$$

and

$$\mathcal{L}_{\rm odd} = \frac{r^2}{16\pi G_{æ}} \sqrt{\frac{f}{h}} (\dot{\vec{\mathcal{X}}}^t \boldsymbol{K} \dot{\vec{\mathcal{X}}} + \dot{\vec{\mathcal{X}}}^t \boldsymbol{R} \vec{\mathcal{X}}' + \vec{\mathcal{X}}'^t \boldsymbol{G} \vec{\mathcal{X}}' + \vec{\mathcal{X}}^t \boldsymbol{M} \vec{\mathcal{X}}). \tag{4.8}$$

The vector field $\vec{\mathcal{X}}$ is given by

$$\vec{\mathcal{X}} = \begin{pmatrix} \chi \\ \delta u \end{pmatrix}, \qquad \vec{\mathcal{X}}^t = (\chi, \delta u), \tag{4.9}$$

where χ and δu are the dynamical perturbations arising from the gravitational and Aether sectors, respectively. $\boldsymbol{K}$, $\boldsymbol{R}$, $\boldsymbol{G}$, and $\boldsymbol{M}$ are the 2×2 symmetric and real matrices, among which only $\boldsymbol{M}$ has off-diagonal components. Nonvanishing components of these matrices are

$$\begin{aligned}
K_{11} &= \frac{4C_1^2 C_{10}}{(L-2)(a^2 C_9^2 - 4C_8 C_{10})}, \\
K_{22} &= \frac{C_1 C_5 - C_2^2}{C_1}, \\
R_{11} &= -\frac{aC_9}{C_{10}} K_{11}, \qquad R_{22} = \frac{C_1 C_6 - 2C_2 C_3}{C_1}, \\
G_{11} &= \frac{C_8}{C_{10}} K_{11}, \qquad G_{22} = \frac{C_1 C_7 - C_3^2}{C_1}, \\
M_{11} &= -C_1, \\
M_{22} &= LC_{12} + \frac{L[C_8 C_{11}^2 + C_9^2 (C_{10} + aC_{11})]}{a^2 C_9^2 - 4C_8 C_{10}},
\end{aligned} \tag{4.10}$$

where the explicit form of $M_{12} (= M_{21})$ is not shown here, and

$$\begin{aligned}
C_5 &= \frac{c_1 + c_4 a^2 f}{r^2 f}, \qquad C_6 = \frac{2c_4 ab}{r^2}, \\
C_7 &= \frac{[c_4(a^2 f - 1) - c_1]h}{r^2}, \\
C_8 &= -\frac{[c_{13}(a^2 f - 1) + 1]h}{2r^4}, \\
C_9 &= \frac{c_{13} b}{r^4}, \qquad C_{10} = \frac{1 - c_{13} a^2 f}{2r^4 f}, \\
C_{11} &= \frac{c_{13} a}{r^4}, \qquad C_{12} = -\frac{c_1}{r^4}.
\end{aligned} \tag{4.11}$$

Since M_{12} does not depend on L, it does not affect the angular propagation speeds in the large l limit (which will be discussed below).

In Ref. [62], the linear stability conditions of BHs against odd-parity perturbations were derived by using the coordinates t and r. As we showed in Sec. III C, unless a proper coordinate is chosen, we may encounter artificial ghosts or Lagrangian instabilities in theories with superluminal propagation. To overcome this problem, we use the Aether-orthogonal frame introduced in Sec. III D, where ϕ defines the causality and chronology: all particles must move along the increasing direction of ϕ [51,77]. As a result, the future light cone defined by each particle with any given speed lies to the future of spacelike hypersurfaces ($\phi =$ constant), as explained explicitly in Ref. [75].

We transform the action (4.6) to that in the coordinate system (3.46). We convert the t and r derivatives of χ and δu to their ϕ and ψ derivatives by exploiting the relations (3.50) and (3.51). We also use Eq. (3.37) to express a and b with respect to α. Then, the second-order action of odd-parity perturbations yields

$$\mathcal{S}_{\rm odd} = \sum_l L \int {\rm d}\phi {\rm d}\psi \hat{\mathcal{L}}_{\rm odd}, \tag{4.12}$$

where

$$\hat{\mathcal{L}}_{\rm odd} = \frac{1}{16\pi G_{æ}} \sqrt{\frac{h}{f}} (\vec{\mathcal{X}}^t_{,\phi} \hat{\boldsymbol{K}} \vec{\mathcal{X}}_{,\phi} + \vec{\mathcal{X}}^t_{,\psi} \hat{\boldsymbol{G}} \vec{\mathcal{X}}_{,\psi} + \vec{\mathcal{X}}^t \hat{\boldsymbol{M}} \vec{\mathcal{X}}). \tag{4.13}$$

Nonvanishing components of the 2×2 matrices $\hat{\boldsymbol{K}}$, $\hat{\boldsymbol{G}}$, and $\hat{\boldsymbol{M}}$ are given by

$$\hat{K}_{11} = \frac{2(1-c_{13})^2 \alpha^2 r^2}{(L-2)(1+f\alpha^2)^2}, \tag{4.14}$$

$$\hat{K}_{22} = \frac{4c_{14}\alpha^2}{(1+f\alpha^2)^2} \frac{f}{h}, \tag{4.15}$$

$$\hat{G}_{11} = -\frac{(1+f\alpha^2)^2}{(1-f\alpha^2)^2} c_T^2 \hat{K}_{11}, \tag{4.16}$$

$$\hat{G}_{22} = -\frac{(1+f\alpha^2)^2}{(1-f\alpha^2)^2} c_V^2 \hat{K}_{22}, \tag{4.17}$$

$$\hat{M}_{11} = -\frac{1}{2}(1-c_{13}), \tag{4.18}$$

$$\hat{M}_{22} = -L \frac{2c_1 - c_{13}(2c_1 - c_{13})}{2(1-c_{13})r^2} \frac{f}{h}, \tag{4.19}$$

besides the off-diagonal components $\hat{M}_{12} = \hat{M}_{21}$ (which are of order L^0). The quantities c_T^2 and c_V^2 are defined, respectively, by Eqs. (2.12) and (2.11). We recall that the Lagrangian (4.8) possesses products of the t and r derivatives, but the Lagrangian (4.13) does not contain products of the ϕ and ψ derivatives.

The absence of ghosts for dynamical perturbations χ and δu requires the two conditions $\hat{K}_{11} > 0$ and $\hat{K}_{22} > 0$. The former is satisfied except for $c_{13} = 1$ (in which case $\hat{K}_{11} = 0$). The second holds under the inequality

$$c_{14} > 0. \tag{4.20}$$

This is equivalent to the no-ghost condition of vector perturbations on the Minkowski background [18,21]. On the other hand, it does not coincide with the no-ghost condition derived on the SSS background (3.23) with the coordinates t and r [62]. The latter coordinate choice is not suitable for discussing the linear stability of BHs, since the constant t hypersurfaces are not always spacelike.

To study the propagation of small-scale perturbations with large angular frequencies ω and momenta k, we assume the solutions to the perturbation equations for χ and δu in the form,

$$\vec{\mathcal{X}} = \vec{\mathcal{X}}_0 e^{-i(\omega\phi - k\psi)}, \tag{4.21}$$

with $\vec{\mathcal{X}}_0 = (\chi_0, \delta u_0)$, where χ_0 and δu_0 are constants.

The radial propagation speeds can be known by considering the modes with $\omega r_{\rm g} \approx k r_{\rm g} \gg l \gg 1$. In this regime, we substitute Eq. (4.21) into the perturbation equations following from Eq. (4.13). This leads to the two dispersion relations

$$\omega^2 = -\frac{\hat{G}_{11}}{\hat{K}_{11}} k^2 = \frac{(1+f\alpha^2)^2}{(1-f\alpha^2)^2} c_T^2 k^2, \tag{4.22}$$

$$\omega^2 = -\frac{\hat{G}_{22}}{\hat{K}_{22}} k^2 = \frac{(1+f\alpha^2)^2}{(1-f\alpha^2)^2} c_V^2 k^2, \tag{4.23}$$

which correspond to those of χ and δu, respectively. Considering a given point P, in the neighborhood of which we can always express the line element (3.46) in the form,

$$\mathrm{d}s^2 = -\mathrm{d}\tilde{\phi}^2 + \mathrm{d}\tilde{\psi}^2 + r^2 \Omega_{pq} \mathrm{d}\vartheta^p \mathrm{d}\vartheta^q, \tag{4.24}$$

where

$$\mathrm{d}\tilde{\phi}^2 = \frac{(1+f\alpha^2)^2}{4\alpha^2} \mathrm{d}\phi^2, \qquad \mathrm{d}\tilde{\psi}^2 = \frac{(1-f\alpha^2)^2}{4\alpha^2} \mathrm{d}\psi^2. \tag{4.25}$$

Note that $\tilde{\phi}$ corresponds to a proper time for this coordinate. Then, the radial propagation speed squared yields

$$c_r^2 = \left(\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}\tilde{\phi}}\right)^2 = \frac{(1-f\alpha^2)^2}{(1+f\alpha^2)^2} \left(\frac{\mathrm{d}\psi}{\mathrm{d}\phi}\right)^2 = \frac{(1-f\alpha^2)^2}{(1+f\alpha^2)^2} \frac{\omega^2}{k^2}, \tag{4.26}$$

where we used $(\mathrm{d}\psi/\mathrm{d}\phi)^2 = \omega^2/k^2$. Then, from Eqs. (4.22) and (4.23), the radial squared propagation speeds of χ and δu are given, respectively, by

$$c_{r1}^2 = c_T^2 = \frac{1}{1 - c_{13}}, \tag{4.27}$$

$$c_{r2}^2 = c_V^2 = \frac{2c_1 - c_{13}(2c_1 - c_{13})}{2c_{14}(1 - c_{13})}. \tag{4.28}$$

Thus, they are identical to the squared tensor and vector propagation speeds on the Minkowski background, respectively. These values are different from those derived on the SSS background (3.23) with the coordinates t and r [62]. To avoid the Laplacian instabilities along the radial direction, we require the two conditions $c_T^2 > 0$ and $c_V^2 > 0$. Under the no-ghost condition (4.20), they amount to the inequalities

$$c_{13} < 1, \tag{4.29}$$

$$2c_1 - c_{13}(2c_1 - c_{13}) > 0. \tag{4.30}$$

Under the inequality (4.29), the other no-ghost condition $\hat{K}_{11} > 0$ is also satisfied.

To derive the angular propagation speeds, we consider the eikonal limit $l \approx \omega r_{\rm g} \gg k r_{\rm g} \gg 1$. We substitute the solution (4.21) into the perturbation equations of motion by noting that the off-diagonal matrix components $\hat{M}_{12} = \hat{M}_{21}$ are of order L^0. Then, in the eikonal limit, we obtain

$$\omega^2 = -\frac{\hat{M}_{11}}{\hat{K}_{11}} = \frac{(L-2)(1+f\alpha^2)^2}{4(1-c_{13})\alpha^2 r^2}, \tag{4.31}$$

$$\omega^2 = -\frac{\hat{M}_{22}}{\hat{K}_{22}} = \frac{L(1+f\alpha^2)^2[2c_1 - c_{13}(2c_1 - c_{13})]}{8c_{14}(1-c_{13})\alpha^2 r^2}, \tag{4.32}$$

which correspond to the dispersion relations of χ and δu, respectively. In terms of the proper time $\tilde{\phi}$ in the coordinate (4.24), the propagation speed squared in the θ direction is given by

$$c_\Omega^2 = \left(\frac{r\mathrm{d}\theta}{\mathrm{d}\tilde{\phi}}\right)^2 = \frac{4\alpha^2}{(1+f\alpha^2)^2} \left(\frac{r\mathrm{d}\theta}{\mathrm{d}\phi}\right)^2 = \frac{4\alpha^2}{(1+f\alpha^2)^2} \frac{r^2\omega^2}{l^2}, \tag{4.33}$$

where we used $\mathrm{d}\theta/\mathrm{d}\phi = \omega/l$. Substituting Eqs. (4.31) and (4.32) into Eq. (4.33) and taking the limit $l \gg 1$, the squared propagation speeds of χ and δu are given, respectively, by

$$c_{\Omega 1}^2 = \frac{1}{1-c_{13}} = c_T^2, \tag{4.34}$$

$$c_{\Omega 2}^2 = \frac{2c_1 - c_{13}(2c_1 - c_{13})}{2c_{14}(1-c_{13})} = c_V^2. \tag{4.35}$$

These values are equivalent to c_{r1}^2 and c_{r2}^2 derived in Eqs. (4.27) and (4.28), respectively. Thus, the perturbations χ and δu propagate with the same sound speeds as those in the Minkowski spacetime both along the radial and angular directions. The linear stability of BHs is ensured under the three conditions (4.20), (4.29), and (4.30).

B. $l = 1$

For the dipole mode ($l = 1$ and $L = 2$), the metric components h_{ab} vanish identically and hence there is a residual gauge degree of freedom to be fixed. We choose the gauge $W = 0$ and introduce the Lagrangian multiplier χ given by Eq. (4.4). For the coordinate system (3.23), the second-order action is expressed in the form $\mathcal{S}_{\rm odd} = \int {\rm d}t{\rm d}r\mathcal{L}_{\rm odd}$, where

$$\mathcal{L}_{\rm odd} = \frac{r^2}{8\pi G_{æ}}\sqrt{\frac{f}{h}}\bigg[C_1\bigg\{2\chi\bigg(-Q' + \frac{2Q}{r} + \frac{C_2\dot{\delta u} + C_3\delta u' + C_4\delta u}{C_1}\bigg) - \chi^2\bigg\} - \frac{(C_2\dot{\delta u} + C_3\delta u' + C_4\delta u)^2}{C_1} + C_5\dot{\delta u}^2 + C_6\dot{\delta u}\delta u' + C_7\delta u'^2 + (2C_{12} + C_{13})\delta u^2\bigg], \tag{4.36}$$

with

$$C_{13} = \frac{\lambda}{r^2} - \frac{c_{13}[(rh' + 2h - 2)f + rhf']}{2r^4 f} - \frac{2c_4(fa' + f'a)ah}{r^3}, \tag{4.37}$$

and λ is given in the Appendix. Varying the Lagrangian (4.36) with respect to Q, we obtain

$$\left(\sqrt{\frac{f}{h}}r^4 C_1\chi\right)' = 0. \tag{4.38}$$

We can choose an appropriate boundary condition at spatial infinity, such that Eq. (4.38) gives $\chi = 0$. Then, the Lagrangian (4.36) reduces to

$$\mathcal{L}_{\rm odd} = \frac{r^2}{8\pi G_{æ}}\sqrt{\frac{f}{h}}\bigg[\bigg(C_5 - \frac{C_2^2}{C_1}\bigg)\dot{\delta u}^2 + \bigg(C_7 - \frac{C_3^2}{C_1}\bigg)\delta u'^2 + \bigg(C_6 - \frac{2C_2C_3}{C_1}\bigg)\delta\dot{u}\delta u' + \mathcal{M}\delta u^2\bigg], \tag{4.39}$$

where

$$\mathcal{M} = 2C_{12} + C_{13} - \frac{C_4^2}{C_1} + \left(\frac{C_3C_4}{C_1}\right)'. \tag{4.40}$$

Now, we convert the Lagrangian (4.39) to that in the Aether-orthogonal frame. For this purpose, we replace the derivatives $\delta\dot{u}$ and $\delta u'$ with $\delta u_{,\phi}$ and $\delta u_{,\psi}$ by using the relations (3.50) and (3.51). Then, the resulting second-order action reduces to $\mathcal{S}_{\rm odd} = \int {\rm d}\phi{\rm d}\psi\hat{\mathcal{L}}_{\rm odd}$, where

$$\hat{\mathcal{L}}_{\rm odd} = \frac{1}{8\pi G_{æ}}\sqrt{\frac{f}{h}}(\mathcal{K}\delta u_{,\phi}^2 + \mathcal{G}\delta u_{,\psi}^2 + r^2\mathcal{M}\delta u^2), \tag{4.41}$$

with

$$\mathcal{K} = \frac{4\alpha^2 c_{14}}{(1+f\alpha^2)^2}, \tag{4.42}$$

$$\mathcal{G} = -\frac{2\alpha^2[2c_1 - c_{13}(2c_1 - c_{13})]}{(1-f\alpha^2)^2(1-c_{13})}. \tag{4.43}$$

Thus, the Aether perturbation δu is the only propagating degree of freedom for $l = 1$. The ghost is absent under the condition $\mathcal{K} > 0$, which translates to

$$c_{14} > 0, \tag{4.44}$$

and is the same as Eq. (4.20) derived for $l \geq 2$. The radial squared propagation speed measured in terms of the proper time $\tilde{\phi}$ reads

$$c_r^2 = -\frac{(1-f\alpha^2)^2}{(1+f\alpha^2)^2}\frac{\mathcal{G}}{\mathcal{K}} = \frac{2c_1 - c_{13}(2c_1 - c_{13})}{2c_{14}(1-c_{13})} = c_V^2, \tag{4.45}$$

which is equivalent to the squared propagation speed of vector perturbations in the Minkowski spacetime. Thus, for $l = 1$, there are no additional stability conditions to those derived for $l \geq 2$.

C. Cases of specific coefficients

For the multiples $l \geq 2$, we consider several specific cases in which some of the coefficients $c_{1,2,3,4}$ are vanishing. From Eq. (4.15), the matrix component $\hat{K}_{22}$ vanishes for $c_{14} = 0$. The matrix components $\hat{G}_{22}$ and $\hat{M}_{22}$ can be expressed as

$$\hat{G}_{22} = -\frac{2[2c_1 - c_{13}(2c_1 - c_{13})]\alpha^2}{(1-c_{13})(1-f\alpha^2)^2}\frac{f}{h}, \tag{4.46}$$

$$\hat{M}_{22} = -\frac{L[2c_1 - c_{13}(2c_1 - c_{13})]}{(1-c_{13})r^2}\frac{f}{h}. \tag{4.47}$$

Then, the Aether field either behaves as a vector-type instantaneous mode or exhibits a strong coupling problem for

$$c_{14} = 0, \quad \text{and} \quad 2c_1 - c_{13}(2c_1 - c_{13}) \neq 0, \tag{4.48}$$

depending on the behavior of the system at nonlinear level, the analysis of which is beyond the scope of the present paper. If we demand that $c_{13} = 0$ for the consistency with the observational bound (2.16), then either a vector-type instantaneous mode or a strong coupling problem may arise for $c_{14} = 0$ and $c_1 \neq 0$, depending on the nonlinear behavior of the system. This is the case for the stealth Schwarzshild BH solution discussed in Refs. [48,62,78].

If we consider Einstein-Aether theory with $c_{14} = 0$, $c_1 = 0$, and $c_{13} = 0$, i.e.,

$$c_1 = 0, \qquad c_2 \neq 0, \qquad c_3 = 0, \qquad c_4 = 0, \tag{4.49}$$

we have the following matrix components

$$\begin{aligned} \hat{K}_{11} &= \frac{2\alpha^2 r^2}{(L-2)(1+f\alpha^2)^2}, \\ \hat{G}_{11} &= -\frac{2\alpha^2 r^2}{(L-2)(1-f\alpha^2)^2}, \\ \hat{M}_{11} &= -\frac{1}{2}, \\ \hat{K}_{22} &= \hat{G}_{22} = \hat{M}_{22} = 0. \end{aligned} \tag{4.50}$$

The number of propagating degrees of freedom in the odd-parity sector is 1 at linear level. This indicates either the absence of vector modes, the presence of a vector-type instantaneous mode or the strong coupling problem for δu, depending on the nonlinear behavior of this system. Fortunately, in this case, we know the nonlinear behavior since Einstein-Aether theory with the coefficients (4.49) is equivalent to a class of cuscuton theories with a quadratic potential [13], provided that the derivative of the expansion $\theta = \nabla_\mu u^\mu$ is nonzero.[5] This means that the absence of the time kinetic term and the gradient term shown above for the specific coefficients (4.49) simply corresponds to the absence of vector modes as far as the equivalence to the cuscuton theory holds on the background with $\partial_\mu \theta \neq 0$. There is a single dynamical degree of freedom χ with the propagation speeds given by

$$c_{r1}^2 = c_{\Omega 1}^2 = 1, \tag{4.51}$$

which are both luminal.

V. CONCLUSIONS

In this paper, we addressed the linear stability of BHs against odd-parity perturbations in Einstein-Aether theory given by the action (2.1). In this theory, there is a preferred threading aligned with a unit timelike vector field. If the background Aether field u_μ has vanishing vorticity, one can introduce a scalar (Khronon) field ϕ whose gradient $\partial_\mu \phi$ is timelike and proportional to u_μ. This property holds for the SSS background given by the line element (3.23).

In Einstein-Aether theory, the constant t hypersurfaces in the coordinate (3.23) are not always spacelike outside the universal horizon, which now is the boundary of a BH and is always inside the metric horizon, when superluminal speeds are allowed [51]. In this sense, the derivation of linear stability conditions using t as a time clock can lead to inconsistent results. The proper coordinate choice for obtaining no-ghost conditions and propagation speeds of dynamical perturbations should be the Aether-orthogonal frame in which the Khronon field ϕ is treated as a time clock, in which case the constant time hypersurfaces are always spacelike over the whole region outside the universal horizon, as shown explicitly by the metric (3.46). In Sec. III C, we argued how the coordinate choice different from the Aether-orthogonal frame can give rise to apparent ghost and Laplacian instabilities.

In Sec. IV, we derived the second-order action of odd-parity perturbations by transforming the action derived for the SSS coordinate (3.23) in Ref. [62] to the one in the Aether-orthogonal frame with the line element (3.46). For this purpose, we exploited transformation properties (3.50)–(3.51) of the derivatives of perturbations between the two sets of coordinates. For the multipoles $l \geq 2$, there are two dynamical perturbations χ and δu arising from the gravitational and vector-field sectors, respectively. The resulting second-order Lagrangian is of the form (4.13), which does not contain products of the ϕ and ψ derivatives [unlike the Lagrangian (4.8) containing products of the t and r derivatives]. The stability analysis of BHs in the Aether-orthogonal frame shows that the ghost is absent

[5]If $\partial_\mu \theta = 0$, then u^μ is undetermined by equations of motion and $\lambda = 0$.

under the inequality $c_{14} > 0$, which is the same no-ghost condition of vector perturbations on the Minkowski background. In large momentum limits, the radial squared propagation speeds of χ and δu are equivalent to those of the tensor and vector perturbations on the Minkowski background. This is also the case for the angular squared propagation speeds of χ and δu in the eikonal limit $l \gg 1$. For $l = 1$, the vector-field perturbation alone propagates with the same stability conditions of δu as those derived for $l \geq 2$.

We thus showed that the proper odd-parity stability analysis of BHs based on the Aether-orthogonal frame gives rise to the same no-ghost conditions and propagation speeds of dynamical perturbations as those on the Minkowski background. In Sec. IV C, we discussed several specific cases of coupling constants in which the strong coupling problem may arise or the number of degrees of freedom reduces. It will be of interest to classify surviving BH solutions free from the linear instability and strong coupling problems. For this purpose, we plan to extend the stability analysis in the Aether-orthogonal frame to perturbations in the even-parity sector.

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APPENDIX: THE LAGRANGE MULTIPLIER IN SPHERICAL SPACETIMES

The quantity λ in Eq. (4.37) is given by

$$\begin{aligned}
\lambda = \frac{1}{4f^2r^2(a^2f-1)}\Big\{ & fr^2\Big[a^2(6c_1+3c_2+2c_{13}-8c_{14})h(f')^2+c_2(2hf''+f'h')\Big] \\
& +2a^2f^4\Big[-2a^2(2(c_2+c_{13})h-c_2rh')+4(c_{14}-c_1)hr^2(a')^2+a(-c_1+c_2+c_{13})r(2hra''+a'(rh'+4h))\Big] \\
& -f^2\Big[rh'(a^2(-2c_1+3c_2+2c_{13})rf'-4c_2) \\
& +2h(a^4(3c_1+c_2+c_{13}-4c_{14})r^2(f')^2+a^2r((-2c_1+3c_2+2c_{13})rf''+2(-2c_1+c_2+2c_{13})f') \\
& -a(11c_1-5c_2-5c_{13}-8c_{14})r^2a'f'+4(c_2+c_{13}))\Big]-(2c_2+c_{13})hr^2(f')^2 \\
& +2f^3\Big[a^4r((-c_1+c_2+c_{13})rf'h'+2h((-c_1+c_2+c_{13})rf''+(-2c_1+c_2+2c_{13})f')) \\
& +a^2(8(c_2+c_{13})h-4c_2rh')-2(-2c_1+c_2+c_{13}+2c_{14})hr^2(a')^2 \\
& +a^3(-11c_1+3c_2+3c_{13}+8c_{14})hr^2a'f'+a(c_1-c_2-c_{13})r(2hra''+a'(rh'+4h))\Big]\Big\}. \qquad (A1)
\end{aligned}$$

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