Symmetry restoration in transverse diffeomorphism invariant scalar field theories

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(Received 21 February 2024; revised 29 May 2024; accepted 24 June 2024; published 2 August 2024)

We explore the idea of restoring the full diffeomorphism (Diff) invariance in theories with only transverse diffeomorphisms (TDiff) by the introduction of additional fields. In particular, we consider in detail the case of a TDiff invariant scalar field and how Diff symmetry can be restored preserving locality by introducing an additional vector field. We reobtain the corresponding dynamics and energy-momentum tensor from the covariantized action and analyze the potential and kinetic domination regimes. For the former, the theory describes a cosmological constant–type behavior, while for the latter we show that the theory can describe an adiabatic perfect fluid whose equation of state and speed of sound is obtained in a straightforward way. Furthermore, the reformulation with the full symmetry allows us to analyze the gravitational properties of the theory beyond those particular regimes. In particular, we find the general expression for the effective speed of sound of the nonadiabatic perfect fluid, which provides us with physically reasonable conditions that should be satisfied by the coupling functions. Finally, we investigate the particular models leading to an adiabatic fluid.

DOI: 10.1103/PhysRevD.110.044009

I. INTRODUCTION

Our current best description of gravitational phenomena is, and has been for over a century, the theory of general relativity (GR). It is not a theory without weaknesses, however, and although it performs remarkably well in numerous tests on Solar System scales, there is reason to believe that it is not the end of the story. For one, it is not a theory that serves to describe gravity in its most extreme regimes, where one expects quantum effects to gain importance, but it is also possible to find problems even while remaining classical. Indeed, the various tensions between theory and observations in cosmology are another hint at the possibility that the theory breaks down at such scales. Sparked by considerations of the sort, together with other theoretical issues such as the cosmological constant problem, modified theories of gravity have been a central object of study in this regard (see e.g. [1] for a review).

In particular, it proves worthwhile to reconsider the fundamental symmetries involved in GR, namely the diffeomorphism (Diff) invariance of the theory. This amounts to the assertion that the physical equations remain invariant under general coordinate transformations. The study of situations where such a symmetry is broken can in fact be traced back to Einstein himself in 1919 with the introduction of unimodular gravity [2], where the metric determinant is reduced to be a nondynamical field fixed to the value $g = 1$. Unimodular gravity is perhaps the most well-known example of a theory with broken Diff invariance, the symmetry group in that case being the union of transverse diffeomorphisms (TDiff) and Weyl rescalings (together dubbed WTDiff; see e.g. [3] for a review). The equations of motion of the theory are the trace-free Einstein equations (see e.g. [4] for a comprehensive introduction), in which any cosmological constant–type contribution does not gravitate, thus providing an elegant solution to the cosmological constant problem.

In more recent years, interest has grown in theories that present TDiff invariance. Simply put, transverse diffeomorphisms are general coordinate transformations in which the Jacobian determinant is required to be $J = 1$. Infinitesimally, if we consider the coordinate transformation $x^\mu \to \hat{x}^\mu = x^\mu + \xi^\mu(x)$ generated by a vector field $\xi^\mu(x)$, then what we do is require the condition $\partial_\mu \xi^\mu = 0$ (see [5] for a concise introduction to transverse diffeomorphisms). The fact that the Jacobian determinant equals unity means that objects that were tensor densities under Diff become actual tensors under TDiff. In particular, the metric determinant becomes a TDiff scalar field, and this has interesting implications. Indeed, on the one hand, the metric determinant becomes a scalar field to which one may endow dynamics. On the other hand, the invariant volume element we find in an action integral is no longer fixed by the symmetry to be $d\text{vol} = \sqrt{g} d^4x$, but can actually take on the more general form $d\text{vol} = f(g) d^4x$, with $f(g)$ being an arbitrary function of the metric determinant, and this opens

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up an enormous range of possibilities for novel couplings. Field theories in which the gravitational sector is TDiff invariant were studied in Refs. [6–10], where cosmological implications were also discussed. One can also study the consequences of breaking the symmetries in the matter sector. TDiff invariant theories with a scalar field were recently considered in Refs. [5,11–13]. Reference [5] considered general scalar field TDiff theories in cosmological contexts, Ref. [12] performed a general study for a scalar field without assuming any background geometry, and Ref. [13] provided a unified description for the dark sector using a particular theory, comparing the results with the latest cosmological observations and data sets.

Now, it is not an uncommon situation in physics to find several equivalent descriptions, or reformulations, of the same theory. Examples include, but are by no means limited to, the well-known equivalence between the Palatini and the metric approach to GR, the scalar-tensor perspective of $f(R)$ gravity (see, for example, [1]), or the correspondence between the field equations of nonlinear Ricci-based metric-affine theories of gravity coupled to scalar matter and GR coupled to a different scalar field Lagrangian [14]. Moreover, ever since the pioneering work of Stueckelberg [15] (see e.g. [16] for a review), it is well known how a theory may recover (or reveal) its broken gauge symmetry via the introduction of additional fields. With these two ideas in mind, we present in this work an alternative formalism for the treatment of a TDiff theory for a scalar field based on previous work done by Henneaux and Teitelboim [17] within the context of unimodular gravity (see also Kuchar [18] for an alternative approach). Following that spirit, in this work we show how a TDiff invariant field theory may be equivalently described as a Diff invariant theory with an additional field. We then work on the particular case of a scalar field theory from both points of view. This idea of finding an equivalent description with symmetry restoration has also been applied on the different gravitational framework of massive gravity [19].

The paper is organized as follows. First of all, in Sec. II we review the TDiff approach for the scalar field and summarize the main results for the potential domination and kinetic domination regimes in Secs. II A and II B. In Sec. III we show how it is possible to reformulate a TDiff invariant field theory in a way that recovers Diff invariance via the introduction of an additional field. Section IV is then devoted to the reformulation of our scalar field theory in a covariantized manner. In Sec. IV A we recover the results of the TDiff approach in the potential domination regime. Then, in Sec. IV B, we not only reobtain the results in the kinetic domination regime, but the use of the covariantized approach allows us to study the stability of the adiabatic fluid in a simple way. Moreover, in Sec. IV C we argue how the use of a particular approach could lead to the study of different kinetic models considered to be more natural. In addition, in the covariantized approach we can find the constraint on the metric when both kinetic and potential terms are present, and this is discussed in Sec. V. This constraint allow us to obtain the effective speed of sound of fluid perturbations, in Sec. V A, which can be used to impose conditions on the physically allowed coupling functions. Then, in Sec. V B, we focus our attention on the particular cases leading to adiabatic models. Section V C is then devoted to analyzing an open question regarding the possibility of a constant kinetic coupling function, while in Sec. V D we consider a particular solution of a family of TDiff theories that yields the same results as GR, and study its stability. Finally, in Sec. VI we present the main conclusions of the work and discuss future work. In Appendix A we discuss a different way of covariantizing our theory, and in Appendix B we include some calculations that are not needed to follow the thread of the main discussion, but which the reader might find useful.

As a final note, we remark that our conventions in this work include the usage of units in which $\hbar = c = 1$, the metric signature $(+,-,-,-)$, and the notation $g = |\det(g_{\mu\nu})|$.

II. THE TDIFF APPROACH

The breaking of Diff invariance down to TDiff in the matter action was recently studied in general backgrounds in Ref. [12]. This work considers a scalar field coupled to gravity via arbitrary functions of the metric determinant, with total action

$$S = S_{\rm EH} + S_m, \tag{1}$$

where

$$S_{\rm EH} = -\frac{1}{16\pi G}\int d^4x\sqrt{g}R, \tag{2}$$

and

$$S_m = \int d^4x\left\{\frac{f_k(g)}{2}g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi - f_v(g)V(\psi)\right\}. \tag{3}$$

This action is indeed seen to be invariant only under transverse diffeomorphisms due to the arbitrary functions of (the absolute value of) the metric determinant $f_k(g)$ and $f_v(g)$. In this work we assume that $f_k \geq 0$ in order to avoid ghost instabilities.

Let us summarize in this section the main results obtained in Ref. [12] following the TDiff approach. The equation of motion (EoM) for the scalar field is

$$\partial_\mu(f_k(g)\partial^\mu\psi) + f_v(g)V'(\psi) = 0, \tag{4}$$

where in general a prime denotes a derivative with respect to its argument. On the other hand, the EoMs for the gravitational field are the usual Einstein equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{5}$$

where the energy-momentum tensor (EMT) for the scalar field is found from its definition

$$T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S_m}{\delta g^{\mu\nu}}, \tag{6}$$

and reads

$$T_{\mu\nu} = \frac{2}{\sqrt{g}} \left\{ \frac{1}{2} f_k(g) \partial_\mu \psi \partial_\nu \psi + g \left[f_v'(g) V(\psi) - \frac{1}{2} f_k'(g) (\partial\psi)^2 \right] g_{\mu\nu} \right\}, \tag{7}$$

where we denote

$$(\partial\psi)^2 \equiv \partial_\alpha \psi \partial^\alpha \psi. \tag{8}$$

Under the assumption of the field derivative $\partial_\mu \psi$ being a timelike vector, it is possible to rewrite the EMT in perfect fluid form. Indeed, defining a unit timelike vector field u^μ through

$$u^\mu = \frac{\partial^\mu \psi}{\sqrt{(\partial\psi)^2}} \equiv \frac{\partial^\mu \psi}{N}, \tag{9}$$

where we denote the normalization as

$$N \equiv \sqrt{(\partial\psi)^2}, \tag{10}$$

together with an energy density

$$\rho = \frac{2}{\sqrt{g}} \left\{ \frac{1}{2} f_k (\partial\psi)^2 + g \left[f_v' V - \frac{1}{2} f_k' (\partial\psi)^2 \right] \right\} \tag{11}$$

and a pressure

$$p = -\frac{2g}{\sqrt{g}} \left[f_v' V - \frac{1}{2} f_k' (\partial\psi)^2 \right], \tag{12}$$

one can rewrite the EMT (7) as

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu - p g_{\mu\nu}. \tag{13}$$

In this paper we also work with the timelike vector assumption whenever we wish to reexpress the analysis as that of a perfect fluid.

One of the main points of study in the TDiff approach is the conservation of the EMT. Indeed, the conservation of this quantity is an automatic consequence of the Noether theorem for a theory with symmetry under diffeomorphisms, but it does not follow trivially when we have less symmetry. Instead, one argues that the conservation of the EMT on the solutions to the EoMs is a consistency requirement of the theory, since the Einstein equations (5) still hold and the Einstein tensor is divergenceless:

$$\nabla_\mu G^{\mu\nu} = 0 \Rightarrow \nabla_\mu T^{\mu\nu} = 0. \tag{14}$$

Within the TDiff approach, this consistency condition allows one to obtain a certain (physical) constraint on the metric.

We now focus on two limiting cases of interest. These are the potential domination regime and the kinetic domination regime, which we review in the following.

A. Potential domination in the TDiff approach

Everything is rather simple in the potential regime, which amounts to ignoring the kinetic contribution in the action. When we do so, the EoM (4) for ψ becomes $f_v V' = 0$, which (for a nonvanishing coupling function f_v) tells us that the field takes on the constant value $\psi = \psi_0$ which is the extremum of the potential: $V(\psi) = V(\psi_0) = \text{const}$. The EMT (7) simplifies to

$$T_{\mu\nu} = 2V f_v' \sqrt{g} g_{\mu\nu}, \tag{15}$$

and its conservation becomes

$$2V g^{\mu\nu} \nabla_\mu (f_v' \sqrt{g}) = 2V g^{\mu\nu} \partial_\mu (f_v' \sqrt{g}) = 0, \tag{16}$$

where in the first term we pulled (covariantly) constant terms out of the covariant derivative and in the second we recognized that the product $f_v' \sqrt{g}$ is a TDiff scalar so that we may use a partial derivative. Since the relation (16) must be met for all metrics, it follows that

$$\partial_\mu (f_v' \sqrt{g}) = \frac{1}{\sqrt{g}} \left(\frac{1}{2} f_v' + g f_v'' \right) \partial_\mu g = 0. \tag{17}$$

The above relation is satisfied whenever the coupling function takes the form

$$f_v(g) = A\sqrt{g} + B, \tag{18}$$

with A and B being constants of integration, but if we wish to leave the coupling function arbitrary (which in principle we do), then it must be the case that

$$\partial_\mu g = 0 \Rightarrow g = \text{const}. \tag{19}$$

This is the constraint on the metric that we obtain in the potential domination regime: the determinant must be constant. It is interesting to note that, as the determinant is a constant quantity, any given function of the determinant will also assume a constant value, for instance the function $f'_v(g)$. This is relevant because, looking back at the EMT (15), it may be written as

$$T_{\mu\nu} \equiv \lambda g_{\mu\nu}, \tag{20}$$

with

$$\lambda \equiv 2V f'_v \sqrt{g} = \text{const}, \tag{21}$$

and we have the behavior of a cosmological constant.

B. Kinetic domination in the TDiff approach

The study of the kinetic domination regime is more involved. The EoM for the field in this regime becomes $\partial_\mu(f_k \partial^\mu \psi) = 0$, and its solution must satisfy

$$(\partial\psi)^2 = C_\psi(x)\left(\frac{\sqrt{g}}{\delta V f_k}\right)^2, \tag{22}$$

where $C_\psi(x)$ is a function such that $u^\mu \partial_\mu C_\psi(x) = 0$, and where we find the cross-sectional volume of the congruence δV, related to the expansion by [20]

$$\nabla_\mu u^\mu = u^\mu \partial_\mu \ln \delta V. \tag{23}$$

Regarding EMT conservation, when working with a perfect fluid it is common to project the conservation equations $\nabla_\mu T^{\mu\nu} = 0$ onto directions longitudinal and transverse to the velocity of the fluid. For the former one contracts with u_ν, and for the latter one acts with the orthogonal projector $h^\mu{}_\nu = \delta^\mu_\nu - u^\mu u_\nu$, obtaining respectively

$$\dot{\rho} + (\rho + p)\nabla_\mu u^\mu = 0, \tag{24a}$$

$$(\rho + p)\dot{u}^\mu - (g^{\mu\nu} - u^\mu u^\nu)\nabla_\nu p = 0, \tag{24b}$$

where we use the dot notation $\dot{} \equiv u^\mu \nabla_\mu$. In the kinetic regime, the perfect fluid quantities take on the form

$$\rho = \frac{(\partial\psi)^2}{\sqrt{g}}(f_k - g f'_k), \tag{25a}$$

$$p = \frac{(\partial\psi)^2}{\sqrt{g}} g f'_k, \tag{25b}$$

and the equation of state (EoS) parameter reads

$$w = \frac{p}{\rho} = \frac{g f'_k}{f_k - g f'_k} \equiv \frac{F}{1 - F}, \tag{26}$$

where we define

$$F \equiv \frac{g f'_k}{f_k}. \tag{27}$$

It is interesting to remark that the EoS parameter (26) is a function of only the metric determinant, $w = w(g)$.

Studying the longitudinal projection (24a) on the solution to the EoM (22) yields the following relation:

$$(2F - 1)\frac{g}{f_k} = C_g(x)\delta V^2, \tag{28}$$

where $C_g(x)$ is a function that must satisfy

$$u^\mu \partial_\mu C_g(x) = 0. \tag{29}$$

Equation (28) is one of the main results in the TDiff approach, and indeed shows how the study of EMT conservation is not a trivial matter but rather imposes a constraint on the metric.

Probing the transverse projection (24b) for further information ends up revealing that the two functions $C_\psi(x)$ and $C_g(x)$ are inversely proportional, i.e.

$$C_\psi C_g = \text{const} \equiv c_\rho. \tag{30}$$

The reason for naming the constant as c_ρ is because it is possible to find the following nice expression for the energy density in the kinetic regime [12]:

$$\rho = \frac{c_\rho}{(w - 1)\sqrt{g}}. \tag{31}$$

Since the EoS is only a function of the metric determinant, it follows that both the energy density and the pressure are functions of only the metric determinant, and this dependence on a single quantity reveals that we are dealing with an adiabatic fluid. The adiabatic speed of sound is defined through $\delta p = c_a^2 \delta\rho$ which, joined with $p = w\rho$, yields

$$c_a^2 = w + w' \frac{\rho}{\rho'}. \tag{32}$$

In the TDiff approach, it takes the form [12]

$$c_a^2 = -\frac{g f_k (f'_k + 2g f''_k)}{f_k^2 + (2g f'_k)^2 - g f_k (5 f'_k + 2g f''_k)}. \tag{33}$$

In this way we conclude our summary of the general framework resulting from the consideration of a TDiff scalar field. For interesting consequences and phenomenology of this theory we refer the reader to Refs. [5,12,13].

III. COVARIANTIZED ACTION

In this section we note that one can rewrite an action with broken Diff invariance in a Diff invariant way via the introduction of additional fields, similar in spirit to the Stueckelberg procedure in gauge theories [15]. In this work we follow Henneaux and Teitelboim [17] and, in order to preserve locality, introduce the new field in the form of a vector. The way to restore the Diff symmetry is not unique, however, and other references preferred to introduce a scalar field (see e.g. [21] and references therein). We refer the reader to Appendix A for a more detailed discussion on this subject.

Having clarified that point, let us now consider a TDiff invariant field theory where each of the terms in the action is of the form

$$S_{\mathrm{TDiff}}[g_{\mu\nu},\Psi]=\int d^4x f(g)\mathcal{L}(g_{\mu\nu},\Psi,\partial_\mu\Psi), \tag{34}$$

where $\mathcal{L}$ is a Diff scalar. The general case could be written as the sum of different terms,

$$S_{\mathrm{TDiff}}[g_{\mu\nu},\Psi]=\int d^4x\sum_i f_i(g)\mathcal{L}_i(g_{\mu\nu},\Psi,\partial_\mu\Psi), \tag{35}$$

but in order not to clutter the treatment we consider the simpler expression (34), which is precisely a general term in the above summation.

In order to perform the covariantization, we first of all introduce a Diff scalar density $\bar{\mu}$ which transforms as $\sqrt{g}$ under general coordinate transformations. Doing so, the Diff invariant theory given by

$$S_{\mathrm{Diff}}[g_{\mu\nu},\Psi,\bar{\mu}]=\int d^4x\sqrt{g}\left[\frac{\bar{\mu}}{\sqrt{g}}f(g/\bar{\mu}^2)\right]\mathcal{L}(g_{\mu\nu},\Psi,\partial_\mu\Psi) \tag{36}$$

agrees with the TDiff theory (34) in the coordinate frame in which $\bar{\mu}=1$, which we refer to as the "TDiff frame" for simplicity. The term in between brackets is a Diff scalar and an arbitrary function of the combination $\bar{\mu}/\sqrt{g}$, which we write as

$$H(\bar{\mu}/\sqrt{g})\equiv\frac{\bar{\mu}}{\sqrt{g}}f(g/\bar{\mu}^2). \tag{37}$$

If, for simplicity, we denote the argument by $\bar{\mu}/\sqrt{g}\equiv Y$, then we have that

$$H(Y)\equiv Yf(Y^{-2}). \tag{38}$$

In the TDiff frame $\bar{\mu}=1$ (equivalently, $Y=1/\sqrt{g}$), we would find

$$H(Y)|_{\bar{\mu}=1}=\frac{f(g)}{\sqrt{g}}. \tag{39}$$

The question now is how the newly introduced scalar density $\bar{\mu}$ is related to the new field (equivalently, how the combination Y is related to the new field). As we mentioned there are different possibilities, but we choose a vector field T^μ as our addition to the theory. A scalar density $\bar{\mu}$ that transforms as $\sqrt{g}$ may be built from a vector field through the simple combination [5,17]

$$\bar{\mu}=\partial_\mu(\sqrt{g}T^\mu). \tag{40}$$

In this way, the variable Y turns out to be related to T^μ through

$$Y=\frac{\bar{\mu}}{\sqrt{g}}=\nabla_\mu T^\mu, \tag{41}$$

and so we see that the newly introduced vector field T^μ enters the theory via its divergence. It is interesting to remark that in the general Stueckelberg procedure the idea is to take the gauge functions and promote them to fields. Since four-dimensional diffeomorphisms are generated by four functions, it is natural to introduce a vector field T^μ. This being said, however, going from TDiff back to Diff actually only involves removing one condition (recall that $J=1$, or $\partial_\mu\xi^\mu=0$), and thus it makes sense that the newly introduced vector field ends up appearing through the scalar combination $Y=\nabla_\mu T^\mu$.

Our covariantized action (36) would then take the form

$$S_{\mathrm{Diff}}[g_{\mu\nu},\Psi,T^\mu]=\int d^4x\sqrt{g}H(Y)\mathcal{L}(g_{\mu\nu},\Psi,\partial_\mu\Psi), \tag{42}$$

where we bear in mind that $Y=\nabla_\mu T^\mu$. We note that in the case $H(Y)=\text{const}$ the dependence on the new field would be lost because it would correspond to the original theory being Diff invariant already [i.e. $f(g)\propto\sqrt{g}$]. In any case, since

$$S_{\mathrm{Diff}}[g_{\mu\nu},\Psi,T^\mu]|_{\bar{\mu}=1}=S_{\mathrm{TDiff}}[g_{\mu\nu},\Psi], \tag{43}$$

we see how one may alternatively work within the TDiff approach or the covariantized approach since they are equivalent, as it is explicitly shown when choosing the gauge $\bar{\mu}=1$ (equivalently, $Y=1/\sqrt{g}$) in the Diff invariant action. Thus, the invariance under full diffeomorphisms has been restored due to the introduction of a Diff vector field T^μ.

IV. THE COVARIANTIZED APPROACH

Let us take the Diff invariant action [5]

$$S_{\mathrm{Diff}}[g_{\mu\nu},\psi,T^\mu]=S_{\mathrm{EH}}+\int d^4x\sqrt{g}[H_k(Y)X-H_v(Y)V] \tag{44}$$

where, for simplicity, we denote the kinetic term by

$$X \equiv \frac{1}{2} g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi = \frac{1}{2} (\partial\psi)^2. \quad (45)$$

If we set $\bar{\mu} = 1$ and recognize the functions $f_k(g)$ and $f_v(g)$, we immediately realize that it has become precisely the action (1) for a TDiff scalar field. Let us keep working now in the covariantized approach, and see what we obtain. Variations of the action (44) with respect to the scalar field ψ yield the following EoM:

$$\nabla_\nu [H_k(Y) g^{\mu\nu} \nabla_\mu \psi] + H_v(Y) V'(\psi) = 0. \quad (46)$$

Recalling that the covariant divergence of a vector V^α may be expressed as

$$\nabla_\alpha V^\alpha = \frac{1}{\sqrt{g}} \partial_\alpha (\sqrt{g} V^\alpha), \quad (47)$$

and using Eq. (39), one can see that in the $\bar{\mu} = 1$ gauge this equation reduces to

$$\frac{1}{\sqrt{g}} \partial_\nu [f_k(g) \partial^\mu \psi] + \frac{f_v(g)}{\sqrt{g}} V'(\psi) = 0, \quad (48)$$

which is equivalent to the EoM (4) we found for ψ in the TDiff approach.

On the other hand, variations of the action (44) with respect to the vector field that has restored the Diff invariance, that is T^μ, provide the following EoM:

$$\partial_\nu [H'_k(Y) X - H'_v(Y) V] = 0. \quad (49)$$

Hence, there is a conserved quantity as a result of the derivative dependence of the field T^μ on the action (44). One could think that these equations are new with respect to the TDiff formalism. However, it should be noted that in said formalism one has the equations coming from the conservation of the EMT that is no longer trivially satisfied in that framework, which are the Eq. (14). So, as we will show, these equations must be equivalent.

Finally, variations with respect to the metric tensor $g_{\mu\nu}$ provide the Einstein equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \quad (50)$$

where the total EMT is found using the definition (6) and takes the form

$$T_{\mu\nu} = H_k(Y) \partial_\mu \psi \partial_\nu \psi - [H_k(Y) X - H_v(Y) V] g_{\mu\nu} + Y [H'_k(Y) X - H'_v(Y) V] g_{\mu\nu} \quad (51)$$

(see Appendix B for the calculation of this quantity). It should be noted that the conservation of the EMT (51) has to be trivially satisfied when considering the EoM of the fields (46) and (49), due to the invariance of the action (44) under general diffeomorphisms. Indeed, it can be explicitly checked that this is the case, and we refer the reader to Appendix B for the calculation.

On the other hand, noting the definition of the $H(Y)$ functions in Eq. (38), one can obtain the relation

$$H'(Y) = f(Y^{-2}) - 2Y^{-2} \frac{df}{d(Y^{-2})}, \quad (52)$$

so that

$$H'(Y)|_{\bar{\mu}=1} = f(g) - 2g f'(g). \quad (53)$$

Taking this result into account, together with Eq. (39), we can immediately verify that the covariantized EMT (51) reduces to the TDiff EMT (7) in the $\bar{\mu} = 1$ frame.

This covariantized formalism has been applied to the Friedmann-Lemaître-Robertson-Walker model in Ref. [5], recovering the information provided by the constraint on the metric in the TDiff case. In the next section we show that this is possible in general.

A. Potential domination in the covariantized approach

We begin with the potential domination regime. Neglecting the kinetic parts, the EoM (46) for ψ becomes

$$H_v(Y) V'(\psi) = 0, \quad (54)$$

and this means that either $H_v = 0$ (which is trivial) or that the field takes on the constant value $\psi(x) = \psi_0$ such that the potential reaches an extremum $V(\psi) = V(\psi_0) = \text{const}$ (exactly as we concluded in the TDiff approach). On the other hand, the EoM (49) for T^μ in the potential regime becomes

$$\partial_\mu [H'_v(Y) V] = 0 \Rightarrow \partial_\mu H'_v(Y) = H''_v(Y) \partial_\mu Y = 0, \quad (55)$$

where we have already used that $V = \text{const}$ to pull it out of the derivative. We now have a couple of options in order for this equation to be satisfied, namely that $H''_v(Y) = 0$ or that $\partial_\mu Y = 0$. The first of these implies

$$H''_v(Y) = 0 \Rightarrow H_v(Y) = BY + A, \quad (56)$$

which in the $\bar{\mu} = 1$ frame, that is $H_v \to f_v / \sqrt{g}$ and $Y \to 1/\sqrt{g}$, reduces to the condition (18) on the coupling function f_v we already found. The second option gives

$$\partial_\mu Y = 0 \Rightarrow Y = \text{const}, \quad (57)$$

which in the $\bar{\mu} = 1$ frame reduces to $g = \text{const}$, and this is precisely the condition (19) on the metric determinant we previously obtained.

Moreover, the EMT (51) in the potential regime takes the form

$$T_{\mu\nu} = V(H_v - YH_v')g_{\mu\nu} \equiv \lambda g_{\mu\nu}, \tag{58}$$

and if we use the above EoM we see that it is of a cosmological constant–type once again, since

$$\lambda \equiv V(H_v - YH_v') = \text{const}. \tag{59}$$

Thus, in the potential limit we have recovered our previous results, as it should be expected.

B. Kinetic domination in the covariantized approach

In the kinetic domination regime we neglect the potential terms. The EoM (49) for T^μ becomes

$$\partial_\mu[H_k'(Y)X] = \frac{1}{2}\partial_\mu[H_k'(Y)(\partial\psi)^2] = 0. \tag{60}$$

It follows from the above that

$$H_k'(Y)(\partial\psi)^2 = \text{const} \equiv -c_\rho, \tag{61}$$

where we have named the arbitrary constant as $-c_\rho$ for future convenience. On the other hand, the EoM (46) for ψ in the kinetic domination regime becomes

$$\nabla_\mu[H_k(Y)\partial^\mu\psi] = \nabla_\mu[H_k(Y)Nu^\mu] = 0, \tag{62}$$

where we have recalled the velocity (9). If we expand the equation and divide through by H_k (which we take to be nonzero), then we obtain

$$Nu^\mu\nabla_\mu \ln H_k + u^\mu\nabla_\mu N + N\nabla_\mu u^\mu = 0. \tag{63}$$

Recalling Eq. (23) for the relation between the expansion $\nabla_\mu u^\mu$ and the cross-sectional volume δV, and using the fact that the covariant derivatives may be changed by partial derivatives when acting on scalar functions, it follows from the above that

$$\begin{aligned} &Nu^\mu\partial_\mu(\ln H_k + \ln N + \ln \delta V) \\ &\quad = Nu^\mu\partial_\mu \ln(H_k N\delta V) = 0. \end{aligned} \tag{64}$$

From this expression, we finally conclude that

$$(\partial\psi)^2 = \frac{C_\psi(x)}{(H_k\delta V)^2}, \tag{65}$$

where $C_\psi(x)$ is a function such that $u^\mu\partial_\mu C_\psi(x) = 0$. Going now to the $\bar{\mu} = 1$ frame and using Eq. (39) for $H_k|_{\bar{\mu}=1}$, we immediately recover the expression (22) for the solution to the EoM in the TDiff approach.

On the other hand, substituting Eq. (65) into (61), it follows that

$$-c_\rho = H_k'(\partial\psi)^2 = H_k'\frac{C_\psi(x)}{(H_k\delta V)^2}, \tag{66}$$

which leads to

$$-\frac{H_k'}{H_k^2} = \frac{c_\rho}{C_\psi(x)}\delta V^2. \tag{67}$$

If we now define the function

$$C_g(x) \equiv \frac{c_\rho}{C_\psi(x)}, \tag{68}$$

which satisfies $u^\mu\partial_\mu C_g(x) = 0$, we may write the above equation as

$$-\frac{H_k'}{H_k^2} = C_g(x)\delta V^2. \tag{69}$$

It is now immediate to see that we recover the constraint (28) when going to the frame $\bar{\mu} = 1$, since

$$-\frac{H_k'}{H_k^2}\bigg|_{\bar{\mu}=1} = -\frac{f_k - 2gf_k'}{f_k^2/g}, \tag{70}$$

and this implies that

$$(2F - 1)\frac{g}{f_k} = C_g(x)\delta V^2, \tag{71}$$

where we have recalled the definition of the function F in Eq. (27). It is worth noting that obtaining this result in the TDiff approach requires a longer calculation [12], while in the covariantized approach we have found it rather directly.

On another note, in the kinetic domination regime the EMT (51) takes the form

$$T_{\mu\nu} = H_k\partial_\mu\psi\partial_\nu\psi - [H_k - YH_k']Xg_{\mu\nu}, \tag{72}$$

which, under the assumption of the field derivative $\partial_\mu\psi$ being a timelike vector, is seen to be equivalent to that of a perfect fluid, using the same definition (9) for the velocity, and with energy density

$$\rho = -\frac{c_\rho}{2}\left(\frac{H_k}{H_k'} + Y\right) \tag{73}$$

and pressure

$$p = -\frac{c_\rho}{2}\left(\frac{H_k}{H_k'} - Y\right), \tag{74}$$

where we have made use of the solution (61) to simplify $H'_k X = -c_\rho/2$. Note that we have obtained that the energy density and pressure are functions of only Y, and this dependence on a single variable means that possible perturbations of the fluid will be adiabatic.

The EoS parameter now reads

$$w = \frac{p}{\rho} = \frac{\frac{H_k}{H'_k} - Y}{\frac{H_k}{H'_k} + Y}, \tag{75}$$

and, as expected, it is seen to be a function of Y only. Note also that in the $\bar{\mu} = 1$ frame we have

$$w|_{\bar{\mu}=1} = \frac{\frac{f_k/\sqrt{g}}{f_k - 2gf'_k} - \frac{1}{\sqrt{g}}}{\frac{f_k/\sqrt{g}}{f_k - 2gf'_k} + \frac{1}{\sqrt{g}}} = \frac{gf'_k}{f_k - gf'_k}, \tag{76}$$

which coincides with the expression (26) for the EoS parameter previously found in the TDiff formalism. Consider now the following quantity:

$$w - 1 = \frac{-2Y}{\frac{H_k}{H'_k} + Y} = \frac{Yc_\rho}{\rho}, \tag{77}$$

where in the second equality we have recalled the expression (73) for the energy density. Rearranging, we obtain the following simple expression for the energy density:

$$\rho = \frac{Yc_\rho}{w - 1}. \tag{78}$$

After evaluation in the frame $\bar{\mu} = 1$, i.e. substituting $Y \to 1/\sqrt{g}$ and recalling from Eq. (76) that the EoS parameter is recovered, it finally yields the simple expression (31) for the energy density that was obtained in the TDiff approach.

Finally, the speed of sound of the adiabatic perturbations follows from the definition (32), which yields

$$c_a^2 = \frac{H_k H''_k}{H_k H''_k - 2(H'_k)^2}, \tag{79}$$

or, finding a common factor,

$$c_a^2 = \frac{1}{1 - 2\frac{(H'_k)^2}{H_k H''_k}}. \tag{80}$$

Note that in obtaining this second expression we have divided by H''_k; so, it will not be valid, for example, for the interesting case of TDiff dark matter that will be commented on in the next section. From this equation it is easy to note that, in order have a stable adiabatic fluid (i.e. a non-negative c_a^2), one needs the coupling function to satisfy[1]

$$\frac{(H'_k)^2}{H_k H''_k} < \frac{1}{2}. \tag{81}$$

Moreover, in order to also avoid the propagation of superluminal perturbations (i.e. to have $c_a^2 \leq 1$), it must be the case that

$$H''_k < 0, \tag{82}$$

where we have taken into account that $H_k > 0$ (so that the kinetic term is positive and there are no ghosts in the theory).

On the other hand, it is possible once again to verify that Eq. (80), when evaluated in the $\bar{\mu} = 1$ frame, reduces to the previously obtained adiabatic speed of sound in Eq. (33). To this end, we must simply recall Eqs. (39) and (53) for $H|_{\bar{\mu}=1}$ and $H'|_{\bar{\mu}=1}$, respectively, and also use that

$$H''(Y) = 2Y^{-3}\left[\frac{df}{d(Y^{-2})} + 2Y^{-2}\frac{d^2 f}{d^2(Y^{-2})}\right], \tag{83}$$

which yields

$$H''(Y)|_{\bar{\mu}=1} = 2g^{3/2}[f'(g) + 2gf''(g)]. \tag{84}$$

After careful substitution, we finally obtain

$$c_a^2|_{\bar{\mu}=1} = -\frac{gf_k(f'_k + 2gf''_k)}{f_k^2 + (2gf'_k)^2 - gf_k(5f'_k + 2gf''_k)}, \tag{85}$$

which indeed recovers the TDiff adiabatic speed of sound in Eq. (33).

Furthermore, evaluated in the $\bar{\mu} = 1$ frame the requirement for stability (81) is translated to

$$\frac{(f_k - 2gf'_k)^2}{gf_k(f'_k + 2gf''_k)} < 1, \tag{86}$$

while the requirement for subluminal propagation (82) becomes

$$f'_k + 2gf''_k < 0. \tag{87}$$

These conditions on the kinetic coupling function f_k can also be seen to follow directly from the TDiff expression for the adiabatic speed of sound (85), as should be expected. In order to see precisely how, it comes in handy to rewrite said expression in a cleaner manner. Inspired by the form of the

[1]The case in which $H''_k = 0$ implies $c_a^2 = 0$, which trivially satisfies both stability and subluminality requirements. Hence, what we present here implicitly assumes that $H''_k \neq 0$.

quantities appearing in Eq. (86), it is possible to see how the denominator of (85) can be written as

$$f_k^2 + (2gf_k')^2 - gf_k(5f_k' + 2gf_k'') \\ = (f_k - 2gf_k')^2 - gf_k(f_k' + 2gf_k''). \tag{88}$$

Knowing this, it turns out that we can rewrite the TDiff adiabatic speed of sound (85) in the rather clean form

$$c_a^2|_{\bar{\mu}=1} = \frac{1}{1 - \frac{(f_k - 2gf_k')^2}{gf_k(f_k' + 2gf_k'')}}, \tag{89}$$

from which, indeed, the same conditions can easily be seen to follow. Nevertheless, it is worth stressing that knowing the results in the covariantized approach and translating them into the TDiff approach is precisely what helped us in finding an appropriate and simple rewrite for the TDiff adiabatic speed of sound. Indeed, working directly from the convoluted expression (85) would have implied a much greater effort in finding the conditions for stability and subluminality, and in fact this work was not done in Ref. [12].

As a general comment before concluding, we remark that the above conditions on either H_k or f_k (which are the requirements for stable perturbations and subluminal propagation) serve as a way of selecting classes of possible coupling functions (and hence particular models) that are physically meaningful. Of course, there is in principle no reason to restrict ourselves to only those two requirements, and one could also consider the study of the energy conditions (as was done in Ref. [12] in the TDiff framework) as a way of further restricting physically viable theories.

C. Simple models in the kinetic regime

Regarding some particular models of interest in the kinetic regime, note that one can obtain a constant EoS parameter for the fluid considering a power-law kinetic function. Indeed, taking into account Eq. (75) one has

$$H_k(Y) = CY^\beta \Rightarrow w = \frac{1-\beta}{1+\beta}. \tag{90}$$

As already discussed in Ref. [12] following the TDiff approach, in this case we have that the propagation speed of the adiabatic field perturbations is

$$c_a^2 = \frac{1-\beta}{1+\beta} = w, \tag{91}$$

which has been obtained taking into account Eq. (80). This is because a power-law kinetic function in the covariantized approach implies a power-law kinetic coupling in the TDiff approach but with a different value of the exponent [note Eq. (38)]. It should be noted that the dark matter model commented on in Ref. [12] in this framework corresponds to $H_k = CY$.

However, the models with a constant EoS parameter are the only family that will have a similar functional expression in both approaches. For example, an exponential kinetic function in the covariantized approach is not equivalent to an exponential kinetic coupling in the TDiff approach according to Eq. (38). In this case, Eq. (75) leads to the EoS parameter

$$H_k(Y) = Ce^{\beta Y} \Rightarrow w = \frac{1-\beta Y}{1+\beta Y}, \tag{92}$$

which could interpolate between the behavior of stiff matter and that of a cosmological constant. Nevertheless, taking into account Eq. (80), one obtains

$$c_a^2 = -1. \tag{93}$$

So, the particular models with an exponential kinetic function in the covariantized approach are unstable, whereas models with an exponential kinetic coupling in the TDiff approach can have interesting phenomenology [13]. Finally, thus, it should be emphasized that models that could be natural to consider in one approach (due to a simple form of the relevant function) could appear unnatural in the other approach (in which the relevant function could be more complicated).

V. GENERAL MODELS

Up to this point we have focused on the two limiting regimes and recovered our previously known results, explicitly confirming the equivalence between the two approaches. However, the covariantized treatment can go further; indeed, it yields in a straightforward manner a very important and previously unknown result, namely, the general constraint on the metric when both kinetic and potential terms are present. Let us now discuss this point.

In the TDiff approach of Ref. [12] the analysis is restricted to the two limiting regimes of kinetic and potential domination for simplicity, because the study of the conservation of the TDiff EMT (7) requires an incredible amount of effort. In this way, the constraints are obtained in each of the two regimes, but the general situation with both kinetic and potential terms is not discussed. However, in the covariantized approach, the EoMs of the new vector field (49) are very simple to study, much more so than the conservation of the TDiff EMT; moreover, as we discussed, they encode the same information. Solving those EoMs, then, we will obtain the metric constraint in the most general situation.

Now, the solution to the EoM (49) of the new field T^μ reads, quite simply,

$$H_k' X - H_v' V = \text{const.} \tag{94}$$

The value of the constant is in principle "arbitrary" (speaking more precisely, it is fixed by the initial conditions), but since we should recover the results we already know from the limiting regimes, we write it as

$$H_k' X - H_v' V = -\frac{c_\rho}{2}, \tag{95}$$

so that the kinetic limit is indeed immediately verified (we remark that the aforementioned "arbitrariness" is not lost; it simply resides in the constant c_ρ which is fixed by the initial conditions[2]). If we now evaluate the above expression in the $\bar{\mu} = 1$ frame, taking into account Eq. (53) for $H'|_{\bar{\mu}=1}$, then we obtain the following:

$$(f_k - 2g f_k')X - (f_v - 2g f_v')V = -\frac{c_\rho}{2}. \tag{96}$$

This novel result is the general expression for the constraint on the metric whenever we have both a kinetic and a potential term present, something that was not previously studied. It is easy to verify that it gives the correct results in the two limiting regimes. Of course, it is a formal expression, meaning that one would need to solve the EoM for the scalar field ψ in order to extract actual information on the metric constraint. Nevertheless, the main point we wish to highlight is the great utility of the covariantized approach, as it has yielded a general and previously unknown result.

On another note, the simple solution (95) to the EoM for T^μ also helps in simplifying the covariantized EMT (51), in particular its last term, so that in the end we may write it as

$$T_{\mu\nu} = H_k \partial_\mu \psi \partial_\nu \psi - \left(H_k X - H_v V + \frac{c_\rho}{2} Y \right) g_{\mu\nu}. \tag{97}$$

Assuming a timelike derivative $\partial_\mu \psi$, we may again express it in perfect fluid form using the usual definition (9) for the velocity and defining the energy density and pressure, respectively, as

$$\rho = H_k X + H_v V - \frac{c_\rho}{2} Y, \tag{98a}$$

$$p = H_k X - H_v V + \frac{c_\rho}{2} Y, \tag{98b}$$

while the EoS parameter reads

$$w = \frac{H_k X - H_v V + Y c_\rho / 2}{H_k X + H_v V - Y c_\rho / 2}. \tag{99}$$

Let us remark that, as these expressions stand, we cannot in general ensure the adiabaticity of our perfect fluid (see the following sections for further comments on this).

Finally, let us briefly discuss the GR limit of our theory, which essentially means $f_k = f_v \equiv f = \sqrt{g}$ in the TDiff approach or, equivalently, $H_k = H_v \equiv H = 1$ in the covariantized approach. For simplicity, we only discuss it within the covariantized approach in order to keep it brief, since they are equivalent. Approaching the GR limit is done by considering small variations around the GR solution, which we write as

$$H(Y) = 1 + \epsilon h(Y), \tag{100}$$

where ϵ is a small parameter for power counting and $h(Y)$ is an arbitrary function, and then taking the limit $\epsilon \to 0$. Now, Eq. (95) tells us that

$$\epsilon (X - V) h' = -\frac{c_\rho}{2}, \tag{101}$$

so that the arbitrary constant is $c_\rho = \mathcal{O}(\epsilon)$. Looking at the energy density and pressure in (98), this means that when approaching the GR limit these quantities behave as

$$\rho = X + V + \mathcal{O}(\epsilon), \tag{102a}$$

$$p = X - V + \mathcal{O}(\epsilon), \tag{102b}$$

and the EoS parameter behaves as

$$w \underset{\epsilon \to 0}{=} \frac{X - V}{X + V}. \tag{103}$$

Thus, we indeed recover for all of these quantities the same expression that we would have for a canonical scalar field in GR. In the subsequent sections, we shall also verify the GR limit at different points.

A. Effective speed of sound

We mentioned previously how we could not in general ensure the adiabaticity of the fluid. In this section we will see that the most general situation is precisely a non-adiabatic fluid. A comment is needed before proceeding, however: we assume from this point onwards that the kinetic coupling function H_k is not constant, so that $H_k' \neq 0$. The particular case of a constant coupling function requires a separate study which the reader may find in the final subsection.

Having cleared that up, let us now begin by considering the solutions (95) to the EoM of T^μ. Solving for the kinetic term and showing every dependence, we obtain

[2]We can see from Eq. (95) how giving initial conditions for the fields and their derivatives fixes the constant c_ρ. Note also that an initial condition for Y really is, in the TDiff frame, just an initial condition for g; this makes more sense in the context of Eq. (96).

$$X = \frac{H_v'(Y)V(\psi) - c_\rho/2}{H_k'(Y)} = X(Y, \psi). \tag{104}$$

Explicitly substituting this expression into Eq. (98) we obtain that the energy density and pressure satisfy

$$\rho = \frac{H_k}{H_k'}\left(H_v' V - \frac{c_\rho}{2}\right) + H_v V - \frac{c_\rho}{2} Y = \rho(Y, \psi), \tag{105a}$$

$$p = \frac{H_k}{H_k'}\left(H_v' V - \frac{c_\rho}{2}\right) - H_v V + \frac{c_\rho}{2} Y = p(Y, \psi), \tag{105b}$$

i.e. they are functions of two variables and so our fluid is not adiabatic. Now, as functions of two variables, the perturbations will be written as

$$\delta\rho = \left.\frac{\partial\rho}{\partial Y}\right|_\psi \delta Y + \left.\frac{\partial\rho}{\partial\psi}\right|_Y \delta\psi, \tag{106a}$$

$$\delta p = \left.\frac{\partial p}{\partial Y}\right|_\psi \delta Y + \left.\frac{\partial p}{\partial\psi}\right|_Y \delta\psi, \tag{106b}$$

and we may join these equations to write[3]

$$\delta p = c_s^2 \delta\rho + \alpha\delta\psi, \tag{107}$$

where we denote

$$c_s^2 \equiv \left.\frac{\partial p/\partial Y}{\partial\rho/\partial Y}\right|_\psi, \tag{108a}$$

$$\alpha \equiv (-c_s^2)\left.\frac{\partial\rho}{\partial\psi}\right|_Y + \left.\frac{\partial p}{\partial\psi}\right|_Y. \tag{108b}$$

Note that c_s^2 will be the effective speed of sound of cosmological perturbations. Indeed, in the reference frame comoving with the fluid (sometimes called the “rest” frame) we would find $\delta\psi = 0$ and also $\delta p_{\rm rest} = c_s^2 \delta\rho_{\rm rest}$ (see Refs. [22,23] for a discussion). On another note, in situations in which $\alpha = 0$, c_s^2 would play the role of the adiabatic speed of sound (indeed, recalling that for adiabatic perturbations $\delta p = c_a^2 \delta\rho$, we would have that $c_a^2 = c_s^2$).

Having mentioned those physical interpretations, let us now compute c_s^2 from its definition in Eq. (108a). To this end, we proceed bit by bit, starting with the numerator. Differentiating the pressure as expressed in Eq. (98), we get

$$\begin{aligned}\left.\frac{\partial p}{\partial Y}\right|_\psi &= H_k' X + H_k \left.\frac{\partial X}{\partial Y}\right|_\psi - H_v' V + \frac{c_\rho}{2} \\ &= H_k \left.\frac{\partial X}{\partial Y}\right|_\psi,\end{aligned} \tag{109}$$

where in the second equality we have used the solutions (95) to the EoM of the vector field T^μ. Now, differentiating Eq. (104) it follows that

$$\left.\frac{\partial X}{\partial Y}\right|_\psi = -\frac{1}{(H_k')^2}\left[V(H_k'' H_v' - H_k' H_v'') - \frac{c_\rho}{2} H_k''\right] \tag{110}$$

and so, finally, we have that the numerator of c_s^2 in Eq. (108a) takes the form

$$\left.\frac{\partial p}{\partial Y}\right|_\psi = -\frac{H_k}{(H_k')^2}\left[V(H_k'' H_v' - H_k' H_v'') - \frac{c_\rho}{2} H_k''\right]. \tag{111}$$

Consider now the denominator of c_s^2 in Eq. (108a). Differentiating the energy density (98), we obtain

$$\begin{aligned}\left.\frac{\partial\rho}{\partial Y}\right|_\psi &= H_k' X + H_k \left.\frac{\partial X}{\partial Y}\right|_\psi + H_v' V - \frac{c_\rho}{2} \\ &= 2\left(H_v' V - \frac{c_\rho}{2}\right) + \left.\frac{\partial p}{\partial Y}\right|_\psi,\end{aligned} \tag{112}$$

where in the second equality we have used the solutions (95) to rewrite $H_k' X$, and also recalled the expression (109). Now that we have both the numerator and the denominator of (108a), we can compute the quantity c_s^2 to be

$$c_s^2 = \frac{A(Y, \psi)}{A(Y, \psi) - B(Y, \psi)}, \tag{113}$$

where we denote

$$A(Y, \psi) \equiv H_k\left[V(H_k'' H_v' - H_k' H_v'') - \frac{c_\rho}{2} H_k''\right], \tag{114a}$$

$$B(Y, \psi) \equiv 2(H_k')^2\left(H_v' V - \frac{c_\rho}{2}\right) = 2(H_k')^3 X(Y, \psi). \tag{114b}$$

[Note that we have used the expression (104) in the final equality in order to write $B(Y, \psi)$ in a more compact manner.] We may rewrite the above expression for c_s^2 in a cleaner form in cases where $A(Y, \psi) \neq 0$ by using it as a common factor, obtaining

$$c_s^2 = \frac{1}{1 - \frac{B(Y,\psi)}{A(Y,\psi)}}. \tag{115}$$

[3]Strictly speaking, we should take into account the perturbations in the value of the constant c_ρ. However, since it is a constant, it will contribute only to the zero mode; as such, it will not affect the subsequent discussion regarding the propagation of the perturbations.

Now, a necessary condition for stability is that $c_s^2 \geq 0$, which in turn implies[4] that

$$\frac{B(Y,\psi)}{A(Y,\psi)} < 1. \tag{116}$$

Requiring also the avoidance of superluminal perturbations ($c_s^2 \leq 1$), one obtains

$$\frac{B(Y,\psi)}{A(Y,\psi)} \leq 0. \tag{117}$$

These are the general expressions, with which one can verify that in the kinetic regime we recover the previous expressions for the stability (81) and subluminal propagation (82). It is worth noting, however, that in the kinetic regime the constant c_ρ cancels out rather early in the process [as early as in the EoS parameter, cf. (75)] and completely disappears from the subsequent treatment. However, when the potential is nonvanishing, it explicitly enters all of the analysis.

We now discuss the simple case of equal coupling functions, the GR limit, and finally translate our results into the TDiff frame.

1. Equal coupling functions, $H_k = H_v \equiv H$

In the case where both coupling functions coincide, the mentioned expressions simplify significantly. In such a case, we obtain

$$c_s^2 = \frac{1}{1 + \frac{4(H')^3}{H}\frac{X(Y,\psi)}{c_\rho H''}}. \tag{118}$$

So, the stability of the perturbations requires

$$\frac{(H')^3}{H}\frac{X(Y,\psi)}{c_\rho H''} > -\frac{1}{4}. \tag{119}$$

Moreover, in order to avoid superluminalities, taking into account that $H > 0$ and $X > 0$, one also needs

$$\frac{H'}{c_\rho H''} > 0. \tag{120}$$

Once again, these inequalities represent physically reasonable conditions which should help in selecting physically allowed coupling functions.

[4]The case $A(Y,\psi) = 0$ implies $c_s^2 = 0$, which trivially satisfies both stability and subluminality for all $B(Y,\psi)$. For this reason, the requirements we present here implicitly assume that we are in the case $A(Y,\psi) \neq 0$.

2. GR limit

Let us now consider the GR limit for the quantities discussed in this section. Approaching this limit, $A(Y,\psi)$ and $B(Y,\psi)$ behave as

$$A(Y,\psi) = (1+\epsilon h)\left[0 - \frac{c_\rho}{2}\epsilon h''\right] = \mathcal{O}(\epsilon^2), \tag{121a}$$

$$B(Y,\psi) = 2(\epsilon h')^2\left(\epsilon h' V - \frac{c_\rho}{2}\right) = \mathcal{O}(\epsilon^3), \tag{121b}$$

where we have recalled that $c_\rho = \mathcal{O}(\epsilon)$. The fraction then behaves as

$$\frac{B(Y,\psi)}{A(Y,\psi)} = \mathcal{O}(\epsilon), \tag{122}$$

which vanishes in the limit $\epsilon \to 0$. As a result, in the GR limit we have that $c_s^2 \to 1$, as it should be for a Diff invariant scalar field with a canonical kinetic term (cf. our results at the end of the $H_k = \text{const}$ discussion, and see also Ref. [23] for more details).

3. Translation to the TDiff frame

We finally translate our general results to the TDiff frame. In the $\bar{\mu} = 1$ frame, the energy density and pressure (98) translate to

$$\rho|_{\bar{\mu}=1} = \frac{1}{\sqrt{g}}\left(f_k X + f_v V - \frac{c_\rho}{2}\right), \tag{123a}$$

$$p|_{\bar{\mu}=1} = \frac{1}{\sqrt{g}}\left(f_k X - f_v V + \frac{c_\rho}{2}\right), \tag{123b}$$

which recalling the general constraint (96) can be seen to be equivalent to the previously presented energy density (11) and pressure (12), respectively. The EoS parameter (99) in the TDiff frame reads

$$w|_{\bar{\mu}=1} = \frac{f_k X - f_v V + c_\rho/2}{f_k X + f_v V - c_\rho/2}. \tag{124}$$

Once again we may see that the fluid is nonadiabatic, as the general metric constraint (96) reveals that

$$X = \frac{(f_v - 2g f_v')V(\psi) - c_\rho/2}{f_k - 2g f_k'} = X(g,\psi), \tag{125}$$

and so the energy density and pressure above are functions of both the metric determinant g and the scalar field ψ.

On the other hand, the expression (113) for c_s^2 evaluated in the TDiff frame may be written as

$$c_s^2|_{\bar{\mu}=1} = \frac{a(g,\psi)}{a(g,\psi) - b(g,\psi)}, \tag{126}$$

where we denote

$$a(g,\psi) \equiv A(Y,\psi)|_{\bar{\mu}=1} = -c_\rho g f_k (f_k' + 2g f_k'') + 2g f_k V[(f_k' + 2g f_k'')(f_v - 2g f_v') - (f_k - 2g f_k')(f_v' + 2g f_v'')], \tag{127a}$$

$$b(g,\psi) \equiv B(Y,\psi)|_{\bar{\mu}=1} = 2(f_k - 2g f_k')^3 X(g,\psi). \tag{127b}$$

Once again, whenever $a(g,\psi) \neq 0$ we may find a common factor and write

$$c_s^2|_{\bar{\mu}=1} = \frac{1}{1 - \frac{b(g,\psi)}{a(g,\psi)}}. \tag{128}$$

The requirement of stable perturbations translates to

$$\frac{b(g,\psi)}{a(g,\psi)} < 1, \tag{129}$$

while in order to also avoid superluminal propagations we must have

$$\frac{b(g,\psi)}{a(g,\psi)} \leq 0. \tag{130}$$

Although these are the general conditions, we can once more find some simplifications when the coupling functions coincide ($f_k = f_v \equiv f$), which allow us to obtain the following requirements for stability

$$\frac{(f - 2gf')^3}{gf} \frac{X(g,\psi)}{c_\rho (f' + 2gf'')} > -\frac{1}{2} \tag{131}$$

and subluminality

$$\frac{f - 2gf'}{c_\rho (f' + 2gf'')} > 0, \tag{132}$$

respectively. In any case, all of the conditions above should be helpful in deciding whether a particular TDiff model is physically reasonable. All in all, we have seen that the covariantized treatment is quite direct, and the subsequent translation of the results to the TDiff framework is fairly straightforward.

Thus concludes the most general situation of our nonadiabatic fluid. In the following section, we will discuss some particular cases of interest in which the fluid is adiabatic.

B. Adiabatic models

We described in the previous section how the most general situation was a nonadiabatic fluid. In this section we perform a more detailed discussion of adiabatic models. We begin by recalling Eq. (107) for δp, where we note that in order to have an adiabatic fluid the second term should vanish, meaning that the general condition for an adiabatic fluid is simply $\alpha = 0$. Now, α is itself a sum of two terms,

$$\alpha = \underbrace{(-c_s^2) \frac{\partial \rho}{\partial \psi}\bigg|_Y}_{\text{(i)}} + \underbrace{\frac{\partial p}{\partial \psi}\bigg|_Y}_{\text{(ii)}} = 0, \tag{133}$$

so there are two possibilities: it could happen that the terms (i) and (ii) vanish separately, or it could happen that they do not vanish separately but their combination does. In the following we consider these possibilities in detail, and also study their implications.

1. Case I

We begin with the case in which the two terms in α vanish separately. In particular, let us begin by studying the conditions under which the term (ii) $= 0$. Differentiating the pressure as it stands in Eq. (105) we obtain

$$\text{(ii)} = V' \left(\frac{H_k}{H_k'} H_v' - H_v \right) = 0. \tag{134}$$

Thus, term (ii) will vanish in situations in which $V' = 0$ (i.e. $V = V_0$ a constant potential) and in situations in which

$$\frac{H_k}{H_k'} H_v' - H_v = 0. \tag{135}$$

There are a couple of options at this point. If $H_v' = 0$ then the equation tells us that $H_v = 0$. If $H_v' \neq 0$ then we may rearrange the above expression as

$$\frac{H_k'}{H_k} = \frac{H_v'}{H_v}, \tag{136}$$

which may be straightforwardly integrated to yield

$$H_v = C H_k, \tag{137}$$

where C is a constant of integration. Thus, we conclude that there are three situations in which term (ii) vanishes:

(1) A constant potential $V = V_0$.
(2) A vanishing potential coupling $H_v = 0$.
(3) A potential coupling of the form $H_v = C H_k$.

Next we move on to the first term, and study the conditions under which (i) $= 0$. Differentiating the energy density as it stands in Eq. (105) we obtain

TABLE I. Summary of the six (independent) adiabatic TDiff models, where the first five correspond to $H_k' \neq 0$ and the last one to $H_k' = 0$. In the table, three dots "$\cdots$" mean that the quantity in question is arbitrary, and the integration constants $\{V_0, C, a, b, c, d, k\}$ are unrestricted unless otherwise specified.

Name	$V(\psi)$	$H_k(Y)$	$H_v(Y)$	c_ρ	Restrictions
I.1. Shift-symmetric model	$\cdots$	$\cdots$	0	$\cdots$	$H_k' \neq 0$
I.2. Constant potential model	V_0	$\cdots$	$\cdots$	$\cdots$	$H_k' \neq 0$
II.1. Unstable model	$\cdots$	e^{aY+b}	d	$\cdots$	$a,\ d \neq 0$
II.2. Constant EoS model	$\cdots$	$a(H_v)^b$	$H_v' \neq 0$	0	$a,\ b \neq 0$
II.3. Constant speed of sound model	$\cdots$	$a(Y+\frac{d}{c})^b$	$cY+d$	$\cdots$	$a,\ b,\ c \neq 0$
III. GR fluid model	$\frac{c_\rho}{2c}$	k	$cY+d$	$\neq 0$	$c \neq 0$

$$(\mathrm{i}) = (-c_s^2)V'\left(\frac{H_k}{H_k'}H_v' + H_v\right) = 0. \tag{138}$$

The simplest option is similar to the previous case: term (i) will vanish when the potential $V = V_0$ is constant. Another possibility is that

$$\frac{H_k}{H_k'}H_v' + H_v = 0, \tag{139}$$

which again admits the solution $H_v = 0$ and also

$$\frac{H_k'}{H_k} = -\frac{H_v'}{H_v}. \tag{140}$$

Integrating, we would find

$$H_v = \frac{C}{H_k}, \tag{141}$$

where C is a constant of integration. The final possibility is the vanishing of c_s^2 or, equivalently, the vanishing of $A(Y,\psi)$ as Eq. (113) reveals. This amounts to

$$V(H_k''H_v' - H_k'H_v'') - \frac{c_\rho}{2}H_k'' = 0. \tag{142}$$

If the potential was identically zero, it would be nothing but a particular case of constant potential, and we already know that both terms (i) and (ii) would vanish. Thus, it would not be necessary to keep probing for information regarding the vanishing of c_s^2. In the following calculations, we assume that $V \neq 0$, so that we may rearrange to get

$$H_k''H_v' - H_k'H_v'' = \frac{c_\rho}{2V}H_k''. \tag{143}$$

At this point, it could happen that $H_k'' = 0$, which on the one hand implies $H_k = aY + b$ and on the other

$$H_v'' = 0 \tag{144}$$

(where we have recalled $H_k' \neq 0$), so that $H_v = cY + d$. On the other hand, in cases where $H_k'' \neq 0$ we could divide through and obtain the following:

$$H_v' - \frac{H_k'}{H_k''}H_v'' = \frac{c_\rho}{2V}. \tag{145}$$

On the one hand, when $c_\rho = 0$ we obtain a particular case of model II. 2. (studied later). On the other hand, when $c_\rho \neq 0$, we remark that the left-hand side of this equation is a function of only Y, while the right-hand side is a function of only ψ. Since we are looking for particular models such that all solutions are adiabatic, we ask both sides to equal the same constant K. In particular, this means that we get a constant potential

$$V = \frac{c_\rho}{2K} = V_0, \tag{146}$$

which takes us to the first case and so we do not need to look for further information. We thus conclude that, in practice, there are four situations in which term (i) vanishes:

(1) A constant potential $V = V_0$.
(2) A vanishing potential coupling $H_v = 0$.
(3) A potential coupling of the form $H_v = \frac{C}{H_k}$.
(4) Both coupling functions are linear: $H_k = aY + b$ and $H_v = cY + d$.

Contrasting these four situations for which $(\mathrm{i}) = 0$ with the three for which $(\mathrm{ii}) = 0$, and bearing in mind that both terms must vanish separately and simultaneously, reveals that in practice we can only have three cases in which both terms (i) and (ii) vanish independently:

(1) A constant potential $V = V_0$.
(2) A vanishing potential coupling $H_v = 0$.
(3) A linear kinetic coupling function $H_k = aY + b$ and at the same time a potential coupling function of the form $H_v = CH_k$ (which will also be linear).

The physical implications of each of these subcases shall be studied with greater detail later on. For now, let us remark that the reader may find in Table I a summary of the results obtained.

2. Case II

We now consider the case in which the two terms in α do not vanish separately but their combination does. Now, since none of the terms in Eq. (133) for α vanishes individually, we can rearrange said expression to obtain

$$c_s^2 = \frac{\partial p/\partial\psi}{\partial\rho/\partial\psi}\bigg|_Y. \tag{147}$$

Differentiating the energy density and pressure in Eq. (105) and recalling the expression (113) for c_s^2 we find that

$$\frac{A(Y,\psi)}{A(Y,\psi)-B(Y,\psi)} = \frac{\frac{H_k}{H_k'}H_v' - H_v}{\frac{H_k}{H_k'}H_v' + H_v}. \tag{148}$$

This would be the general relation that establishes the adiabaticity of the fluid in the case where both terms in α do not vanish separately but only through their combination.

In cases where $H_v' = 0$ we would on the one hand have that $H_v = \text{const}$ (different from zero, since $H_v = 0$ belongs to the two terms in α vanishing separately) and on the other it would follow from the above equation that

$$c_s^2 = -1. \tag{149}$$

This would thus be an unstable adiabatic fluid. Despite this fact, we could carry the analysis further and find

$$B(Y,\psi) = 2A(Y,\psi). \tag{150}$$

Substituting the expressions (114), and using that $H_v' = 0$, one obtains

$$c_\rho (H_k')^2 = c_\rho H_k H_k'', \tag{151}$$

which, after canceling a common c_ρ [nonzero, since in our $H_v' = 0$ study it would imply that $A(Y,\psi) = 0$ and that is not the case], may be rearranged to find

$$\frac{H_k'}{H_k} = \frac{H_k''}{H_k'}. \tag{152}$$

Integrating twice, this would yield a kinetic coupling function of the form

$$H_k = e^{aY+b}, \tag{153}$$

where a and b are constants of integration.

Consider now the cases where $H_v' \neq 0$. This means that we could find a common factor on the right-hand side of Eq. (148) and write it as

$$\frac{1}{1-\frac{B(Y,\psi)}{A(Y,\psi)}} = \frac{\frac{H_k}{H_k'} - \frac{H_v}{H_v'}}{\frac{H_k}{H_k'} + \frac{H_v}{H_v'}}, \tag{154}$$

where we have also simplified the left-hand side recalling that $A(Y,\psi) \neq 0$ (if it vanished we would be in a particular case of the previous study). We can rearrange this equation to find

$$\frac{B(Y,\psi)}{A(Y,\psi)} = 1 - \frac{\frac{H_k}{H_k'} + \frac{H_v}{H_v'}}{\frac{H_k}{H_k'} - \frac{H_v}{H_v'}} \equiv \varphi(Y). \tag{155}$$

[Note that in the situation under study $\varphi(Y) \neq 0$, because its vanishing would imply $H_v = 0$ as follows from the above definition of $\varphi(Y)$, and this is incompatible with $H_v' \neq 0$.] Now, the right-hand side is a function only of Y, while the left-hand side depends on both Y and ψ. Let us try to better localize these dependencies and see what we obtain. We first of all write the equation as

$$B(Y,\psi) = \varphi(Y)A(Y,\psi). \tag{156}$$

Substituting now the expressions for $A(Y,\psi)$ and $B(Y,\psi)$ from Eq. (114) and rearranging the result to group terms with V on one side, we obtain

$$V\{2(H_k')^2 H_v' - \varphi(Y)H_k(H_k''H_v' - H_k'H_v'')\} = \frac{c_\rho}{2}[2(H_k')^2 - \varphi(Y)H_k H_k'']. \tag{157}$$

Now, if the term inside curly brackets was nonzero, it would mean that we could solve for the potential $V(\psi)$ and it would be a function of only Y. Once again we would ask for both sides to be constant, but a constant potential is part of the previous situation and does not belong to the present study. Hence, it must be the case that the term in between curly brackets on the left-hand side of the above equation vanishes, implying in turn that the full right-hand side must vanish as well. We thus conclude that

$$2(H_k')^2 H_v' - \varphi(Y)H_k(H_k''H_v' - H_k'H_v'') = 0, \tag{158a}$$

$$\frac{c_\rho}{2}[2(H_k')^2 - \varphi(Y)H_k H_k''] = 0. \tag{158b}$$

Let us focus on the first expression. Dividing by the nonzero H_v' and rearranging, we find that

$$2(H_k')^2 - \varphi(Y)H_k H_k'' = -\varphi(Y)H_k H_k'\frac{H_v''}{H_v'}. \tag{159}$$

Substituting this result into the second expression, we obtain

$$-\frac{c_\rho}{2}\varphi(Y)H_k H_k' \frac{H_v''}{H_v'} = 0, \tag{160}$$

which implies

$$c_\rho H_v'' = 0. \tag{161}$$

Thus, the two possibilities for the cases where $H_v' \neq 0$ are $c_\rho = 0$ and $H_v'' = 0$. The analysis for $c_\rho = 0$ deserves greater care, but regarding $H_v'' = 0$ we can already say that (since $H_v' \neq 0$) we must have $H_v = cY + d$ where $c \neq 0$ and d are constants of integration.

Let us now pause for a moment and make a list of the three distinct situations in which the two terms in α do not vanish identically but their combination does:

(1) $H_v = v \neq 0$ and $H_k = e^{aY+b}$ (unstable model).
(2) $H_v' \neq 0$ and $c_\rho = 0$.
(3) $H_v = cY + d$, with $c \neq 0$.

The first case yields an unstable model, as we have discussed. Regarding the other two situations, we still do not know what the form of the kinetic coupling function H_k is, so let us focus on that now.

The case in which the constant c_ρ vanishes (while $H_v' \neq 0$) turns out to be quite simple, as many convenient cancellations take place. Indeed, evaluating $A(Y,\psi)$ and $B(Y,\psi)$ from Eq. (114) whenever $c_\rho = 0$ results in

$$\frac{B(Y,\psi)}{A(Y,\psi)} = 2\frac{\frac{H_k'}{H_k}}{\frac{H_k''}{H_k'} - \frac{H_v''}{H_v'}}, \tag{162}$$

and we see that the potential V completely disappears from the ratio. Referring back to Eq. (155), it follows that

$$2\frac{\frac{H_k'}{H_k}}{\frac{H_k''}{H_k'} - \frac{H_v''}{H_v'}} = (-2)\frac{\frac{H_v}{H_v'}}{\frac{H_k}{H_k'} - \frac{H_v}{H_v'}}. \tag{163}$$

This equation may be simplified and rearranged carefully to find

$$\frac{H_k''}{H_k'} - \frac{H_k'}{H_k} = \frac{H_v''}{H_v'} - \frac{H_v'}{H_v}. \tag{164}$$

Integrating once, it follows that

$$\frac{H_k'}{H_k} = b\frac{H_v'}{H_v}, \tag{165}$$

where b is an integration constant (nonzero, recall that all throughout $H_k' \neq 0$). Integrating a second time, we finally conclude that

$$H_k = a(H_v)^b, \tag{166}$$

where a is another (nonzero) integration constant. In this way, given any potential coupling function H_v that satisfies $H_v' \neq 0$ and a kinetic coupling function of the above form, the fluid will be adiabatic.

Now we move on to the other option, i.e. a linear potential coupling function $H_v = cY + d$, with $c \neq 0$. Substituting this form into the expression (159) we find the following equation for the kinetic coupling function:

$$2(H_k')^2 = \varphi(Y)H_k H_k'' = (-2)\frac{Y + d/c}{\frac{H_k}{H_k'} - (Y + d/c)} H_k H_k'', \tag{167}$$

where in the second equality we have used the definition (155) for $\varphi(Y)$ particularized to our case. The above expression may be simplified and rearranged to find

$$\frac{H_k'}{H_k} - \frac{H_k''}{H_k'} = \frac{1}{Y + d/c}. \tag{168}$$

Integrating once, it follows that

$$\frac{H_k'}{H_k} = \frac{b}{Y + d/c}, \tag{169}$$

where b is a (nonzero) integration constant, and integrating a second time one may finally obtain

$$H_k = a(Y + d/c)^b, \tag{170}$$

where a is another (nonzero) integration constant.

Thus, the three possibilities for adiabatic models that we find in Case II would be:

(1) $H_v = d \neq 0$ and $H_k = e^{aY+b}$ (unstable model).
(2) $c_\rho = 0$, with $H_v' \neq 0$ and $H_k = a(H_v)^b$.
(3) $H_v = cY + d$ and $H_k = a(Y + d/c)^b$.

Thus concludes the possible ways in which the models may be adiabatic. For simplicity and an easy reference, we also include these results in Table I.

We now move on to the analysis of the physical implications of each of the six adiabatic subcases we have found.

3. Subcase I.1.—Shift-symmetric model

We begin by noting that the particular model with $H_v = 0$ is shift symmetric, and its results are fundamentally those found in the kinetic domination regime (which we already knew to be adiabatic). If we compute the energy density, pressure, EoS parameter, and adiabatic speed of sound, we straightforwardly reobtain the same results as those presented in Sec. IV B. Since this case has been previously analyzed in detail, we shall not reproduce it again here.

4. Subcase I.2.—Constant potential model

Let us continue by considering the case in which we have both kinetic and potential contributions but the potential is $V = V_0 = \text{const}$. The general EoM (46) for ψ actually reduces to the kinetic EoM (62), whose solution is known and is given by (65). Substituting into the solutions (95) for the EoM of T^μ, we obtain

$$\frac{H_k'}{2}\frac{C_\psi(x)}{(H_k\delta V)^2} - H_v' V_0 = -\frac{c_\rho}{2}, \tag{171}$$

and going to the $\bar{\mu} = 1$ frame it follows that

$$\frac{(f_k - 2gf_k')}{2}\frac{gC_\psi(x)}{(f_k\delta V)^2} - (f_v - 2gf_v')V_0 = -\frac{c_\rho}{2}. \tag{172}$$

This expression is the general constraint on the metric whenever we have a kinetic term and a constant potential.

Now, the energy density and pressure are found from (105) to be

$$\rho = \frac{H_k}{H_k'}\left(H_v' V_0 - \frac{c_\rho}{2}\right) + H_v V_0 - \frac{c_\rho}{2}Y = \rho(Y), \tag{173a}$$

$$p = \frac{H_k}{H_k'}\left(H_v' V_0 - \frac{c_\rho}{2}\right) - H_v V_0 + \frac{c_\rho}{2}Y = p(Y). \tag{173b}$$

Their dependence on a single variable reveals that we are dealing with an adiabatic fluid, with EoS parameter

$$w = \frac{c_\rho(H_k - YH_k') + 2V_0(H_v H_k' - H_k H_v')}{c_\rho(H_k + YH_k') - 2V_0(H_v H_k' + H_k H_v')}. \tag{174}$$

We may also calculate the speed of sound of the adiabatic perturbations using the relation (32), which yields

$$c_a^2 = \frac{A(Y)}{A(Y) - B(Y)}, \tag{175}$$

where we denote

$$A(Y) \equiv V_0 H_k(H_k'' H_v' - H_k' H_v'') - \frac{c_\rho}{2}H_k H_k'', \tag{176a}$$

$$B(Y) \equiv 2(H_k')^2\left(H_v' V_0 - \frac{c_\rho}{2}\right). \tag{176b}$$

Before proceeding any further, let us stress a couple of points. First, the above expression (175) reduces to the adiabatic speed of sound (79) in the kinetic regime, as one would expect. Second, and perhaps more interestingly, let us remark that we have obtained Eq. (175) for the adiabatic speed of sound via the relation (32) (valid precisely for adiabatic perturbations). Nevertheless, in the case of a constant potential it is immediately seen that the ψ dependence in the quantities of the previous section completely disappears: from Eqs. (114) and (176) we see that $A(Y,\psi)|_{V_0} = A(Y)$ and $B(Y,\psi)|_{V_0} = B(Y)$, while from (115) and (175) we verify that $c_s^2|_{V_0} = c_a^2$. So, consistently, this result could have been obtained by noting that for a constant potential we would have $\alpha = 0$ in Eq. (107), as we previously mentioned.

Finally, requiring the stability of the perturbations translates to $B(Y)/A(Y) < 1$, and if we also wish to have subluminality then it must be the case that $B(Y)/A(Y) \leq 0$.

All in all, the case of a constant potential is a simple but nontrivial example which the TDiff approach was unable to reach due to the difficulty related with the study of EMT conservation, but which the covariantized approach allows to study in a straightforward manner.

5. Subcase I.3.—Constant pressure model

Another case in which the fluid is adiabatic is the one in which the two couplings are linear and related by

$$H_k = aY + b, \tag{177a}$$

$$H_v = CH_k. \tag{177b}$$

The energy density and pressure are found from (105) to be

$$\rho = 2C(aY + b)V - \frac{c_\rho}{2}\left(2Y + \frac{b}{a}\right), \tag{178a}$$

$$p = -\frac{c_\rho b}{2a} = \text{const}. \tag{178b}$$

Looking at the above quantities, a couple of features immediately jump out. The first one is that the pressure is constant, and as a result we find that the fluid will have no pressure perturbations, i.e. $\delta p = 0$. If we compute the speed of sound (113) for this model, it consistently reveals that $c_s^2 = 0$ (satisfying both stability and subluminality).

The second feature is that the energy density depends in general on both Y and ψ, and so one could wonder how the adiabatic speed of sound could be calculated given that we have a dependence on two variables here. This is solved by again remarking that the pressure is constant, so that from the definition of the adiabatic speed of sound as

$$c_a^2 = \frac{\dot{p}}{\dot{\rho}} = \frac{u^\alpha \partial_\alpha p}{u^\beta \partial_\beta \rho} \tag{179}$$

we consistently find $c_a^2 = 0 = c_s^2$, and the two coincide (as they should for our adiabatic fluid).

6. Subcase II.1.—Unstable model

Although the model defined by the coupling functions

$$H_k = e^{aY+b}, \tag{180a}$$

$$H_v = d, \tag{180b}$$

yields unstable perturbations, we nevertheless discuss it for completeness. The energy density and pressure follow from (105) and read

$$\rho = -\frac{c_\rho}{2}\left(\frac{1}{a} + Y\right) + Vd, \tag{181a}$$

$$p = -\frac{c_\rho}{2}\left(\frac{1}{a} - Y\right) - Vd. \tag{181b}$$

Once again they depend on a couple of variables, and so the correct way of finding the adiabatic speed of sound is from its definition as

$$c_a^2 = \frac{\dot{p}}{\dot{\rho}} = -1, \tag{182}$$

which is an unstable result as we expected, coinciding with the speed of sound $c_s^2 = -1$ as obtained from (113).

7. Subcase II.2.—Constant EoS model

The main feature of this subcase is the vanishing of the constant c_ρ (which greatly simplifies the treatment) and we recall that the coupling functions are related by

$$H_k = a(H_v)^b, \tag{183}$$

such that $a, b, H_v' \neq 0$. The energy density and pressure are found substituting these coupling functions in (105), and take the form

$$\rho = \frac{1+b}{b} H_v V, \tag{184a}$$

$$p = \frac{1-b}{b} H_v V, \tag{184b}$$

and the EoS parameter turns out to be constant:

$$w = \frac{1-b}{1+b}. \tag{185}$$

Finally, the speed of sound (113) reads

$$c_s^2 = \frac{1-b}{1+b}, \tag{186}$$

and it indeed coincides with the adiabatic speed of sound since, for the case of a constant EoS parameter, we have $c_a^2 = w = c_s^2$. The requirement of stability translates to the possibilities

$$-1 < b < 0 \quad \text{and} \quad 0 < b \leq 1, \tag{187}$$

and also requiring subluminality gives

$$b > 0. \tag{188}$$

In obtaining the above results we have recalled that we are in a model in which $b \neq 0$ from the start, so we must remove the possibility $b = 0$ from them.

8. Subcase II.3.—Constant speed of sound model

Another particular model yielding an adiabatic fluid is that in which the coupling functions take the forms

$$H_k = a\left(Y + \frac{d}{c}\right)^b, \tag{189a}$$

$$H_v = cY + d, \tag{189b}$$

with all constants different from zero except perhaps d. A trivial rewriting of the kinetic coupling function as

$$H_k = ac^b(cY+d)^b = ac^b(H_v)^b \tag{190}$$

reveals that this case is closely related to the previous one, where now H_v is linear, but with the drawback that now the constant c_ρ does not necessarily vanish. If the initial conditions were chosen such that $c_\rho = 0$, however, we would be in a particular realization of the previous case. Now, our choice of initial conditions affects the particular value of our fluid quantities (energy density, pressure, EoS) at any given time, but it should not affect the perturbations, and this is what we shall see.

The energy density and pressure for this model are found from (105) to be

$$\rho = \frac{1+b}{b}\left(-\frac{c_\rho}{2}Y + Vd + cYV\right) - \frac{c_\rho d}{2bc}, \tag{191a}$$

$$p = \frac{1-b}{b}\left(-\frac{c_\rho}{2}Y + Vd + cYV\right) - \frac{c_\rho d}{2bc}, \tag{191b}$$

and the EoS parameter takes the form

$$w = \frac{(1-b)\left(-\frac{c_\rho}{2}Y + Vd + cYV\right) - \frac{c_\rho d}{2c}}{(1+b)\left(-\frac{c_\rho}{2}Y + Vd + cYV\right) - \frac{c_\rho d}{2c}}. \tag{192}$$

All of these fluid quantities would reduce to the ones in the previous case when the initial conditions were chosen such that $c_\rho = 0$. Now, both the energy density and pressure depend on two variables, and so in order to find the adiabatic speed of sound we must resort to the following definition:

$$c_a^2 = \frac{\dot{p}}{\dot{\rho}} = \frac{1-b}{1+b}. \tag{193}$$

Calculating the speed of sound from Eq. (113) yields the same result, so that indeed $c_s^2 = c_a^2$. Moreover, we may immediately see that the adiabatic speed of sound (193) is identical to that from the previous section, given in (186). So, indeed, even though our choice of initial conditions affects the "background physics" of our fluid, it does not affect its perturbations. Also, since the form of the adiabatic speed of sound is the same, we already know the requirements for stability and subluminality: (187) and (188), respectively.

As a final note, the reader may recognize that the particular model given by $b = 1$ (which means that the two coupling functions are linear and proportional to each other) reduces to the constant pressure situation I.3. we previously studied. Therefore, in a way, the present model is more general as it encompasses the aforementioned one.

With this last case, we have finally finished the discussion of the six situations in which the perturbations of our (generally nonadiabatic) fluid became adiabatic. Nevertheless, as we anticipated, we are still missing the analysis of the situations in which the kinetic coupling function H_k is constant, so that $H_k' = 0$. For completeness, this detail shall be studied in the following section.

C. Constant kinetic coupling function

After all of our general discussion, we come back to a question that remains open. Let us consider the case in which the potential coupling function is a constant, i.e.

$$H_k = k = \text{const}. \tag{194}$$

What happens in this situation? To begin with, since $H_k' = 0$, the solutions (95) for the EoM of T^μ tell us that

$$H_v' V = \frac{c_\rho}{2}. \tag{195}$$

Possible solutions to this equation are $V = 0$ and $H_v' = 0$, which both imply that $c_\rho = 0$. Beyond these trivial cases, we may divide through by the potential and find

$$H_v' = \frac{c_\rho}{2V}. \tag{196}$$

Now, it can be checked that the only case leading to an adiabatic model is the one in which both sides equal the same constant c. On the one hand, this tells us that the potential takes on the constant value

$$V = \frac{c_\rho}{2c}, \tag{197}$$

and on the other that the potential coupling is linear,

$$H_v = cY + d, \tag{198}$$

where d is an integration constant. In summary, then, we in principle have three situations to study:

(1) $V = \frac{c_\rho}{2c} \neq 0$ and $H_v = cY + d$, with $c \neq 0$.
(2) $V = 0$, with H_v' arbitrary.
(3) $H_v' = 0 (\Rightarrow H_v = v = \text{const})$, with V arbitrary.

A couple of comments are due at this point. On the one hand, the second model gives simply

$$S_{\text{Diff}} = S_{\text{EH}} + \int d^4x \sqrt{g} kX, \tag{199}$$

and this is nothing but the standard GR case of a shift-symmetric scalar field. On the other hand, the third model gives

$$S_{\text{Diff}} = S_{\text{EH}} + \int d^4x \sqrt{g} (kX - vV), \tag{200}$$

and this is nothing but the standard GR case of a general scalar field. Thus, since the second and third models are nothing but GR, we shall not study them in detail here, but rather we refer the reader to Refs. [22,23]. Instead, we focus on the first case, which is the one that yields "true" TDiff results. Its analysis is done in the following subsection, and the reason for the numbering we have chosen will be understood once it is finished.

1. Subcase III—GR fluid model

Substituting the particular expressions $V = \frac{c_\rho}{2c}$ and $H_v = cY + d$ into Eq. (98), one finds some convenient cancellations which yield

$$\rho = kX + \frac{c_\rho d}{2c} = \rho(X), \tag{201a}$$

$$p = kX - \frac{c_\rho d}{2c} = p(X), \tag{201b}$$

and the fluid is thus adiabatic once again. It is interesting to remark that, recalling the form of the potential, the above expressions may be rewritten as

$$\rho = kX + Vd, \tag{202a}$$

$$p = kX - Vd, \tag{202b}$$

and so the energy density and pressure would fundamentally be identical to those of a canonical scalar field with constant potential in GR (modulo superfluous renormalizations of ψ). In this manner, even though our theory is not GR, we would obtain the same phenomenology. In particular, the EoS parameter reads

$$w = \frac{kX - Vd}{kX + Vd}, \tag{203}$$

which is the typical form in GR, and interpolates between $w = -1$ (potential domination) and $w = 1$ (kinetic domination). Moreover, if we compute the adiabatic speed of sound from (32) we obtain $c_a^2 = 1$, as is the case in GR.

We have thus studied the situation whenever the kinetic coupling function H_k is constant, and this completes any of the questions left open in the previous analysis (which was carried out assuming $H_k' \neq 0$). Since we have obtained an adiabatic fluid, we have numbered this case as III and we include it in Table I for ease of reference.

D. Particular Diff solution in TDiff theories

As a final comment, we discuss a family of general TDiff models that can yield Diff solutions to the theory. A word of caution: we perform the study in the covariantized approach, where strictly speaking the theories are (by construction) Diff invariant already; hence, whenever we speak of a "TDiff theory" in this context, we fundamentally mean that the coupling functions $H(Y)$ are not constant (meaning that the theory when going to the TDiff frame was not already Diff invariant to start with, or in other words, that we actually needed to introduce a new field to restore the symmetry).

Having cleared that up, let us consider the family of TDiff theories in which the coupling functions have a common extremum at $Y = Y_0$, i.e.

$$H_k'(Y_0) = 0 = H_v'(Y_0). \tag{204}$$

Now, the solution $Y = Y_0 = \text{const}$ is a valid (and trivial) solution to the EoM (49) of the vector field T^μ. Substituting this solution into the EoM (46) for ψ yields

$$H_k(Y_0)\nabla_\mu\nabla^\mu\psi + H_v(Y_0)V'(\psi) = 0. \tag{205}$$

In this way, up to superfluous constant factors, the scalar field ψ follows the usual Diff equation. Moreover, the EMT (51) on these solutions reads

$$T_{\mu\nu} = H_k(Y_0)\partial_\mu\psi\partial_\nu\psi - [H_k(Y_0)X - H_v(Y_0)V]g_{\mu\nu}, \tag{206}$$

and so we also recover the standard expression for a Diff theory with a scalar field (up to constant factors).

We stress that we are not setting the coupling functions $H(Y)$ to be constant from the start, which indeed amounts to the assertion that the original theory was Diff invariant already. Instead, we are working with nonconstant, arbitrary coupling functions $H(Y)$ such that they have an extremum at $Y = Y_0$. These TDiff theories are explicitly not GR, but what we are seeing is that they admit a particular solution that reproduces the same results as GR. Thus, in principle, we would not be able to distinguish GR from a particular solution of a TDiff theory where the coupling functions reach an extremum at the same point. A valid and useful question in order to discern this would be: are these solutions stable under perturbations? By this we mean the following: were we to slightly perturb the solutions ($Y = Y_0 + \delta Y$, etc.), would the perturbation δY decay so that we would again reach the $Y = Y_0$ behavior? We shall address this question in the following.

1. Perturbations

Let us begin by considering the evolution of the perturbations in general, before particularizing to our family of theories. The EoM (49) for T^μ may be written as

$$\partial_\mu \mathcal{S}(X, Y, \psi) = 0, \tag{207}$$

where for simplicity we denote

$$\mathcal{S}(X, Y, \psi) \equiv H_k'(Y)X - H_v'(Y)V(\psi). \tag{208}$$

The solutions (95) are the level surfaces

$$\mathcal{S}(X, Y, \psi) = \text{const} \equiv -\frac{c_\rho}{2}, \tag{209}$$

where the particular constant (the particular "level") is fixed through the initial conditions on the fields and their derivatives. Now, suppose we have found a solution $\bar{\Phi} \equiv \{\bar{X}, \bar{Y}, \bar{\psi}\}$ to the theory (which we refer to as our "background"). Slightly perturbing it and demanding that the perturbed fields $\Phi = \bar{\Phi} + \delta\Phi$ are also a solution to the theory reveals that, in general, we will change what level surface our new (perturbed) solution lives on. In particular, demanding

$$\partial_\mu \mathcal{S}(\Phi) = \partial_\mu \mathcal{S}(\bar{\Phi} + \delta\Phi) = 0 \tag{210}$$

yields the following equation for the perturbations:

$$\begin{aligned} &H_k'(\bar{Y})\delta X + \mathcal{B}(\bar{X}, \bar{Y}, \bar{\psi})\delta Y - H_v'(\bar{Y})V'(\bar{\psi})\delta\psi \\ &\quad = \text{const} \equiv -\frac{\delta c_\rho}{2}, \end{aligned} \tag{211}$$

where, for ease of viewing, we defined the background quantity

$$\mathcal{B}(\bar{X}, \bar{Y}, \bar{\psi}) \equiv H_k''(\bar{Y})\bar{X} - H_v''(\bar{Y})V(\bar{\psi}). \tag{212}$$

In the above equation, the constant δc_ρ would now be fixed through the initial conditions on the perturbed fields $\Phi = \bar{\Phi} + \delta\Phi$ and their derivatives.

Note that this would not be the only equation for the perturbations; we would also have the equations coming from perturbing the EoM of the scalar field ψ and the Einstein equations. Nevertheless, in order to keep it brief, it will suffice to simply consider this one.

2. Stability of the Diff solution in TDiff theories

Having presented the general case, let us now consider the family of TDiff theories for which the coupling functions $H(Y)$ reach a common extremum at $H'_k(Y_0)=0=H'_v(Y_0)$. As we saw, the solution $\bar{Y}=Y_0=\text{const}$ is a valid solution to the EoM of T^μ (it is a trivial one in fact) which yields a behavior identical to GR. Now, is this particular solution stable? Let us consider Eq. (211) above, particularized to these theories. The extremum condition helps to simplify it to

$$\mathcal{B}(\bar{X},Y_0,\bar{\psi})\delta Y=-\frac{\delta c_\rho}{2}. \qquad (213)$$

Assuming that the background quantity $\mathcal{B}(\bar{X},Y_0,\bar{\psi})$ is nonzero[5] and solving for the perturbation, we find that

$$\delta Y=\frac{-\delta c_\rho/2}{\mathcal{B}(\bar{X},Y_0,\bar{\psi})}. \qquad (214)$$

In the particular situation in which we choose the initial conditions $\delta Y|_{\text{initial}}=0$ (essentially meaning that the perturbed solution crosses $Y=Y_0$ at some point, which we take as the "initial" one), and thus $\delta c_\rho=0$ and the perturbation δY would always be zero. Such a theory would thus be "stable" in the sense that the perturbation decays (it actually stays zero constantly) and we recover the initial situation of $Y=Y_0$. Nevertheless, in general, we will be in a situation in which $\delta c_\rho\neq 0$, and so if we wish to shed some light on the stability of our solution we must study the evolution of the background quantity $\mathcal{B}(\bar{X},Y_0,\bar{\psi})$. Since this is difficult to tackle in general, let us consider the two limiting regimes of potential and kinetic domination and see if we can gain some intuition about what might happen.

Potential domination.—In this case, the background quantity reads $\mathcal{B}=-H''_v(Y_0)V(\bar{\psi})$. Now, the EoM for ψ in the potential regime tells us that the scalar field takes on the constant value $\bar{\psi}=\psi_0$ such that the potential reaches an extremum. As a result,

$$\mathcal{B}=-H''_v(Y_0)V(\psi_0)=\text{const}. \qquad (215)$$

[5]If it vanished, it would on the one hand imply $\delta c_\rho=0$ (meaning we do not change level surface), and on the other hand the equation would turn into a $0=0$ identity from which no further information could be extracted. As a result, we would have to consider the perturbed Einstein equations and the perturbed equations for the scalar field to extract some information.

Therefore, the perturbation stays constant as well[6]:

$$\delta Y=\text{const}. \qquad (216)$$

This means that the solution $\bar{Y}=Y_0=\text{const}$ is *not stable* in the potential regime, since the perturbation does not decay but stays frozen.

Kinetic domination.—In this case, the background quantity reads $\mathcal{B}=H''_k(Y_0)\bar{X}$. Now, from Eq. (65), the EoM for ψ in the kinetic regime tells us that

$$2\bar{X}=\frac{C_\psi(x)}{(H_k(Y_0)\delta V)^2}. \qquad (217)$$

As a result, the background quantity behaves as

$$\mathcal{B}=\frac{H''_k(Y_0)C_\psi(x)}{2H_k^2(Y_0)(\delta V)^2}. \qquad (218)$$

Substituting this result back into the expression for the perturbation, we find that

$$\delta Y=\frac{-\delta c_\rho H_k^2(Y_0)}{H''_k(Y_0)C_\psi(x)}(\delta V)^2\equiv C_Y(x)(\delta V)^2, \qquad (219)$$

where we have grouped the term multiplying $(\delta V)^2$ into $C_Y(x)$ such that $\dot{C}_Y(x)=u^\mu\partial_\mu C_Y(x)=0$ (i.e. the quantity C_Y remains constant in the direction parallel to the velocity). In a cosmological scenario we have $\delta V=a^3$ and $C_Y=\text{const}$, so the perturbation grows as $\delta Y\propto a^6$. This rapid growth with the scale factor reveals that the particular solution $Y=Y_0=\text{const}$ is *not stable in a cosmological scenario* (at least in models without recollapse). Without directly going to the cosmological scenario, one can see from the above equation that the stability of the solution is related to the background cross-sectional volume diminishing along the scalar field's direction. We can make this statement more mathematically precise by taking the dot derivative,

$$(\delta Y)^\bullet=2\delta Y(\ln\delta V)^\bullet, \qquad (220)$$

which, recalling Eq. (23), reveals that it is proportional to the expansion. In this way, we confirm that for a model with a positive expansion the perturbation will grow (making the solution unstable), whereas for a model with a negative expansion it will decay (making the solution stable).

As a final note, let us remark that even though in Subcase III from the previous section we also obtained a GR behavior, in that case said behavior is general for any

[6]This is actually a general result for the perturbation δY in the potential regime, not limited to the particular family of theories we are studying. This may be seen by evaluating Eq. (211) in the potential regime.

solution to the equations of motion. Indeed, in Subcase III we are actually fixing the coupling functions and hence the theory, so all solutions of the theory will have that behavior (we do not have any "particular solution whose stability we should study"). In the present case, however, we are dealing with a particular solution of a family of theories, without actually fixing the coupling functions.

VI. CONCLUSIONS

In this work we explored the idea of restoring the Diff invariance to a theory in which it was previously broken down to TDiff. Inspired by the work of Stueckelberg [15] for gauge fields and the treatment by Henneaux and Teitelboim [17] for unimodular gravity, we reformulated a TDiff invariant theory for a scalar field ψ in a way in which the invariance of the theory under general diffeomorphisms is recovered at the expense of introducing an additional vector field T^μ. We highlighted that the two approaches are indeed equivalent ways of dealing with the same problem. In the TDiff approach, the main tool was the study of the EMT conservation. Indeed, due to the lack of Diff invariance, it was not an automatic consequence of the symmetry, but rather a consistency condition which should be satisfied by the solutions of the theory. On the other hand, in the covariantized approach the symmetry is restored and the EMT is once again trivially conserved. In this case, then, the additional information is provided by the EoM of the newly incorporated vector field T^μ.

We have not only shown the equivalence of the two approaches at the level of general actions, but we have also recovered the results in the potential and kinetic regimes. Furthermore, in the kinetic regime the covariantized treatment has shed light on the conditions to be satisfied by the coupling function to obtain stable adiabatic fluids. We have also discussed how simple models in the kinetic regime can be found following both approaches. Apart from models leading to a constant EoS parameter, which appear as a power law for the coupling function of the corresponding approach, a natural model in one approach will not necessarily be simple in the other.

On the other hand, we have shown that the covariantized treatment also yields a novel and important result which was too difficult to obtain in the TDiff approach, namely, the general constraint on the metric whenever both kinetic and potential terms are present. The study of this general result reveals that we may describe our matter content as a nonadiabatic perfect fluid. We calculated the effective speed of sound of this fluid and found conditions on the allowed coupling functions in order to have physically viable models. Then, we easily translated all of these new results to the TDiff framework as well. Now, although the most general situation is nonadiabatic, we can find some particular subcases that give adiabatic perturbations. We found a total of six adiabatic models and performed a detailed analysis of their physical implications. One of them was already known (in particular, the shift-symmetric model was known to be adiabatic), but we were able to find and study the other models thanks to the simplified treatment provided by the covariantized approach. Finally, we also discussed a particular solution to a family of TDiff theories that yields the same results as GR. Moreover, we studied the stability of this solution, finding that it is unstable in the potential regime, while in the kinetic regime the stability depends on the expansion.

Overall, we conclude that the covariantized treatment carried out in this work is a useful tool for several reasons. From the fundamental point of view, the restoration of a symmetry via the introduction of additional quantities can be useful in discerning which are the truly dynamical components of a theory. This shall all be further pursued in future work. From the more practical point of view, the covariantized approach has proven to be faster for particular calculations and, most importantly, it has also revealed a previously unknown result of general character (the implications of which have been thoroughly studied). Nevertheless, we have seen how the more natural models in one approach are not necessarily those that one would consider in the other approach in the first place. So, one could be missing interesting phenomenology by focusing on only one of the approaches. Moreover, it is intriguing to consider what other problems and analyses, perhaps difficult in one approach, the other approach might help simplify. In addition, having two separate yet equivalent ways of tackling the same problem provides one with a consistency check of sorts, in the sense that a result obtained via one approach may be compared with that obtained through the other. For all of these reasons, we think that combining both approaches will allow us to better understand different aspects of the theory.

ACKNOWLEDGMENTS

The authors would like to thank an anonymous referee of Ref. [5] for useful comments and suggestions which sparked the analysis in this work, as well as an anonymous referee of the present paper for motivating us to develop its content further. We also wish to thank Javier Rubio for useful discussions and for bringing Ref. [21] to our attention. This work has been supported by the MICIU (Spain) project PID2022-138263NB-I00 funded by MICIU/AEI/10.13039/501100011033 and by ERDF/EU. D. J.-G. also acknowledges financial support from the Instituto de Física de Partículas y del Cosmos de la Universidad Complutense de Madrid (IPARCOS-UCM) through the "Ayudas de Doctorado IPARCOS-UCM/2023".

APPENDIX A: DIFFERENT COVARIANTIZATIONS

As we mention in Sec. III, the way of restoring the Diff symmetry is not unique. We chose to do so with a vector field T^μ following Henneaux and Teitelboim [17] because, in this way, locality is preserved in the covariantized action, but we could very well follow the route taken by Blas *et al.* in Ref. [21], where they directly considered introducing a new scalar field. There are some differences, however, the discussion of which has been reserved for the present appendix.

We begin by noting that we can add an arbitrary constant C_0 to our TDiff Lagrangian without affecting the EoM of the theory, i.e. we may work equally well with

$$S_{\rm TDiff} = S_{\rm EH} + \int d^4x [f_k(g)X - f_v(g)V + C_0]. \quad \text{(A1)}$$

Let us now covariantize this action in the way we present in Sec. III. We obtain the following expression:

$$S_{\rm Diff} = S_{\rm EH} + \int d^4x \sqrt{g}[H_k(Y)X - H_v(Y)V + C_0 Y], \quad \text{(A2)}$$

where we still have not specified how the scalar density $\bar{\mu}$ (equivalently, the combination $Y = \bar{\mu}/\sqrt{g}$) is related to the new field. Let us now do precisely that: we shall introduce a new scalar field σ to the theory, and a simple scalar density which we may construct with it is

$$\bar{\mu} = \sigma\sqrt{g}, \quad \text{(A3)}$$

which transforms as wanted. As a result of this choice, we have

$$Y = \sigma, \quad \text{(A4)}$$

and so our Diff action becomes

$$S_{\rm Diff} = S_{\rm EH} + \int d^4x \sqrt{g}[H_k(\sigma)X - H_v(\sigma)V + C_0\sigma], \quad \text{(A5)}$$

which is a theory for $g_{\mu\nu}$ and two scalar fields (ψ and σ). If we now take variations with respect to the new scalar field σ we find that its EoM is

$$H_k'(\sigma)X - H_v'(\sigma)V + C_0 = 0, \quad \text{(A6)}$$

which is practically identical in form to the integrated EoM (95) for T^μ.

Despite their formal similarities, there are a couple of subtle differences which we wish to stress at this point. The first is that, even though we started by including a superfluous constant in the TDiff action (where it had no effect on the EoM of the theory), in the covariantized approach this arbitrary constant is no longer superfluous as it explicitly enters the EoM of the theory, as shown by Eq. (A6). As explained in Ref. [21], there is no longer a one-to-one correspondence between the EoM in the TDiff approach and those in the covariantized approach: those obtained from the TDiff action (A1) correspond to a whole family of the ones obtained from the covariantized action (A2). The constant C_0 represents a global degree of freedom which appears explicitly in the action, not fixed in any way by the fields in the theory (recall that, when we included a vector T^μ, the constant c_ρ could be fixed with the initial conditions on the fields and their derivatives). The nonlocality in the covariantized action implied by this new global degree of freedom is what we were trying to avoid when we introduced a vector field. Beyond this fact, the second difference we wish to highlight is that the EoM (A6) for σ is not really dynamical [as opposed to (95), where derivatives of T^μ appear]. As a result, the newly included scalar field σ would be a spectator field.

Finally, we stress that although the first difference (i.e. having a new global degree of freedom) is a general feature of covariantizing with a scalar field, the second difference (i.e. σ being a nondynamical, spectator field) is a result of our particular field theory. Indeed, recall that we are not breaking the symmetry in the gravitational sector but only in the matter sector, where the determinant appears in the volume element. If we, for instance, decided to break the symmetry in the gravitational sector by including a kinetic term of the form $\partial^\mu g \partial_\mu g$ then, after covariantizing, the scalar field σ would have dynamics. All of these points were studied in greater detail in Ref. [21].

APPENDIX B: COVARIANTIZED EMT

We include in this appendix a couple of calculations regarding the covariantized EMT, which are not needed to follow the main thread of the discussion, but which could come in handy in order to clear up some of the computations. The first one has to do with explicitly obtaining the expression (51) for the covariantized EMT, and the second one with its (trivial) conservation.

Let us begin by obtaining it. As we know from the definition (6), we must find the functional derivative of the matter action

$$S_m = \int d^4x \sqrt{g}[H_k(Y)X - H_v(Y)V(\psi)] \quad \text{(B1)}$$

with respect to the metric. To this end, we find the variation δS_m with respect to the metric, which reads

$$\delta S_m = \int d^4x \{(H_k X - H_v V)\delta\sqrt{g} + \sqrt{g} X \delta H_k + \sqrt{g} H_k \delta X - \sqrt{g} V \delta H_v - \sqrt{g} H_v \delta V\}. \quad \text{(B2)}$$

The needed variations with respect to the metric are the following:

$$\delta V = 0, \quad \text{(B3a)}$$

$$\delta X = \frac{1}{2}\partial_\mu\psi\partial_\nu\psi\delta g^{\mu\nu}, \tag{B3b}$$

$$\delta\sqrt{g} = -\frac{1}{2}\sqrt{g}g_{\mu\nu}\delta g^{\mu\nu}, \tag{B3c}$$

$$\delta H = H'\left[\frac{1}{2}Yg_{\mu\nu}\delta g^{\mu\nu} + \frac{1}{\sqrt{g}}\partial_\alpha(T^\alpha\delta\sqrt{g})\right]. \tag{B3d}$$

The usage of the first three is simple enough, but the fourth one deserves a bit more care. After substituting these expressions into the variation (B2) of the matter action, the integrand will have a term of the form

$$\sqrt{g}X\delta H_k = \sqrt{g}XH_k'\frac{1}{2}Yg_{\mu\nu}\delta g^{\mu\nu} - T^\alpha\delta\sqrt{g}\partial_\alpha(H_k'X) + \partial_\alpha(T^\alpha H_k'X\delta\sqrt{g}) \tag{B4}$$

and a term of the form

$$\sqrt{g}V\delta H_v = \sqrt{g}VH_v'\frac{1}{2}Yg_{\mu\nu}\delta g^{\mu\nu} - T^\alpha\delta\sqrt{g}\partial_\alpha(H_v'V) + \partial_\alpha(T^\alpha H_v'V\delta\sqrt{g}). \tag{B5}$$

A couple of steps are now needed for simplification. The first step is to, as usual, discard the boundary terms appearing in the last lines of the above two expressions, i.e. the ones arising from $\partial_\alpha(T^\alpha H_k'X\delta\sqrt{g})$ and $\partial_\alpha(T^\alpha H_v'V\delta\sqrt{g})$. The second step is to recognize that after performing the combination $\sqrt{g}X\delta H_k - \sqrt{g}V\delta H_v$ we obtain (among others) a term of the form

$$T^\alpha\delta\sqrt{g}\partial_\alpha(H_v'V - H_k'X) = 0, \tag{B6}$$

which vanishes by virtue of the EoM (49) for the vector field T^μ. Carefully substituting and taking everything into account, we finally obtain

$$T_{\mu\nu} = H_k(Y)\partial_\mu\psi\partial_\nu\psi - [H_k(Y)X - H_v(Y)V]g_{\mu\nu} + Y[H_k'(Y)X - H_v'(Y)V]g_{\mu\nu}, \tag{B7}$$

which is precisely the covariantized EMT (51).

The second comment we wish to make is regarding the conservation of this EMT, which should be trivial given that the theory has Diff symmetry, and it can be explicitly checked that this is so. Proceeding piece by piece, we have that

$$\nabla_\mu(H_k\nabla^\mu\psi\nabla^\nu\psi) = \nabla^\nu\psi\nabla_\mu(H_k\nabla^\mu\psi) + H_k\nabla^\mu\psi(\nabla_\mu\nabla^\nu\psi), \tag{B8a}$$

$$\nabla_\mu[(H_kX - H_vV)g^{\mu\nu}] = H_k\nabla^\mu\psi(\nabla_\mu\nabla^\nu\psi) + \nabla^\nu Y(H_k'X - H_v'V) - H_vV'\nabla^\nu\psi, \tag{B8b}$$

$$\nabla_\mu[Y(H_k'X - H_v'V)g^{\mu\nu}] = \nabla^\nu Y(H_k'X - H_v'V) + Y\nabla^\nu(H_k'X - H_v'V). \tag{B8c}$$

Joining all the pieces together we can see some very convenient cancellations, and also that

$$0 = \nabla_\mu T^{\mu\nu} = \nabla^\nu\psi\underbrace{[\nabla_\mu(H_k\nabla^\mu\psi) + H_vV']}_{=0\text{ by }\psi\text{ EoM (46)}} + Y\underbrace{\nabla^\nu(H_k'X - H_v'V)}_{=0\text{ by }T^\mu\text{ EoM (49)}} = 0, \tag{B9}$$

i.e. we indeed obtain that the conservation of the EMT is trivially satisfied on the solutions to the EoM of the theory, as we expected.

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