Quasi-two-body decays $B \to Pf_0(500) \to P\pi^+\pi^-$ in the perturbative QCD approach

Jia-Wei Zhang[1], Bo-Yan Cui[1], Xing-Gang Wu[2], Hai-Bing Fu[3,*] and Ya-Hui Chen[4,†]

[1]*Department of Physics, Chongqing University of Science and Technology, Chongqing 401331, People's Republic of China*

[2]*Department of Physics, Chongqing Key Laboratory for Strongly Coupled Physics, Chongqing University, Chongqing 401331, People's Republic of China*

[3]*Department of Physics, Guizhou Minzu University, Guiyang 550025, People's Republic of China*

[4]*Department of Physics, College of Basic Medical Sciences, Army Medical University, Chongqing 400038, People's Republic of China*

In this paper, we study the quasi-two-body decays $B \to Pf_0(500) \to P\pi^+\pi^-$ [with $P = (\pi, K, \eta, \eta')$] within framework of perturbative QCD (PQCD) factorization approach. With the help of $\pi - \pi$ distribution amplitude and scalar form factor $F_{\pi\pi}(\omega^2)$, we calculate the CP averaged branching fraction and the CP asymmetry for the quasi-two-body decays $B \to Pf_0(500) \to P\pi^+\pi^-$. Taking the quasi-two-body decay $B^+ \to \pi^+ f_0(500) \to \pi^+\pi^+\pi^-$ as an explicit example, we present the behavior of differential branching fraction and direct CP violation versus the π-π invariant mass. The total branching fraction and direct CP violation are $\mathcal{B}(B^+ \to \pi^+[\sigma \to]\pi^+\pi^-) = (1.78 \pm 0.41 \pm 0.51) \times 10^{-6}$ and $\mathcal{A}_{CP}(B^+ \to \pi^+[\sigma \to]\pi^+\pi^-) = (29.8 \pm 11.1 \pm 13.0)\%$ respectively. Our results could be tested by further experiments.

I. INTRODUCTION

Three-body B-meson decays are considerably more challenging than that of two-body-decays, mainly due to the entangled resonant and nonresonant contributions, the complex interplay between the weak and strong dynamics [1], and other possible final-state-interactions (FSI) [2–4] in the three-body B meson decays. Traditional approaches for the two-body-decays are no longer satisfactory in the three-body decay processes [5]. Practically, the hadronic three-body B meson decay processes, in most cases, are considered to be dominated by the low-energy S-, P- and D-wave resonant states, which could be treated in the quasi-two-body framework. By neglecting the FSI between the meson pair originated from the resonant states and the bachelor particle, the factorization procedure can be applied [5,6]. Substantial theoretical efforts for different quasi-two-body B meson decays has been made within different theoretical approaches, cf. Refs. [7–29]. As well, the contributions from various intermediate resonant states for the

three-body B-meson decays in the context of perturbative QCD (PQCD) approach [30–32] have been investigated in Refs. [33–43].

Compared with vector and tensor mesons, the identification of the scalar mesons is long-standing puzzle [44]. Scalar resonances are hard to be resolved, since some of them have large decay widths which make us difficult to distinguish between resonance and background. In the theoretical point of view, their masses do not fit the expectation in the naive quark model. For the lightest scalar meson $f_0(500)$ (also refereed as σ) meson, which is, in the present state, not a ordinary meson in the sense that it cannot be interpreted as predominantly made of quark and antiquark [45]. After more than 60 years of study of $f_0(500)$, various of interpretations have been proposed. More explicitly, a light scalar-isoscalar field was first postulated [46] for explanation of the inter-nucleon attraction. Then, Linear Sigma Model was proposed [47] to describe the chiral symmetry in pion-pion interaction, this explains why $f_0(500)$ is usually called as the σ meson. The linear sigma model plays a relevant role, in history, for the understanding of spontaneous chiral symmetry breaking, where all fields become Goldstone bosons, i.e., pions, except σ. When the $f_0(500)$ and $f_0(980)$ are considered as $q\bar{q}$ state, the mixing of light and strange quark may appear [48], it can be characterized by a 2×2 rotation matrix with a single parameter, i.e., the mixing angle ϕ. If the $q\bar{q}$ state does exist, the mixing angle can be constrained by the scalar decay channel which is exactly what we study.

*Contact author: fuhb@gzmu.edu.cn
†Contact author: cyh@tmmu.edu.cn

Comparison between our theoretical prediction and the experimental data may allow us to probe the inner structure of light scalar mesons. Tetraquarks is also a popular interpretation [49] which is bounded to form a color neutral resonance by two valence quark and two antiquark [45]. There are also some different explanation of quark level dynamics to describe the formation of scalar mesons, such as the bag model with additional one gluon exchange [49], the diquark-antidiquark configurations [50], the large-N quantum chromodynamics [51], moreover, including instanton effects [52]. In this theories, tetraquark states are unmixed, as assumed in Ref. [48], with a constrain of mixing angle less than 5 degree [53]. Consequently, how to distinguish $q\bar{q}$ and $q\bar{q}q\bar{q}$ states becomes a significant issue. In Ref. [54], authors proposed a method to distinguish two kinds of scalar mesons based on sum rule technique, it may becomes a important criterion of phenomenological study and may provide more detail of the inner structure of scalar mesons.

Within the framework of PQCD approach, quasi-two body decays involving resonant state $f_0(500)$ have been studied [34,55–57], we would like to extend to our previous studies to the quasi-two-body B meson decays $B \to Pf_0(500) \to P\pi\pi$, where the bachelor particle P denotes the light pseudoscalar π, K, η, or η'. Typical diagrams for the $B \to Pf_0(500) \to P\pi\pi$ decay processes are shown in Fig. 1. Inspired by the generalized parton distribution (GPD) in hard exclusive two pion production [58–61], the two-meson distribution amplitude was introduced in three-body hadronic B decays in the frame work of PQCD approach [62,63] as the universal nonperturbative input. The decay amplitude for the quasi-two-body decays $B \to Pf_0(500) \to P\pi\pi$ can be expressed as the convolution of the nonperturbative wave function and hard kernel [33,62,63]

$$\mathcal{A} = \phi_B \otimes H \otimes \phi_P \otimes \phi_{\pi\pi}^{S\text{-wave}}, \qquad (1)$$

where hard kernel H contains one hard gluon, and the distribution amplitudes ϕ_B, ϕ_P and $\phi_{\pi\pi}^{S\text{-wave}}$ absorb the nonperturbative dynamics in the decay processes.

The rest of this paper are organized as follows. We give a brief introduction of the theoretical framework in Sec. II. Numerical results and some discussions are shown in Sec. III, and a brief conclusion will be summarized in Sec. IV. The relevant factorization formulas for the decay amplitudes are collected in the Appendix.

II. FRAMEWORK

In the light-cone coordinate, the B meson momentum p_B, σ meson momentum p, bachelor particle momentum p_3, and the corresponding quark momenta k_B, k, k_3 in the rest frame of B meson are defined as

$$p_B = \frac{m_B}{\sqrt{2}}(1, 1, \mathbf{0}_T), \quad k_B = \left(0, \frac{m_B}{\sqrt{2}}x_B, \mathbf{k}_{BT}\right),$$

$$p = \frac{m_B}{\sqrt{2}}(1, \eta, \mathbf{0}_T), \quad k = \left(\frac{m_B}{\sqrt{2}}z, 0, \mathbf{k}_T\right),$$

$$p_3 = \frac{m_B}{\sqrt{2}}(0, 1 - \eta, \mathbf{0}_T), \quad k_3 = \left(0, \frac{m_B}{\sqrt{2}}(1 - \eta)x_3, \mathbf{k}_{3T}\right),$$

$$(2)$$

where the variable η is defined as $\eta = \omega^2/m_B^2$ with $\omega = \sqrt{p^2}$ stand for invariant mass of dipion. Terms x_B, z, and x_3 are the momentum fraction of k_B, k, and k_3 respectively.

The B meson can be treated as heavy-light system, whose wave function can be written as [64]

$$\Phi_B(x, b) = \frac{1}{\sqrt{2N_c}}(\not{p}_B + m_B)\gamma_5\phi_B(x, b), \qquad (3)$$

where b is the conjugate space coordinate of the transverse momentum of the valence quark of meson, N_c is the color factor, the distribution amplitude $\phi_B(x, b)$ are chosen as

$$\phi_B(x, b) = N_B x^2(1 - x^2)\exp\left[-\frac{1}{2}\left(\frac{xm_B}{\omega_B}\right)^2 - \frac{\omega_B^2 b^2}{2}\right], \qquad (4)$$

where the ω_B is the shape parameter and N_B being the normalization factor. The shape parameter ω_B is mainly fixed from the fit to the $B \to \pi$ form factors derived from lattice QCD [65] and LCSR [66]. The normalization constant N_B is related to the decay constant f_B through the relation

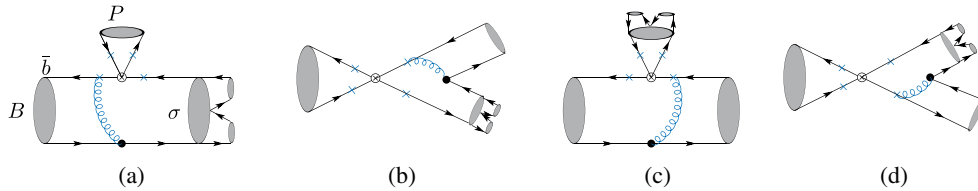$$\int_0^1 dx\phi_B(x, b = 0) = \frac{f_B}{2\sqrt{6}}. \qquad (5)$$

FIG. 1. Typical diagrams for the quasi-two-body decays $B_{(s)} \to P\sigma \to P\pi\pi$. The diagram (a) for the $B \to \sigma$ transition, and diagram (c) for the $B \to P$ transition, as well as the diagrams (b) and (d) for annihilation contributions. The symbol $\otimes$ stands for the weak vertex and $\times$ denotes possible attachments of hard gluons.

For the final state of light pseudoscalar meson, whose wave function being the

$$\Phi_P(p,z) = \frac{i}{\sqrt{2N_c}} \gamma_5 [\not{p}\phi^A(z) + m_0\phi^P(z) + m_0(\not{v}\not{n} - 1)\phi^T(z)], \qquad (6)$$

where m_0 is chiral mass of corresponding pseudoscalar meson. The p and z are associated momentum and momentum fraction respectively. The explicit expression of relevant distribution amplitude in light-cone sum rule up to twist-3 are shown [66–69]

$$\phi^A(x) = \frac{f_P}{2\sqrt{2N_c}} 6x(1-x)[1 + a_1^P C_1^{3/2}(2x-1) + a_2^P C_2^{3/2}(2x-1) + a_4^P C_4^{3/2}(2x-1)],$$

$$\phi^P(x) = \frac{f_P}{2\sqrt{2N_c}} \left[1 + \left(30\eta_3 - \frac{5}{2}\rho_P^2 \right) C_2^{1/2}(2x-1) - 3\left[\eta_3\omega_3 + \frac{9}{20}\rho_P^2(1 + 6a_2^P) \right] C_4^{1/2}(2x-1) \right],$$

$$\phi^T(x) = \frac{f_P}{2\sqrt{2N_c}} (1-2x)\left[1 + 6\left(5\eta_3 - \frac{1}{2}\eta_3\omega_3 - \frac{7}{20}\rho_P^2 - \frac{3}{5}\rho_P^2 a_2^P \right) (1 - 10x + 10x^2) \right], \qquad (7)$$

with the Gegenbauer moments are $a_1^{\pi,\eta_{q,s}} = 0, a_1^K = 0.06, a_2^{\pi,K} = 0.25, a_2^{\eta_{q,s}} = 0.115, a_4^{\pi,\eta_{q,s}} = -0.015$. Meanwhile, the parameters are $\rho_\pi = m_\pi/m_0^\pi, \rho_K = m_K/m_0^K, \rho_{\eta_q} = 2m_q/m_0^q, \rho_{\eta_s} = 2m_s/m_0^s, \eta_3 = 0.015, \omega_3 = -3$ with $m_0^\pi = (1.4 \pm 0.1)$ GeV, $m_0^K = (1.6 \pm 0.1)$ GeV, $m_0^{\eta_q} = 1.07$ GeV, $m_0^{\eta_s} = 1.92$ GeV. The definition of Gegenbauer polynomials can be found in Refs. [68,69].

For the η and η' mesons, its components has been studied extensively, two major mixing mechanism were adopt in most study, the quark flavor basis and singlet-octet basis. Different process under different assumption are studied to determine the mixing angle, we prefer to choose the so-called Feldmann-Kroll-Stech (FKS) formalism [70,71] in which it was considered as mixing of η_q and η_s, which made of $n\bar{n} = (u\bar{u} + d\bar{d})/\sqrt{2}$ and $s\bar{s}$ respectively, the physical state η and η' related to flavor state η_q and η_s through a two by two rotation matrix with a single parameter, the mixing angle ϕ

$$\begin{pmatrix} |\eta\rangle \\ |\eta'\rangle \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} |\eta_q\rangle \\ |\eta_s\rangle \end{pmatrix} \qquad (8)$$

with $\phi = 39.3° \pm 1.0°$. For the possible glueball effect are considered to be small [72], we will neglect this contribution. The distribution amplitude of $n\bar{n}$ and $s\bar{s}$ contain same Lorentz structure with pion except for the difference decay constant and chiral parameters, we collect the relation [72]

$$f_n = (1.07 \pm 0.02)f_\pi = 139.1 \pm 2.6 \text{ MeV},$$
$$f_s = (1.34 \pm 0.06)f_\pi = 174.2 \pm 7.8 \text{ MeV}. \qquad (9)$$

The S-wave π-π distribution amplitude takes the following form of [58–61,73]

$$\Phi_{\pi\pi}^{S\text{-wave}} = \frac{1}{\sqrt{2N_c}} [\not{p}\phi_{v\nu=-}^{I=0}(z,\zeta,\omega^2) + \omega\phi_s^{I=0}(z,\zeta,\omega^2) + \omega(\not{n}\not{v} - 1)\phi_{t\nu=+}^{I=0}(z,\zeta,\omega^2)], \qquad (10)$$

where

$$\phi_{v\nu=-}^{I=0}(z,\zeta,\omega^2) = \phi^0 = \frac{9F_s(\omega^2)}{\sqrt{2N_c}} a_2^{I=0} z(1-z)(1-2z),$$

$$\phi_s^{I=0}(z,\zeta,\omega^2) = \phi^s = \frac{F_s(\omega^2)}{2\sqrt{2N_c}},$$

$$\phi_{t\nu=+}^{I=0}(z,\zeta,\omega^2) = \phi^t = \frac{F_s(\omega^2)}{2\sqrt{2N_c}} (1-2z), \qquad (11)$$

where the expression of scalar form factor and associated auxiliary functions can be found in [55,74].

According to the typical Feynman diagrams as shown in Fig. 1 and the quark currents for each decays, the decay amplitudes for considered quasi-two-body decays $B \to P f_0(500) \to P\pi\pi$ are given as

$$\mathcal{A}(B^+ \to \pi^+[\sigma \to]\pi^+\pi^-) = \frac{G_F}{2}\left\{ V_{ub}^* V_{ud} \left[\left(\frac{C_1}{3} + C_2\right)(F_{T\sigma}^{LL} + F_{A\sigma}^{LL} + F_{AP}^{LL}) + C_1(M_{T\sigma}^{LL} + M_{A\sigma}^{LL} + M_{AP}^{LL}) + C_2 M_{TP}^{LL} \right] \right.$$

$$- V_{tb}^* V_{td} \left[\left(\frac{C_3}{3} + C_4 + \frac{C_9}{3} + C_{10}\right)(F_{T\sigma}^{LL} + F_{A\sigma}^{LL} + F_{AP}^{LL}) + \left(\frac{C_5}{3} + C_6 + \frac{C_7}{3} + C_8\right) \right.$$

$$\times (F_{T\sigma}^{SP} + F_{A\sigma}^{SP} + F_{AP}^{SP}) + (C_3 + C_9)(M_{T\sigma}^{LL} + M_{A\sigma}^{LL} + M_{AP}^{LL}) + (C_5 + C_7)(M_{T\sigma}^{LR} + M_{A\sigma}^{LR} + M_{AP}^{LR})$$

$$+ \left(C_3 + 2C_4 - \frac{C_9}{2} + \frac{C_{10}}{2}\right)M_{TP}^{LL}\left(C_5 - \frac{C_7}{2}\right)M_{TP}^{LR} + \left(2C_6 + \frac{C_8}{2}\right)M_{TP}^{SP}$$

$$\left. + \left(\frac{C_5}{3} + C_6 - \frac{C_7}{6} - \frac{C_8}{2}\right)F_{TP}^{SP} \right]\right\}, \tag{12}$$

$$\mathcal{A}(B^+ \to K^+[\sigma \to]\pi^+\pi^-) = \frac{G_F}{2}\left\{ V_{ub}^* V_{us} \left[\left(\frac{C_1}{3} + C_2\right)(F_{T\sigma}^{LL} + F_{A\sigma}^{LL}) + C_1(M_{T\sigma}^{LL} + M_{A\sigma}^{LL}) + C_2 M_{TP}^{LL} \right] \right.$$

$$- V_{tb}^* V_{ts} \left[\left(\frac{C_3}{3} + C_4 + \frac{C_9}{3} + C_{10}\right)(F_{T\sigma}^{LL} + F_{A\sigma}^{LL}) + \left(\frac{C_5}{3} + C_6 + \frac{C_7}{3} + C_8\right)(F_{T\sigma}^{SP} + F_{A\sigma}^{SP}) \right.$$

$$+ (C_3 + C_9)(M_{T\sigma}^{LL} + M_{A\sigma}^{LL}) + (C_5 + C_7)(M_{T\sigma}^{LR} + M_{A\sigma}^{LR}) + \left(2C_4 + \frac{C_{10}}{2}\right)M_{TP}^{LL}$$

$$\left. + \left(2C_6 + \frac{C_8}{2}\right)M_{TP}^{SP} \right]\right\}, \tag{13}$$

$$\mathcal{A}(B^0 \to \pi^0[\sigma \to]\pi^+\pi^-) = \frac{G_F}{2\sqrt{2}}\left\{ V_{ub}^* V_{ud} \left[\left(C_1 + \frac{C_2}{3}\right)(F_{T\sigma}^{LL} + F_{A\sigma}^{LL} + F_{AP}^{LL}) + C_2(M_{T\sigma}^{LL} + M_{A\sigma}^{LL} + M_{TP}^{LL} + M_{AP}^{LL}) \right] \right.$$

$$- V_{tb}^* V_{td} \left[\left(-\frac{C_3}{3} - C_4 - \frac{3C_7}{2} - \frac{C_8}{2} + \frac{5C_9}{3} + C_{10}\right)(F_{T\sigma}^{LL} + F_{A\sigma}^{LL} + F_{AP}^{LL}) \right.$$

$$+ \left(-\frac{C_5}{3} - C_6 + \frac{C_7}{6} + \frac{C_8}{2}\right)(F_{T\sigma}^{SP} + F_{A\sigma}^{SP} + F_{TP}^{SP} + F_{AP}^{SP}) + \left(-C_3 + \frac{C_9}{2} + \frac{3C_{10}}{2}\right)$$

$$\times (M_{T\sigma}^{LL} + M_{A\sigma}^{LL} + M_{AP}^{LL}) + \left(-C_3 - 2C_4 + \frac{C_9}{2} - \frac{C_{10}}{2}\right)M_{TP}^{LL} + \frac{3C_8}{2}(M_{T\sigma}^{SP} + M_{A\sigma}^{SP} + M_{AP}^{SP})$$

$$\left. + \left(-2C_6 - \frac{C_8}{2}\right)M_{TP}^{SP} + \left(-C_5 + \frac{C_7}{2}\right)(M_{T\sigma}^{LR} + M_{A\sigma}^{LR} + M_{TP}^{LR} + M_{AP}^{LR}) \right]\right\}, \tag{14}$$

$$\mathcal{A}(B^0 \to K^0[\sigma \to]\pi^+\pi^-) = \frac{G_F}{2}\left\{ V_{ub}^* V_{us}[C_2 M_{TP}^{LL}] - V_{tb}^* V_{ts} \left[\left(\frac{C_3}{3} + C_4 - \frac{C_9}{6} - \frac{C_{10}}{2}\right)(F_{T\sigma}^{LL} + F_{A\sigma}^{LL}) \right. \right.$$

$$+ \left(\frac{C_5}{3} + C_6 - \frac{C_7}{6} - \frac{C_8}{2}\right)(F_{T\sigma}^{SP} + F_{A\sigma}^{SP}) + \left(C_3 - \frac{C_9}{2}\right)(M_{T\sigma}^{LL} + M_{A\sigma}^{LL})$$

$$\left. + \left(C_5 - \frac{C_7}{2}\right)(M_{T\sigma}^{LR} + M_{A\sigma}^{LR}) + \left(2C_4 + \frac{C_{10}}{2}\right)M_{TP}^{LL} + \left(2C_6 + \frac{C_8}{2}\right)M_{TP}^{SP} \right]\right\}, \tag{15}$$

$$\mathcal{A}(B_s^0 \to \bar{K}^0[\sigma \to]\pi^+\pi^-) = \frac{G_F}{2}\left\{ V_{ub}^* V_{ud}[C_2 M_{TP}^{LL}] - V_{tb}^* V_{td} \left[\left(\frac{C_5}{3} + C_6 - \frac{C_7}{6} - \frac{C_8}{2}\right)(F_{TP}^{SP} + F_{AP}^{SP}) \right. \right.$$

$$+ \left(\frac{C_3}{3} + C_4 - \frac{C_9}{6} - \frac{C_{10}}{2}\right)F_{AP}^{LL} + \left(C_3 + 2C_4 - \frac{C_9}{2} + \frac{C_{10}}{2}\right)M_{TP}^{LL} + \left(C_5 - \frac{C_7}{2}\right)M_{TP}^{LR}$$

$$\left. + \left(2C_6 - \frac{C_8}{2}\right)M_{TP}^{SP} + \left(C_3 - \frac{C_9}{2}\right)M_{AP}^{LL} + \left(C_5 - \frac{C_7}{2}\right)M_{AP}^{LR} \right]\right\}, \tag{16}$$

$$\mathcal{A}(B^0 \to \eta_q[\sigma \to]\pi^+\pi^-) = \frac{G_F}{2\sqrt{2}}\left\{V_{ub}^*V_{ud}\left[\left(C_1 + \frac{C_2}{3}\right)(F_{T\sigma}^{LL} + F_{A\sigma}^{LL} + F_{AP}^{LL}) + C_2(M_{T\sigma}^{LL} + M_{A\sigma}^{LL} + M_{TP}^{LL} + M_{AP}^{LL})\right]\right.$$

$$- V_{tb}^*V_{td}\left[\left(C_5 + \frac{C_6}{3} - \frac{C_7}{6} - \frac{C_8}{2}\right)(F_{T\sigma}^{SP} + F_{TP}^{SP}) + \left(C_6 - \frac{C_8}{2}\right)(M_{T\sigma}^{SP} + M_{TP}^{SP})\right.$$

$$+ \left(\frac{7C_3}{3} + \frac{7C_4}{3} - 2C_5 - \frac{2C_6}{3} - \frac{C_7}{2} - \frac{C_8}{6} + \frac{C_9}{3} - \frac{C_{10}}{3}\right)(F_{T\sigma}^{LL} + F_{A\sigma}^{LL} + F_{AP}^{LL})$$

$$+ \left(C_3 + 2C_4 - \frac{C_9}{2} + \frac{C_{10}}{2}\right)(M_{T\sigma}^{LL} + M_{TP}^{LL}) + \left(C_5 + C_6 - \frac{C_7}{2} + C_8\right)(M_{T\sigma}^{LR} + M_{TP}^{LR})$$

$$+ \left(\frac{C_5}{3} + C_6 - \frac{C_7}{6} - \frac{C_8}{2}\right)(F_{A\sigma}^{SP} + M_{AP}^{SP}) + (C_4 + C_{10})(M_{A\sigma}^{LL} + M_{AP}^{LL})$$

$$\left.\left.+ (C_6 + C_8)(M_{A\sigma}^{SP} + M_{AP}^{SP})\right]\right\}, \tag{17}$$

$$\mathcal{A}(B^0 \to \eta_s[\sigma \to]\pi^+\pi^-) = \frac{G_F}{2}\left\{-V_{tb}^*V_{td}\left[\left(C_4 - \frac{C_{10}}{2}\right)M_{T\sigma}^{LL} + \left(C_6 - \frac{C_8}{2}\right)M_{T\sigma}^{SP}\right.\right.$$

$$\left.\left.+ \left(C_3 + \frac{C_4}{3} - C_5 - \frac{C_6}{3} + \frac{C_7}{2} + \frac{C_8}{6} - \frac{C_9}{2} - \frac{C_{10}}{6}\right)F_{T\sigma}^{LL}\right]\right\}, \tag{18}$$

$$\mathcal{A}(B^0 \to \eta[\sigma \to]\pi^+\pi^-) = \mathcal{A}(B^0 \to \eta_q[\sigma \to]\pi^+\pi^-)\cos\phi - \mathcal{A}(B^0 \to \eta_s[\sigma \to]\pi^+\pi^-)\sin\phi, \tag{19}$$

$$\mathcal{A}(B^0 \to \eta'[\sigma \to]\pi^+\pi^-) = \mathcal{A}(B^0 \to \eta_q[\sigma \to]\pi^+\pi^-)\sin\phi + \mathcal{A}(B^0 \to \eta_s[\sigma \to]\pi^+\pi^-)\cos\phi, \tag{20}$$

$$\mathcal{A}(B_s^0 \to \eta_q[\sigma \to]\pi^+\pi^-) = \frac{G_F}{2\sqrt{2}}\left\{V_{ub}^*V_{us}\left[\left(C_1 + \frac{C_2}{3}\right)(F_{A\sigma}^{LL} + F_{AP}^{LL}) + C_2(M_{A\sigma}^{LL} + M_{AP}^{LL})\right]\right.$$

$$- V_{tb}^*V_{ts}\left[\left(2C_3 + \frac{2C_4}{3} - 2C_5 - \frac{2C_6}{3} - \frac{C_7}{2} - \frac{C_8}{6} + \frac{C_9}{2} + \frac{C_{10}}{6}\right)(F_{A\sigma}^{LL} + F_{AP}^{LL})\right.$$

$$\left.\left.+ \left(2C_4 + \frac{C_{10}}{2}\right)(M_{A\sigma}^{LL} + M_{AP}^{LL}) + \left(2C_6 + \frac{C_8}{2}\right)(M_{A\sigma}^{SP} + M_{AP}^{SP})\right]\right\}, \tag{21}$$

$$\mathcal{A}(B_s^0 \to \eta_s[\sigma \to]\pi^+\pi^-) = \frac{G_F}{2}\{V_{ub}^*V_{us}[C_2 M_{TP}^{LL}] - V_{tb}^*V_{ts}[(C_4 + C_{10})M_{TP}^{LL} + (C_6 + C_8)M_{TP}^{SP}]\}, \tag{22}$$

$$\mathcal{A}(B_s^0 \to \eta[\sigma \to]\pi^+\pi^-) = \mathcal{A}(B_s^0 \to \eta_q[\sigma \to]\pi^+\pi^-)\cos\phi - \mathcal{A}(B_s^0 \to \eta_s[\sigma \to]\pi^+\pi^-)\sin\phi, \tag{23}$$

$$\mathcal{A}(B_s^0 \to \eta'[\sigma \to]\pi^+\pi^-) = \mathcal{A}(B_s^0 \to \eta_q[\sigma \to]\pi^+\pi^-)\sin\phi + \mathcal{A}(B_s^0 \to \eta_s[\sigma \to]\pi^+\pi^-)\cos\phi. \tag{24}$$

where G_F is the Fermi constant, V_{ij} is the CKM matrix element, and the combinations of the Wilson coefficients $a_{1,2}$ are defined as $a_1 = C_1/3 + C_2$ and $a_2 = C_2/3 + C_1$. The expressions of individual amplitudes $F_{T\sigma}^{LL}$, $F_{T\sigma}^{SP}$, $F_{A\sigma}^{LL}$, $F_{A\sigma}^{SP}$, $M_{T\sigma}^{LL}$, $M_{T\sigma}^{LR}$, $M_{T\sigma}^{SP}$, $M_{A\sigma}^{LL}$, $M_{A\sigma}^{LR}$, $M_{A\sigma}^{SP}$, F_{TP}^{SP}, M_{TP}^{LL}, M_{TP}^{LR}, M_{TP}^{SP}, F_{AP}^{LL}, F_{AP}^{SP}, M_{AP}^{LL}, M_{AP}^{LR} and M_{AP}^{SP} from different subdiagrams in Fig. 1, which are collected in the Appendix.

Differential branching ratio of $B \to P f_0(500) \to P\pi\pi$ can be written as

$$\frac{d\mathcal{B}}{d\eta} = \tau_B \frac{|\vec{p}_1||\vec{p}|B_0^2 C^2}{32\pi^3 m_B m_0^2}\overline{|\mathcal{A}|^2}, \tag{25}$$

where B_0 is proportional to quark condensate, in $\pi\pi$ scalar system, it can be parameterized as $B_0 \simeq m_\pi^2/(m_u + m_d)$.

Following the definition from Ref. [74], the constant C takes the form of $C = g_{\sigma\pi\pi}\bar{f}_\sigma/(\sqrt{2}B_0 m_0)$. Meanwhile, τ_B is the mean lifetime of B meson, $|\vec{p}_1|$ and $|\vec{p}|$ are the three momenta of $f_0(500)$ resonance and light pseudoscalar meson respectively in the center-of-mass frame of π-π, and can be written as

$$|\vec{p}_1| = \frac{\sqrt{\lambda(\omega^2, m_\pi^2, m_\pi^2)}}{2\omega}, \qquad |\vec{p}| = \frac{\sqrt{\lambda(m_B^2, m_P^2, \omega^2)}}{2\omega}, \tag{26}$$

with m_B, m_P and m_π are the masses of the B, light pseudoscalar and π mesons respectively, and the Källén function $\lambda(a, b, c) = a^2 + b^2 + c^2 - 2(ab + ac + bc)$.

TABLE I.   Numerical values of the theory input parameters employed in the PQCD predictions of the quasi-two-body decays $\to Pf(500) \to P\pi\pi$ as well as the subsequent phenomenological analysis for the quasi-two-body decay observables.

Parameter	Value	Unit	Reference	Parameter	Value	Unit	Reference
$m_{B^\pm}$	5.279	GeV	[44]	$\tau_{B^\pm}$	1.638	ps	[44]
m_{B^0}	5.280	GeV	[44]	τ_{B^0}	1.519	ps	[44]
m_{B_s}	5.367	GeV	[44]	τ_{B_s}	1.527	ps	[44]
$f_B\|_{N_f=2+1+1}$	190.0	MeV	[75]	$f_{B_s}\|_{N_f=2+1+1}$	230.3	MeV	[75]
$m_{\pi^\pm}$	0.140	GeV	[44]	m_{π^0}	0.135	GeV	[44]
$m_{K^\pm}$	0.494	GeV	[44]	m_{K^0}	0.498	GeV	[44]
m_η	0.548	GeV	[44]	$m_{\eta'}$	0.958	GeV	[44]
f_π	0.130	GeV	[44]	f_K	0.156	GeV	[44]
λ	0.2250		[44]	$\bar{\rho}$	0.159		[44]
A	0.826		[44]	$\bar{\eta}$	0.348		[44]

III. RESULTS

For the numerical calculation, we adopt QCD scale at $\mu = 0.25$ GeV in modified minimal subtraction scheme. The decay constant of $B^{0,\pm}$, B_s and light pseudoscalar mesons come from FLAG working group's result [75]. The masses and mean life times of $B^{0,\pm}$ and B_s mesons, the masses of light pseudoscalar mesons and Wolfenstein parameters are all come from recent updated *Review of Particle Physics* [44]. We summarize explicitly the numerical values of the necessary standard model inputs and the hadronic parameters in Table I. Here the subscript N_f in decay constant $f_{B_{(s)}}$ represents the number of dynamical quark flavor in lattice simulation. $N_f = 2 + 1 + 1$ is for $m_u = m_d < m_s < m_c$ with four flavors dynamical quarks. The result for $N_f = 2 + 1 + 1$ is considered to be the most realistic one in comparing with $N_f = 2$ and $N_f = 2 + 1$, which can be found by the Flavor Lattice Averaging Group in detail [75].

By using the formula of differential branching fraction in Eq. (25) and explicit decay amplitude in the Appendix, we obtain the PQCD predictions of CP averaged branching fractions and direct CP violations in Table II for quasi-two-body $B \to Pf(500) \to P\pi\pi$ processes. The first uncertainty comes from shape parameter ω_B in the B meson distribution amplitude, we vary it value at 10% magnitude, that is $\omega_{B^{0,\pm}} = 0.40 \pm 0.04$ GeV and $\omega_{B_s} = 0.5 \pm 0.05$ GeV for $B^{0,\pm}$ and B_s mesons respectively. The second uncertainty comes from the Gegenbauer moment $a_2^{I=0} = 0.20 \pm 0.20$ in the S-wave π-π distribution amplitude. We ignore the uncertainties of parameters in distribution amplitude of light pseudoscalar mesons and Wolfenstein parameters since this uncertainties are considered to be small. It is interesting to see that the ω_B produce the largest theoretical uncertainty in PQCD predictions of quasi-two-body B meson decays [33–43], but in this study and previous quasi-two-body B meson decays involving

S-wave π-π contributions [34,55–57], the Gegenbauer moment $a_2^{I=0}$ plays more important role.

From the numerical results as listed in Table II, we have the following comments:

(i) In the $B \to PR \to P\pi\pi$ decays, we can extract the two-body branching fractions $\mathcal{B}(B \to PR)$ by using the relation under the quasi-two-body approximation

$$\mathcal{B}(B \to PR \to P\pi\pi) = \mathcal{B}(B \to PR) \cdot \mathcal{B}(R \to \pi\pi).$$
(27)

TABLE II.   PQCD predictions of CP-averaged branching fraction and direct CP violation for the quasi-two-body decays $B \to Pf(500) \to P\pi\pi$.

Decay modes		Quasi-two-body results
$B^+ \to \pi^+[\sigma \to]\pi^+\pi^-$	$\mathcal{B}(10^{-6})$	$1.78 \pm 0.41(\omega_B) \pm 0.51(a_2)$
	$\mathcal{A}_{CP}(\%)$	$29.8 \pm 11.1(\omega_B) \pm 13.0(a_2)$
$B^+ \to K^+[\sigma \to]\pi^+\pi^-$	$\mathcal{B}(10^{-7})$	$8.14 \pm 1.26(\omega_B) \pm 1.10(a_2)$
	$\mathcal{A}_{CP}(\%)$	$-63.5 \pm 8.2(\omega_B) \pm 11.3(a_2)$
$B^0 \to \pi^0[\sigma \to]\pi^+\pi^-$	$\mathcal{B}(10^{-7})$	$2.02 \pm 0.55(\omega_B) \pm 1.23(a_2)$
	$\mathcal{A}_{CP}(\%)$	$-64.6 \pm 21.8(\omega_B) \pm 38.7(a_2)$
$B^0 \to K^0[\sigma \to]\pi^+\pi^-$	$\mathcal{B}(10^{-7})$	$3.90 \pm 0.26(\omega_B) \pm 2.68(a_2)$
	$\mathcal{A}_{CP}(\%)$	$1.31 \pm 18.0(\omega_B) \pm 10.6(a_2)$
$B_s^0 \to \bar{K}^0[\sigma \to]\pi^+\pi^-$	$\mathcal{B}(10^{-7})$	$1.18 \pm 0.22(\omega_B) \pm 0.72(a_2)$
	$\mathcal{A}_{CP}(\%)$	$2.26 \pm 8.61(\omega_B) \pm 8.13(a_2)$
$B^0 \to \eta[\sigma \to]\pi^+\pi^-$	$\mathcal{B}(10^{-8})$	$5.11 \pm 1.11(\omega_B) \pm 4.60(a_2)$
	$\mathcal{A}_{CP}(\%)$	$-88.9 \pm 1.3(\omega_B) \pm 47.24(a_2)$
$B^0 \to \eta'[\sigma \to]\pi^+\pi^-$	$\mathcal{B}(10^{-8})$	$2.29 \pm 0.61(\omega_B) \pm 1.59(a_2)$
	$\mathcal{A}_{CP}(\%)$	$-42.3 \pm 13.4(\omega_B) \pm 5.4(a_2)$
$B_s^0 \to \eta[\sigma \to]\pi^+\pi^-$	$\mathcal{B}(10^{-9})$	$2.00 \pm 1.04(\omega_B) \pm 1.25(a_2)$
	$\mathcal{A}_{CP}(\%)$	$15.2 \pm 35.1(\omega_B) \pm 17.1(a_2)$
$B_s^0 \to \eta'[\sigma \to]\pi^+\pi^-$	$\mathcal{B}(10^{-8})$	$2.33 \pm 0.72(\omega_B) \pm 1.05(a_2)$
	$\mathcal{A}_{CP}(\%)$	$10.4 \pm 3.9(\omega_B) \pm 6.8(a_2)$

TABLE III. Predictions about CP-averaged branching fraction and direct CP violation for the quasi-two-body decays $B^+ \to \pi^+ f_0(500) \to \pi^+\pi^+\pi^-$ decay. Meanwhile, we also listed the QCDF, *BABAR*, LHCb results as a comparison.

References	$\mathcal{B}(10^{-6})$	$\mathcal{A}_{CP}(\%)$
This Work	$1.78 \pm 0.41 \pm 0.51$	$29.8 \pm 11.1 \pm 13.0$
QCDF [76]	$1.65^{+0.42}_{-0.37}$	14.7 ± 0.1
BABAR [77]	<4.1	-
LHCb [78,79]	3.83 ± 0.84	$14.9^{+0.5}_{-0.6}$

FIG. 2. The $\pi\pi$ invariant mass-dependent differential branching fraction for quasi-two-body decay $B^+ \to \pi^+ f_0(500) \to \pi^+\pi^+\pi^-$, the uncertainties generated from ω_B and a_2 are shown by the shaded band.

Combined with results listed in Table II, one can obtain the related two-body branching fractions for the two-body decays $B \to P f_0(500)$. It is interesting to see that the finite width effect are prominent in $B \to P f_0(500)$ decay [76], which means the extraction of two-body results by using quasi-two-body approximation will be greatly underestimated.

(ii) The predicted CP-averaged branching fractions for the considered decay processes are in the range of 10^{-9}–10^{-6}. For the $B^+ \to \pi^+ f_0(500) \to \pi^+\pi^+\pi^-$ decay, our prediction $(1.78 \pm 0.41 \pm 0.51) \times 10^{-6}$ is agree with Ref. [76] from the QCDF prediction by considering the finite width effect, both of two predictions satisfy the upper limit 4.1×10^{-6} from *BABAR* measurement [77]. However, most resent LHCb measurement [78,79] for the $B^+ \to \pi^+ f_0(500) \to \pi^+\pi^+\pi^-$ decay in the context of isobar model yields $(3.83 \pm 0.84) \times 10^{-6}$, the PQCD predicted branching fraction is smaller than the LHCb measurement by a factor of about 2. To have a clear look at the above comparison, we present the predictions in Table III. Furthermore, we show the curve of π-π invariant mass dependent differential branching fraction for quasi-two-body decay $B^+ \to \pi^+ f_0(500) \to \pi^+\pi^+\pi^-$ in Fig. 2. It is found that the main portion of branching fraction for $B^+ \to \pi^+ f_0(500) \to \pi^+\pi^+\pi^-$ received from the region of $[2m_\pi, 1 \text{ GeV}]$, the contributions from the $m_{\pi\pi} > 2.0$ GeV is evaluated about 2.4% and can be neglected safely.

(iii) For the quasi-two-body decay $B^+ \to \pi^+ f_0(500) \to \pi^+\pi^+\pi^-$, our prediction of CP violation reads $(29.8 \pm 11.1 \pm 13.0)\%$, which is much greater than that $14.9^{+0.5}_{-0.6}\%$ from LHCb measurements [78,79] in the context of isobar model for the S-wave π-π, fortunately, this result is still in the range of our prediction by considering the uncertainty from two nonperturbative parameters, which can also be seen in Table III. We also mention that QCDF prediction [76] is in excellent agreement with LHCb result. Besides, LHCb collaboration found that the interference between the S- and P-waves can also generate CP violation effect, unfortunately, CP
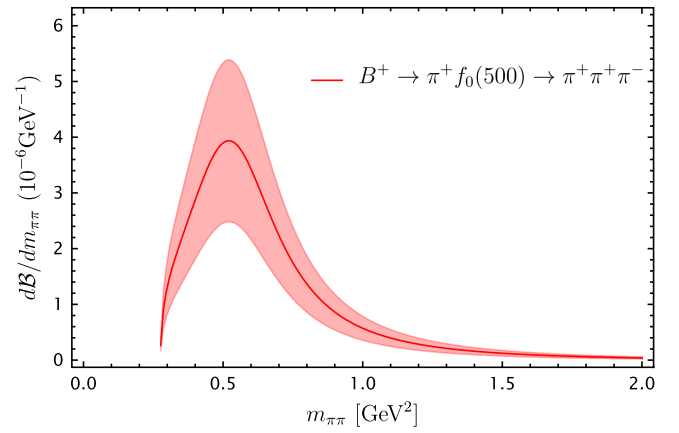
violation can only produced from interference of tree and penguin diagrams in the state of art of PQCD calculations, the S- and P-waves cannot generate extra strong phase, the predictions of CP violation from interference of S- and P-waves is still absent and very challenging in PQCD. For sake of illustrate π-π invariant mass dependent CP asymmetry, we show the π-π invariant mass-dependent CP asymmetry for quasi-two-body decay $B^+ \to \pi^+ f_0(500) \to \pi^+\pi^+\pi^-$ as an example, it is interesting to see that the direct CP asymmetry is decrease as the π-π invariant mass increases.

(iv) $f_0(500)$ was often parameterized by Bugg model [80] in partial wave analysis and it was applied in LHCb measurements [81,82]. However, S-wave π-π are considered to be broad, overlapping resonances, K-matrix model was applied for parametrization of S-wave components as a alternative scheme [78,79]. The rigorous theoretical calculation for nonresonance contribution in the context of PQCD framework is still absent [41], comparison between experiment measurements and theoretical predictions is still challenging. More attempts can be make in future study to parametrize the nonresonance contribution for sake of giving a more reliable result.

(v) It is also worth mentioning that the Gegenbauer moment a_2 plays very different roles in different decay mode. First, the annihilation diagram will generate very large uncertainty from Gegenbauer moment a_2. The flavor changing charged current mode such as $B^+ \to K^+[\sigma \to]\pi^+\pi^-$ decay processes are transition diagram dominated, which can be seen in subdiagram (a) of Fig. 1. Meanwhile, the flavor changing neutral current mode such as $B^0 \to K^0[\sigma \to]\pi^+\pi^-$ are annihilation diagram dominated,

which can be seen in subdiagram (b) of Fig. 1. Taking decay process $B^+ \to K^+[\sigma \to]\pi^+\pi^-$ and $B^0 \to K^0[\sigma \to]\pi^+\pi^-$ as example, we can make a comparison for contributions from different diagrams. It is found that annihilation diagram will give about 72.3% contribution for branching fraction in decay mode $B^0 \to K^0[\sigma \to]\pi^+\pi^-$, therefore, the uncertainty generated by a_2 will very large. On the contrary, annihilation diagram will give just about 35.0% contribution for branching fraction in decay mode $B^+ \to K^+[\sigma \to]\pi^+\pi^-$. Thus, the large uncertainty from Gegenbauer moment a_2 will be suppressed and become much lower than that of annihilation dominated mode. Subsequently, the flavor changing charged current mode are tree dominated, while the neutral mode are penguin dominated. The very complicated complex interference from CKM matrix, Wilson coefficients and form factors will enlarge or reduce the uncertainty caused by a_2, which encoding the very different behaviors in different mode.
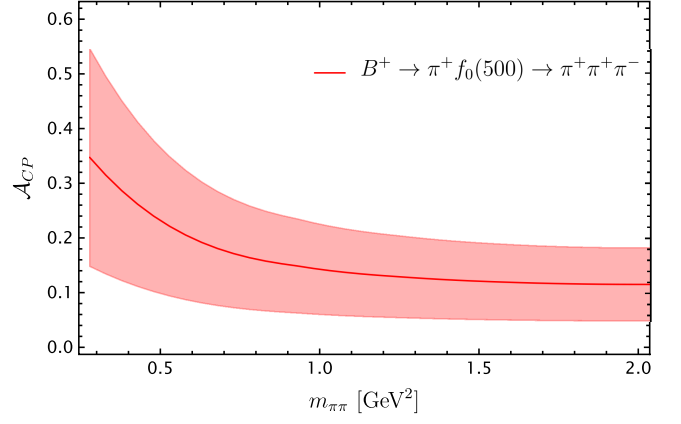

FIG. 3. The $\pi\pi$ invariant mass-dependent CP asymmetry for quasi-two-body decay $B^+ \to \pi^+ f_0(500) \to \pi^+\pi^+\pi^-$, the uncertainties generated from ω_B and a_2 are shown by the shaded band.

IV. CONCLUSION

We have studied the quasi-two-body decays $B \to Pf_0(500) \to P\pi^+\pi^-$ in the framework of PQCD factorization approach by using the scalar form factor $F_{\pi\pi}(\omega^2)$ as the nonperturbative input. Here the bachelor particle P denotes π, K, η and η'. The CP averaged branching fractions and the CP asymmetries for the considered quasi-two-body decay modes have been calculated by using the quasi-two-body approximation. It is found that the branching fractions are within the range of 10^{-9}–10^{-6}. Our predicted branching fraction for $B^+ \to \pi^+ f_0(500) \to \pi^+\pi^+\pi^-$ agrees with the upper limit issued by the *BABAR* collaboration but is smaller than the LHCb measurement by a factor of ~ 2. In Figs. 2 and 3, we have shown the differential branching fraction over the π-π invariant mass and the direct CP violation for the channel $B^+ \to \pi^+ f_0(500) \to \pi^+\pi^+\pi^-$. We hope all the predictions

could be tested by further experiments, it may deeper our understanding for the B meson decay mechanism in the context of PQCD factorization approach, and help us to probe the inner structure of $f_0(500)$.

APPENDIX: DECAY AMPLITUDES

The factorization formulas for decay amplitudes from Fig. 1 are collected below

$$F_{T\sigma}^{LL} = 8\pi C_F m_B^4 f_P \int dx_B dz \int b_B db_B b db \phi_B(x_B, b_B)(1-\eta)\{[\sqrt{\eta}(1-2z)(\phi^s + \phi^t) + (1+z)\phi^0]$$

$$\times E_{1ab}(t_{1a})h_{1a}(x_B, z, b_B, b)S_t(z) + \sqrt{\eta}(2\phi^s - \sqrt{\eta}\phi^0)E_{1ab}(t_{1b})h_{1b}(x_B, z, b_B, b)S_t(x_B)\}, \tag{A1}$$

$$F_{T\sigma}^{SP} = -16\pi C_F m_B^4 r_0 f_P \int dx_B dz \int b_B db_B b db \phi_B(x_B, b_B)\{[\sqrt{\eta}(2+z)\phi^s - \sqrt{\eta}z\phi^t + (1+\eta(1-2z))\phi^0]$$

$$\times E_{1ab}(t_{1a})h_{1a}(x_B, z, b_B, b)S_t(z) + [2\sqrt{\eta}(1-x_B+\eta)\phi^s + (x_B-2\eta)\phi^0]E_{1ab}(t_{1b})h_{1b}(x_B, z, b_B, b)S_t(x_B)\}, \tag{A2}$$

$$M_{T\sigma}^{LL} = 32\pi C_F m_B^4 / \sqrt{2N_c} \int dx_B dz dx_3 \int b_B db_B b_3 db_3 \phi_B(x_B, b_B)\phi^A(1-\eta)$$

$$\times \{[\sqrt{\eta}z(\phi^t - \phi^s) + ((1-\eta)(1-x_3) - x_B + z\eta)\phi^0]E_{1cd}(t_{1c})h_{1c}(x_B, z, x_3, b_B, b_3)$$

$$+ [z(\sqrt{\eta}(\phi^s + \phi^t) - \phi^0) - (x_3(1-\eta) - x_B)\phi^0]E_{1cd}(t_{1d})h_{1d}(x_B, z, x_3, b_B, b_3)\}, \tag{A3}$$

$$M_{T\sigma}^{LR} = -32\pi C_F r_0 m_B^4/\sqrt{2N_c} \int dx_B dz dx_3 \int b_B db_B b_3 db_3 \phi_B(x_B, b_B)\{[\sqrt{\eta}z(\phi^P - \phi^T)(\phi^s + \phi^t)$$
$$+ \sqrt{\eta}((1-x_3)(1-\eta)-x_B)(\phi^P + \phi^T)(\phi^s - \phi^t) + ((1-x_3)(1-\eta)-x_B)(\phi^P + \phi^T)\phi^0$$
$$+ \eta z(\phi^P - \phi^T)\phi^0]E_{1cd}(t_{1c})h_{1c}(x_B, z, x_3, b_B, b_3) + [-\sqrt{\eta}z(\phi^P + \phi^T)(\sqrt{\eta}\phi^0 + (\phi^t + \phi^s))$$
$$+ (x_B - x_3(1-\eta))(\phi^P - \phi^T)(\sqrt{\eta}(\phi^s - \phi^t) + \phi^0)]E_{1cd}(t_{1d})h_{1d}(x_B, z, x_3, b_B, b_3)\}, \tag{A4}$$

$$M_{T\sigma}^{SP} = 32\pi C_F m_B^4/\sqrt{2N_c} \int dx_B dz dx_3 \int b_B db_B b_3 db_3 \phi_B(x_B, b_B)\phi^A(\eta - 1)$$
$$\times \{[\sqrt{\eta}z(\phi^t + \phi^s) + ((\eta-1)(1-x_3) + x_B - z\eta)\phi^0]E_{1cd}(t_{1c})h_{1c}(x_B, z, x_3, b_B, b_3)$$
$$+ [z(\sqrt{\eta}(\phi^s - \phi^t) - \eta\phi^0) - (x_3(1-\eta) - x_B)\phi^0]E_{1cd}(t_{1d})h_{1d}(x_B, z, x_3, b_B, b_3)\}, \tag{A5}$$

$$F_{A\sigma}^{LL} = 8\pi C_F m_B^4 f_B \int dz dx_3 \int b db b_3 db_3 \{[2r_0\sqrt{\eta}\phi^P((2-z)\phi^s + z\phi^t) - (1-\eta)(1-z)\phi^A\phi^0]E_{1ef}(t_{1e})$$
$$\times h_{1e}(z, x_3, b, b_3)S_t(z) + [2r_0\sqrt{\eta}[(1-x_3)(1-\eta)\phi^T - (1+x_3 + (1-x_3)\eta)\phi^P]\phi^s$$
$$+ (x_3(1-\eta) + \eta)(1-\eta)\phi^A\phi^0]E_{1ef}(t_{1f})h_{1f}(z, x_3, b, b_3)S_t(x_3)\}, \tag{A6}$$

$$F_{A\sigma}^{SP} = 16\pi C_F m_B^4 f_B \int dz dx_3 \int b db b_3 db_3 \{[\sqrt{\eta}(1-\eta)(1-z)\phi^A(\phi^s + \phi^t) - 2r_0(1 + (1-z)\eta)\phi^P\phi^0]$$
$$\times E_{1ef}(t_{1e})h_{1e}(z, x_3, b, b_3)S_t(z) + [2\sqrt{\eta}(1-\eta)\phi^A\phi^s - r_0(2\eta\phi^P + x_3(1-\eta)(\phi^P - \phi^T))\phi^0]$$
$$\times E_{1ef}(t_{1f})h_{1f}(z, x_3, b, b_3)S_t(x_3)\}, \tag{A7}$$

$$M_{A\sigma}^{LL} = 32\pi C_F m_B^4/\sqrt{2N_c} \int dx_B dz dx_3 \int b_B db_B b_3 db_3 \phi_B(x_B, b_B)\{[(\eta-1)[x_3(1-\eta) + x_B + \eta(1-z)]\phi^A\phi^0$$
$$+ r_0\sqrt{\eta}(x_3(1-\eta) + x_B + \eta)(\phi^P + \phi^T)(\phi^s - \phi^t) + r_0\sqrt{\eta}(1-z)(\phi^P - \phi^T)(\phi^s + \phi^t)$$
$$+ 2r_0\sqrt{\eta}(\phi^P\phi^s + \phi^T\phi^t)]E_{1gh}(t_{1g})h_{1g}(x_B, z, x_3, b_B, b_3) + [(1-\eta^2)(1-z)\phi^A\phi^0$$
$$+ r_0\sqrt{\eta}(x_B - x_3(1-\eta) - \eta)(\phi^P - \phi^T)(\phi^s + \phi^t) - r_0\sqrt{\eta}(1-z)(\phi^P + \phi^T)(\phi^s - \phi^t)]$$
$$\times E_{1gh}(t_{1h})h_{1h}(x_B, z, x_3, b_B, b_3)\}, \tag{A8}$$

$$M_{A\sigma}^{LR} = -32\pi C_F m_B^4/\sqrt{2N_c} \int dx_B dz dx_3 \int b_B db_B b_3 db_3 \phi_B(x_B, b_B)\{[\sqrt{\eta}(1-\eta)(1+z)\phi^A(\phi^s - \phi^t)$$
$$+ r_0(2 - x_B - x_3(1-\eta))(\phi^P + \phi^T)\phi^0 + r_0\eta(z\phi^P - (2+z)\phi^T)\phi^0]E_{1gh}(t_{1g})h_{1g}(x_B, z, x_3, b_B, b_3)$$
$$+ [\sqrt{\eta}(1-\eta)(1-z)\phi^A(\phi^s - \phi^t) + r_0((x_3(1-\eta) - x_B)(\phi^P + \phi^T) + \eta((2-z)\phi^P + z\phi^T))\phi^0]$$
$$\times E_{1gh}(t_{1h})h_{1h}(x_B, z, x_3, b_B, b_3)\}, \tag{A9}$$

$$M_{A\sigma}^{SP} = 32\pi C_F m_B^4/\sqrt{2N_c} \int dx_B dz dx_3 \int b_B db_B b_3 db_3 \phi_B(x_B, b_B)\{[\phi^0\phi^A(\eta-1)(z(1+\eta) - 1)$$
$$+ r_0\sqrt{\eta}(\phi^T - \phi^P)(\phi^s + \phi^t)((\eta-1)(1-x_3) + x_B) + r_0\sqrt{\eta}(\phi^T + \phi^P)(\phi^s - \phi^t)z - 4r_0\sqrt{\eta}\phi^P\phi^s]$$
$$\times E_{1gh}(t_{1g})h_{1g}(x_B, z, x_3, b_B, b_3) + [\phi^0\phi^A(\eta-1)(x3(1-\eta) + \eta(2-z) - x_B)$$
$$+ (\phi^T + \phi^P)(\phi^s - \phi^t)r_0\sqrt{\eta}(x_3(1-\eta) + \eta - x_B) + (\phi^P - \phi^T)(\phi^s + \phi^t)r_0\sqrt{\eta}(1-z)]$$
$$\times E_{1gh}(t_{1h})h_{1h}(x_B, z, x_3, b_B, b_3)\}, \tag{A10}$$

$$F_{TP}^{SP} = 16\pi C_F F_s(\omega^2)\sqrt{\eta}m_B^4 \int dx_B dx_3 \int b_B db_B b_3 db_3 \phi_B(x_B, b_B)\{[(\eta-1)\phi^A + r_0 x_3(\eta-1)(\phi^P - \phi^T)$$
$$- 2r_0\phi^P]E_{2ab}(t_{2a})h_{2a}(x_B, x_3, b_B, b_3)S_t(x_3) + [x_B(\eta-1)\phi^A + 2r_0(\eta + x_B - 1)\phi^P]$$
$$\times E_{2ab}(t_{2b})h_{2b}(x_B, x_3, b_B, b_3)S_t(x_B)\}, \tag{A11}$$

$$M_{TP}^{LL} = 32\pi C_F m_B^4/\sqrt{2N_c} \int dx_B dz dx_3 \int b_B db_B b db \phi_B(x_B, b_B)\phi^0\{[(1-x_B-z)(1-\eta^2)\phi^A - r_0 x_3(1-\eta)(\phi^P - \phi^T)$$
$$+ r_0(x_B+z)\eta(\phi^P + \phi^T) - 2r_0\eta\phi^P]E_{2cd}(t_{2c})h_{2c}(x_B, z, x_3, b_B, b) - [(z-x_B+x_3(1-\eta))(1-\eta)\phi^A$$
$$+ r_0(x_B-z)\eta(\phi^P - \phi^T) - r_0 x_3(1-\eta)(\phi^P + \phi^T)]E_{2cd}(t_{2d})h_{2d}(x_B, z, x_3, b_B, b)\}, \tag{A12}$$

$$M_{TP}^{LR} = 32\pi C_F m_B^4 \sqrt{\eta}/\sqrt{2N_c} \int dx_B dz dx_3 \int b_B db_B b db \phi_B(x_B, b_B)\{[(1-x_B-z)(1-\eta)(\phi^s + \phi^t)\phi^A$$
$$+ r_0(1-x_B-z)(\phi^s + \phi^t)(\phi^P - \phi^T) + r_0(x_3(1-\eta)+\eta)(\phi^s - \phi^t)(\phi^P + \phi^T)]$$
$$\times E_{2cd}(t_{2c})h_{2c}(x_B, z, x_3, b_B, b) - [(z-x_B)(1-\eta)(\phi^s - \phi^t)\phi^A + r_0(z-x_B)(\phi^s - \phi^t)(\phi^P - \phi^T)$$
$$+ r_0 x_3(1-\eta)(\phi^s + \phi^t)(\phi^P + \phi^T)]E_{2cd}(t_{2d})h_{2d}(x_B, z, x_3, b_B, b)\}, \tag{A13}$$

$$M_{TP}^{SP} = 32\pi C_F m_B^4/\sqrt{2N_c} \int dx_B dz dx_3 \int b_B db_B b db \phi_B(x_B, b_B)\phi^0\{[(1+\eta-x_B-z+x_3(1-\eta))(1-\eta)\phi^A$$
$$+ r_0\eta(x_B+z)(\phi^P - \phi^T) - r_0 x_3(1-\eta)(\phi^P + \phi^T) - 2r_0\eta\phi^P]E_{2cd}(t_{2c})h_{2c}(x_B, z, x_3, b_B, b)$$
$$- [(z-x_B)(1-\eta^2)\phi^A - r_0 x_3(1-\eta)(\phi^P - \phi^T) + r_0\eta(x_B-z)(\phi^P + \phi^T)]E_{2cd}(t_{2d})h_{2d}(x_B, z, x_3, b_B, b)\}, \tag{A14}$$

$$F_{AP}^{LL} = 8\pi C_F m_B^4 f_B \int dz dx_3 \int b db b_3 db_3\{[(x_3(1-\eta)-1)(1-\eta)\phi^A\phi^0 + 2r_0\sqrt{\eta}(x_3(1-\eta)(\phi^P - \phi^T) - 2\phi^P)\phi^s]$$
$$\times E_{2ef}(t_{2e})h_{2e}(z, x_3, b, b_3)S_t(x_3) + [z(1-\eta)\phi^A\phi^0 + 2r_0\sqrt{\eta}\phi^P((1-\eta)(\phi^s - \phi^t) + z(\phi^s + \phi^t))]$$
$$\times E_{2ef}(t_{2f})h_{2f}(z, x_3, b, b_3)S_t(z)\}, \tag{A15}$$

$$F_{AP}^{SP} = 16\pi C_F m_B^4 f_B \int dz dx_3 \int b db b_3 db_3\{[2\sqrt{\eta}(1-\eta)\phi^A\phi^s + r_0(1-x_3)(\phi^P + \phi^T)\phi^0 + r_0\eta((1+x_3)\phi^P$$
$$- (1-x_3)\phi^T)\phi^0]E_{2ef}(t_{2e})h_{2e}(z, x_3, b, b_3)S_t(x_3) + [2r_0(1-\eta(1-z))\phi^P\phi^0 + z\sqrt{\eta}((1-\eta)\phi^A(\phi^s - \phi^t)]$$
$$\times E_{2ef}(t_{2f})h_{2f}(z, x_3, b, b_3)S_t(z)\}, \tag{A16}$$

$$M_{AP}^{LL} = 32\pi C_F m_B^4/\sqrt{2N_c} \int dx_B dz dx_3 \int b_B db_B b_3 db_3 \phi_B(x_B, b_B)\{[(\eta-1)(-\eta+(1+\eta)(x_B+z))\phi^A\phi^0$$
$$+ r_0\sqrt{\eta}(x_3(1-\eta)+\eta)(\phi^P + \phi^T)(\phi^s - \phi^t) + r_0\sqrt{\eta}(1-x_B-z)(\phi^P - \phi^T)(\phi^s + \phi^t) - 4r_0\sqrt{\eta}\phi^P\phi^s]$$
$$\times E_{2gh}(t_{2g})h_{2g}(x_B, z, x_3, b_B, b_3) + [(1-\eta)((1-x_3)(1-\eta) - \eta(x_B-z))\phi^A\phi^0 - r_0\sqrt{\eta}(x_B-z)$$
$$\times (\phi^P + \phi^T)(\phi^s - \phi^t) + r_0\sqrt{\eta}(1-\eta)(1-x_3)(\phi^P - \phi^T)(\phi^s + \phi^t)]E_{2gh}(t_{2h})h_{2h}(x_B, z, x_3, b_B, b_3)\}, \tag{A17}$$

$$M_{AP}^{LR} = 32\pi C_F m_B^4/\sqrt{2N_c} \int dx_B dz dx_3 \int b_B db_B b_3 db_3 \phi_B(x_B, b_B)\{[\sqrt{\eta}(1-\eta)(2-x_B-z)\phi^A(\phi^s + \phi^t)$$
$$- r_0(1+x_3)(\phi^P - \phi^T)\phi^0 - r_0\eta[(1-x_B-z)(\phi^P + \phi^T) - x_3(\phi^P - \phi^T) + 2\phi^P]\phi^0]E_{2gh}(t_{2g})h_{2g}(x_B, z, x_3, b_B, b_3)$$
$$- [r_0(1-\eta)(x_3-1)(\phi^P - \phi^T)\phi^0 + \sqrt{\eta}(x_B-z)[r_0\sqrt{\eta}(\phi^P + \phi^T)\phi^0 + (1-\eta)\phi^A(\phi^s + \phi^t)]]$$
$$\times E_{2gh}(t_{2h})h_{2h}(x_B, z, x_3, b_B, b_3)\}, \tag{A18}$$

$$M_{AP}^{SP} = 32\pi C_F m_B^4/\sqrt{2N_c} \int dx_B dz dx_3 \int b_B db_B b_3 db_3 \phi_B(x_B, b_B)\{[(\eta-1)\phi^0\phi^A(\eta(x_3+x_B+z-2) - x_3+1)$$
$$+ r_0\sqrt{\eta}[(1-\eta)(1-x_3)(\phi^P - \phi^T)(\phi^s + \phi^t) + (z+x_B)(\phi^P + \phi^T)(\phi^s - \phi^t) + 2(\phi^P\phi^s + \phi^T\phi^t)]]$$
$$\times E_{2gh}(t_{2g})h_{2g}(x_B, z, x_3, b_B, b_3) - [r_0\sqrt{\eta}(\phi^P + \phi^T)(\phi^s - \phi^t)(1-\eta)(x_3-1) + r_0\sqrt{\eta}(\phi^P - \phi^T)(\phi^s + \phi^t)(x_B-z)$$
$$+ \phi^0\phi^A(1-\eta^2)(x_B-z)]E_{2gh}(t_{2h})h_{2h}(x_B, z, x_3, b_B, b_3)\}. \tag{A19}$$

where the hard functions are defined as

$$h_i(x_1, x_2, (x_3,)b_1, b_2) = h_1(\beta, b_2) \times h_2(\alpha, b1, b_2)$$

$$h_1(\beta, b_2) = \begin{cases} K_0(\sqrt{\beta}b_2), & \beta \geq 0, \\ \frac{i\pi}{2} H_0^{(1)}(\sqrt{-\beta}b_2), & \beta < 0, \end{cases}$$

$$h_2(\alpha, b1, b_2) = \begin{cases} \theta(b_2 - b_1)K_0(\sqrt{\alpha}b_2)I_0(\sqrt{\alpha}b_1), & \alpha \geq 0, \\ \theta(b_2 - b_1)\frac{i\pi}{2} H_0^{(1)}(\sqrt{-\alpha}b_2)J_0(\sqrt{-\alpha}b_1), & \alpha < 0, \end{cases} \tag{A20}$$

where E_{1mn}, E_{2mn}, E_{3mn}, E_{4mn} ($m = a, c, e, g$ and $n = b, d, f, h$) are the evolution factors,

$$
\begin{aligned}
E_{1ab}(t) &= \alpha(t) \exp[-S_B(t) - S_\sigma(t)], \\
E_{1cd}(t) &= \alpha(t) \exp[-S_B(t) - S_\sigma(t) - S_P(t)]_{b=b_B}, \\
E_{1ef}(t) &= \alpha(t) \exp[-S_P(t) - S_\sigma(t)], \\
E_{1gh}(t) &= \alpha(t) \exp[-S_B(t) - S_\sigma(t) - S_P(t)]_{b=b_3}, \\
E_{2ab}(t) &= \alpha(t) \exp[-S_B(t) - S_P(t)], \\
E_{2cd}(t) &= \alpha(t) \exp[-S_B(t) - S_\sigma(t) - S_P(t)]_{b_3=b_B}, \\
E_{2ef}(t) &= E_{1ef}(t), \\
E_{2gh}(t) &= E_{1gh}(t), \\
E_{3ab}(t) &= \alpha(t) \exp[-S_B(t) - S_\sigma(t)], \\
E_{3cd}(t) &= \alpha(t) \exp[-S_B(t) - S_\sigma(t) - S_D(t)]_{b=b_B}, \\
E_{3ef}(t) &= \alpha(t) \exp[-S_D(t) - S_\sigma(t)], \\
E_{3gh}(t) &= \alpha(t) \exp[-S_B(t) - S_\sigma(t) - S_D(t)]_{b=b_3}, \\
E_{4ef}(t) &= E_{3ef}(t), \\
E_{4gh}(t) &= E_{3gh}(t).
\end{aligned} \tag{A21}
$$

The Sudakov form factors are defined as

$$
\begin{aligned}
S_B(t) &= s\left(\frac{x_B m_B}{\sqrt{2}}, b_B\right) + \frac{5}{3}\int_{1/b_B}^t \frac{d\bar{\mu}}{\bar{\mu}} \gamma_q(\alpha_s(\bar{\mu})), \\
S_\sigma(t) &= s\left(\frac{z(1-r^2)m_B}{\sqrt{2}}, b\right) + s\left(\frac{(1-z)(1-r^2)m_B}{\sqrt{2}}, b\right) + 2\int_{1/b}^t \frac{d\bar{\mu}}{\bar{\mu}} \gamma_q(\alpha_s(\bar{\mu})), \\
S_{D,P}(t) &= s\left(\frac{x_3 m_B}{\sqrt{2}}, b_3\right) + s\left(\frac{(1-x_3)m_B}{\sqrt{2}}, b_3\right) + 2\int_{1/b_3}^t \frac{d\bar{\mu}}{\bar{\mu}} \gamma_q(\alpha_s(\bar{\mu})).
\end{aligned} \tag{A22}
$$

Practically, the mentioned hard scales are chosen as

$$
\begin{aligned}
t_{1a} &= \text{Max}\{\sqrt{|\alpha_{1a}|}, \sqrt{|\beta_{1a}|}, 1/b_B, 1/b\}, & t_{1b} &= \text{Max}\{\sqrt{|\alpha_{1b}|}, \sqrt{|\beta_{1b}|}, 1/b_B, 1/b\}, \\
t_{1c} &= \text{Max}\{\sqrt{|\alpha_{1c}|}, \sqrt{|\beta_{1c}|}, 1/b_B, 1/b_3\}, & t_{1d} &= \text{Max}\{\sqrt{|\alpha_{1d}|}, \sqrt{|\beta_{1d}|}, 1/b_B, 1/b_3\}, \\
t_{1e} &= \text{Max}\{\sqrt{|\alpha_{1e}|}, \sqrt{|\beta_{1e}|}, 1/b_3, 1/b\}, & t_{1f} &= \text{Max}\{\sqrt{|\alpha_{1f}|}, \sqrt{|\beta_{1f}|}, 1/b_3, 1/b\}, \\
t_{1g} &= \text{Max}\{\sqrt{|\alpha_{1g}|}, \sqrt{|\beta_{1g}|}, 1/b_B, 1/b_3\}, & t_{1h} &= \text{Max}\{\sqrt{|\alpha_{1h}|}, \sqrt{|\beta_{1h}|}, 1/b_B, 1/b_3\}, \\
t_{2a} &= \text{Max}\{\sqrt{|\alpha_{2a}|}, \sqrt{|\beta_{2a}|}, 1/b_B, 1/b_3\}, & t_{2b} &= \text{Max}\{\sqrt{|\alpha_{2b}|}, \sqrt{|\beta_{2b}|}, 1/b_B, 1/b_3\}, \\
t_{2c} &= \text{Max}\{\sqrt{|\alpha_{2c}|}, \sqrt{|\beta_{2c}|}, 1/b_B, 1/b\}, & t_{2d} &= \text{Max}\{\sqrt{|\alpha_{2d}|}, \sqrt{|\beta_{2d}|}, 1/b_B, 1/b\},
\end{aligned}
$$

$$t_{2e} = \text{Max}\{\sqrt{|\alpha_{2e}|}, \sqrt{|\beta_{2e}|}, 1/b_3, 1/b\}, \qquad t_{2f} = \text{Max}\{\sqrt{|\alpha_{2f}|}, \sqrt{|\beta_{2f}|}, 1/b_3, 1/b\},$$

$$t_{2g} = \text{Max}\{\sqrt{|\alpha_{2g}|}, \sqrt{|\beta_{2g}|}, 1/b_B, 1/b_3\}, \qquad t_{2h} = \text{Max}\{\sqrt{|\alpha_{2h}|}, \sqrt{|\beta_{2h}|}, 1/b_B, 1/b_3\},$$

$$t_{3a} = \text{Max}\{\sqrt{|\alpha_{3a}|}, \sqrt{|\beta_{3a}|}, 1/b_B, 1/b\}, \qquad t_{3b} = \text{Max}\{\sqrt{|\alpha_{3b}|}, \sqrt{|\beta_{3b}|}, 1/b_B, 1/b\},$$

$$t_{3c} = \text{Max}\{\sqrt{|\alpha_{3c}|}, \sqrt{|\beta_{3c}|}, 1/b_B, 1/b_3\}, \qquad t'_{3c} = \text{Max}\{\sqrt{|\alpha_{3c}|}, \sqrt{|\beta'_{1c}|}, 1/b_B, 1/b_3\},$$

$$t_{3d} = \text{Max}\{\sqrt{|\alpha_{3d}|}, \sqrt{|\beta_{3d}|}, 1/b_B, 1/b_3\}, \qquad t'_{3d} = \text{Max}\{\sqrt{|\alpha_{3d}|}, \sqrt{|\beta'_{1d}|}, 1/b_B, 1/b_3\},$$

$$t_{4e} = \text{Max}\{\sqrt{|\alpha_{4e}|}, \sqrt{|\beta_{4e}|}, 1/b_3, 1/b\}, \qquad t_{4f} = \text{Max}\{\sqrt{|\alpha_{4f}|}, \sqrt{|\beta_{4f}|}, 1/b_3, 1/b\},$$

$$t_{4g} = \text{Max}\{\sqrt{|\alpha_{4g}|}, \sqrt{|\beta_{4g}|}, 1/b_B, 1/b_3\}, \qquad t_{4h} = \text{Max}\{\sqrt{|\alpha_{4h}|}, \sqrt{|\beta_{4h}|}, 1/b_B, 1/b_3\}. \tag{A23}$$

The jet function S_t resums the threshold double logarithm and can be parameterized as

$$\frac{2^{1+2c}\Gamma(3/2 + c)}{\sqrt{\pi}\Gamma(1 + c)}[x(1 - x)]^c, \tag{A24}$$

with $c = 0.4$ for numerical calculation. The parameters in Eq. (A23) take the form of:

$$\alpha_{1a} = zm_B^2$$
$$\beta_{1a} = x_B z m_B^2 = \beta_{1b} = \alpha_{1c} = \alpha_{1d}$$
$$\alpha_{1b} = (x_B - \eta)m_B^2$$
$$\beta_{1c} = -z[(1 - \eta)(1 - x_3) - x_B]m_B^2$$
$$\beta_{1d} = -z[(1 - \eta)x_3 - x_B]m_B^2$$
$$\alpha_{1e} = (z - 1)m_B^2$$
$$\beta_{1e} = -(1 - z)[\eta + (1 - \eta)x_3]m_B^2 = \beta_{1f} = \alpha_{1g} = \alpha_{1h}$$
$$\alpha_{1f} = -(\eta + (1 - \eta)x_3)m_B^2$$
$$\beta_{1g} = \{1 - z[(1 - \eta)(1 - x_3) - x_B]\}m_B^2$$
$$\beta_{1h} = -(1 - z)[(1 - \eta)x_3 + \eta - x_B]m_B^2$$
$$\alpha_{2a} = x_3(1 - \eta)m_B^2$$
$$\beta_{2a} = x_3 x_B(1 - \eta)m_B^2 = \beta_{2b} = \alpha_{2c} = \alpha_{2d}$$
$$\alpha_{2b} = x_B(1 - \eta)m_B^2$$
$$\beta_{2c} = (1 - x_B - z)[(\eta - 1)x_3 - \eta]m_B^2$$
$$\beta_{2d} = (1 - \eta)(x_B - z)m_B^2$$
$$\alpha_{2e} = -[1 - x_3(1 - \eta)]m_B^2$$
$$\beta_{2e} = -z(1 - \eta)(1 - x_3)m_B^2 = \beta_{2f} = \alpha_{2g} = \alpha_{2h}$$
$$\alpha_{2f} = z(\eta - 1)m_B^2$$
$$\beta_{2g} = \{1 - (1 - z - x_B)[\eta + (1 - \eta)x_3]\}m_B^2$$
$$\beta_{2h} = (x_B - z)(1 - \eta)(1 - x_3)m_B^2$$
$$\alpha_{3a} = z(1 - r^2)m_B^2$$
$$\beta_{3a} = x_B z(1 - r^2)m_B^2 = \beta_{3b} = \alpha_{3c} = \alpha_{3d} = \alpha'_{3c} = \alpha'_{3d}$$
$$\alpha_{3b} = (1 - r^2)(x_B - \eta)m_B^2$$

$$\beta_{3c} = -[z(1 - r^2) + r^2(1 - x_3)][(1 - \eta)(1 - x_3) - x_B]m_B^2$$

$$\beta_{3d} = \{r_c^2 - [z(1 - r^2) + x_3 r^2][(1 - \eta)x_3 - x_B]\}m_B^2$$

$$\beta'_{3c} = \{r_c^2 - [z(1 - r^2) + r^2(1 - x_3)][(1 - \eta)(1 - x_3) - x_B]\}m_B^2$$

$$\beta'_{3d} = -[z(1 - r^2) + x_3 r^2][(1 - \eta)x_3 - x_B]m_B^2$$

$$\alpha_{3e} = -[1 - z(1 - r^2) - r_c^2]m_B^2$$

$$\beta_{3e} = -[(1 - r^2)(1 - z) + x_3 r^2][\eta + (1 - \eta)x_3]m_B^2 = \beta_{3f} = \alpha_{3g} = \alpha_{3h}$$

$$\alpha_{3f} = -(1 - r^2 + x_3 r^2)(\eta + (1 - \eta)x_3)m_B^2$$

$$\beta_{3g} = \{1 - [z(1 - r^2) + r^2(1 - x_3)][(1 - \eta)(1 - x_3) - x_B]\}m_B^2$$

$$\beta_{3h} = -[(1 - z)(1 - r^2) + r^2 x_3][(1 - \eta)x_3 + \eta - x_B]m_B^2$$

$$\alpha_{4e} = -(1 - x_3 r^2)[1 - x_3(1 - \eta)]m_B^2$$

$$\beta_{4e} = -[(1 - x_3)r^2 + z(1 - r^2)](1 - \eta)(1 - x_3)m_B^2 = \beta_{4f} = \alpha_{4g} = \alpha_{4h}$$

$$\alpha_{4f} = \{r_c^2 - [r^2 + z(1 - r^2)](1 - \eta)\}m_B^2$$

$$\beta_{4g} = \{1 - [(1 - r^2)(1 - z) + x_3 r^2 - x_B][\eta + (1 - \eta)x_3]\}m_B^2$$

$$\beta_{4h} = -[r^2(1 - x_3) + z(1 - r^2) - x_B](1 - \eta)(1 - x_3)m_B^2 \qquad (A25)$$

[1] J. Charles, S. Descotes-Genon, J. Ocariz, and A. Pérez Pérez, Eur. Phys. J. C **77**, 561 (2017).

[2] I. Bediaga and P. C. Magalhães, arXiv:1512.09284.

[3] I. Bediaga, T. Frederico, and P. C. Magalhães, Phys. Lett. B **780**, 357 (2018).

[4] Y. Yu, H.-B. Fu, H. Zhang, and B.-C. Ke, arXiv:2405.16054.

[5] J. H. Alvarenga Nogueira et al., arXiv:1605.03889.

[6] D. Boito, J. P. Dedonder, B. El-Bennich, R. Escribano, R. Kaminski, L. Lesniak, and B. Loiseau, Phys. Rev. D **96**, 113003 (2017).

[7] M. Gronau and J. L. Rosner, Phys. Lett. B **564**, 90 (2003).

[8] G. Engelhard, Y. Nir, and G. Raz, Phys. Rev. D **72**, 075013 (2005).

[9] M. Gronau and J. L. Rosner, Phys. Rev. D **72**, 094031 (2005).

[10] M. Imbeault and D. London, Phys. Rev. D **84**, 056002 (2011).

[11] M. Gronau, Phys. Lett. B **727**, 136 (2013).

[12] B. Bhattacharya, M. Gronau, and J. L. Rosner, Phys. Lett. B **726**, 337 (2013).

[13] B. Bhattacharya, M. Gronau, M. Imbeault, D. London, and J. L. Rosner, Phys. Rev. D **89**, 074043 (2014).

[14] D. Xu, G.-N. Li, and X.-G. He, Phys. Lett. B **728**, 579 (2014).

[15] D. Xu, G.-N. Li, and X.-G. He, Int. J. Mod. Phys. A **29**, 1450011 (2014).

[16] X.-G. He, G.-N. Li, and D. Xu, Phys. Rev. D **91**, 014029 (2015).

[17] A. Furman, R. Kaminski, L. Lesniak, and B. Loiseau, Phys. Lett. B **622**, 207 (2005).

[18] B. El-Bennich, A. Furman, R. Kaminski, L. Lesniak, and B. Loiseau, Phys. Rev. D **74**, 114009 (2006).

[19] B. El-Bennich, A. Furman, R. Kaminski, L. Lesniak, B. Loiseau, and B. Moussallam, Phys. Rev. D **79**, 094005 (2009); **83**, 039903(E) (2011).

[20] O. Leitner, J. P. Dedonder, B. Loiseau, and R. Kaminski, Phys. Rev. D **81**, 094033 (2010); **82**, 119906(E) (2010).

[21] J. P. Dedonder, A. Furman, R. Kaminski, L. Lesniak, and B. Loiseau, Acta Phys. Pol. B **42**, 2013 (2011).

[22] H.-Y. Cheng, C.-K. Chua, and A. Soni, Phys. Rev. D **72**, 094003 (2005).

[23] H.-Y. Cheng, C.-K. Chua, and A. Soni, Phys. Rev. D **76**, 094006 (2007).

[24] H.-Y. Cheng and C.-K. Chua, Phys. Rev. D **88**, 114014 (2013).

[25] H.-Y. Cheng and C.-K. Chua, Phys. Rev. D **89**, 074025 (2014).

[26] Y. Li, Phys. Rev. D **89**, 094007 (2014).

[27] H.-Y. Cheng, C.-K. Chua, and Z.-Q. Zhang, Phys. Rev. D **94**, 094015 (2016).

[28] S. Kränkl, T. Mannel, and J. Virto, Nucl. Phys. **B899**, 247 (2015).

[29] R. Klein, T. Mannel, J. Virto, and K. K. Vos, J. High Energy Phys. 10 (2017) 117.

[30] Y.-Y. Keum, H.-N. Li, and A. I. Sanda, Phys. Lett. B **504**, 6 (2001).

[31] Y. Y. Keum, H.-N. Li, and A. I. Sanda, Phys. Rev. D **63,** 054008 (2001).

[32] C.-D. Lü, K. Ukai, and M.-Z. Yang, Phys. Rev. D **63,** 074009 (2001).

[33] W.-F. Wang and H.-N. Li, Phys. Lett. B **763,** 29 (2016).

[34] A.-J. Ma, Y. Li, W.-F. Wang, and Z.-J. Xiao, Chin. Phys. C **41,** 083105 (2017).

[35] A.-J. Ma, Y. Li, W.-F. Wang, and Z.-J. Xiao, Phys. Rev. D **96,** 093011 (2017).

[36] Z. Rui and W.-F. Wang, Phys. Rev. D **97,** 033006 (2018).

[37] Y. Li, A.-J. Ma, Z. Rui, W.-F. Wang, and Z.-J. Xiao, Phys. Rev. D **98,** 056019 (2018).

[38] W.-F. Wang and J. Chai, Phys. Lett. B **791,** 342 (2019).

[39] A.-J. Ma, W.-F. Wang, Y. Li, and Z.-J. Xiao, Eur. Phys. J. C **79,** 539 (2019).

[40] B.-Y. Cui, Y.-Y. Fan, F.-H. Liu, and W.-F. Wang, Phys. Rev. D **100,** 014017 (2019).

[41] W.-F. Wang, J. Chai, and A.-J. Ma, J. High Energy Phys. 03 (2020) 162.

[42] Y.-Y. Fan and W.-F. Wang, Eur. Phys. J. C **80,** 815 (2020).

[43] J. Chai, S. Cheng, and A.-J. Ma, Phys. Rev. D **105,** 033003 (2022).

[44] R. L. Workman *et al.* (Particle Data Group), Prog. Theor. Exp. Phys. **2022,** 083C01 (2022).

[45] J. R. Pelaez, Phys. Rep. **658,** 1 (2016).

[46] M. H. Johnson and E. Teller, Phys. Rev. **98,** 783 (1955).

[47] M. Gell-Mann and M. Levy, Nuovo Cimento **16,** 705 (1960).

[48] S. Stone and L. Zhang, Phys. Rev. Lett. **111,** 062001 (2013).

[49] R. L. Jaffe, Phys. Rev. D **15,** 267 (1977).

[50] L. Maiani, F. Piccinini, A. D. Polosa, and V. Riquer, Phys. Rev. Lett. **93,** 212002 (2004).

[51] S. Weinberg, Phys. Rev. Lett. **110,** 261601 (2013).

[52] G. 't Hooft, G. Isidori, L. Maiani, A. D. Polosa, and V. Riquer, Phys. Lett. B **662,** 424 (2008).

[53] R. Fleischer, R. Knegjens, and G. Ricciardi, Eur. Phys. J. C **71,** 1832 (2011).

[54] W. Wang and C.-D. Lu, Phys. Rev. D **82,** 034016 (2010).

[55] W.-F. Wang, H.-N. Li, W. Wang, and C.-D. Lü, Phys. Rev. D **91,** 094024 (2015).

[56] X. Liu, Z.-T. Zou, Y. Li, and Z.-J. Xiao, Phys. Rev. D **100,** 013006 (2019).

[57] Y. Xing and Z.-P. Xing, Chin. Phys. C **43,** 073103 (2019).

[58] M. Diehl, T. Gousset, B. Pire, and O. Teryaev, Phys. Rev. Lett. **81,** 1782 (1998).

[59] D. Müller, D. Robaschik, B. Geyer, F. M. Dittes, and J. Hořejši, Fortschr. Phys. **42,** 101 (1994).

[60] M. V. Polyakov, Nucl. Phys. **B555,** 231 (1999).

[61] P. Hägler, B. Pire, L. Szymanowski, and O. V. Teryaev, Phys. Lett. B **535,** 117 (2002); **540,** 324(E) (2002).

[62] C.-H. Chen and H.-N. Li, Phys. Lett. B **561,** 258 (2003).

[63] C.-H. Chen and H.-N. Li, Phys. Rev. D **70,** 054006 (2004).

[64] H.-N. Li, Prog. Part. Nucl. Phys. **51,** 85 (2003).

[65] K. C. Bowler, L. Del Debbio, J. M. Flynn, L. Lellouch, V. Lesk, C. M. Maynard, J. Nieves, and D. G. Richards (UKQCD Collaborations), Phys. Lett. B **486,** 111 (2000).

[66] P. Ball, J. High Energy Phys. 09 (1998) 005.

[67] P. Ball, J. High Energy Phys. 01 (1999) 010.

[68] P. Ball and R. Zwicky, Phys. Rev. D **71,** 014015 (2005).

[69] P. Ball, V. M. Braun, and A. Lenz, J. High Energy Phys. 05 (2006) 004.

[70] T. Feldmann, P. Kroll, and B. Stech, Phys. Rev. D **58,** 114006 (1998).

[71] T. Feldmann, P. Kroll, and B. Stech, Phys. Lett. B **449,** 339 (1999).

[72] Y.-Y. Charng, T. Kurimoto, and H.-N. Li, Phys. Rev. D **74,** 074024 (2006); **78,** 059901(E) (2008).

[73] U.-G. Meißner and W. Wang, Phys. Lett. B **730,** 336 (2014).

[74] W.-F. Wang, Phys. Lett. B **788,** 468 (2019).

[75] Y. Aoki *et al.* (Flavour Lattice Averaging Group (FLAG) Collaborations), Eur. Phys. J. C **82,** 869 (2022).

[76] H.-Y. Cheng, C.-W. Chiang, and C.-K. Chua, Phys. Rev. D **103,** 036017 (2021).

[77] B. Aubert *et al.* (*BABAR* Collaborations), Phys. Rev. D **72,** 052002 (2005).

[78] R. Aaij *et al.* (LHCb Collaborations), Phys. Rev. D **101,** 012006 (2020).

[79] R. Aaij *et al.* (LHCb Collaborations), Phys. Rev. Lett. **124,** 031801 (2020).

[80] D. V. Bugg, J. Phys. G **34,** 151 (2007).

[81] R. Aaij *et al.* (LHCb Collaborations), Phys. Lett. B **742,** 38 (2015).

[82] R. Aaij *et al.* (LHCb Collaborations), Phys. Rev. D **92,** 032002 (2015).