Multiple Mellin-Barnes integrals and triangulations of point configurations

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(Received 8 March 2024; accepted 23 May 2024; published 2 August 2024)

Mellin-Barnes (MB) integrals are a well-known type of integrals appearing in diverse areas of mathematics and physics, such as in the theory of hypergeometric functions, asymptotics, quantum field theory, solid-state physics, etc. Although MB integrals have been studied for more than a century, it is only recently that, due to a remarkable connection found with conic hulls, N-fold MB integrals can be computed analytically for $N > 2$ in a systematic way. In this article, we present an alternative novel technique by unveiling a new connection between triangulations of point configurations and MB integrals, to compute the latter. To make it ready to use, we have implemented our new method in the *Mathematica* package MBConicHulls.wl, an already existing software dedicated to the analytic evaluation of MB integrals using conic hulls. The triangulation method is remarkably faster than the conic hull approach and can thus be used for the calculation of higher-fold MB integrals, as we show here by testing our code on the case of the off-shell massless scalar one-loop N-point Feynman integral up to $N = 15$, for which the MB representation has 104 folds. Among other examples of applications, we present new simpler solutions for the off-shell one-loop massless conformal hexagon and two-loop double-box Feynman integrals, as well as for some complicated 8-fold MB integrals contributing to the hard diagram of the two-loop hexagon Wilson loop in general kinematics.

DOI: 10.1103/PhysRevD.110.036002

I. INTRODUCTION

Mellin-Barnes (MB) integrals are a special class of integrals whose integrand, in its general form, consists of a ratio of products of Euler gamma functions and parameters raised to the power of integration variables. A typical N-fold MB integral reads

$$I(x_1,\ldots,x_N) = \int_{-i\infty}^{+i\infty}\frac{dz_1}{2\pi i}\cdots\int_{-i\infty}^{+i\infty}\frac{dz_N}{2\pi i}\frac{\prod_{i=1}^{k}\Gamma^{a_i}(s_i(\mathbf{z}))}{\prod_{j=1}^{l}\Gamma^{b_j}(t_j(\mathbf{z}))}x_1^{z_1}\cdots x_N^{z_N} \tag{1}$$

where $\mathbf{z}=(z_1,\ldots,z_N)$, a_i and b_j are positive integers, $k \geq N$[1] and the variables $x_1,\ldots,x_N$ can be complex-valued. The arguments of the gamma functions in the MB integrand are of the form

$$s_i(\mathbf{z}) = \sum_{k=1}^{N} e_{ik}z_k + f_i, \qquad t_j(\mathbf{z}) = \sum_{k=1}^{N} g_{jk}z_k + h_j \tag{2}$$

where f_i and h_j are real or complex numbers, and the coefficients e_{ik} and g_{jk} are usually integers especially in Feynman integral calculus. In general, the integration contours in Eq. (1) satisfy the following property: they do not split the set of poles of each gamma function of the numerator into different subsets. This is easy to visualize in the 1-fold case where such a rule dictates that the contour separates the left-handed poles of $\Gamma(\cdots+z)$ from the right-handed poles of $\Gamma(\cdots-z)$.[2]

MB integrals are of paramount importance due to their wide-range of applications in different branches of physics and mathematics. These include areas as diverse as quantum field theory [2,3], electromagnetic wave propagation in

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[1]Cancellations between numerator and denominator gamma functions are tacitly excluded.

[2]In the cases where one would have MB integrals with straight contours which do not separate the left and right-handed poles, one can perform appropriate transformations on the integration variables to separate them, as shown in [1].

turbulence [4], detector physics [5], condensed matter [6,7], but also in less expected fields such as option pricing [8]. In mathematics, apart from their important role in the theory of multivariable hypergeometric functions [9–11], MB integrals also appear in asymptotics, where they have a different form. In particle physics, MB integrals frequently appear in the evaluation of multi-loop Feynman integrals [2,3]. In this context, the original Feynman integral is first converted into an MB integral using standard procedures [2,3,12,13], and subsequently the MB integral can be used in many ways, such as resolving ϵ singularities of dimensionally regularized Feynman integrals [14–17], finding analytic expressions in terms of hypergeometric functions [18] and special functions [2,19], performing numerical integration [20], counting master integrals (in some cases) [21], deriving partial differential equations without relying on integration-by-part identities [22], etc. An important breakthrough of MB integrals in the context of Feynman integrals calculation concerns the derivation of the first analytic results for the two-loop box integrals in the planar [14] and nonplanar [15] cases. As another notable application, one can mention the evaluation of the planar master integrals appearing in Bhabha scattering [23]. More recently, it has also been used to compute the full two-loop electroweak corrections to the Z-boson production and decay [24] and, in a seminal work, to compute the Higgs boson gluon-fusion production cross section at three loops [25].

Although N-fold MB integrals are widely used, there was no efficient and systematic computational technique for their analytic calculation for the case $N > 2$, until recently in [1,26]. In the latter works, it was shown that a given N-fold MB integral (with fixed N) can be solved by associating a set of conic hulls with the MB integrand and subsequently studying their intersections. This approach was automated in the form of a *Mathematica* package called MBConicHulls.wl [26] which was used to obtain the first analytic solutions of the off-shell massless one-loop hexagon and two-loop double-box conformal Feynman integrals [27] that involved nine-fold MB representations. However, although very useful, the MBConicHulls.wl package becomes limited in speed when it comes to the computation of complicated objects such as those considered in [27].

In the present paper, we propose an alternative novel geometrical approach for the analytic evaluation of multi-fold MB integrals, based on triangulations of configurations of points, which, in addition to the potentially new insights in the theory of MB integrals that it can offer, is computationally much more efficient than the conic hull approach. We show this by developing an updated version of the MBConicHulls.wl [28] package where we have implemented the triangulation procedure by introducing a new module using the TOPCOM software [29] in the background. As an example of application, we are now able to find all possible series representations of the double-box and hexagon conformal Feynman integrals in a very short time, whereas it would have taken more than a lifetime using the former conic hull approach. This allowed us to discover that simpler series representations than those previously published in [27] can be obtained. As another example, we can now compute a complicated part of the hard diagram of the two-loop six-edged Wilson loop in general kinematics [30] and show that 1471926 different series representations can be obtained from its MB representation. Furthermore, the triangulation approach also makes possible the computation of much more complicated objects than the above-mentioned integrals, as we have checked by testing the code on higher-fold MB integrals. As an example, in this paper, we show that the computation of triangulations associated with the scalar off-shell massless one-loop N-point Feynman integral, for N going up to 15 (for which the corresponding MB representation has 104 folds), is possible.

II. THE TRIANGULATION METHOD

Before we delve into the triangulation method, we first perform a change of the integration variables in Eq. (1), to rewrite it as

$$I(x_1,\ldots,x_N) = \int_{-i\infty}^{+i\infty}\frac{dz_1}{2\pi i}\cdots\int_{-i\infty}^{+i\infty}\frac{dz_N}{2\pi i} \times \frac{\Gamma(-z_1)\cdots\Gamma(-z_N)\prod_{i=N+1}^{k'}\Gamma^{a'_i}(s'_i(\mathbf{z}))}{\prod_{j=1}^{l}\Gamma^{b'_j}(t'_j(\mathbf{z}))} \times x_1'^{z_1}\cdots x_1'^{z_N} \tag{3}$$

where we have pulled out the factors $\Gamma(-z_1)\cdots\Gamma(-z_N)$ in the numerator. This change of variables always exists for $k\geq N$ and we call Eq. (3) the *canonical form* of the MB representation. In the rest of this article, we assume that MB integrals are written in this form, where the gamma function arguments are now

$$s'_i(\mathbf{z}) = \sum_{k=1}^{N} e'_{ik}z_k + f'_i, \qquad t'_j(\mathbf{z}) = \sum_{k=1}^{N} g'_{jk}z_k + h'_j \tag{4}$$

For the purpose of analytic evaluation of the MB representation in Eq. (3) with our triangulation method, we assign to this integral a set of $N+\Sigma_{i=N+1}^{k'}a'_i$ points which can be readily extracted from the arguments of the gamma functions of the numerator of its integrand, i.e., $s'_i(\mathbf{z})$. This set consists of N points whose homogeneous coordinates in the $\sum_{i=N+1}^{k'}a'_i$ dimensional Euclidean space are built from the coefficients of the z_i $(i=1,\ldots,N)$ integration variables of the arguments of numerator's (nonpulled out) gamma functions, i.e.

$$P_1 = e'_{l1}, \qquad P_2 = e'_{l2}, \quad \cdots \quad P_N = e'_{lN} \tag{5}$$

and $\sum_{i=N+1}^{k'}a'_i$ additional points corresponding to the unit vectors of dimension $\sum_{i=N+1}^{k'}a'_i$.

$$P_{N+1} = \begin{pmatrix} 1 \\ 0 \\ . \\ 0 \\ 0 \end{pmatrix}, \quad P_{N+2} = \begin{pmatrix} 0 \\ 1 \\ . \\ 0 \\ 0 \end{pmatrix}, \quad \cdots \quad P_{N+\Sigma_{i=N+1}^{k'} a'_i} = \begin{pmatrix} 0 \\ 0 \\ . \\ 0 \\ 1 \end{pmatrix} \tag{6}$$

Therefore, the point configuration associated with Eq. (3) can be written as a $(\Sigma_{i=N+1}^{k'} a'_i) \times (N + \Sigma_{i=N+1}^{k'} a'_i)$ matrix where the columns are made of the points P_i $(i = 1, \ldots, N + \Sigma_{i=N+1}^{k'} a'_i)$ and which we denote as the A-matrix of the MB integral

$$A = (P_1 \quad P_2 \quad \cdots \quad P_{N+\Sigma_{i=N+1}^{k'} a'_i}) \tag{7}$$

In the next step we find all the possible *regular triangulations* of the point configuration $P = \{P_1, \ldots, P_{N+\Sigma_{i=N+1}^{k'} a'_i}\}$. These triangulations are built from a set of simplices, each simplex being, in fact, dual to a conic hull in the conic hull approach [26]. We then observe a remarkable bijective correspondence between the set of possible regular triangulations and the set of relevant intersections of conic hulls in the conic hull approach [26]. Therefore, this allows us to assign a set of poles to each triangulation and to sum their multivariate residues, as in [26], to obtain series solutions. Therefore, in the end, we have several series solutions of the original MB integral, each derived from a specific triangulation. These solutions (when the method is applied to the computation of Feynman integrals or to the derivation of linear transformations of multivariable hypergeometric functions) are, in general, analytic continuations of each other and converge in different regions of the $(x'_1, \ldots, x'_N)$ N-dimensional complex space.

We have implemented the triangulation method in a new version of the *Mathematica* package MBConicHulls.wl [28] which can be used for the analytic calculation of N-fold MB integrals with an arbitrary (but fixed) number of folds $N \geq 1$, as described in the Supplemental Material [31]. We also illustrate in [31] the procedure in detail, by computing the simple two-fold MB integral associated with the Appell F_1 double hypergeometric function.

III. APPLICATIONS TO FEYNMAN INTEGRALS

In this section, we compute higher-fold MB integrals associated with Feynman integrals. We begin with a comparison between the computation times of v.1.1 of MBConicHulls.wl [28], which is based on conic hull intersections, and those obtained from the triangulation approach implemented in v.1.2, for several examples that have up to nine-fold MB representations. Then we consider MB integrals with a very large number of folds, and test our package by computing triangulations and series representations of the off-shell massless scalar one-loop N-point integral, for several values of N going as high as 15 which, in the latter case, yields a 104-fold MB representation. To our knowledge, the corresponding results are new, even for the simplest $N = 4$ case. Some details of these results can be found in the ancillary *Mathematica* notebook `Examples.nb` [28].

A. Comparison of computation times

We perform the comparison of calculation times on five different Feynman integrals: the off-shell massive conformal triangle which has a three-fold MB representation [32], the off-shell massless pentagon in $4 - 2\epsilon$ dimensions, whose MB representation has four folds [1], the off-shell massless hexagon and double-box conformal fishnet Feynman integrals in the generic nonresonant D dimensional case and unit resonant four dimensional case [33] (for interesting recent results on two dimensional fishnet integrals see Ref. [34]) which both have nine-fold MB representations[3] [27], and the hard diagram of the two loop six-edged Wilson loop [30] (see Fig. 1 for the Feynman diagrams of these five examples). For explicit expressions of their MB representations, we refer the reader to the quoted references. We only give here, as an example, the A-matrix of the double-box, which reads

$$A_{\mathrm{DB}} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{8}$$

[3]These last two Feynman integrals are related to one another by a differential equation which allows one to check the obtained results.

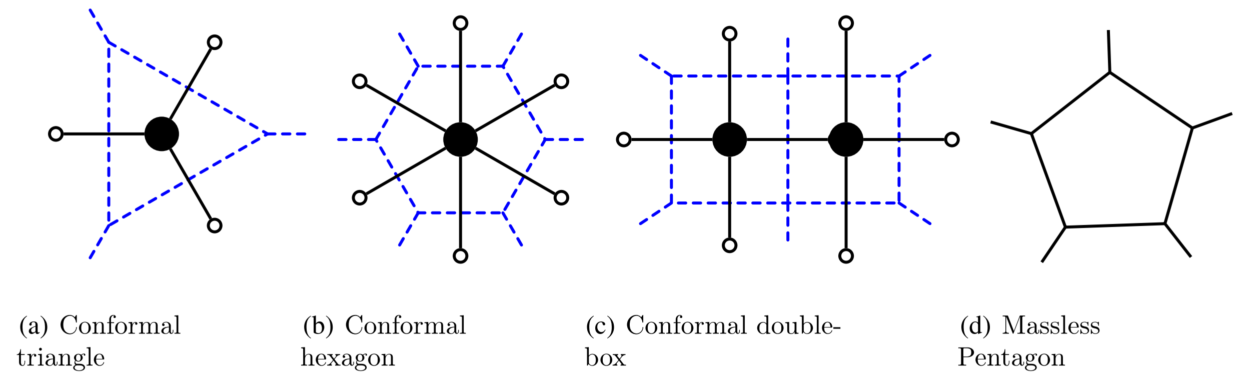

FIG. 1. One-loop and two-loop Feynman diagrams evaluated using the conic hulls and triangulation methods in Table I. (a) Conformal triangle, (b) Conformal hexagon, (c) Conformal doublebox, (d) Massless pentagon.

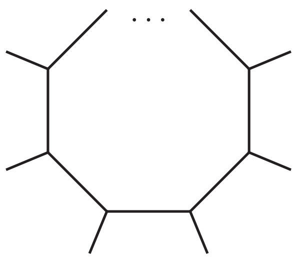

FIG. 2. One-loop N-point massless Feynman integral.

The results presented in Table I show the computation times[4] needed for the calculation of one (sometimes all) series representation(s) by the two methods. They clearly prove the huge improvement that the triangulation method provides in the analytic calculations of multifold MB integrals.

We note that the hexagon and double-box were solved using the conic hull method in [27] as sums of, respectively, 26 and 44 multivariable hypergeometric series, for generic values of the powers of their propagators satisfying the conformal constraint. However, due to computational limitations of the conic hull approach, only very few of all the possible series representations of these integrals could be derived. This is no longer the case due to the efficiency of the triangulation method. In fact, we can find the total numbers of their different series representations, which are 194160 for the hexagon and 243186 for the double box. It is then possible to find in these sets simpler series solutions than those of [27], as sums of 25 hypergeometric series for both the hexagon and double box. These series solutions are presented in the ancillary `Examples.nb` [28] notebook together with the resonant $D = 4$ results.

B. Higher-fold MB integrals: One-loop scalar massless N-point integral

We next consider the computation of MB integrals with a higher number of folds, and test our method on the class of one-loop scalar massless N-point Feynman integrals (see Fig. 2 for the corresponding Feynman diagram) with generic powers of the propagators, whose general MB representation for arbitrary N is known for more than three decades (see Eq. (3.8) in [35] for the notation):

$$\begin{aligned} J^{(N)}(\{\nu_j\}|\{p_j\};0) &= \pi^{D/2} i^{1-D} (k_{1N}^2)^{D/2-\sum_i \nu_i} \frac{1}{\Gamma(D-\sum_i \nu_i)\prod_i \Gamma(\nu_i)} \frac{1}{(2\pi i)^{N(N-1)/2-1}} \\ &\times \int_{-i\infty}^{+i\infty} \cdots \int_{-i\infty}^{+i\infty} \prod_{\substack{j<l \\ (j,l)\neq(1,N)}} \left\{ ds_{jl} \left(\frac{k_{jl}^2}{k_{1N}^2}\right)^{s_{jl}} \Gamma(-s_{jl}) \right\} \Gamma\left(\sum_i \nu_i - D/2 + \sum_{\substack{j<l \\ (j,l)\neq(1,N)}} \sum_l s_{jl} \right) \\ &\times \Gamma\left(D/2 - \sum_i \nu_i + \nu_1 - \sum_{\substack{j<l \\ j\neq 1}} \sum_l s_{jl} \right) \Gamma\left(D/2 - \sum_i \nu_i + \nu_N - \sum_{\substack{j<l \\ l\neq N}} \sum_l s_{jl} \right) \\ &\times \prod_{i=2}^{N-1} \Gamma\left(\nu_i + \sum_{j<i} s_{ji} + \sum_{l>i} s_{il} \right) \end{aligned} \quad (9)$$

[4]On `Ubuntu 22.04.2` with AMD Ryzen Threadripper Pro 5965WX (24-cores 48-threads) and 128 GB RAM using *Mathematica* `13.2.1`.

TABLE I. Speed comparison of the conic hulls and triangulation methods.

			Conic hulls method		Triangulation method	
Feynman integral	MB folds	Total solution number	One solution	All solutions	One solution	All solutions
Conformal triangle	3	14	0.186 sec.	1.44 sec.	0.205 sec.	0.483 sec.
Massless pentagon	5	70	1.276 sec.	1.25 h.	0.318 sec.	2.78 sec.
Conformal hexagon	9	194160	1 min.	…	0.489 sec.	40 min.
Conformal double-box	9	243186	1.9 min.	…	0.635 sec.	1.8 h.
Hard diagram	8	1471926	6 min.	…	1.4 sec.	…

TABLE II. Computation times of the one-loop N-point integral with the triangulation method. We show only the time taken to find a single triangulation and the corresponding set of poles associated to its series solution.

N	Number of folds	Number of terms of the series solution	Computation time
4	5	11	0.384 sec.
5	9	26	0.574 sec.
10	44	1013	1.35 min.
13	77	8178	55.4 min.
15	104	32752	8.9 h.

The analytic expression of the $N = 3$ case is well-known as a combination of four Appell F_4 double hypergeometric functions [36]. However, for $N \geq 4$, due to the intricate structure of the poles in the MB integrand, it is indicated in [35] that it is considerably more complicated to obtain analytic results, and to the best of our knowledge, no such results have been published in the literature since then. However, it is easy to derive these results with the new version of MBConicHulls.wl [28] even for larger values of N, as we have checked by considering the cases until $N = 15$ (the latter having a MB representation with 104 folds); see Table II for a few examples of computation times.

Thanks to the master series of these series representations, which can also be obtained from our package, we have checked numerically these expressions (see `Examples.nb` notebook) against the direct numerical integration of the corresponding MB integrals using the MB.m package [16], for $N = 4$ and $N = 5$.

IV. CONCLUSION AND DISCUSSION

We have presented a new geometrical approach for the analytic evaluation of multifold MB integrals, which is based on the triangulation of point configurations. As described in Sec. II, we assign a set of points to a given MB integral, the triangulations of which yield series solutions of the MB integral. Along with this method, we have shown in Sec. III how this approach considerably improves the computational speed compared to previous techniques, with the resulting fact that MB integrals with a very high number of folds can now be handled analytically in a reasonable computational time. This is possible due to the implementation of the triangulation technique in a new version of the *Mathematica* package MBConicHulls.wl [28].

As practical applications of this method, we have presented new simpler analytic series solutions of the conformal hexagon and double box Feynman integrals than the ones previously obtained in [27]. Among other examples, we computed some contributions to the hard diagram of the two-loop six-edged Wilson loop in general kinematics and the one-loop N-point Feynman integrals for which new analytic results have been derived for the first time due to this new technique. Many other applications will come in the future as it is clear that this novel approach and its powerful *Mathematica* implementation open an entirely new computational perspective not only in Feynman integral calculus, but also in all the fields where MB integrals appear.

ACKNOWLEDGMENTS

We thank J. Rambau for useful discussions on TOPCOM. The work of S. B. is supported by the Swiss National Science Foundation Grant No. PP00P21_76884.

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