Lorentz-violating extension of Wigner function formalism and chiral kinetic theory

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The quantum kinetic equation for the gauge-invariant Wigner function, constructed from spinor fields that obey the Dirac equation modified by CPT and Lorentz symmetry-violating terms, is presented. The equations for the components of the Wigner function in the Clifford algebra basis are accomplished. Focusing on the massless case, an extended semiclassical chiral kinetic theory in the presence of external electromagnetic fields is developed. We calculate the chiral currents and establish the anomalous magnetic and separation effects in a Lorentz-violating background. The chiral anomaly within the context of extended quantum electrodynamics is elucidated. Finally, we derive the semiclassical Lorentz-violating extended chiral transport equation.

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I. INTRODUCTION

The fermionic constituents of the Standard Model of particle physics are mainly the charged Dirac and Weyl particles which also appear in condensed matter as quasiparticles. Various methods have been employed to study the dynamical features of spin-$1/2$ fermions. Nevertheless, an intuitive understanding of physical phenomena involving them is offered by kinetic theory. A systematic construction of kinetic theory begins with the quantum kinetic equation satisfied by the relativistic Wigner function [1,2]. It is the relativistic quantum field theory counterpart of the classical distribution function. In high-energy physics, quantum kinetic theory has predominately been investigated to elucidate phenomena arising in the heavy-ion collisions due to the chiral nature of quarks in a novel phase of quark-gluon plasma. A comprehensive review of quantum transport theory derived by means of the Wigner function method and its applications is provided in [3].

One of the main properties of quantum field theory models is the invariance under Lorentz transformation. However, there are debates about the violation of Lorentz symmetry under certain conditions, such as very high energies. For instance, it may arise from the string theory as outlined in [4,5]. On the other hand, Lorentz symmetry violation may stem from anisotropy in spacetime. A brief historical overview and current status of ideas regarding potential sources of Lorentz violation in particle physics can be found in [6]. Our discussion is based on the Standard Model extension (SME), an effective field theory that incorporates Lorentz- and CPT-violating terms into the Standard Model of particle physics [7,8]. The SME assumes that a fundamental theory, which may be string theory, undergoes spontaneous Lorentz symmetry breaking. This results in a low-energy effective action with explicit Lorentz symmetry-violating (LSV) terms while maintaining microcausality, positivity of energy, and energy-momentum conservation. Notably, in SME particle and observer Lorentz transformations are not treated on equal footing. Observer frames differ in orientation and velocity, hence they are related to coordinate changes. The fundamental theory's Lorentz symmetry is spontaneously broken, so that observer Lorentz symmetry is conserved. However, particle Lorentz symmetry is broken because it is defined by rotations and boosts of the localized field while keeping the expectation values of tensor fields unchanged. Restricting the minimal SME to Abelian gauge theory, one obtains the extended quantum electrodynamics (QED) whose fermionic sector is given by the Dirac spinors coupled to electromagnetic gauge fields in the presence of Lorentz- and CPT-violating terms. This framework is also utilized to discuss violation of emergent Lorentz symmetry in Dirac and Weyl semimetals [9].

We deal with the modified Dirac equation with a certain set of LSV terms: a_μ and $c_{\mu\nu}$. The Boltzmann equation for this choice of LSV coefficients has been examined in [10]. The spinor fields obeying the modified Dirac equation can be quantized using the customary methods of field theory, and the extended QED is invariant under the gauge transformations. Thus, one can construct the extended relativistic gauge-invariant Wigner function and derive the quantum

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kinetic equation as in the ordinary relativistic kinetic theory. We expand the Wigner function in the Clifford algebra basis where the coefficients are scalar, pseudoscalar, vector, axial-vector, and tensor field variables. The quantum kinetic equation yields the coupled equations of these fields, which simplify when the mass is set to zero. We then consider the massless particles and explore the chiral kinetic theory which is manifestly observer Lorentz invariant. One of the peculiar aspects of chiral fermions is the emergence of anomalous effects like chiral magnetic [11,12] and separation [13–15] effects. By choosing the distribution function appropriately we will be able to calculate vector and axial-vector particle currents and show that these anomalous effects get contribution from the LSV terms. Moreover, by studying their conservation we will demonstrate that the chiral anomaly is also altered.

In the next section we discuss the relativistic gauge-invariant Wigner function and the quantum kinetic equation for Dirac particles within extended QED. In Sec. III the quantum kinetic equations of the chiral vector fields will be presented. Two of the three coupled equations are solved in a semiclassical approach. In Sec. IV we employ the modified Fermi-Dirac distribution function [10] and calculate the chiral vector fields, which are then used to establish four-currents generating anomalous magnetic and separation effects. The vector current is shown to be conserved, while the axial-vector four-current leads to the chiral anomaly. In Sec. V, an observer Lorentz-invariant semiclassical chiral kinetic equation is derived. It is followed by a discussion of results and potential future studies in the final section.

II. QUANTUM KINETIC EQUATION

We deal with the Dirac fermion of mass m and charge q in the presence of external electromagnetic fields. There may be several Lorentz symmetry-violating terms that are coordinate reparametrization and gauge invariant [16]. However, we consider a restricted set of Lorentz symmetry-violating coefficients described by the Lagrangian density of the Dirac spinors ψ coupled to the electromagnetic gauge field A_μ, given as ($c = 1$)

$$\mathcal{L} = \bar{\psi}(x)[\Gamma^\mu(i\hbar\partial_\mu - qA_\mu) - M]\psi(x), \tag{1}$$

where $\partial_\mu \equiv \partial/\partial x^\mu$ and

$$\Gamma_\mu = \gamma_\mu + c_{\nu\mu}\gamma^\nu, \qquad M = m + a_\mu\gamma^\mu.$$

The Minkowski metric is $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, and γ_μ are the ordinary γ matrices satisfying $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$. The LSV coefficients a_μ and $c_{\mu\nu}$ are real and constant. The former violates also the CPT invariance. By varying (1) with respect to $\bar{\psi}$ and ψ, one derives the equations of motion as

$$[\Gamma^\mu(i\hbar\partial_\mu - qA_\mu) - M]\psi(x) = 0, \tag{2}$$

$$\bar{\psi}(x)[\Gamma^\mu(i\hbar\partial_\mu^\dagger + qA_\mu) + M] = 0. \tag{3}$$

In [7,16] it has been demonstrated that the spinor operators can be defined as in the ordinary case: In the "concordant frame" where the LSV coefficients are small, there exists a transformation of the spinors $\psi(x)$ that leads to a free Hamiltonian possessing two positive and two negative eigenvalues. Hence, one introduces plane wave solutions and define wave packets. Then, one proceeds as in the ordinary quantum field theory and introduces the spinor operators $\hat{\psi}(x)$ and $\hat{\bar{\psi}}(x)$. Therefore, by means of the four-momentum p^μ one can define the Wigner operator as

$$\hat{W}_{\alpha\beta}(x,p) = \int \frac{d^4y}{(2\pi\hbar)^4} e^{-ip\cdot y/\hbar} \hat{\bar{\psi}}_\beta\left(x + \frac{1}{2}y\right)\hat{\psi}_\alpha\left(x - \frac{1}{2}y\right).$$

By normal ordering $(::)$ and ensemble averaging $(\langle\cdots\rangle)$ the Wigner operator, one acquires the Wigner function $W(x,p) = \langle :\hat{W}(x,p): \rangle$. In the presence of electromagnetic interactions, one can introduce the gauge-invariant Wigner function by means of the gauge link as in the ordinary formalism [1,2]. The derivation of the quantum kinetic equation satisfied by the Wigner function relies only on the Dirac equations. Therefore, employing the modified Dirac equations (2) and (3), it is accomplished as follows:

$$[\Gamma_\mu K^\mu - M]W(x,p) = 0, \tag{4}$$

where

$$K^\mu = \left(\pi^\mu + \frac{i\hbar}{2}\mathcal{D}^\mu\right). \tag{5}$$

We defined

$$\mathcal{D}^\mu \equiv \partial^\mu - qj_0(\Delta)F^{\mu\nu}\partial_{p\nu},$$
$$\pi^\mu \equiv p^\mu - \frac{\hbar q}{2}j_1(\Delta)F^{\mu\nu}\partial_{p\nu},$$

where $\partial_p^\mu \equiv \partial/\partial p_\mu$, and $j_0(\Delta)$, $j_1(\Delta)$ are the spherical Bessel functions in $\Delta \equiv \partial_p^\mu\partial_\mu$. Written in the Clifford algebra basis the Wigner function becomes

$$W = \frac{1}{4}\left(\mathcal{F} + i\gamma^5\mathcal{P} + \gamma^\mu\mathcal{V}_\mu + \gamma^5\gamma^\mu\mathcal{A}_\mu + \frac{1}{2}\sigma^{\mu\nu}\mathcal{S}_{\mu\nu}\right). \tag{6}$$

Let us choose $c_{\mu\nu} = c_{\nu\mu}$ without loss of generality and introduce

$$G_{\mu\nu} = g_{\mu\nu} + c_{\mu\nu} = G_{\nu\mu}, \tag{7}$$

which is a nondiagonal metric [17].

One can show that (4) leads to the following set of coupled equations:

$$G^{\lambda\mu}K_\lambda\mathcal{V}_\mu - m\mathcal{F} - a^\mu\mathcal{V}_\mu = 0, \tag{8}$$

$$iG^{\lambda\mu}K_\lambda\mathcal{A}_\mu + m\mathcal{P} + a^\mu\mathcal{A}_\mu = 0, \tag{9}$$

$$G_{\lambda\mu}K^\lambda\mathcal{F} - iG^{\nu\lambda}K_\lambda\mathcal{S}_{\nu\mu} - m\mathcal{V}_\mu - a_\mu\mathcal{F} = 0, \tag{10}$$

$$iG_{\lambda\mu}K^\lambda\mathcal{P} + \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}G^{\nu\lambda}K_\lambda S^{\alpha\beta} - m\mathcal{A}_\mu - a_\mu\mathcal{P} = 0, \tag{11}$$

$$iG_{\lambda\mu}K^\lambda\mathcal{V}_\nu - iG_{\lambda\nu}K^\lambda\mathcal{V}_\mu - \epsilon_{\mu\nu\rho\sigma}G^{\rho\lambda}K_\lambda\mathcal{A}^\sigma - m\mathcal{S}_{\mu\nu} - i(a_\mu\mathcal{V}_\nu - a_\nu\mathcal{V}_\mu) - \epsilon_{\mu\nu\rho\sigma}a^\sigma\mathcal{A}^\sigma = 0. \tag{12}$$

In the subsequent sections we consider the chiral fermions in the semiclassical limit.

III. CHIRAL VECTOR FIELDS

Observe that for $m = 0$ the equations of the fields $\mathcal{V}_\mu$, $\mathcal{A}_\mu$, (8), (9), (12), decouple from the rest,

$$G^{\lambda\mu}K_\lambda\mathcal{V}_\mu - a^\mu\mathcal{V}_\mu = 0, \tag{13}$$

$$iG^{\lambda\mu}K_\lambda\mathcal{A}_\mu + a^\mu\mathcal{A}_\mu = 0, \tag{14}$$

$$iG_{\lambda\mu}K^\lambda\mathcal{V}_\nu - iG_{\nu\lambda}K^\lambda\mathcal{V}_\mu - \epsilon_{\mu\nu\rho\sigma}G^{\rho\lambda}K_\lambda\mathcal{A}^\sigma - i(a_\mu\mathcal{V}_\nu - a_\nu\mathcal{V}_\mu) - \epsilon_{\mu\nu\rho\sigma}a^\sigma\mathcal{A}^\sigma = 0. \tag{15}$$

In fact, we will deal with the vector field $\mathcal{V}_\mu$ and the axial-vector field $\mathcal{A}_\mu$ for $m = 0$ in the semiclassical limit, keeping at most the $\hbar$-dependent terms. Thus, in terms of

$$D^\mu \equiv \partial_x^\mu - qF^{\mu\nu}\partial_{p\nu}, \tag{16}$$

(5) becomes

$$K_\mu = p_\mu - \frac{i\hbar}{2}D_\mu.$$

Now, by introducing

$$\bar{p}^\mu = G^{\lambda\mu}p_\lambda - a^\mu, \tag{17}$$

$$\bar{D}^\mu = G^{\lambda\mu}D_\lambda, \tag{18}$$

the real parts of (13)–(15) can be expressed as

$$\bar{p}\cdot\mathcal{V} = 0, \tag{19}$$

$$\frac{\hbar}{2}\bar{D}\cdot\mathcal{A} = 0, \tag{20}$$

$$\frac{\hbar}{2}(\bar{D}_\mu\mathcal{V}_\nu - \bar{D}_\nu\mathcal{V}_\mu) - \epsilon_{\mu\nu\alpha\beta}\bar{p}^\alpha\mathcal{A}^\beta = 0. \tag{21}$$

Moreover, the imaginary parts of (13)–(15) are written as

$$\hbar\bar{D}\cdot\mathcal{V} = 0, \tag{22}$$

$$\bar{p}\cdot\mathcal{A} = 0, \tag{23}$$

$$\bar{p}_\mu\mathcal{V}_\nu - \bar{p}_\nu\mathcal{V}_\mu + \frac{\hbar}{2}\epsilon_{\mu\nu\alpha\beta}\bar{D}^\alpha\mathcal{A}^\beta = 0. \tag{24}$$

These two sets of coupled equations can be unified by launching the chiral vector fields

$$\mathcal{J}_\chi = \frac{1}{2}(\mathcal{V} + \chi\mathcal{A}), \tag{25}$$

where $\chi = \pm$ labels the right- and left-handed vector fields. Then (19)–(24) can be combined into

$$\bar{p}\cdot\mathcal{J}_\chi = 0, \tag{26}$$

$$\bar{D}\cdot\mathcal{J}_\chi = 0, \tag{27}$$

$$\frac{\hbar}{2}\epsilon_{\mu\nu\alpha\beta}\bar{D}^\alpha\mathcal{J}_\chi^\beta = -\chi(\bar{p}_\mu\mathcal{J}_{\chi\nu} - \bar{p}_\nu\mathcal{J}_{\chi\mu}). \tag{28}$$

We would like to solve these equations by expanding the chiral vector fields in $\hbar$ as $\mathcal{J}_\chi = \mathcal{J}_\chi^{(0)} + \hbar\mathcal{J}_\chi^{(1)}$. Although in the ordinary case these solutions have been discussed in several papers like [18–20], a comprehensive presentation can be found in [21]. Therefore, we mainly follow the formulation of [21]. The zeroth order solution of (26) and (28) can easily be identified as

$$\mathcal{J}_{\chi\mu}^{(0)} = \bar{p}_\mu f_\chi^{(0)}\delta(\bar{p}^2), \tag{29}$$

where $f_\chi^{(0)}$ is a general distribution function. Equation (29) should also satisfy (27),

$$\begin{aligned}\bar{D}^\mu\left[\bar{p}_\mu f_\chi^{(0)}\delta(\bar{p}^2)\right] &= G^{\mu\lambda}(\partial_{x\lambda} - qF_{\lambda\kappa}\partial_p^\kappa)\\ &\quad\times\left[(G_{\mu\sigma}p^\sigma - a_\mu)f_\chi^{(0)}\delta(\bar{p}^2)\right]\\ &= qG^{\mu\lambda}F_{\lambda\kappa}G_\mu^\kappa f_\chi^{(0)}\delta(\bar{p}^2)\\ &\quad + 2qG^{\mu\lambda}\bar{p}_\mu\bar{p}_\alpha F_{\lambda\kappa}G^{\alpha\kappa}f_\chi^{(0)}\delta'(\bar{p}^2)\\ &\quad + \bar{p}_\mu\delta(\bar{p}^2)\bar{D}^\mu f_\chi^{(0)} = 0.\end{aligned} \tag{30}$$

It is convenient to define

$$\bar{F}^{\mu\nu} = G^{\mu\alpha}F_{\alpha\beta}G^{\beta\nu}, \tag{31}$$

which is antisymmetric

$$\bar{F}^{\nu\mu} = G^{\nu\alpha}F_{\alpha\beta}G^{\beta\mu} = G^{\alpha\nu}F_{\alpha\beta}G^{\mu\beta} = -G^{\mu\beta}F_{\beta\alpha}G^{\alpha\nu} = -\bar{F}^{\mu\nu}. \tag{32}$$

The first and the second terms in (30) vanish due to $\bar{F}^{\mu}_{\mu} = 0$, and $\bar{p}_{\mu}\bar{p}_{\nu}\bar{F}^{\mu\nu} = 0$. Hence it yields the modified Vlasov equation

$$\bar{p}_{\mu}\delta(\bar{p}^2)\bar{D}^{\mu}f_{\chi}^{(0)} = \bar{p}_{\mu}\delta(\bar{p}^2)G^{\mu\lambda}(\partial_{x\lambda} - qF_{\lambda\kappa}\partial_{p}^{\kappa})f_{\chi}^{(0)} = 0. \tag{33}$$

At the $\hbar$ order (26)–(28) yield

$$\bar{p}\cdot\mathcal{J}_{\chi}^{(1)} = 0, \tag{34}$$

$$\bar{D}\cdot\mathcal{J}_{\chi}^{(1)} = 0, \tag{35}$$

$$\epsilon_{\mu\nu\alpha\beta}\bar{D}^{\alpha}\mathcal{J}_{\chi}^{(0)\beta} = -2\chi\left(\bar{p}_{\mu}\mathcal{J}_{\chi\nu}^{(1)} - \bar{p}_{\nu}\mathcal{J}_{\chi\mu}^{(1)}\right). \tag{36}$$

To solve (36), note that

$$\bar{D}^{\alpha}\left[\bar{p}^{\beta}f_{\chi}^{(0)}\delta(\bar{p}^2)\right] = \delta(\bar{p}^2)\bar{p}^{\beta}\bar{D}^{\alpha}f_{\chi}^{(0)} - q\bar{F}^{\alpha\beta}f_{\chi}^{(0)}\delta(\bar{p}^2) - 2q\bar{F}^{\alpha\rho}\bar{p}_{\rho}\bar{p}^{\beta}f_{\chi}^{(0)}\delta'(\bar{p}^2).$$

By making use of the Schouten identity

$$k_{\mu}\epsilon_{\nu\rho\sigma\lambda} + k_{\nu}\epsilon_{\rho\sigma\lambda\mu} + k_{\rho}\epsilon_{\sigma\lambda\mu\nu} + k_{\sigma}\epsilon_{\lambda\mu\nu\rho} + k_{\lambda}\epsilon_{\mu\nu\rho\sigma} = 0, \tag{37}$$

and defining the dual of $\bar{F}$ as

$$\tilde{\bar{F}}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\bar{F}^{\rho\sigma}, \tag{38}$$

we get

$$-\epsilon_{\mu\nu\alpha\beta}\bar{F}^{\alpha\rho}\bar{p}_{\rho}\bar{p}^{\beta} = -2\tilde{\bar{F}}_{\nu\beta}\bar{p}_{\mu}\bar{p}^{\beta} - 2\tilde{\bar{F}}_{\beta\mu}\bar{p}_{\nu}\bar{p}^{\beta} + \bar{p}_{\alpha}\epsilon_{\beta\rho\mu\nu}\bar{F}^{\alpha\rho}\bar{p}^{\beta} - 2\tilde{\bar{F}}_{\mu\nu}\bar{p}^2. \tag{39}$$

The third term on the right-hand side is equal to the term on the left-hand side up to a minus sign, thus (39) yields

$$\epsilon_{\mu\nu\alpha\beta}\bar{F}^{\alpha\rho}\bar{p}_{\rho}\bar{p}^{\beta}f_{\chi}^{(0)}\delta'(\bar{p}^2) = (\tilde{\bar{F}}_{\nu\beta}\bar{p}_{\mu} + \tilde{\bar{F}}_{\beta\mu}\bar{p}_{\nu})\bar{p}^{\beta}f_{\chi}^{(0)}\delta'(\bar{p}^2) - \tilde{\bar{F}}_{\mu\nu}f_{\chi}^{(0)}\delta(\bar{p}^2), \tag{40}$$

where we employed the identity $\delta'(\bar{p}^2) = -\delta(\bar{p}^2)/\bar{p}^2$. Therefore, the left-hand side of (36) is expressed as

$$\epsilon_{\mu\nu\alpha\beta}\bar{D}^{\alpha}\mathcal{J}_{\chi}^{(0)\beta} = \epsilon_{\mu\nu\alpha\beta}\delta(\bar{p}^2)\bar{p}^{\beta}\bar{D}^{\alpha}f_{\chi}^{(0)} - 2q(\tilde{\bar{F}}_{\nu\beta}\bar{p}_{\mu} + \tilde{\bar{F}}_{\beta\mu}\bar{p}_{\nu})\bar{p}^{\beta}f_{\chi}^{(0)}\delta'(\bar{p}^2). \tag{41}$$

Now, we plug (41) into (36) and multiply it with $\bar{p}^{\nu}$,

$$-2\bar{p}^2(q\tilde{\bar{F}}_{\beta\mu}\bar{p}^{\beta}f_{\chi}^{(0)}\delta'(\bar{p}^2) - 2\chi\mathcal{J}_{\mu}^{(1)}) = 0. \tag{42}$$

Its general solution can be written as

$$\mathcal{J}_{\mu}^{(1)} = \chi q\tilde{\bar{F}}_{\beta\mu}\bar{p}^{\beta}f_{\chi}^{(0)}\delta'(\bar{p}^2) + \bar{p}_{\mu}f_{\chi}^{(1)}\delta(\bar{p}^2) + H_{\mu}\delta(\bar{p}^2), \tag{43}$$

where $f_{\chi}^{(1)}$ is a general distribution function and H_{μ} is a vector field satisfying

$$\bar{p}^{\mu}H_{\mu}\delta(\bar{p}^2) = 0, \tag{44}$$

and

$$\epsilon_{\mu\nu\alpha\beta}\delta(\bar{p}^2)\bar{p}^{\beta}\bar{D}^{\alpha}f_{\chi}^{(0)}\delta(\bar{p}^2) = -2\chi(\bar{p}_{\mu}H_{\nu} - \bar{p}_{\nu}H_{\mu})\delta(\bar{p}^2). \tag{45}$$

By introducing the four-vector n_{μ}, which is defined to satisfy $n^2 = 1$, one can solve (44) and (45) as

$$H_{\mu} = \frac{\chi}{2\bar{p}\cdot n}\epsilon_{\mu\nu\alpha\beta}\bar{p}^{\nu}n^{\alpha}\bar{D}^{\beta}f_{\chi}^{(0)}. \tag{46}$$

By means of the Schouten identity (37) and the modified Vlasov equation (33), we can show that it indeed satisfies (45),

$$\begin{aligned}
&\frac{1}{\bar{p}\cdot n}(\bar{p}_{\nu}\epsilon_{\mu\rho\alpha\beta} - \bar{p}_{\mu}\epsilon_{\nu\rho\alpha\beta})\bar{p}^{\rho}n^{\alpha}\delta(\bar{p}^2)\bar{D}^{\beta}f_{\chi}^{(0)} \\
&= \frac{-1}{\bar{p}\cdot n}(\bar{p}_{\mu}\epsilon_{\rho\alpha\beta\nu} + \bar{p}_{\rho}\epsilon_{\alpha\beta\nu\mu} + \bar{p}_{\alpha}\epsilon_{\beta\nu\mu\rho} \\
&\quad + \bar{p}_{\beta}\epsilon_{\nu\mu\rho\alpha} + \bar{p}_{\mu}\epsilon_{\nu\rho\alpha\beta})\bar{p}^{\rho}n^{\alpha}\delta(\bar{p}^2)\bar{D}^{\beta}f_{\chi}^{(0)} \\
&= \frac{-1}{\bar{p}\cdot n}(\bar{p}^2\epsilon_{\alpha\beta\nu\mu}n^{\alpha} + \bar{p}\cdot n\epsilon_{\beta\nu\mu\rho}\bar{p}^{\rho})\delta(\bar{p}^2)\bar{D}^{\beta}f_{\chi}^{(0)} \\
&= \epsilon_{\mu\nu\beta\rho}\bar{p}^{\rho}\delta(\bar{p}^2)\bar{D}^{\beta}f_{\chi}^{(0)}.
\end{aligned}$$

Therefore, we conclude that the semiclassical solution of (26) and (28) is

$$\mathcal{J}_{\chi}^{\mu} = \bar{p}^{\mu}f_{\chi}\delta(\bar{p}^2) + \hbar q\chi\tilde{\bar{F}}^{\mu\nu}\bar{p}_{\nu}f_{\chi}^{(0)}\delta'(\bar{p}^2) + \frac{\hbar\chi}{2\bar{p}\cdot n}\epsilon^{\mu\nu\alpha\beta}\bar{p}_{\nu}n_{\alpha}\delta(\bar{p}^2)\bar{D}_{\beta}f_{\chi}^{(0)}, \tag{47}$$

where we defined the distribution function $f_{\chi} = f_{\chi}^{(0)} + \hbar f_{\chi}^{(1)}$. Obviously, it should also satisfy (27). We will come back to it in Sec. V. Now, we will specify the equilibrium distribution function and calculate chiral currents.

IV. FERMI-DIRAC DISTRIBUTION AND THE CHIRAL CURRENTS

The equilibrium distribution function of the particles which obey Fermi-Dirac statistics in the LSV background is studied in [10]. There, the standard relativistic kinetic theory formulation of gases [22] is employed by incorporating the LSV modifications of the equations of motion: The massless particles obey the extended dispersion relation

$$\bar{p}\cdot\bar{p}=0, \tag{48}$$

and the constraint

$$G^{-1}_{\rho\mu}u^{\mu}(G^{-1})^{\rho\nu}u_{\nu}-1=0. \tag{49}$$

$u_\mu \equiv dx_\mu/d\tau$ is the four-velocity of the fluid where τ is the proper time in the absence of LSV. By introducing

$$\tilde{u}_\mu \equiv G^{-1}_{\mu\nu}u^{\nu}, \tag{50}$$

(49) can be expressed as

$$\tilde{u}_\mu\tilde{u}^{\mu}=1. \tag{51}$$

The Boltzmann and relativistic Uehling-Uhlenbeck equations have been derived following the ordinary relativistic theory in terms of the one-particle phase space distribution function $f(x,p)$, and $\tilde{p}^{\mu}=G^{\mu\nu}\bar{p}_{\nu}$. Then, the H-theorem is demonstrated by making use of the extended transport equations and introducing the entropy-density four-current

$$s^{\mu}=\frac{1}{(2\pi\hbar)^3}\int\frac{d^3p}{\tilde{p}^0}\tilde{p}^{\mu}[(f(x,p)-1)\ln(1-f(x,p)) - f(x,p)\ln f(x,p)], \tag{52}$$

which yields the total entropy

$$S=\int d^3x s^0=\frac{1}{(2\pi\hbar)^3}\int d^3x d^3p[(f(x,p)-1) \times\ln(1-f(x,p))-f(x,p)\ln f(x,p)]. \tag{53}$$

The Boltzmann constant is set $k=1$. Observe that (53) is independent of the LSV coefficients. Thus, as in the ordinary case, one can observe that for particles with momentum p_1, p_2 scattered to particles with momentum p_3, p_4, the total entropy (53) is stationary when $\phi(x,p)=-\ln[f(x,p)/(1-f(x,p))]$ satisfies the condition

$$\phi(x,p_3)+\phi(x,p_4)-\phi(x,p_1)-\phi(x,p_2)=0. \tag{54}$$

For momentum conserving scatterings, the most general solution of (54) can be shown to be

$$f(x,p)=\frac{1}{e^{-\alpha(x)+\beta(x)\cdot p}+1}, \tag{55}$$

where $\alpha(x)$ and $\beta_\mu(x)$ are arbitrary. Equation (55) should obey the Uehling-Uhlenbeck equation which is solved by the Fermi-Dirac distribution function

$$f_{\mathrm{FD}}(x,p)=\frac{1}{e^{(-\mu+u\cdot p)/T}+1}, \tag{56}$$

for $\partial u_\mu/\partial x^\nu=0$. On mass-shell $\bar{p}^2=0$, thus the equilibrium distribution function for chiral fermions and antifermions is given by

$$f^{\mathrm{eq}}_{\chi}=\frac{2}{(2\pi\hbar)^3}\left[\frac{\theta(\tilde{u}\cdot\bar{p})}{e^{(u\cdot p-\mu_\chi)/T}+1}+\frac{\theta(-\tilde{u}\cdot\bar{p})}{e^{-(u\cdot p-\mu_\chi)/T}+1}\right]. \tag{57}$$

Now by expressing it in $\bar{p}$ we get

$$f^{\mathrm{eq}}_{\chi}=\frac{2}{(2\pi\hbar)^3}\left[\frac{\theta(\tilde{u}\cdot\bar{p})}{e^{(\tilde{u}\cdot\bar{p}-\mu_\chi-\tilde{u}\cdot a)/T}+1}+\frac{\theta(-\tilde{u}\cdot\bar{p})}{e^{-(\tilde{u}\cdot\bar{p}-\mu_\chi-\tilde{u}\cdot a)/T}+1}\right]. \tag{58}$$

μ_χ are given by the total chemical potential μ and the chiral chemical potential μ_5 as $\mu_{R,L}=\mu\pm\mu_5$. Observe that the chemical potentials μ_χ are effectively shifted with $\tilde{u}\cdot a$, reminiscent of expressing the distribution function (57) in terms of $\bar{p}_\mu$, which is the LSV extended momentum appearing in the dispersion relation (48).

We would like to study the chiral vector field (47) by choosing f_χ as the modified Fermi-Dirac distribution function (58) and setting $n=\tilde{u}$. In this frame we introduce $\bar{E}_\mu=\bar{F}_{\mu\nu}\tilde{u}^{\nu}$, and $\bar{B}_\mu=(1/2)\epsilon_{\mu\nu\alpha\beta}\tilde{u}^{\nu}\bar{F}^{\alpha\beta}$, so that the field strength and its dual can be expressed as

$$\bar{F}^{\mu\nu}=\bar{E}^{\mu}\tilde{u}^{\nu}-\bar{E}^{\nu}\tilde{u}^{\mu}+\epsilon^{\mu\nu\alpha\beta}\tilde{u}_\alpha\bar{B}_\beta, \tag{59}$$

$$\tilde{\bar{F}}^{\mu\nu}=\bar{B}^{\mu}\tilde{u}^{\nu}-\bar{B}^{\nu}\tilde{u}^{\mu}+\epsilon^{\mu\nu\alpha\beta}\tilde{u}_\beta\bar{E}_\alpha. \tag{60}$$

At the zeroth order in Planck constant f^{eq}_{χ} should satisfy the LSV Vlasov equation (33), which for constant temperature T, leads to

$$G^{\nu\mu}(\partial_\mu-qF_{\mu\alpha}\partial^{\alpha}_{p})f^{\mathrm{eq}}_{\chi}=(\bar{\partial}^{\nu}\mu_\chi+q\bar{F}^{\nu\alpha}\tilde{u}_\alpha)\frac{\partial f^{\mathrm{eq}}_{\chi}}{\partial\mu_\chi}=0.$$

It is satisfied by letting $\mu=(\mu_R+\mu_L)/2$ and $\mu_5=(\mu_R-\mu_L)/2$, to fulfil the conditions

$$\bar{\partial}_\sigma\mu=-\bar{E}_\sigma,\qquad \bar{\partial}_\sigma\mu_5=0. \tag{61}$$

Observe that $\bar{E}_\sigma = G_{\sigma\mu}E^\mu$ where E_μ is the electric field, hence (61) coincides with $\partial_\sigma \mu = -E_\sigma, \partial_\sigma \mu_5 = 0$, which are the relations obeyed in the ordinary chiral kinetic theory [18].

The chiral vector field can be expressed as

$$\begin{aligned}\mathcal{J}^{\mathrm{eq}}_{\chi\mu} &= \bar{p}_\mu f^{\mathrm{eq}}_\chi \delta(\bar{p}^2) + \hbar q\chi \tilde{\bar{F}}_{\mu\nu}\bar{p}^\nu f^{\mathrm{eq}}_\chi \delta'(\bar{p}^2) + \frac{\hbar\chi}{2\bar{p}\cdot\tilde{u}}\epsilon_{\mu\nu\alpha\beta}\bar{p}^\nu \tilde{u}^\alpha \delta(\bar{p}^2)\bar{D}^\beta f^{\mathrm{eq}}_\chi \\ &= \bar{p}_\mu f^{\mathrm{eq}}_\chi \delta(\bar{p}^2) + \hbar q\chi(\bar{B}_\mu \tilde{u}_\nu - \bar{B}_\nu \tilde{u}_\mu)\bar{p}^\nu f^{\mathrm{eq}}_\chi \delta'(\bar{p}^2) + \hbar q\chi \epsilon_{\mu\nu\alpha\beta}\tilde{u}^\beta \bar{E}^\alpha \bar{p}^\nu f^{\mathrm{eq}}_\chi \delta'(\bar{p}^2) \\ &\quad + \frac{\hbar\chi}{2\bar{p}\cdot\tilde{u}}\epsilon_{\mu\nu\alpha\beta}\bar{p}^\nu\tilde{u}^\alpha\delta(\bar{p}^2)(\bar{\partial}^\beta\tilde{u}^\kappa)(\bar{p}_\kappa + a_\kappa)\frac{\partial f^{\mathrm{eq}}_\chi}{\partial(\tilde{u}\cdot\bar{p} - \tilde{u}\cdot a)},\end{aligned} \tag{62}$$

where the last term can also be written as follows:

$$\frac{\hbar\chi}{2\bar{p}\cdot\tilde{u}}\delta(\bar{p}^2)\epsilon_{\mu\nu\alpha\beta}\bar{p}^\nu\tilde{u}^\alpha(\bar{\partial}^\beta u^\kappa)p_\kappa\frac{\partial f^{\mathrm{eq}}_\chi}{\partial(u\cdot p)}.$$

Let us deal with vanishing vorticity $\partial^\beta u^\kappa = 0$ and constant electromagnetic fields E_μ, B_μ. Then, by employing the identity $\bar{p}^2\delta''(\bar{p}^2) = -2\delta'(\bar{p}^2)$ and the relation

$$\begin{aligned}\epsilon^{\mu\nu\rho\beta}\epsilon_{\beta\sigma\delta\alpha} &= \delta^\mu_\sigma(\delta^\nu_\delta\delta^\rho_\alpha - \delta^\rho_\delta\delta^\nu_\alpha) + \delta^\nu_\sigma(\delta^\rho_\delta\delta^\mu_\alpha - \delta^\mu_\delta\delta^\rho_\alpha) \\ &\quad + \delta^\rho_\sigma(\delta^\mu_\delta\delta^\nu_\alpha - \delta^\nu_\delta\delta^\mu_\alpha),\end{aligned} \tag{63}$$

one can easily observe that (62) satisfies the remaining equation (27).

Chiral current is defined as

$$j^\mu_\chi(x) = \int d^4p\,\mathcal{J}^\mu_\chi(x,p). \tag{64}$$

By the change of variables $p_\mu \to \bar{p}_\mu$, we get

$$j^\mu_\chi(x) = \frac{1}{|G|}\int d^4\bar{p}\,\mathcal{J}^\mu_\chi(x,p), \tag{65}$$

where $|G| = \det G^\mu_\nu$.

To perform the integrals over $\bar{p}^\mu = (\bar{p}^0, \vec{\bar{p}})$, let us designate $|\vec{\bar{p}}| \equiv \bar{P}$ and note that

$$\begin{aligned}&\int_0^\infty d\bar{P}\,\bar{P}^k\frac{1}{e^{\left[\bar{P}-\mu_\chi-\tilde{u}\cdot a\right]/T}+1} \\ &= -T^{k+1}k!Li_{k+1}\left(-e^{(\mu_\chi+\tilde{u}\cdot a)/T}\right),\end{aligned} \tag{66}$$

where $Li_s(x)$ are the polylogarithms whose properties which we use in the calculations can be found in [23].

By performing the integrals we get

$$j^\mu_\chi = n_\chi\tilde{u}^\mu + \xi_\chi\bar{B}^\mu, \tag{67}$$

with

$$n_\chi = \frac{1}{6\pi^2\hbar^3|G|}\left[(\mu_\chi + \tilde{u}\cdot a)^3 + \pi^2T^2(\mu_\chi + \tilde{u}\cdot a)\right], \tag{68}$$

$$\xi_\chi = \frac{1}{4\pi^2\hbar^2|G|}(\mu_\chi + \tilde{u}\cdot a). \tag{69}$$

Then, the vector and axial-vector currents defined by $j^\mu = j^\mu_R + j^\mu_L$ and $j^\mu_5 = j^\mu_R - j^\mu_L$ are established as

$$j^\mu = n\tilde{u}^\mu + q\xi\bar{B}^\mu, \tag{70}$$

$$j^\mu_5 = n_5\tilde{u}^\mu + q\xi_5\bar{B}^\mu, \tag{71}$$

where

$$\begin{aligned}n &= \frac{1}{3\pi^2\hbar^3|G|}\left[\mu^3 + (\tilde{u}\cdot a)^3\right] \\ &\quad + \frac{1}{\pi^2\hbar^3|G|}\left[\mu\mu_5^2 + (\mu^2+\mu_5^2)(\tilde{u}\cdot a) + \mu(\tilde{u}\cdot a)^2\right] \\ &\quad + \frac{T^2}{3\hbar^3|G|}(\mu + \tilde{u}\cdot a),\end{aligned} \tag{72}$$

$$n_5 = \frac{1}{6\pi^2\hbar^3|G|}\left[(\mu_\chi + \tilde{u}\cdot a)^3 + \pi^2T^2(\mu_\chi + \tilde{u}\cdot a)\right], \tag{73}$$

$$\xi = \frac{\mu_5}{2\pi^2\hbar^2|G|}, \tag{74}$$

$$\xi_5 = \frac{\mu + \tilde{u}\cdot a}{2\pi^2\hbar^2|G|}. \tag{75}$$

In the absence of LSV terms, the magnetic parts of the currents (70) and (71) are known as chiral magnetic effect (CME) and chiral separation effect (CSE). We found that these effects are modified in two ways. First, the magnetic field is replaced by $\bar{B}$, whose dependence on the magnetic field $B_\mu = (1/2)\epsilon_{\mu\nu\alpha\beta}u^\nu F^{\alpha\beta}$ is not direct. Second, the coefficients (74) and (75) depend on the LSV terms.

By making use of (61) one observes that the vector current is conversed but the axial-vector current is anomalous,

$$\partial_\mu j^\mu = 0, \tag{76}$$

$$\partial_\mu j_5^\mu = \frac{q^2}{2\pi^2\hbar^2|G|} E\cdot\bar{B}. \tag{77}$$

The chiral anomaly in this context has been studied in [24,25]. In [24] $a=0$, and the chiral current was defined as $G_{\mu\nu}J_5^\nu$ where J_5 is the ordinary axial-vector current. Thus, its magnetic part is similar to (70). By studying the index theorem, they concluded that the chiral anomaly is the same as in the ordinary Lorentz-invariant theory. The same conclusion is drawn in [26]. However, in [25], the chiral anomaly was calculated by employing the Fujikawa method as

$$\bar{\partial}_\mu j_5^\mu = \frac{q^2}{2\pi^2\hbar^2|G|}\bar{E}\cdot\bar{B}.$$

This coincides with our result (77).

V. CHIRAL TRANSPORT EQUATION

As we have already mentioned, (47) should also satisfy (27),

$$\bar{D}\cdot\mathcal{J}_\chi = \bar{D}^\mu\left[\bar{p}_\mu f_\chi\delta(\bar{p}^2) + \hbar q\chi\tilde{\bar{F}}_{\mu\beta}\bar{p}^\beta f_\chi^{(0)}\delta'(\bar{p}^2) + \frac{\hbar\chi}{2\bar{p}\cdot n}\epsilon_{\mu\nu\alpha\beta}\bar{p}^\nu n^\alpha\delta(\bar{p}^2)\bar{D}^\beta f_\chi^{(0)}\right] = 0. \tag{78}$$

Following the approach in [21], we will show that (78) leads to the LSV extended chiral transport equation. Since $\bar{F}_{\mu\nu}$ is antisymmetric, the first term can easily be shown to be

$$\bar{D}^\mu[\bar{p}_\mu f_\chi\delta(\bar{p}^2)] = \delta(\bar{p}^2)\bar{p}_\mu\bar{D}^\mu f_\chi. \tag{79}$$

Let us focus on the second term of (78). First, note that

$$4\tilde{\bar{F}}^{\mu\beta}\bar{p}_\beta\bar{F}_{\mu\nu}\bar{p}^\nu = \tilde{\bar{F}}^{\mu\beta}\bar{F}_{\mu\beta}\bar{p}^2, \tag{80}$$

which is derived by means of the Schouten identity (37). By employing the identity (37) and the relations $\bar{p}^2\delta'(\bar{p}^2) = -\delta(\bar{p}^2)$, $\bar{p}^2\delta''(\bar{p}^2) = -2\delta'(\bar{p}^2)$, we get

$$\bar{D}^\mu[\tilde{\bar{F}}_{\mu\beta}\bar{p}^\beta f_\chi^{(0)}\delta'(\bar{p}^2)] = \tilde{\bar{F}}_{\mu\beta}\bar{p}^\beta\delta'(\bar{p}^2)\bar{D}^\mu f_\chi^{(0)}. \tag{81}$$

The third term of (78) can be expressed as

$$\begin{aligned}\bar{D}^\mu\left[\frac{1}{2\bar{p}\cdot n}\epsilon_{\mu\nu\alpha\beta}\bar{p}^\nu n^\alpha\delta(\bar{p}^2)\bar{D}^\beta f_\chi^{(0)}\right] ={}& -q\tilde{\bar{F}}^{\mu\nu}\bar{p}_\nu\delta'(\bar{p}^2)\bar{D}_\mu f_\chi^{(0)} - \frac{q}{\bar{p}\cdot n}\bar{p}_\mu\tilde{\bar{F}}^{\mu\nu}n_\nu\delta'(\bar{p}^2)\bar{p}\cdot\bar{D}f_\chi^{(0)} \\ &- \frac{q}{2(\bar{p}\cdot n)^2}\epsilon_{\mu\nu\alpha\beta}\bar{F}^{\mu\rho}n_\rho n^\nu\bar{p}^\alpha\delta(\bar{p}^2)\bar{D}^\beta f_\chi^{(0)} + \frac{q}{4\bar{p}\cdot n}\epsilon_{\mu\nu\alpha\beta}\bar{p}^\nu n^\alpha\delta(\bar{p}^2)[G_\rho^\beta(\bar{\partial}^\mu F^{\rho\sigma}) \\ &- G_\rho^\mu(\bar{\partial}^\beta F^{\rho\sigma})]\partial_{p\sigma}f_\chi^{(0)} - \frac{1}{2(\bar{p}\cdot n)^2}\epsilon_{\mu\nu\alpha\beta}(\bar{\partial}^\mu n^\rho)\bar{p}_\rho n^\nu\bar{p}^\alpha\delta(\bar{p}^2)\bar{D}^\beta f_\chi^{(0)} \\ &+ \frac{1}{2\bar{p}\cdot n}\epsilon_{\mu\nu\alpha\beta}(\bar{\partial}^\mu n^\nu)\bar{p}^\alpha\delta(\bar{p}^2)\bar{D}^\beta f_\chi^{(0)}.\end{aligned} \tag{82}$$

Here, we utilized the following equality, derived by making use of the Schouten identity (37):

$$\epsilon^{\mu\nu\lambda\rho}\bar{F}_{\mu\sigma}\bar{p}^\sigma n_\nu\bar{p}_\lambda\delta'(\bar{p}^2)\bar{D}_\rho f_\chi^{(0)} = (\bar{p}\cdot n)\tilde{\bar{F}}^{\lambda\rho}\bar{p}_\lambda\delta'(\bar{p}^2)\bar{D}_\rho f_\chi^{(0)} + n_\nu\tilde{\bar{F}}^{\nu\lambda}\bar{p}_\lambda\bar{p}^\rho\delta'(\bar{p}^2)\bar{D}_\rho f_\chi^{(0)} + \tilde{\bar{F}}^{\rho\nu}n_\nu\bar{p}^2\delta'(\bar{p}^2)\bar{D}_\rho f_\chi^{(0)}, \tag{83}$$

and the commutator relation

$$\epsilon^{\mu\nu\lambda\rho}[\bar{D}_\mu,\bar{D}_\rho]f_\chi^{(0)} = q\epsilon^{\mu\nu\lambda\rho}G_\mu^\alpha G_\rho^\beta[(\partial_\alpha F_{\beta\sigma}) - (\partial_\beta F_{\alpha\sigma})]\partial_p^\sigma f_\chi^{(0)}. \tag{84}$$

Now, by inserting (79), (81), and (82) into (78), one establishes

$$\begin{aligned}&\delta\left(\bar{p}^2 - \frac{\hbar q\chi}{\bar{p}\cdot n}\bar{p}_\mu\tilde{\bar{F}}^{\mu\nu}n_\nu\right)\left\{\bar{p}\cdot\bar{D}f_\chi - \frac{\hbar q\chi}{2(\bar{p}\cdot n)^2}\epsilon_{\mu\nu\alpha\beta}\bar{F}^{\mu\rho}n_\rho n^\nu\bar{p}^\alpha\delta(\bar{p}^2)\bar{D}^\beta f_\chi^{(0)} + \frac{\hbar q\chi}{4\bar{p}\cdot n}\epsilon_{\mu\nu\alpha\beta}\bar{p}^\nu n^\alpha\delta(\bar{p}^2)[G_\rho^\beta(\bar{\partial}^\mu F^{\rho\sigma})\right. \\ &\left. - G_\rho^\mu(\bar{\partial}^\beta F^{\rho\sigma})]\partial_{p\sigma}f_\chi^{(0)} - \frac{\hbar\chi}{2(\bar{p}\cdot n)^2}\epsilon_{\mu\nu\alpha\beta}(\bar{\partial}^\mu n^\rho)\bar{p}_\rho n^\nu\bar{p}^\alpha\delta(\bar{p}^2)\bar{D}^\beta f_\chi^{(0)} + \frac{\hbar\chi}{2\bar{p}\cdot n}\epsilon_{\mu\nu\alpha\beta}(\bar{\partial}^\mu n^\nu)\bar{p}^\alpha\delta(\bar{p}^2)\bar{D}^\beta f_\chi^{(0)}\right\} = 0.\end{aligned} \tag{85}$$

Obviously, one should keep only the terms up to the $\hbar$ order. Equation (85) is the semiclassical chiral transport equation, which is observer Lorentz invariant. The δ function dictates the mass-shell condition.

VI. DISCUSSIONS

We studied the fermionic part of the extended QED which is the modified Dirac equation with a set of LSV terms, in the presence of external electromagnetic fields. It is observer Lorentz symmetric as is reminiscent of the spontaneously broken Lorentz invariance of the fundamental theory which may be string theory. We proceed by quantizing spinor fields using standard methods and utilizing these spinor operators to construct a relativistic Wigner function through conventional quantum field theory techniques.

The quantum kinetic equation governing the Wigner function is accomplished by following the well-established formulation of [2]. By decomposing the Wigner function in terms of the Clifford algebra generators constructed from the γ matrices, we derive a set of coupled equations satisfied by the scalar, pseudoscalar, vector, axial-vector, and antisymmetric tensor fields. Notably, in the massless limit, the equations of the vector and axial-vector fields are decoupled from the rest. In fact, we focus on the chiral fermions and adopt the semiclassical approximation where the terms up to the first order in Planck constant are considered.

Within this semiclassical approximation we solve two of the three kinetic equations of the chiral vector fields. Subsequently, we compute particle four-currents by adopting the analog of the Dirac-Fermi distribution proposed in [10]. The extended magnetic and separation effects are established. The vector current is conserved and the axial-vector current is anomalous, where the chiral anomaly depends on the LSV coefficients. Then, by imposing the third equation which should be satisfied by the chiral vector fields, the chiral semiclassical kinetic equation is accomplished.

How can one experimentally test the results of the modified chiral kinetic theory? This may be achieved through the currents (70) and (71), which yield the CME and CSE. In high-energy physics, experimental evidence for the CME is investigated in heavy-ion collisions. The current experimental status has been recently reviewed in [27]. We have seen that LSV parameters modify the coefficient of the magnetic field (74), by $|G|^{-1}$, which would be nearly impossible to detect. However, note that the modified current is along $\bar{\boldsymbol{B}}$, hence the direction of the CME is altered by LSV parameters. Although this effect is small, it is the unique source in the deviation in the direction. This can even help in experimentally observing the CME, which has been investigated only along the direction of the magnetic field.

In principle, one could derive semiclassical kinetic equations for massive fermions by examining the defining equations (8)–(12). However, even in the case of ordinary Dirac fermions, this task is intricate, as evidenced by the complexities encountered when employing the approaches outlined in [28,29].

Integration of (85) over the zeroth component of momentum to obtain the nonrelativistic (3D) chiral kinetic theory is a desired step, albeit complicated by the presence of $G_{\mu\nu}$. A pragmatic solution involves setting $a_0 = 0, G_{0\mu} = \delta_{0\mu}$, yielding $\bar{p}_0 = p_0$, which facilitates to get the mass-shell condition dictated by the Dirac δ function.

By integrating the equations governing the components of the Wigner function (8)–(12), with respect to the zeroth component of momentum, one can derive what is known as the equal-time formulation, a technique elucidated in detail in [30–33].

While our study primarily focuses on establishing chiral currents under conditions of vanishing vorticity, the presence of vorticity may necessitate modifications to the theory, akin to conventional scenarios as discussed in [34,35].

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