# Neutrino phenomenology in the modular $S_3$ seesaw model

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We have studied neutrino phenomenology in the supersymmetric type-I seesaw model endowed with the $\Gamma_2 \simeq S_3$ modular symmetry. We have identified different realizations of the $S_3$ modular symmetry, referred to as models A, B, C, and D. The four models are compatible with neutrino mass being inverted ordering (IO). Moreover, models A, B, and D can also accommodate normal ordering (NO) neutrino masses. We identify parameter space for each model compatible with neutrino oscillation at the 2-$\sigma$ level. We then proceed to study the neutrino phenomenology of each model. We find that the lightest neutrino mass can be as light as 0.64 meV in the case of NO in model A and 50 meV in the case of IO in model D. The smallest effective electron neutrino mass attainable in our analysis is 8.8 meV in the case of NO (model A) and 50 meV for IO (model D). Finally, we note that the effective Majorana mass can be as small as 0.33 meV in the case of NO (model A) and 22 meV for IO (model D).

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## I. INTRODUCTION

The Standard Model (SM) initially posited neutrinos as massless particles, yet experimental evidence from neutrino oscillation data suggests otherwise, indicating they possess tiny but nonzero masses. This revelation solidifies the presence of neutrino mixing, implying that at least two neutrinos have nonzero masses [1–3]. Unlike other fermions in the SM, neutrinos lack right-handed counterparts, precluding them from acquiring masses through the Higgs mechanism. However, the introduction of a dimension-five Weinberg operator [4–6] offers a plausible mechanism for neutrino mass generation, although the origin and flavor structure of this operator remains debated. Hence, exploring scenarios beyond the SM becomes essential to accommodate the observed nonzero neutrino masses.

Various models have been proposed to elucidate neutrino masses and mixing patterns, including the seesaw model [7–12], radiative neutrino mass generation [13–18], and models involving extra dimensions [19–23]. Understanding neutrino mixing patterns has prompted the adoption of symmetry-based approaches, wherein discrete flavor symmetries [24–31], are introduced to enforce specific mixing patterns among leptons and possibly other fields. While these approaches have some early successes, recent experimental data have revealed their deficiencies, requiring departures from minimal models. Incorporating the measured value of parameters such as the reactor angle ($\theta_{13}$) and $CP$-violating phase often leads to nontrivial corrections and challenging predictability. Therefore, new methods are to be explored to tackle the above drawbacks, to which modular symmetry serves the purpose.

Modular flavor symmetry [32–35] offers a new tool for model building. It promotes Yukawa couplings and mass parameters to modular forms, which transform in a nontrivial representation of a flavor symmetry group. The idea of modular flavor symmetry has been widely applied to construct viable neutrino mass models with symmetry groups such as $S_3$ [33,36–39], $A_4$ [40–58], $S_4$ [59–65], and $A_5$ [66,67], along with more expansive groups [68], double covering of $A_4$ [69–71], $S_4$ [72,73], and $A_5$ [74–76]. However, $S_3$, the smallest finite modular group, has not been extensively studied. This will be the focus of our paper.

In this work, we apply modular $S_3$ symmetry to the supersymmetric type-I seesaw model. This scenario, despite

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its simplicity, has so far been overlooked in the literature.[1] Since the irreducible representations of $S_3$ are one and two dimensional, there are many possibilities in assigning lepton chiral supermultiplets into $S_3$ representations. We characterize each different realization of $S_3$ symmetry by the number of two-dimensional representations employed. We have identified four different scenarios, referred to as models A, B, C and D, which are compatible with the current neutrino oscillation data and contain the least number of free parameters.

The manuscript is organized as follows: We briefly describe $S_3$ modular symmetry in Sec. II. In Sec. III, we classify different realizations of $S_3$ modular symmetry which give rise to viable neutrino oscillations. We then explore the neutrino phenomenology for each class of model in Sec. IV. Finally, we conclude in Sec. V.

## II. THE $\Gamma_2 \cong S_3$ MODULAR GROUP

The modular group $\bar{\Gamma}$ consists of linear fractional transformations, denoted by $\gamma$, acting on the modulus $\tau$ in the upper-half complex plane. The transformation is defined as follows:

$$\tau \longrightarrow \gamma\tau = \frac{a\tau + b}{c\tau + d}, \tag{1}$$

where $a$, $b$, $c$, and $d$ are integers satisfying $ad - bc = 1$ and $\mathrm{Im}[\tau] > 0$. This transformation is isomorphic to $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\{I, -I\}$, where $I$ is the identity transformation. The modular transformation is generated by two fundamental operations, $S$ and $T$:

$$S\colon \tau \longrightarrow -\frac{1}{\tau}, \qquad T\colon \tau \longrightarrow \tau + 1. \tag{2}$$

These operations satisfy the algebraic relations:

$$S^2 = \mathbb{I}, \qquad (ST)^3 = \mathbb{I}, \tag{3}$$

where $\mathbb{I}$ represents the identity transformation.

We define a series of groups, denoted as $\Gamma(N)$ for $N = 1, 2, 3, \ldots$, as follows:

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\}. \tag{4}$$

For $N = 2$, we denote $\bar{\Gamma}(2)$ as $\Gamma(2)/\{I, -I\}$. Since the element $-I$ is not in $\Gamma(N)$ for $N > 2$, we have $\bar{\Gamma}(N) = \Gamma(N)$, and these are infinite normal subgroups of $\bar{\Gamma}$, referred to as principal congruence subgroups.

The quotient groups, defined as $\Gamma_N \equiv \bar{\Gamma}/\bar{\Gamma}(N)$, are finite modular groups. In these groups, the condition $T^N = \mathbb{I}$ is imposed. Specifically, the groups $\Gamma_N$ with $N = 2, 3, 4, 5$ are isomorphic to $S_3$, $A_4$, $S_4$, and $A_5$, respectively [35].

Modular forms of level $N$ are holomorphic functions $f(\tau)$, which transform under the action of $\Gamma(N)$ as

$$f(\gamma\tau) = (c\tau + d)^{k_I} f(\tau), \qquad \gamma \in \Gamma(N), \tag{5}$$

where $k_I \geq 0$ is referred to as the modular weight. For $k_I = 0$, the modular form is a constant. Since $T^N = \mathbb{I}$, it follows that $f(\tau + N) = f(\tau)$ and $f(\tau)$ admits a Fourier expansion ($q$ expansion)

$$f(\tau) = \sum_i a_i q_N^i, \quad \text{where } q_N = e^{2i\pi\tau/N}. \tag{6}$$

In this work, we will focus on the group $S_3$, which corresponds to $N = 2$.

The group $S_3$ exhibits three irreducible representations: the doublet **2**, the singlet **1**, and the pseudosinglet **1′**. The lowest even-weighted modular forms arise when $k_I = 2$ and the corresponding modular coupling is represented as $Y_{\mathbf{2}}^{(2)} = (Y_1(\tau), Y_2(\tau))$ and is given in terms of the Dedekind eta function $[\eta(\tau)]$ as given in Ref. [33]. However, for the practical numerical calculations purpose, we use the $q$-expansion [33] form as given below

$$\begin{aligned} Y_1(\tau) &= \frac{1}{8} + 3q^2 + 3q^4 + 12q^6 + 3q^8 \cdots, \\ Y_2(\tau) &= \sqrt{3}q(1 + 4q^2 + 6q^4 + 8q^6 \cdots), \end{aligned} \tag{7}$$

where $q \equiv e^{i\pi\tau}$.

One crucial thing to note is that the higher-order modular forms utilized in model building discussed in Sec. III are obtained by applying $S_3$ product rules. Consider two $S_3$ doublets $x = (x_1, x_2)$ and $y = (y_1, y_2)$. The $S_3$ product rule yields $x \otimes y = (x_2y_2 - x_1y_1, x_1y_2 + x_2y_1)_2 \oplus (x_1y_1 + x_2y_2)_1 \oplus (x_1y_2 - x_2y_1)_{1'}$ where the subscript denote the $S_3$ representation [26]. Hence, the higher weight modular forms ($Y_{\mathrm{a}}^{(k_I)}$), where "a" represents the $S_3$ charge and $k_I$

[1] Modular $S_3$ symmetry in the supersymmetric context has been studied in Ref. [33] where neutrino masses are generated by the dimension-five Weinberg operator. Our work can be regarded as a UV completion of the scenarios considered there.

is the modular weight, used in this paper are shown below

$$
\begin{aligned}
&Y_{\mathbf{1}}^{(4)} = Y_1^2 + Y_2^2, \qquad Y_{\mathbf{1}}^{(6)} = 3Y_1^2 Y_2 - Y_1^3, \qquad Y_{\mathbf{1'}}^{(6)} = 3Y_1 Y_2^2 - Y_2^3, \\
&Y_{\mathbf{2}}^{(4)} = \begin{bmatrix} Y_2^2 - Y_1^2 \\ 2Y_1 Y_2 \end{bmatrix} = \begin{bmatrix} (Y_{\mathbf{2}}^{(4)})_1 \\ (Y_{\mathbf{2}}^{(4)})_2 \end{bmatrix}, \qquad Y_{\mathbf{2}}^{(6)} = \begin{bmatrix} Y_1^3 + Y_1 Y_2^2 \\ Y_2^3 + Y_1^2 Y_2 \end{bmatrix} = \begin{bmatrix} (Y_{\mathbf{2}}^{(6)})_1 \\ (Y_{\mathbf{2}}^{(6)})_2 \end{bmatrix} \\
&Y_{\mathbf{1}}^{(8)} = (Y_1^2 + Y_2^2)^2.
\end{aligned} \tag{8}
$$

## III. MODEL FRAMEWORK

In this section, we describe the models considered in our analysis. Our models are based on the supersymmetric type-I seesaw model in which the minimal supersymmetric Standard Model is extended by three electroweak singlet chiral supermultiplets $N_i^c$. To establish our convention and notation, the supermultiplets in the leptonic sector are $L_i(2,-1/2)$, $E_i^c(1,-1)$, and $N_i^c(1,0)$, where the numbers in parenthesis are the electroweak charges, and the subscript $i$ is the flavor index. For completeness, the Higgs supermultiplets are $H_u(1/2,-1/2)$ and $H_d(1/2,1/2)$ with the vacuum expectation value (vev) $v_u$ and $v_d$, respectively. The electroweak vev is $v = \sqrt{v_u^2 + v_d^2} = 246$ GeV. For later convenience, we define $\tan\beta = v_d/v_u$.

The superpotential for the lepton sector can be written as

$$
\mathcal{W} \supset (Y_\ell)_{ij} E_i^c L_j H_d + (Y_\nu)_{ij} N_i^c L_j H_u + \frac{1}{2}(M_R)_{ij} N_i^c N_j^c, \tag{9}
$$

where $Y_\ell$ and $Y_\nu$ are the Yukawa coupling matrices, and $M_R$ is the Majorana mass matrix for $N_i^c$. After electroweak symmetry breaking, the superpotential induces charged lepton masses $M_\ell = Y_\ell v_d/\sqrt{2}$. The charged lepton mass matrix is diagonalized by unitary matrix $U_\ell$, with $U_\ell^\dagger M_\ell^\dagger M_\ell U_\ell = \mathrm{diag}(m_e^2, m_\mu^2, m_\tau^2)$. In the neutral lepton sector, $\mathcal{N} = (\nu, N^c)$, the induced mass matrix is given by

$$
M_{\mathcal{N}} = \begin{pmatrix} 0 & M_D^T \\ M_D & M_R \end{pmatrix}, \tag{10}
$$

where

$$
M_D = \frac{Y_\nu v_u}{\sqrt{2}}. \tag{11}
$$

In the limit where $|M_R| \gg |M_D|$, the light neutrino mass matrix is given by

$$
M_\nu = M_D^T M_R^{-1} M_D. \tag{12}
$$

$M_\nu$ can be diagonalized by a unitary matrix $U_\nu$, with $U_\nu^\dagger M_\nu^\dagger M_\nu U_\nu = \mathrm{diag}(m_{\nu_1}^2, m_{\nu_2}^2, m_{\nu_3}^2)$. The mismatch between $U_\ell$ and $U_\nu$,

$$
U \equiv U_\ell^\dagger U_\nu, \tag{13}
$$

is the Pontecovo-Maki-Nakagawa-Sakata (PMNS) matrix [77]. The modular $S_3$ symmetry is imposed on the leptonic sector. As a result, the structures of $Y_{\ell,\nu}$ and $M_R$ are constrained by the modular $S_3$ invariant. Since the $S_3$ symmetry can be realized in many different ways, we systematically classify them by specifying which supermultiplet transforms in the doublet representation of $S_3$. Then, for each class of model, we identify the $S_3$ representation and modular weight of each supermultiplet,

TABLE I. In this table, we depict the particle content of the different models and their charges under $S_3$ modular symmetry, where $k_I$ is the modular weight. Also, the number of free real parameters, in addition of the complex modulus $\tau$, and the possible ordering of neutrino masses are provided for each model.

| | Fields | $E_1^c$ | $E_2^c$ | $E_3^c$ | $L_1$ | $L_2$ | $L_3$ | $N_1^c$ | $N_2^c$ | $N_3^c$ | Free parameters | NO | IO |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| MODEL A | $S_3$ | 2 | | 1 | 2 | | 1 | 2 | | 1 | 7 | ✓ | ✓ |
| | $k_I$ | 1 | | −1 | 1 | | 1 | 1 | | 3 | | | |
| MODEL B | $S_3$ | 2 | | 1 | 2 | | 1 | 1 | 1 | 1′ | 7 | ✓ | ✓ |
| | $k_I$ | 0 | | 0 | 2 | | 0 | 0 | 2 | 0 | | | |
| MODEL C | $S_3$ | 1 | 1′ | 1 | 2 | | 1 | 2 | | 1 | 4 | ✗ | ✓ |
| | $k_I$ | 1 | 1 | −1 | 1 | | 1 | 1 | | 1 | | | |
| MODEL D | $S_3$ | 2 | | 1′ | 1 | 1′ | 1′ | 2 | | 1′ | 9 | ✓ | ✓ |
| | $k_I$ | 0 | | 0 | 2 | 2 | 0 | 0 | | 4 | | | |

which leads to viable neutrino masses and mixing with the least number of free parameters. For our analysis, we focus on scenarios where two or more lepton families contain the doublet representation. Having only one lepton species transforming in the doublet generally allows more free parameters, reducing the predictive power of the model. All different modular $S_3$ realizations considered in this work are shown in Table I.

### A. Model A

We first consider a scenario where all three lepton species, $E_i^c$, $L_i$, and $N_i^c$, transform in the doublet representation of $S_3$. For definiteness, we take $\Psi \equiv (\Psi_1, \Psi_2)$, where $\Psi = E^c$, $L$, and $N^c$, respectively, as the doublet. The superpotential in the charged lepton sector consistent with modular $S_3$ symmetry is given by

$$\mathcal{W}_\ell = \alpha_\ell (E^c L)_2 Y_{\mathbf{2}}^{(2)} H_d + \beta_\ell (E^c Y_{\mathbf{2}}^{(2)})_1 L_3 H_d + \gamma_\ell E_3^c L_3 H_d, \tag{14}$$

where a subscript on the parentheses indicates its $S_3$ representation. This superpotential leads to a nondiagonal charged lepton mass matrix

$$M_\ell = \frac{v_d}{\sqrt{2}} \begin{pmatrix} -\alpha_\ell Y_1 & \alpha_\ell Y_2 & 0 \\ \alpha_\ell Y_2 & \alpha_\ell Y_1 & 0 \\ \beta_\ell Y_1 & \beta_\ell Y_2 & \gamma_\ell \end{pmatrix}. \tag{15}$$

The Yukawa couplings $\alpha_\ell$, $\beta_\ell$, and $\gamma_\ell$ are, in general, complex. However, one can perform arbitrary phase redefinition on $E_i^c$ to make them real. Hence, they are completely determined by the charged lepton masses through the relations

$$\mathrm{Tr}(M_\ell M_\ell^\dagger) = m_e^2 + m_\mu^2 + m_\tau^2, \tag{16}$$

$$\mathrm{Det}(M_\ell M_\ell^\dagger) = m_e^2 m_\mu^2 m_\tau^2, \tag{17}$$

$$\frac{1}{2}[\mathrm{Tr}(M_\ell M_\ell^\dagger)]^2 - \frac{1}{2}\mathrm{Tr}[(M_\ell M_\ell^\dagger)^2] = m_e^2 m_\mu^2 + m_\mu^2 m_\tau^2 + m_\tau^2 m_e^2. \tag{18}$$

For the neutral leptons, the superpotential is given by

$$\begin{aligned}\mathcal{W}_\nu = {} & \alpha_D (N^c L)_2 Y_{\mathbf{2}}^{(2)} H_u + \beta_D N_3^c (L Y_{\mathbf{2}}^{(4)})_1 H_u \\ & + \gamma_D (N^c Y_{\mathbf{2}}^{(2)}) L_3 H_u + \omega_D N_3^c L_3 Y_{\mathbf{1}}^{(4)} H_u \\ & + \alpha_R M (N^c N^c)_2 Y_{\mathbf{2}}^{(2)} + \beta_R M N_3^c (N^c Y_{\mathbf{2}}^{(4)})_1 \\ & + M N_3^c N_3^c Y_{\mathbf{1}}^{(6)}.\end{aligned} \tag{19}$$

The first three terms give rise to a Dirac matrix

$$M_D = \frac{\omega_D v_u}{\sqrt{2}} \begin{pmatrix} -\bar{\alpha}_D Y_1 & \bar{\alpha}_D Y_2 & \bar{\gamma}_D Y_1 \\ \bar{\alpha}_D Y_2 & \bar{\alpha}_D Y_1 & \bar{\gamma}_D Y_2 \\ \bar{\beta}_D (Y_{\mathbf{2}}^{(4)})_1 & \bar{\beta}_D (Y_{\mathbf{2}}^{(4)})_2 & Y_{\mathbf{1}}^{(4)} \end{pmatrix}, \tag{20}$$

where we introduce a shorthand notation $\bar{x} = x/\omega_D$. The last three terms of $\mathcal{W}_\nu$ give rise to the Majorana mass matrix for $N^c$,

$$M_R = M \begin{pmatrix} -2\alpha_R Y_1 & 2\alpha_R Y_2 & \beta_R (Y_{\mathbf{2}}^{(4)})_1 \\ 2\alpha_R Y_2 & 2\alpha_R Y_1 & \beta_R (Y_{\mathbf{2}}^{(4)})_2 \\ \beta_R (Y_{\mathbf{2}}^{(4)})_1 & \beta_R (Y_{\mathbf{2}}^{(4)})_2 & Y_{\mathbf{1}}^{(6)} \end{pmatrix}. \tag{21}$$

The above Dirac and Majorana mass matrices lead to the light neutrino mass matrix $M_\nu = M_D^T M_R^{-1} M_D$. Note that the complex combination $m = \omega_D^2 v_u^2 / M$ sets the scale of light neutrino mass. The phase of $m$ has no effect on neutrino oscillations. Moreover, four of the five parameters $\bar{\alpha}_D$, $\bar{\beta}_D$, $\bar{\gamma}_D$, $\alpha_R$, and $\beta_R$ can be made real by redefining the phase of $L_i$ and $E_i^c$. Hence, in this scenario, in addition to the modulus $\tau$, the light neutrino oscillations depend on 7 real parameters: $|m|$, $|\bar{\alpha}_D|$, $|\bar{\beta}_D|$, $|\bar{\gamma}_D|$, $|\alpha_R|$, $|\beta_R|$, and one phase.

### B. Model B

In this scenario, we take $E^c = (E_1^c, E_2^c)$ $L = (L_1, L_2)$ to be doublets of $S_3$. The complete $S_3$ charged assignment and the modular weight of each lepton supermultiplet are shown in model B of Table I. The superpotential for the charged leptons is given by

$$\mathcal{W}_\ell = \alpha_\ell (E^c L)_2 Y_{\mathbf{2}}^{(2)} H_d + \beta_\ell E_3^c (Y_{\mathbf{2}}^{(2)} L)_1 H_d + \gamma_\ell E_3^c L_3 H_d. \tag{22}$$

This leads to a charged lepton mass matrix

$$M_\ell = \frac{v_d}{\sqrt{2}} \begin{pmatrix} -\alpha_\ell Y_1 & \alpha_\ell Y_2 & 0 \\ \alpha_\ell Y_2 & \alpha_\ell Y_1 & 0 \\ \beta_\ell Y_1 & \beta_\ell Y_2 & \gamma_\ell \end{pmatrix}. \tag{23}$$

The Yukawa couplings $\alpha_\ell$, $\beta_\ell$, and $\gamma_\ell$ can be made real by an arbitrary phase rotation on the superfields $E_i^c$ and $L_3$. Hence, they are completely determined by charged lepton masses; see Eq. (18).

For the neutral leptons, the superpotential is given by

$$\begin{aligned}\mathcal{W}_\nu = {} & \alpha_D N_1^c (L Y_{\mathbf{2}}^{(2)})_1 H_u + \beta_D N_2^c (L Y_{\mathbf{2}}^{(4)})_1 H_u \\ & + \gamma_D N_3^c (L Y_{\mathbf{2}}^{(2)})_{1'} H_u + \omega_D N_1^c L_3 H_u \\ & + \alpha_R M N_1^c N_1^c + \beta_R M N_2^c N_2^c Y_{\mathbf{1}}^{(4)} + M N_3^c N_3^c.\end{aligned} \tag{24}$$

The first four terms in $\mathcal{W}_\nu$ lead to the Dirac mass matrix

$$M_D = \frac{\omega_D v_u}{\sqrt{2}} \begin{pmatrix} \bar{\alpha}_D Y_1 & \bar{\alpha}_D Y_2 & 1 \\ \bar{\beta}_D (Y_{\mathbf{2}}^{(4)})_1 & \bar{\beta}_D (Y_{\mathbf{2}}^{(4)})_2 & 0 \\ \bar{\gamma}_D Y_2 & -\bar{\gamma}_D Y_1 & 0 \end{pmatrix}, \quad (25)$$

where we have employed a short-hand notation $\bar{x} = x/\omega_D$. The last three terms of $\mathcal{W}_\nu$ are the Majorana masses for $N_i^c$

$$M_R = M \begin{pmatrix} \alpha_R & 0 & 0 \\ 0 & \beta_R Y_{\mathbf{1}}^{(4)} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (26)$$

As is the case for model A, the complex combination $m = \omega_D^2 v_u^2/M$ determines the scale of light neutrino masses. Also, four of the five complex parameters, $\bar{\alpha}_D$, $\bar{\beta}_D$, $\bar{\gamma}_D$, $\alpha_R$ and $\beta_R$, can be made real by a phase redefinition of $L_i$ and $N_i^c$. Hence, in addition to the modulus $\tau$, there are 7 free real parameters in the light neutrino sector: $|m|$, $|\bar{\alpha}_D|$, $|\bar{\beta}_D|$, $|\bar{\gamma}_D|$, $|\alpha_R|$, $|\beta_R|$, and one phase.

### C. Model C

In this scenario, we take $L = (L_1, L_2)$ and $N^c = (N_1^c, N_2^c)$ as $S_3$ doublets. The complete $S_3$ charge assignment and modular weight can be found in model C of Table I.

The superpotential for the charged lepton sector is

$$\mathcal{W}_\ell = \alpha_\ell E_1^c (L Y_{\mathbf{2}}^{(2)})_1 H_d + \beta_\ell E_2^c (L_2 Y_{\mathbf{2}}^{(2)})_{1'} H_d + \gamma_\ell E_3^c L_3 H_d. \quad (27)$$

This leads to a charged lepton mass matrix

$$M_\ell = \frac{v_d}{\sqrt{2}} \begin{pmatrix} \alpha_\ell Y_1 & \alpha_\ell Y_2 & 0 \\ \beta_\ell Y_2 & -\beta_\ell Y_1 & 0 \\ 0 & 0 & \gamma_\ell \end{pmatrix}. \quad (28)$$

Again, as in the case of model A and model B above, the couplings $\alpha_\ell$, $\beta_\ell$ and $\gamma_\ell$ can be made real by redefining the phase of $E_i^c$. Hence they are completely determined by the charged lepton masses via Eq. (18).

In the neutral lepton sector, the superpotential is given by

$$\begin{aligned} \mathcal{W}_\nu = {} & \alpha_D \left[(N^c L)_2 Y_{\mathbf{2}}^{(2)}\right]_1 H_u + \beta_D (N^c Y_{\mathbf{2}}^{(2)})_1 L_3 H_u \\ & + \omega_D N_3^c (L Y_{\mathbf{2}}^{(2)})_1 H_u + M\left(\left[(N^c N^c)_2 Y_{\mathbf{2}}^{(2)}\right]_1 \right. \\ & \left. + \alpha_R N_3^c (N^c Y_{\mathbf{2}}^{(2)})_1\right). \end{aligned} \quad (29)$$

It leads to the Dirac mass matrix

$$M_D = \frac{\omega_D v_u}{\sqrt{2}} \begin{pmatrix} -\bar{\alpha}_D Y_1 & \bar{\alpha}_D Y_2 & \bar{\beta}_D Y_1 \\ \bar{\alpha}_D Y_2 & \bar{\alpha}_D Y_1 & \bar{\beta}_D Y_2 \\ Y_1 & Y_2 & 0 \end{pmatrix}, \quad (30)$$

and the Majorana mass matrix

$$M_R = M \begin{pmatrix} -Y_1 & Y_2 & \alpha_R Y_1 \\ Y_2 & Y_1 & \alpha_R Y_2 \\ \alpha_R Y_1 & \alpha_R Y_2 & 0 \end{pmatrix}. \quad (31)$$

The light neutrino mass scale in model C, similar to models A and B, depends on the combination $m = \omega_D^2 v_u^2/M$. Moreover, by performing a phase redefinition of $L_i$ and $N_i^c$, one can make all the complex parameters $\bar{\alpha}_D$, $\bar{\beta}_D$, and $\alpha_R$ real. Hence, in addition to the complex modulus $\tau$, neutrino oscillations are determined by four real parameters: $|m|$, $|\bar{\alpha}_D|$, $|\bar{\beta}_D|$, and $|\alpha_R|$.

Finally, we note that the light neutrino mass matrix in model C has a vanishing 3–3 element. As a result, the neutrino masses are inverted ordering in this scenario.

### D. Model D

In this scenario, we take $E^c = (E_1^c, E_2^c)$ and $N^c = (N_1^c, N_2^c)$ as $S_3$ doublets. The complete $S_3$ charge assignment and modular weight can be found in model D of Table I.

The superpotential in the charged lepton sector can be written as

$$\mathcal{W}_\ell = \alpha_\ell (E^c Y_{\mathbf{2}}^{(2)})_1 L_1 H_d + \beta_\ell (E^c Y_{\mathbf{2}}^{(2)})_{1'} L_2 H_d + \gamma_\ell E_3^c L_3 H_d. \quad (32)$$

The above superpotential leads to a charged lepton mass matrix

$$M_\ell = \frac{v_d}{\sqrt{2}} \begin{pmatrix} \alpha_\ell Y_1 & \beta_\ell Y_2 & 0 \\ \alpha_\ell Y_2 & -\beta_\ell Y_1 & 0 \\ 0 & 0 & \gamma_\ell \end{pmatrix}. \quad (33)$$

As in the previous scenarios, the couplings $\alpha_\ell$, $\beta_\ell$, and $\gamma_\ell$ can be made real by arbitrary phase redefinition of the $E_i^c$ and one of the $L_i$ supermultiplet. Thus, they are determined by the charged lepton masses via Eq. (18).

In the neutral lepton sector, the superpotential is given by

$$\begin{aligned} \mathcal{W}_\nu = {} & \omega_D (N^c Y_{\mathbf{2}}^{(2)})_1 L_1 H_u + \alpha_D (N^c Y_{\mathbf{2}}^{(2)})_{1'} L_2 H_u \\ & + \beta_D N_3^c L_1 Y_{\mathbf{1'}}^{(6)} H_u + \gamma_D N_3^c L_2 Y_{\mathbf{1}}^{(6)} H_u \\ & + \eta_D N_3^c L_3 Y_{\mathbf{1}}^{(4)} H_u + M\left[(N^c N^c)_1 + \alpha_R N_3^c (N^c Y_{\mathbf{2}}^{(4)})_{1'} \right. \\ & \left. + \beta_R N_3^c N_3^c Y_{\mathbf{1}}^{(8)}\right]. \end{aligned} \quad (34)$$

The first five terms in the superpotential give rise to a Dirac mass matrix

$$M_D = \frac{\omega_D v_u}{\sqrt{2}} \begin{pmatrix} Y_1 & \bar{\alpha}_D Y_2 & 0 \\ Y_2 & -\bar{\alpha}_D Y_1 & 0 \\ \bar{\beta}_D Y_{\mathbf{1}'}^{(6)} & \bar{\gamma}_D Y_{\mathbf{1}}^{(6)} & \bar{\eta}_D Y_{\mathbf{1}}^{(4)} \end{pmatrix}, \quad (35)$$

and the last three terms give the Majorana mass matrix

$$M_R = M \begin{pmatrix} 1 & 0 & \alpha_R (Y_{\mathbf{2}}^{(4)})_2 \\ 0 & 1 & -\alpha_R (Y_{\mathbf{2}}^{(4)})_1 \\ \alpha_R (Y_{\mathbf{2}}^{(4)})_2 & -\alpha_R (Y_{\mathbf{2}}^{(4)})_1 & \beta_R Y_{\mathbf{1}}^{(8)} \end{pmatrix}. \quad (36)$$

Concerning the light neutrinos, the combination $m = \omega_D^2 v_u^2 / M$ sets their masses scale. Moreover, one can redefine the phase of $L_i$ and $N_i^c$ to rotate away all but two phases in the complex parameters $\bar{\alpha}_D$, $\bar{\beta}_D$ $\bar{\gamma}_D$, $\bar{\eta}_D$, $\alpha_R$, and $\beta_R$. Hence, in this model the light neutrino phenomenology depends on 9 real parameters: $|m|$, $|\bar{\alpha}_D|$, $|\bar{\beta}_D|$ $|\bar{\gamma}_D|$, $|\bar{\eta}_D|$, $|\alpha_R|$, $|\beta_R|$, and two phases. The large number of free parameters in this model makes it somewhat less interesting than the other models considered above. However, we find that in this scenario, neutrino masses are compatible with both normal ordering and inverted ordering.

## IV. NEUTRINO PHENOMENOLOGY

In this section, we numerically analyze the neutrino mass matrices depicted in the previous section. This is achieved by performing a scan on the model parameter space. For each model, we identify the region of parameter space consistent with neutrino oscillation data at the 2-$\sigma$ level; see Table II. In our analysis, the solar mass splitting is taken to be $\Delta m_{\rm sol}^2 = m_{\nu_2}^2 - m_{\nu_1}^2$, while the atmospheric mass splitting is $\Delta m_{atm}^2 = m_{\nu_3}^2 - m_{\nu_1}^2$ ($m_{\nu_2}^2 - m_{\nu_3}^2$) for NO (IO).

In our scan, the parameters $\bar{\alpha}_D$, $\bar{\beta}_D$, $\bar{\gamma}_D$, $\bar{\eta}_D$, $\alpha_R$, and $\beta_R$ are varied within the range $[10^{-4}, 10^4]$. The range is chosen to mimic the hierarchy between the electron and the top Yukawa couplings. Note that a similar range has been widely employed in the literature; see, e.g., [59,60,79–82]. The combination $\omega_D^2 v_u^2 / M$, which sets the neutrino mass scale, is taken to be in the range (0, 1 eV]. The modulus $\tau$ is varied within the fundamental domain defined by

$$\mathrm{Im}(\tau) > 0, \qquad |\mathrm{Re}(\tau)| \leq \frac{1}{2}, \quad \text{and} \quad |\tau| \geq 1. \quad (37)$$

For each set of parameters, we first numerically diagonalize both the charged lepton and the neutrino mass matrices to obtain the PMNS matrix $U$ and the mass splitting $\Delta m_{\rm sol}^2$, $\Delta m_{\rm atm}^2$. Then, the oscillation angles are extracted from

$$\sin^2\theta_{13} = |U_{13}|^2, \qquad \sin^2\theta_{12} = \frac{|U_{12}|^2}{1 - |U_{13}|^2},$$
$$\sin^2\theta_{23} = \frac{|U_{23}|^2}{1 - |U_{13}|^2}, \quad (38)$$

and the Jarlskog invariant ($J_{CP}$) is determined from

$$J_{CP} = \mathrm{Im}(U_{12} U_{23} U_{13}^* U_{22}^*). \quad (39)$$

Our analysis shows that models A, B, and D can accommodate NO and IO neutrino masses. However, model C is only compatible with IO due to its minimal structure. Figure 1 shows the modulus $\tau$ in each model, which produces neutrino oscillation parameters within 2-$\sigma$ of the best-fit values. It is interesting to note that for both NO and IO in model B, the modulus tends to cluster in the bottom left or bottom right regions of the fundamental domain. Similarly, for model C (IO), the modulus is clustered around the bottom left corner of the fundamental domain.

In addition to the oscillation parameters, there are constraints on neutrino masses from cosmology and direct searches. The sum of neutrino masses is constrained by the cosmic microwave background and the baryonic acoustic

TABLE II. The best-fit values for the neutrino oscillation parameters and their 1-$\sigma$ ranges as extracted from Ref. [78].

| Parameter | Best-fit value and 1-$\sigma$ range | |
|---|---|---|
| | *NO* | *IO* |
| $\Delta m_{\rm sol}^2/(10^{-5}\ \mathrm{eV}^2)$ | $7.41_{-0.20}^{+0.21}$ | $7.41_{-0.20}^{+0.21}$ |
| $\Delta m_{\rm atm}^2/(10^{-3}\ \mathrm{eV}^2)$ | $2.507_{-0.027}^{+0.028}$ | $2.486_{-0.028}^{+0.025}$ |
| $\sin^2\theta_{12}$ | $0.303_{-0.011}^{+0.012}$ | $0.303_{-0.012}^{+0.012}$ |
| $\sin^2\theta_{13}$ | $0.02225_{-0.00059}^{+0.00056}$ | $0.02223_{-0.00058}^{+0.00058}$ |
| $\sin^2\theta_{23}$ | $0.451_{-0.016}^{+0.019}$ | $0.569_{-0.021}^{+0.016}$ |
| $J_{\rm CP}$ | $-0.0254_{-0.0080}^{+0.0115}$ | $-0.0330_{-0.0011}^{+0.0044}$ |

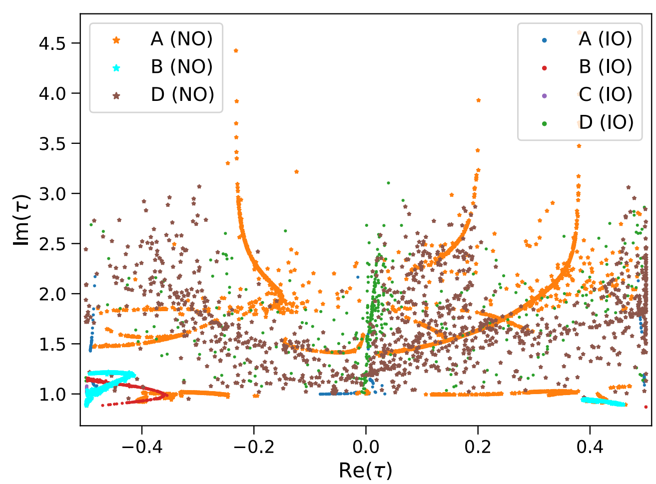


FIG. 1. The modulus $\tau$ that leads to viable neutrino oscillation parameters for models A, B, C, and D.

TABLE III. The range of $m_{\rm light}$, $m_{\nu_e}^{\rm eff}$, and $m_{\beta\beta}^{\rm eff}$ in each scenario.

| | Ordering | $m_{\rm light}$ [eV] | $m_{\nu_e}^{\rm eff}$ [eV] | $m_{\beta\beta}^{\rm eff}$ [eV] |
|---|---|---|---|---|
| MODEL A | NO | $(6.4\times10^{-3}, 0.12)$ | $(8.8\times10^{-3}, 0.12)$ | $(3.3\times10^{-3}, 5.7\times10^{-2})$ |
| | IO | $(5.8\times10^{-2}, 0.22)$ | $(5.8\times10^{-2}, 0.22)$ | $(2.6\times10^{-2}, 0.11)$ |
| MODEL B | NO | $(2.3\times10^{-3}, 7.7\times10^{-2})$ | $(9.0\times10^{-3}, 7.7\times10^{-2})$ | $(1.6\times10^{-3}, 7.7\times10^{-2})$ |
| | IO | $(5\times10^{-2}, 0.14)$ | $(4.9\times10^{-2}, 0.13)$ | $(4.8\times10^{-2}, 0.13)$ |
| MODEL C | IO | $(7.1\times10^{-2}, 7.2\times10^{-2})$ | $(7.1\times10^{-2}, 7.2\times10^{-2})$ | $(5.5\times10^{-2}, 6.2\times10^{-2})$ |
| MODEL D | NO | $(7.5\times10^{-3}, 0.11)$ | $(1.2\times10^{-2}, 0.11)$ | $(8.2\times10^{-3}, 0.10)$ |
| | IO | $(5\times10^{-2}, 0.15)$ | $(5\times10^{-2}, 0.15)$ | $(2.2\times10^{-2}, 0.14)$ |

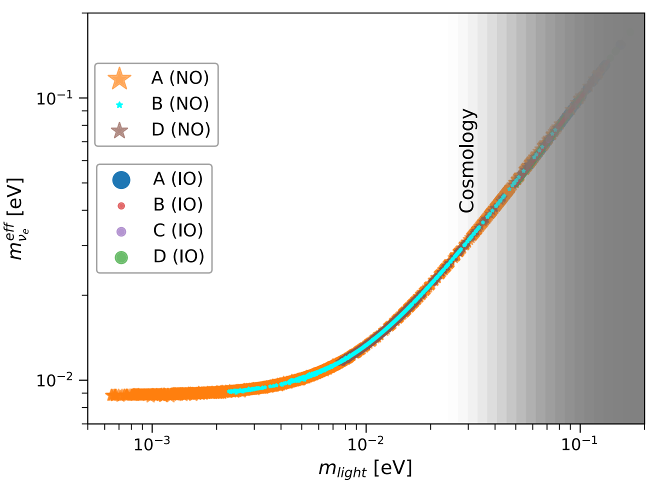


FIG. 2. The effective electron neutrino mass as a function of the lightest neutrino mass. The vertical shaded region is excluded by cosmological measurements.

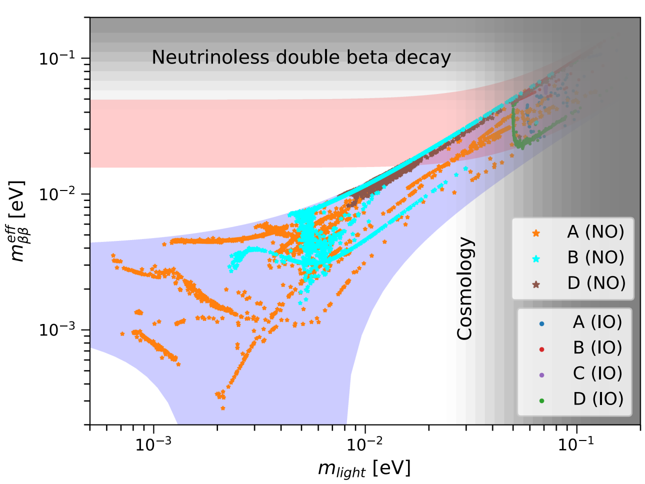


FIG. 3. The effective Majorana mass for $\nu_e$ as a function of the lightest neutrino mass. The vertical shaded region is excluded by cosmological measurements. The horizontal shaded region is excluded by the neutrinoless double beta decay experiments.

oscillations measurements, with $\sum_i m_{\nu_i} \lesssim 0.12$–$0.52$ eV at 95% confidence level (CL) [83]. The range in the upper limit reflects uncertainties due to the assumed cosmological models. These limits translate to the upper bound on the lightest neutrino mass, $m_{\rm light} \lesssim 0.02$–$0.16$ eV. For the models considered in this paper, the NO scenarios are compatible with the most stringent upper bound $m_{\rm light} \lesssim 0.02$ eV. However, the IO scenarios are only compatible with the more relaxed constraint $m_{\rm light} \lesssim 0.16$ eV. The range of $m_{\rm light}$ in each scenario is shown in Table III.

For direct probes of neutrino mass, the effective electron neutrino mass, defined as $m_{\nu_e}^{\rm eff} = \sqrt{\sum_i |U_{ei}|^2 m_{\nu_i}^2}$, is determined from the endpoint of beta decay spectrum. The most stringent upper bound on $m_{\nu_e}^{\rm eff}$ is provided by the KATRIN experiment, with $m_{\nu_e}^{\rm eff} \lesssim 0.8$ eV at 90% CL [84]. This upper bound is roughly an order of magnitude above the expected $m_{\nu_e}^{\rm eff}$ for models A, B, C, and D; see Fig. 2. On the other hand, the neutrinoless double beta decay experiments probe the effective Majorana mass of $\nu_e$, defined by $m_{\beta\beta}^{\rm eff} = |\sum_i U_{ei}^2 m_{\nu_i}|$. The strongest bound on $m_{\beta\beta}^{\rm eff}$ is provided by the KamLAND-Zen experiment, with $m_{\beta\beta}^{\rm eff} \leq 0.036$–$0.156$ eV at 90% CL [85]. The range in the upper bounds reflects uncertainties in nuclear matrix element estimations. With the most stringent estimation of the bound, the IO scenario of models B and C are ruled out; see Fig 3. For convenience, possible values of $m_{\nu_e}^{\rm eff}$ and $m_{\beta\beta}^{\rm eff}$ in each scenario are shown in Table III.

## V. CONCLUSION AND DISCUSSION

We have constructed viable models of neutrino masses with the $S_3$ modular symmetry in a type-I supersymmetric framework. Our approach introduces three $SU(2)_L$ singlet chiral supermultiplets ($N^c$) to facilitate the type-I seesaw mechanism. The Yukawa couplings are constructed using the $S_3$ modular forms. Flavor $S_3$ symmetry is broken by the vev of the complex modulus $\tau$. This discrete symmetry proves instrumental in determining the structure of the neutrino mass matrix. We have identified 4 different realizations of the modular $S_3$ symmetry. All four of them are compatible with IO, while models A, B, and D are also compatible with NO.

We perform a parameter scan for all four models to identify the model parameter space consistent with neutrino oscillation data at the 2-$\sigma$ level. We then determined the lightest neutrino mass, the effective electron neutrino mass, and the effective Majorana mass of $\nu_e$ in each scenario and compared them against experimental bounds. It turns out the IO scenario of each model is tightly constrained by cosmological bound on the lightest neutrino mass. Moreover, the IO scenarios of models B and C are also strongly constrained by the effective Majorana mass. On the other hand, the effective electron neutrino mass in all scenarios considered in this work is well below the current experimental limit.

The next generation of neutrino mass measurement can probe the parameter space of the model proposed in this work even further. On the cosmology front, the Simon Observatory can measure the sum of neutrino mass with a sensitivity of 0.04 eV [86]. This will cover the entire parameter space of all models considered in this work. For direct measurements, the planned phase II of the LEGEND experiment is projected to constrain $m_{\beta\beta}^{\rm eff} \leq$ 0.013–0.029 eV [87]. This will cover all the parameter space in the IO scenarios and a large portion of parameter space in the NO cases. On the other hand, KATRIN and HOLMES experiments are projected to provide the limit $m_{\nu_e}^{\rm eff} < 0.20$ eV [88,89], which only cover a small portion of the IO scenario of model A parameter space.

As a final remark, we note that it is experimentally possible to distinguish between models A, B, C, and D. This can be done by measuring the neutrino Majorana phases. In model C, there is only one physical phase, thus the two Majorana phases are related to the Dirac phase. On the other hand, models A and B contain two physical phases. As a result, there will be one relation among the Dirac and Majorana phases. Lastly, model D contains three physical phases; thus, there will be no relation in the Dirac and Majorana phases.

*Note added.* While this work is being completed, Ref. [90] appeared. The work features a seesaw mechanism under modular $S_3$ symmetry, employing two right-handed neutrinos and predicting a normal ordering of neutrino masses.

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