Hamiltonian formulation of the Rothe-Stamatescu model and field mixing

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While neutrino oscillations have led to attention and research on field mixing arising from quadratic interactions, the field mixing inherent in clothed particles is more fundamental, serving as a significant source of complexity and nonperturbative challenges in quantum field theory. We present an example of an analytical solution for field mixing involving a three-point interaction between a bosonic field and a fermionic field. Specifically, we study the Rothe-Stamatescu (RS) model and utilize lattice regularization to provide a well-defined Hamiltonian that is absent in the original continuous RS model. Because of the complexity introduced by three-point interactions compared to quadratic interactions, the Fock representation commonly used in discussions of field mixing does not work well; instead, we define a representation based on real space to investigate the physical vacuum and clothed particles. These eigenstates not only reveal the field mixing between the bosonic and fermionic fields but also allow us to directly observe the spatial entanglement structure.

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I. INTRODUCTION

Traditional perturbative quantum field theory fails in nonperturbative regimes. To address this issue, lattice quantum field theories have been proposed as a means to regulate quantum field theory before encountering any ill-defined formal calculations. These theories have well-defined path integrals or Hamiltonians [1–10]. While correlation functions in the path integral formulation of a lattice theory can be numerically computed using computers, the calculations are performed in Euclidean spaces after a Wick rotation. This choice limits our ability to directly observe real-time dynamics, such as time-dependent processes like string breaking phenomena, and also introduces challenges associated with the sign problem [10–12]. The Hamiltonian formulation of a lattice theory, on the other hand, provides a well-defined Hamiltonian and allows for explicit discussions about quantum states, entanglement structure, and time evolution [8,9,13]. In addition, Hamiltonian simulations of relativistic lattice field theories, based on the tool of Hamiltonian lattice field theory, have recently garnered significant attention [14].

When it comes to studying the Hamiltonian formulation of a theory, we immediately encounter two major issues. The first is defining the quantum version of the Hamiltonian, and the second is finding the eigenstates (and eigenvalues) of the Hamiltonian. Because of the complexity of dealing directly with the full $3+1$-dimensional Standard Model, researchers often first consider simpler models in lower dimensions, such as the Schwinger model. The Schwinger model, which is QED in $1+1$ dimensions, was initially proposed by Schwinger in 1962 [15]. Subsequently, in 1971, Lowenstein and Swieca defined and solved the equations of motion for the Schwinger model [16]. Following this development, the Hamiltonian version of the Schwinger model emerged [17–20]. The energy spectrum and eigenstates of the quantum Hamiltonian were also determined, enabling in-depth exploration of the nonperturbative effects. These studies often utilized techniques such as heat kernel regularization or ζ-function regularization. On the other hand, lattice regularization was employed to construct the Hamiltonian of the lattice Schwinger model [21–24]. Recently, it became popular to employ lattice regularization based on Kogut-Susskind staggered fermions [2,3,21] to explore aspects such as the structure of the vacuum state, quark confinement, energy spectra, entanglement structure, gauge symmetries, topology, and real-time dynamics [13,25–36].

However, in the Schwinger model, electromagnetic waves are absent due to Gauss's law, leaving only a fermion field. As for the quadratic interaction between two fields, it has been thoroughly investigated in studies related to neutrino oscillations. Considering two different

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flavor neutrinos, ν_e and ν_μ, the Lagrangian for fermion mixing [37] is given by

$$\mathcal{L}_f = \bar{\nu}_e(i\gamma^\alpha\partial_\alpha - m_e)\nu_e + \bar{\nu}_\mu(i\gamma^\alpha\partial_\alpha - m_\mu)\nu_\mu - m_{e\mu}(\bar{\nu}_e\nu_\mu + \bar{\nu}_\mu\nu_e).$$

Because of the interaction term $\mathcal{L}_I = -m_{e\mu}(\bar{\nu}_e\nu_\mu + \bar{\nu}_\mu\nu_e)$, the creation operator corresponding to an individual flavor field cannot annihilate the vacuum, nor can it generate the eigenstates of the Hamiltonian from the vacuum. The creation and annihilation operators for mass eigenstates are a combination of the original flavor creation and annihilation operators. In addition to fermion mixing, boson mixing has also been studied [38]. The Lagrangian for boson mixing is given by

$$L_b = L_{0,\alpha} + L_{0,\beta} - \lambda(\phi_\alpha^\dagger\phi_\beta + \phi_\beta^\dagger\phi_\alpha),$$

where $L_{0,\alpha(\beta)}$ represents the free Lagrangian for the bosonic fields, and the interaction term $L_I = -\lambda(\phi_\alpha^\dagger\phi_\beta + \phi_\beta^\dagger\phi_\alpha)$ leads to mixing of the bosonic fields. The nonperturbative vacuum states, nonperturbative effects due to field mixing, and the entanglement resulting from field mixing have all been extensively researched and discussed in the context of both fermion and boson mixing [39–47].

It is worth noting that the previously mentioned $\mathcal{L}_I$ and L_I are simple quadratic interactions involving fields of the same type. However, in general cases, interactions can be of higher order and can involve the coupling between fermionic and bosonic fields, which leads to more complex field mixing. Pauli and Fierz introduced a transformation to the fundamental equations of nonrelativistic QED [where the bare electron is described by $\frac{1}{2m}(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A})^2$], effectively replacing the electron with its own field plus the electron itself [48]. This transformed entity came to be known as the "dressed electron." Ref. [49] describes how the concept of the dressed electron inspired the birth of renormalization and its significant implications for condensed matter physics. With the development of quantum field theory, a concept similar to the dressed electron emerged in relativistic QED known as a "dressed state," which describes a charged particle dressed with an infrared "cloud of soft photons" [50–53]. In QED, the interaction does not fall off asymptotically, which requires choosing dressed states involving the mixing of bosonic and fermionic fields as in and out states for the S matrices, ensuring the absence of infrared divergences. Additionally, a concept similar to the dressed state is the "clothed particle." In quantum field theory, a one-particle state is an eigenstate of the Hamiltonian, and it is referred to as a clothed particle to distinguish it from a bare particle [54–56]. The concept of clothed particles demonstrates a more general and fundamental field mixing, which can reflect the process of renormalization nonperturbatively. Because of the interaction between the fermionic field and bosonic field in the Hamiltonian, the excitation of the one-fermion state involves both the fermionic field and the bosonic field, rather than just the fermionic field alone. The same applies to a one-boson state. Furthermore, the physical vacuum encompasses entanglement between various interacting fields.

For the previously discussed case of quadratic interactions ($\mathcal{L}_I$ and L_I), there is a significant difference between the physical vacuum and the bare vacuum, and the unitary inequivalence of the Fock space of base (unmixed) eigenstates and the physical mixed eigenstates has been demonstrated. The specific structure of the physical vacuum for quadratic interactions has been precisely determined, and the condensate structure of the physical vacuum can lead to nonperturbative effects. For example, approximating the physical vacuum as the bare vacuum results in different neutrino oscillation formulas compared to nonperturbative oscillation formulas. As for the three-point interactions, which are more general interactions, Ref. [54] solves the Hamiltonians of three solvable models to discuss issues related to clothed particles. However, in these models, the free part of the fermionic field is oversimplified to $H_0 = m\int \mathrm{d}p b_p^\dagger b_p$ instead of the relativistic form $H_0 = \int \mathrm{d}p E_p b_p^\dagger b_p$. These models are nonrelativistic, and due to the absence of pair effects, the bare vacuum is equivalent to the physical vacuum, and bare bosons are equivalent to clothed bosons. Reference [53] discusses dressed states in a Hamiltonian formulation, where the fermionic field is described by the nonrelativistic expression $H_0 = \int \mathrm{d}p \frac{p^2}{2m} b_p^\dagger b_p$, and the bare vacuum is equivalent to the physical vacuum. Therefore, the model we aim to study is the Hamiltonian formulation of a relativistic theory involving a three-point interaction between bosonic and fermionic fields, where the physical vacuum and clothed particles can be solved in a nonperturbative manner to reveal the mixing structure of bosonic and fermionic fields within the eigenstates.

In this paper, we choose the Rothe-Stamatescu (RS) model to investigate its eigenstates, explicitly demonstrating the mixing of fermionic and bosonic fields. The RS model is a solvable $(1+1)$-dimensional model introduced by Rothe and Stamatescu [57]. In the original variables of the RS model, there is a bosonic field ϕ_0 with mass m_0 and a massless fermionic field Ψ_0 that interact through the term $\Delta\mathcal{L} = -g_0\partial_u\phi_0\bar{\Psi}_0\gamma^5\gamma^\mu\Psi_0$ in the Lagrangian. At first glance, it appears that there is a three-point "interaction" vertex in the Lagrangian. However, it is important to emphasize that this interaction is, in fact, spurious and falls within the Borchers class of the free field [58], as it can be eliminated through a field redefinition. This field redefinition will be illustrated in subsequent discussions.

For convenience, we will continue to refer to this three-point vertex as an interaction, but it should be kept in mind that it is spurious. Rothe and Stamatescu regularized the equations of motion for the RS model and obtained the correlation functions. Later, Ref. [59] propose the possibility of describing the RS model using a Hamiltonian framework. However, it only provides a classical Hamiltonian without the ability to demonstrate the renormalization process and compute energy eigenvalues or eigenstates. As mentioned earlier, it is currently popular to use lattice regularization with Kogut-Susskind staggered fermions to construct the Hamiltonian formulation of the Schwinger model [25,26,28–34], which can reveal the real space structure of quantum states and avoid the fermion doubling problem (in $1+1$ dimensions). Therefore, we adopt the same regularization method to deal with the RS model. We provide the Hamiltonian of the lattice RS model with staggered fermions, solve the operator equations of motion for the Hamiltonian, and compare them with the original RS model. We also derive the correlation functions and demonstrate that the correlation functions of the lattice RS model can recover those of the original RS model in the continuum limit. This confirms that the lattice RS model presented in this paper is indeed equivalent to the original RS model in the continuum limit. Since we are dealing with a relativistic theory with a three-point interaction, using the traditional Fock representation would make the representation of quantum states excessively complex, obscuring the structure of the quantum states. Therefore, we introduce a representation based on real space to represent the physical vacuum and clothed particles. This representation not only directly reveals how the bosonic field degrees of freedom mix with the fermionic field degrees of freedom but also illustrates the spatial entanglement structure.

This paper is organized as follows. In Sec. II, we provide a brief introduction to the original RS model and some fundamental results related to it using our notation system. In Sec. III, we present the Hamiltonian of the lattice RS model. In Sec. IV, we employ new field variables to decouple the bosonic and fermionic parts of the lattice Hamiltonian. We then derive the eigenstates of the bosonic part in a representation expanded by the eigenstates of new field operator, while the eigenstates of the fermionic part are expressed in a specially defined representation. In Sec. V, we derive the correlation functions for the lattice RS theory and define renormalized fields, masses, and coupling constants. Notably, the field-strength renormalization constant of the fermionic field tends to zero in the continuum limit. We also show that the lattice correlation functions approach those of the original RS model in the continuum limit. In Sec. VI, we introduce a representation corresponding to the original field variables and express the physical vacuum and clothed particles in this representation. This not only reveals the entanglement between the bosonic and fermionic components of these eigenstates but also allows us to directly observe the spatial entanglement structure of quantum states. In Appendix A, we derive the equations of motion for the bosonic field in the lattice RS model and compare them with those of the original RS model. Similarly, in Appendix B, we derive the equations of motion for the fermionic field in the lattice RS model and compare them to the original RS model. Section VII concludes the paper with a summary and some discussions, along with an outlook on this work.

II. A BRIEF REVIEW OF THE RS MODEL

In this section, we offer a concise overview of the RS model, as proposed in the work by Rothe and Stamatescu [57]. We will refer to this continuous RS model as the “original RS model” in the following to distinguish it from the lattice RS model introduced later. We present the main results using our notation system, and for more detailed derivations and conclusions, please refer to Ref. [57].

The Lagrangian of the RS model can be written as

$$\mathcal{L} = i\bar{\Psi}_0\gamma^\mu\partial_\mu\Psi_0 + \frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 - \frac{1}{2}m_0^2\phi_0^2 - g_0\bar{\Psi}_0\gamma^5\gamma^\mu\Psi_0\partial_\mu\phi_0. \tag{1}$$

In the given expression, ϕ_0 stands for the bare bosonic field, characterized by its bare mass denoted as m_0. The bare fermionic field is identified as Ψ_0, while the bare coupling between the bosonic and fermionic fields is represented by g_0. Additionally, the γ matrices used here are taken as

$$\gamma^0 = \gamma_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \gamma^1 = -\gamma_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$
$$\gamma^5 = \gamma^0\gamma^1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2}$$

It is imperative to emphasize that the Lagrangian (1) is only a formal representation, the precise formulation of the theory necessitates a suitable regularization. Later on, we will perform regularization at the level of Hamiltonian, while the regularization is carried out at the level of equations of motion in the original RS model [57]. The regularized equation of motion governing the bosonic field is expressed as

$$(\partial_\mu\partial^\mu + m_0^2)\phi_0(x) = g_0\partial_\mu \lim_{\epsilon\to 0}\left\{\bar{\Psi}_0(x+\epsilon)\gamma^5\gamma^\mu\Psi_0(x) - \frac{g_0}{\pi}\epsilon^{\mu\lambda}\epsilon^{\nu\rho}\frac{\epsilon_\lambda\epsilon_\rho}{\epsilon^2}\partial_\nu\phi_0(x)\right\}, \tag{3}$$

where ϵ^{μ} is a spacelike vector satisfying $\epsilon^2 < 0$ and $\epsilon^{\mu\nu}$ is the Levi-Civita tensor in 2D. The regularized fermionic field equation of motion can be written as

$$i\gamma^{\mu}\partial_{\mu}\Psi_r(x) = g_r \lim_{\epsilon\to 0}\gamma^5\left[\gamma^{\mu}\Psi_r(x)\partial_{\mu}\phi_r(x-\epsilon) - i\frac{g_r}{2\pi}\frac{\not{\epsilon}}{\epsilon^2}\Psi_r(x)\right], \tag{4}$$

where we introduce the renormalized fermionic field Ψ_r, the renormalized bosonic field ϕ_r, and the renormalized coupling constant g_r, which are connected to the bare parameters as follows:

$$\phi_r = \left(1-\frac{g_0^2}{\pi}\right)^{\frac{1}{2}}\phi_0, \tag{5}$$

$$\Psi_r(x) = \lim_{\epsilon\to 0}\exp\left(\frac{1}{2}g_0^2\langle\Omega|\phi_0(\epsilon)\phi_0(0)|\Omega\rangle\right)\Psi_0(x), \tag{6}$$

$$g_r = \left(1-\frac{g_0^2}{\pi}\right)^{-\frac{1}{2}}g_0. \tag{7}$$

The regularized equations of motion (3) and (4) establish the precise framework for the RS model. Through the utilization of (3) and (4), we gain the ability to compute various observables, including the correlation functions. The two-point correlation function for the bosonic field is expressed as

$$\begin{aligned}&\langle\Omega|\mathrm{T}\{\phi_r(x_1,t_1)\phi_r(x_2,t_2)\}|\Omega\rangle\\ &\quad=\int \mathrm{d}q\mathrm{d}\omega_q\frac{i}{\omega_q^2-q^2-m_r^2+i\epsilon}\mathrm{e}^{-i\omega_q(t_1-t_2)+i(x_1-x_2)q},\end{aligned} \tag{8}$$

where we introduce the renormalized mass parameter as $m_r = (1-g_0^2/\pi)^{-1/2}m_0$. The two-point correlation for the fermionic field can be formulated as

$$\begin{aligned}\langle\Omega|\Psi_r(x)\bar{\Psi}_r(y)|\Omega\rangle &= -\frac{i}{2\pi}\frac{\gamma_{\mu}(x-y)^{\mu}}{(x-y)^2}\\ &\quad\times\exp\left(g_r^2\langle\Omega|\phi_r(x)\phi_r(y)|\Omega\rangle\right).\end{aligned} \tag{9}$$

It is crucial to highlight that, for the sake of simplicity and without sacrificing generality, we have assumed a particular time order $x^0 > y^0$. Consequently, we have omitted the time order operator T in this context. Turning to the interaction, the three-point correlation function can be expressed as follows:

$$\begin{aligned}\langle\Omega|\phi_r(x)\Psi_r(y)\bar{\Psi}_r(0)|\Omega\rangle &= ig_r\gamma^5\langle\Omega|\Psi_r(y)\bar{\Psi}_r(0)|\Omega\rangle\\ &\quad\times\left[\langle\Omega|\phi_r(x)\phi_r(y)|\Omega\rangle\right.\\ &\quad\left.-\langle\Omega|\phi_r(x)\phi_r(0)|\Omega\rangle\right].\end{aligned} \tag{10}$$

Once again, we omit the time-ordering operator, as we are focusing on a specific time-ordering sequence where $x^0 > y^0 > 0$. Please note that, for the sake of brevity, we will represent the operator $\hat{A}$ as A. However, if any ambiguity arises, we will include the operator hat symbol for clarity.

In the subsequent sections, we adopt lattice regularization to construct the Hamiltonian formulation of the RS model which was originally defined only at the level of equations of motion.

III. THE LATTICE RS MODEL

In this study, we adopt a real-time lattice approach, which implies that we refrain from employing Wick rotation, instead focusing solely on discretizing the spatial direction. The discrete field operators, denoted as $\phi_n, \pi_n, \Psi_{\alpha}(n)$, maintain a relationship with the original continuous field operators $\phi(x), \pi(x), \Psi_{c\alpha}(x)$ as follows:

$$\begin{aligned}&\phi_n = \frac{1}{a}\int_{na-\frac{a}{2}}^{na+\frac{a}{2}}\mathrm{d}x\,\phi(x), \qquad \pi_n = \int_{na-\frac{a}{2}}^{na+\frac{a}{2}}\mathrm{d}x\,\pi(x),\\ &\Psi_{\alpha}(n) = \frac{1}{\sqrt{a}}\int_{na-\frac{a}{2}}^{na+\frac{a}{2}}\mathrm{d}x\,\Psi_{c\alpha}(x).\end{aligned} \tag{11}$$

Based on the commutation relations of the continuous fields, we can derive the commutation relations of the discrete fields,

$$\begin{aligned}&[\phi(x),\pi(y)] = i\delta(x-y) \Rightarrow [\phi_n,\pi_m] = i\delta_{n,m}.\\ &\left\{\Psi_{c\alpha}(x),\Psi_{c\beta}^{\dagger}(y)\right\} = \delta_{\alpha,\beta}\delta(x-y) \Rightarrow \left\{\Psi_{\alpha}(n),\Psi_{\beta}^{\dagger}(m)\right\}\\ &\qquad = \delta_{\alpha,\beta}\delta_{n,m}.\\ &[\Psi_{\alpha}(n),\phi_m] = 0, \qquad [\Psi_{\alpha}(n),\pi_m] = 0.\end{aligned} \tag{12}$$

It is worth noting that in $(1+1)$ dimensions, the fermionic field has only two components. We label the components with $\alpha = u, d$.

We are now ready to present the lattice Hamiltonian for the RS model,

$$
\begin{aligned}
H = {} & \frac{1}{2}\sum_n \left\{ -i\left[\Psi_u(n)+\Psi_u(n+1)\right]^\dagger \left[\Psi_u(n+1)-\Psi_u(n)\right]\frac{1}{a} + i\left[\Psi_d(n)+\Psi_d(n+1)\right]^\dagger\left[\Psi_d(n+1)-\Psi_d(n)\right]\frac{1}{a}\right\} \\
& + \frac{1}{2}\sum_n \left\{ i\left[\Psi_d(n+1)-\Psi_d(n)\right]^\dagger \left[\Psi_u(n+1)-\Psi_u(n)\right]\frac{1}{a} + i\left[\Psi_u(n)-\Psi_u(n+1)\right]^\dagger\left[\Psi_d(n+1)-\Psi_d(n)\right]\frac{1}{a}\right\} \\
& + \sum_n a\left[\frac{1}{2F}\left(\frac{\pi_n}{a}\right)^2 + \frac{F}{2}\left(\frac{\phi_{n+1}-\phi_n}{a}\right)^2 + \frac{1}{2}m^2\phi_n^2\right] \\
& + \sum_n \frac{1}{2aF}\left\{-2g\left[\Psi_u^\dagger(n)\Psi_u(n)-\Psi_d^\dagger(n)\Psi_d(n)\right]\pi_n + g^2\left[\Psi_u^\dagger(n)\Psi_u(n)-\Psi_d^\dagger(n)\Psi_d(n)\right]^2\right\} \\
& + \sum_n \frac{1}{2a}\left[i(\mathrm{e}^{ig\phi_{n+1}-ig\phi_n}-1)\Psi_u^\dagger(n+1)\Psi_u(n) + i(\mathrm{e}^{ig\phi_{n+1}-ig\phi_n}-1)\Psi_d^\dagger(n)\Psi_d(n+1) + \mathrm{H.c.}\right] \\
& + \sum_n \frac{1}{2a}\Big[2i(\mathrm{e}^{-2ig\phi_n}-1)\Psi_d^\dagger(n)\Psi_u(n) + \mathrm{H.c.} \\
& + i(\mathrm{e}^{ig\phi_{n+1}+ig\phi_n}-1)\Psi_u^\dagger(n+1)\Psi_d(n) + i(\mathrm{e}^{ig\phi_n+ig\phi_{n+1}}-1)\Psi_u^\dagger(n)\Psi_d(n+1) + \mathrm{H.c.}\Big] - E_0, \qquad (13)
\end{aligned}
$$

where H.c. denotes the Hermitian conjugate of all terms in the same row, and E_0 is introduced to ensure that the energy of the ground state is 0. Furthermore, it will be revealed later through the analysis of correlation functions that if we require the continuum limit of the lattice's bare field ϕ_n to be the bare field $\phi_0(x)$ of the original RS model, then the coefficient F takes the value of $F = 1 - \frac{g^2}{\pi}$.

We require that space forms a circle, and let this circle have N lattice points with each lattice point separated by a distance a. Thus, the total length of the circle is $L = Na$. Starting from index 0, we number the lattice points consecutively, such that lattice point $n = 0$ and lattice point $n = N$ correspond to the same point. This is equivalent to imposing periodic boundary conditions. Therefore, the summation over n is defined to range from $n = 0$ to $n = N - 1$, denoted as $\sum_n \equiv \sum_{n=0}^{N-1}$. Keeping the lattice spacing a constant, as the number of lattice points N approaches infinity, the length of the circle becomes infinitely large, and the boundary conditions can be neglected. This returns our model to the lattice field theory on infinite space. Furthermore, if we let the lattice spacing a approach zero, the lattice field theory eventually becomes a continuous field theory. The process of converting a lattice theory with finite volume (length) into a theory of infinite volume is referred to as taking the "continuum limit," denoted as the limit of $\lim_{a\to 0}\lim_{N\to\infty}$. However, for convenience, we sometimes refer to $\lim_{a\to 0}$ as the continuum limit in the subsequent text. It is important to note that, before taking the limit $\lim_{a\to 0}$, we have already taken the limit $\lim_{N\to\infty}$, so referring to both $\lim_{a\to 0}$ and $\lim_{a\to 0}\lim_{N\to\infty}$ as the continuum limit in the following text will not cause any confusion.

Although the lattice Hamiltonian (13) may appear complex, it can be shown that, in the continuum limit, the lattice Hamiltonian transforms into the classical Hamiltonian H_{cl} corresponding to the Lagrangian (1) when we treat all the field operators of the lattice Hamiltonian as classical fields and set $F = 1, m = m_0, g = g_0$. Specifically, in the continuum limit, the first two lines of (13) tend toward $H_f = \int \mathrm{d}x(-i\Psi_c^\dagger\gamma^0\gamma^1\partial_1\Psi_c)$, the third line tends toward $H_b = \int \mathrm{d}x[\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_1\phi)^2 + \frac{1}{2}m^2\phi^2]$, the fourth line tends toward $V_1 = \int \mathrm{d}x[gJ_5^0\pi + \frac{1}{2}g^2(J_5^0)^2]$, and the remaining lines tend toward $V_2 = \int \mathrm{d}x gJ_5^1\partial_1\phi$, where Ψ_c is the continuous classical fermionic field and $J_5^\mu \equiv \bar{\Psi}_c\gamma^5\gamma^\mu\Psi_c$. Formally, the naive continuum limit of the lattice Hamiltonian (13) retrieves the classical Hamiltonian $H_{cl} \equiv H_f + H_b + V_1 + V_2$ of the classical RS model. However, it is important to note that the fields in the Hamiltonian (13) are originally operators, and the original RS model is a quantum field theory, not a classical field theory. Therefore, verifying that the continuum limit of the lattice RS model corresponds to the original RS model requires further work. In the following sections, we will demonstrate that the continuum limit of lattice RS model correlation functions matches the correlation functions of the original RS model. Additionally, in the Appendixes, we calculate the equations of motion for the lattice RS model and compare them with those of the original RS model.

Starting from the lattice Hamiltonian (13), we can derive the equations of motion for the operators ϕ_n and $\Psi(n)$ based on the Heisenberg equations. Appendix A derives the continuum limit of the equations of motion for the operator ϕ_n and compares them to the equations of motion for the original RS model (3). In Appendix B, the continuum limit of the equations of motion for the operator $\Psi(n)$ is derived and compared to the equations of motion for the original RS model (4). A careful analysis reveals that the behavior exhibited by taking the continuum limit and subsequently letting the field separation tend to zero in our lattice theory is consistent with the original RS model.

For a more detailed derivation and analysis, please refer to Appendixes A and B.

IV. SOLVING THE LATTICE RS MODEL

In this section, we focus on resolving the Hamiltonian (13). To simplify the Hamiltonian expression, we introduce a new set of variables,

$$\begin{aligned} \varphi_n &= F^{\frac{1}{2}}\phi_n, \\ \pi'_n &= \left(\frac{1}{F}\right)^{\frac{1}{2}}\left[\pi_n - g\left(\Psi_u^\dagger(n)\Psi_u(n) - \Psi_d^\dagger(n)\Psi_d(n)\right)\right], \\ \psi(n) &= \mathrm{e}^{-ig\gamma^5\phi_n}\Psi(n). \end{aligned} \tag{14}$$

It can be shown that these new variables still satisfy the canonical commutation relations,

$$\begin{aligned} &[\varphi_m, \pi'_n] = i\delta_{nm}, \quad \{\psi_u(m), \psi_u^\dagger(n)\} = \delta_{nm}, \\ &[\psi_u(m), \pi'_n] = 0, \quad [\psi_u(m), \varphi_n] = 0, \\ &\{\psi_d(m), \psi_d^\dagger(n)\} = \delta_{nm}, \quad [\psi_d(m), \pi'_n] = 0, \\ &[\psi_d(m), \varphi_n] = 0, \quad \{\psi_u(m), \psi_d(n)\} = 0. \end{aligned} \tag{15}$$

With the aid of these recently introduced variables, we can now reformulate the Hamiltonian (13) into a more concise expression,

$$\begin{aligned} H = &\frac{1}{2}\sum_n \bigg\{ -i\left[\psi_u(n) + \psi_u(n+1)\right]^\dagger \left[\psi_u(n+1) - \psi_u(n)\right]\frac{1}{a} \\ &+ i\left[\psi_d(n) + \psi_d(n+1)\right]^\dagger\left[\psi_d(n+1) - \psi_d(n)\right]\frac{1}{a}\bigg\} \\ &+ \frac{1}{2}\sum_n \bigg\{ i\left[\psi_d(n+1) - \psi_d(n))\right]^\dagger \left[\psi_u(n+1) - \psi_u(n)\right]\frac{1}{a} \\ &+ i\left[\psi_u(n) - \psi_u(n+1)\right]^\dagger\left[\psi_d(n+1) - \psi_d(n)\right]\frac{1}{a}\bigg\} \\ &+ \sum_n a\left[\frac{1}{2}\left(\frac{\pi'_n}{a}\right)^2 + \frac{1}{2}\left(\frac{1}{a}\right)^2(\varphi_{n+1} - \varphi_n)^2 \right. \\ &\left. + \frac{1}{2}(F^{-\frac{1}{2}}m)^2\varphi_n^2\right] - E_0. \end{aligned} \tag{16}$$

Clearly, Eq. (16) can be decomposed into two separate components: $H = H_B + H_F$. Here, H_B encompasses contributions solely from the bosonic field and its conjugate momentum, while H_F consists of contributions solely from the fermionic fields $\psi_u(n)$ and $\psi_d(n)$. In conclusion, as stated in the Introduction, the vertex in the original Hamiltonian (13) can be precisely eliminated through the field redefinition (14), resulting in a free theory Hamiltonian (16). Therefore, upon mentioning the interaction in the following discussions, it becomes evident that these interactions are spurious and can be eliminated using (14).

A. Solving the bosonic part

As previously mentioned, the Hamiltonian has been divided into two distinct components. We will initially focus exclusively on the contributions originating from H_B, which can be specifically expressed as

$$\begin{aligned} H_B = &\sum_n a\left[\frac{1}{2}\left(\frac{\pi'_n}{a}\right)^2 + \frac{1}{2}\left(\frac{1}{a}\right)^2(\varphi_{n+1} - \varphi_n)^2 \right. \\ &\left. + \frac{1}{2}\left(F^{-\frac{1}{2}}m\right)^2\varphi_n^2\right] - E_{0B}, \end{aligned} \tag{17}$$

where E_{0B} is chosen to ensure the ground state has zero energy.

For convenience in calculation, we stipulate that N is an odd number. Similar to the solution in continuous field theory, we can solve H_B using creation and annihilation operators that satisfy the commutation relation $[a_q, a_l^\dagger] = \delta_{ql}$,

$$\begin{aligned} H_B &= \sum_q \left(a_q^\dagger a_q + \frac{1}{2}\right)\omega_q - E_{0B}, \\ \omega_q &\equiv \sqrt{\left(\frac{2}{a}\sin\frac{aq}{2}\right)^2 + (F^{-\frac{1}{2}}m)^2}. \end{aligned} \tag{18}$$

Here, we define E_{0B} as $E_{0B} \equiv \frac{1}{2}\sum_q \omega_q$, such that the ground state energy of H_B is zero. The annihilation operators can be expressed in terms of the field ϕ_n and its canonical conjugate π'_n as

$$a_q = \sqrt{\frac{L\omega_q}{2}}\frac{1}{N}\sum_n\left(\varphi_n + \frac{i}{a\omega_q}\pi'_n\right)\mathrm{e}^{-inaq}. \tag{19}$$

Considering the imposed periodic boundary condition, all admissible values for q are determined as follows: $q = \frac{2\pi k}{Na}$, where $k = -\frac{N-1}{2}, -\frac{N-1}{2}+1, \ldots, \frac{N-1}{2}-1, \frac{N-1}{2}$. Consequently, the summation over q equivalently translates into a summation over integer k, ranging from $k = -\frac{N-1}{2}$ to $k = \frac{N-1}{2}$, which is expressed as $\sum_q \equiv \sum_{q=-(N-1)\frac{\pi}{Na}}^{(N-1)\frac{\pi}{Na}} = \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}}$.

Having successfully diagonalized H_B, we can now proceed to compute the eigenstates associated with this Hamiltonian. We denote the ground state of H_B as $|\Omega\rangle_B$, thus $a_q|\Omega\rangle_B = 0$. Then a one-boson state can be represented as

$$|q\rangle_B = a_q^\dagger|\Omega\rangle_B. \tag{20}$$

To illustrate how the bosonic and fermionic fields interact in the upcoming discussion, we need to employ a representation based on real space. For the bosonic part, we can choose the representation expanded by the eigenstates of bosonic field operators. The eigenstates of the field operator $\hat{\phi}_n$ denoted as $|\phi\rangle_B$ satisfy $\hat{\phi}_n|\phi\rangle_b = \phi_n|\phi\rangle_B$ for any spatial point n. We will subsequently refer to the representation constructed from these field operator eigenstates as $\{|\phi\rangle_B\}$. According to Eq. (19), Eq. (14), and the properties $a_q|\Omega\rangle_B = 0$, we can derive the expressions for the ground state and one-boson states in the representation $\{|\phi\rangle_B\}$,

$$|\Omega\rangle_B = \mathcal{N} F^{\frac{1}{2}} \int \mathrm{d}\phi \mathrm{e}^{-\frac{F}{2}\sum_{n,m}\mathcal{E}_{nm}\phi_n\phi_m}|\phi\rangle_B, \tag{21}$$

$$|q\rangle_B = \mathcal{N} F \sqrt{2L\omega_q}\frac{1}{N} \times \int \mathrm{d}\phi \left(\sum_n \mathrm{e}^{inaq}\phi_n\right) \mathrm{e}^{-\frac{F}{2}\sum_{n,m}\mathcal{E}_{nm}\phi_n\phi_m}|\phi\rangle_B, \tag{22}$$

where

$$\mathcal{E}_{nm} \equiv \frac{1}{N}\sum_q a\omega_q \mathrm{e}^{i(n-m)aq}, \quad \int \mathrm{d}\phi \equiv \int \mathrm{d}\phi_1 \int \mathrm{d}\phi_2 \cdots \int \mathrm{d}\phi_N. \tag{23}$$

We can also derive the expression for ϕ_n in terms of creation and annihilation operators,

$$\phi_n(t) = F^{-\frac{1}{2}}\varphi_n = F^{-\frac{1}{2}}\sum_q \sqrt{\frac{1}{2L\omega_q}}(a_q^\dagger \mathrm{e}^{i\omega_q t - inaq} + a_q \mathrm{e}^{-i\omega_q t + inaq}) = \phi_n^-(t) + \phi_n^+(t). \tag{24}$$

Employing the commutation relation between creation and annihilation operators, we can compute the commutation relation between ϕ_n^+ and ϕ_n^- at $t = 0$,

$$f(n-n') \equiv [\phi_n^+, \phi_{n'}^-] = F^{-1}\sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} \frac{1}{2L\sqrt{(\frac{2}{a}\sin\frac{\pi k}{N})^2 + (F^{-\frac{1}{2}}m)^2}} \times \cos\left[(n-n')\frac{2\pi k}{N}\right]. \tag{25}$$

It is worth noting that $f(n)$ exhibits even symmetry in relation to n: $f(-n) = f(n)$. Furthermore, in the continuum limit, both $f(0)$ and $f(1)$ tend toward infinity. However, the disparity between $f(0)$ and $f(1)$ can remain finite: $\lim_{a\to 0}\lim_{N\to\infty}[f(0) - f(1)] = \frac{1}{F\pi}$.

B. Solving the fermionic part

According to (16), the Hamiltonian for the fermionic part is

$$\begin{aligned} H_F = &\frac{1}{2}\sum_n \Big\{ -i[\psi_u(n) + \psi_u(n+1)]^\dagger [\psi_u(n+1) - \psi_u(n)]\frac{1}{a} \\ &+ i[\psi_d(n) + \psi_d(n+1)]^\dagger [\psi_d(n+1) - \psi_d(n)]\frac{1}{a} \Big\} \\ &+ \frac{1}{2}\sum_n \Big\{ i[\psi_d(n+1) - \psi_d(n))]^\dagger [\psi_u(n+1) - \psi_u(n)]\frac{1}{a} \\ &+ i[\psi_u(n) - \psi_u(n+1)]^\dagger [\psi_d(n+1) - \psi_d(n)]\frac{1}{a} \Big\} \\ &- E_{0F}, \end{aligned} \tag{26}$$

where E_{0F} is chosen such that the ground state energy is zero. In fact, H_F is equivalent to the Hamiltonian of free fermionic fields regularized on a lattice using the Kogut-Susskind staggered fermions [2,3,21]. The formulation of H_F as presented in Eq. (26) adopts the standard representation for the γ matrices given by (2). However, the lattice Hamiltonian with staggered fermions proposed by Kogut and Susskind employs a different representation of the γ matrices: $\gamma'^0 = R\gamma^0R^\dagger, \gamma'^1 = R\gamma^1R^\dagger, \gamma'^5 = \gamma'^0\gamma'^1$, where the transformation matrix R is

$$R = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}. \tag{27}$$

In the γ' representation, we denote the fermionic field operators corresponding to ψ_u and ψ_d as ψ'_u and ψ'_d. Their relationship with the original field operators ψ_u and ψ_d can be expressed as follows:

$$\psi'_u(n) = \frac{1}{\sqrt{2}}[\psi_u(n) + \psi_d(n)], \quad \psi'_d(n) = \frac{1}{\sqrt{2}}[-\psi_u(n) + \psi_d(n)]. \tag{28}$$

By employing the ψ'_u and ψ'_d operators, we can offer a more concise representation of the fermionic component of the Hamiltonian,

$$H_F = \sum_n \Big\{ i\psi'^\dagger_u(n)[\psi'_d(n+1) - \psi'_d(n)]\frac{1}{a} + i\psi'^\dagger_d(n+1)[\psi'_u(n+1) - \psi'_u(n)]\frac{1}{a} \Big\} - E_{0F}. \tag{29}$$

This is precisely the lattice Hamiltonian with staggered fermions formulated by Kogut and Susskind. They employed the Jordan-Wigner transformation to rewrite the Hamiltonian (29) into the Hamiltonian of a one-dimensional quantum antiferromagnetic spin chain. Our approach differs from theirs in that we do not rewrite the Hamiltonian (29), but instead directly solve the equations of motion from (29). Then we can express the solutions in terms of creation and annihilation operators that satisfy the anticommutation relations $\{d_q, d_l^\dagger\} = \delta_{ql}$ and $\{b_q, b_l^\dagger\} = \delta_{ql}$,

$$\psi'_u(n,t) = \sum_{q\neq 0} \frac{1}{\sqrt{2N}} \left[d_q \mathrm{e}^{-iE_q t + i(n+\frac{1}{2})aq} + \mathrm{sgn}(q) b_q^\dagger \mathrm{e}^{iE_q t - i(n+\frac{1}{2})aq} \right] + \frac{1}{\sqrt{N}} d_0, \tag{30}$$

$$\psi'_d(n,t) = \sum_{q\neq 0} \frac{1}{\sqrt{2N}} \left[-\mathrm{sgn}(q) d_q \mathrm{e}^{-iE_q t + inaq} - b_q^\dagger \mathrm{e}^{iE_q t - inaq} \right] - \frac{1}{\sqrt{N}} b_0^\dagger, \tag{31}$$

where $E_q \equiv |\frac{2}{a}\sin\frac{aq}{2}|$ and $\mathrm{sgn}(q)$ is the sign function.

By employing Eqs. (30) and (31), we can reformulate the Hamiltonian (26) using the creation and annihilation operators with $E_{0F} \equiv -\sum_{q\neq 0} \frac{2}{a} |\sin(\frac{a}{2}q)|$,

$$H_F = \sum_{q\neq 0} E_q (d_q^\dagger d_q + b_q^\dagger b_q). \tag{32}$$

Then we can employ the diagonalized Hamiltonian (32) to determine the ground state $|\Omega\rangle_F$ that is annihilated by b_q and d_q for all q: $b_q|\Omega\rangle_F = d_q|\Omega\rangle_F = 0$. Now, we are going to introduce a representation for the fermion. First, define a quantum state $|0\rangle_F$ as follows:

$$|0\rangle_F \equiv \prod_q b_{-q}^\dagger |\Omega\rangle_F. \tag{33}$$

It is straightforward to demonstrate the normalization of $|0\rangle_F$: ${}_F\langle 0|0\rangle_F = {}_F\langle\Omega| \prod_{q'} b_{q'} \prod_q b_{-q}^\dagger |\Omega\rangle_F = 1$. Based on the properties $b_q^\dagger b_q^\dagger = 0$ and $d_q|\Omega\rangle_F = 0$, we can infer that $|0\rangle_F$ possesses the following crucial attributes:

$$\psi'_u(n,t)|0\rangle_F = 0, \quad \psi'_d(n,t)|0\rangle_F = 0. \tag{34}$$

Next, we act on $|0\rangle_F$ with any number of fermionic fields $\psi'^\dagger(n)$ to generate a sequence of quantum states

$$\psi'^\dagger_{\alpha_1}(n_1)\psi'^\dagger_{\alpha_2}(n_2)\psi'^\dagger_{\alpha_3}(n_3)\cdots\psi'^\dagger_{\alpha_s}(n_s)|0\rangle_F, \tag{35}$$

where

$$s = 1,2,\ldots,N, \quad \alpha_i = u,d, \quad n_i \neq n_j, \quad \forall\ \alpha_i = \alpha_j.$$

According to Eq. (34) and the commutation relations, it is easy to prove that the quantum states (35) are orthogonal and normalized.

It is worth noting that the expression $\psi'^\dagger_{\alpha_1}(n_1) \times \psi'^\dagger_{\alpha_2}(n_2)\psi'^\dagger_{\alpha_3}(n_3)\cdots\psi'^\dagger_{\alpha_s}(n_s)|0\rangle_F$ exhibits similarities with the traditional Fock space basis $B^\dagger_{\alpha_1}(q_1)B^\dagger_{\alpha_2}(q_2)B^\dagger_{\alpha_3}(q_3)\cdots B^\dagger_{\alpha_s}(q_s)|\Omega\rangle_F$, where $B_1(q) = b_q, B_2(q) = d_q$. However, while momentum labels the creation operators in the Fock space, our "creation" operators $\psi'^\dagger_\alpha(n)$ are labeled by spatial coordinates. Here, we define a new representation by utilizing the set of quantum states (35) as our basis and denote this representation as $\{\prod \psi'^\dagger|0\rangle_F\}$. Our subsequent calculations will be built upon this representation.

Next, let us derive the representation of the ground state in the representation $\{\prod \psi'^\dagger|0\rangle_F\}$. According to the definition of $|0\rangle_F$ in Eq. (33) and the commutation relation $\{b_q, b_l^\dagger\} = \delta_{ql}$, we can express the ground state as

$$|\Omega\rangle_F = \prod_q b_q |0\rangle_F. \tag{36}$$

Combining Eqs. (30) and (31), we can express the creation and annihilation operators in terms of the field operators ψ'_u and ψ'_d,

$$\begin{aligned} d_q &= \mathrm{e}^{-i\frac{1}{2}aq} \frac{1}{\sqrt{2N}} \sum_n \mathrm{e}^{-ianq} \psi'_u(n) - \mathrm{sgn}(q) \frac{1}{\sqrt{2N}} \sum_n \mathrm{e}^{-ianq} \psi'_d(n), \\ d_0 &= \frac{1}{\sqrt{N}} \sum_n \psi'_u(n), \end{aligned} \tag{37}$$

$$\begin{aligned} b_q &= \mathrm{sgn}(q) \mathrm{e}^{-i\frac{1}{2}aq} \frac{1}{\sqrt{2N}} \sum_n \mathrm{e}^{-ianq} \psi'^\dagger_u(n) - \frac{1}{\sqrt{2N}} \sum_n \mathrm{e}^{-ianq} \psi'^\dagger_d(n), \\ b_0 &= -\frac{1}{\sqrt{N}} \sum_n \psi'^\dagger_d(n). \end{aligned} \tag{38}$$

By substituting (38) into (36), we can express $|\Omega\rangle_F$ using the basis (35),

$$|\Omega\rangle_F = \sqrt{2} \prod_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} \frac{1}{\sqrt{2N}} \left[\mathrm{sgn}(k) \mathrm{e}^{-i\frac{1}{2}\frac{2\pi k}{N}} \sum_n \mathrm{e}^{-in\frac{2\pi k}{N}} \psi'^\dagger_u(n) - \sum_n \mathrm{e}^{-in\frac{2\pi k}{N}} \psi'^\dagger_d(n) \right] |0\rangle_F. \tag{39}$$

The creation operators for fermions and antifermions are $d_q^\dagger$ and $b_q^\dagger$, respectively. The quantum state of a single fermion is denoted as $|q;+\rangle_F$, while the quantum state of a single antifermion is denoted as $|q;-\rangle_F$. We can express $|q;+\rangle_F$ and $|q;-\rangle_F$ (where $q \neq 0$) in the representation $\{\prod \psi'^\dagger|0\rangle_F\}$ as follows:

$$
\begin{aligned}
|q;+\rangle_F = \frac{1}{\sqrt{N}}\bigg[& \mathrm{e}^{i\frac{1}{2}\frac{2\pi k}{N}}\sum_n \mathrm{e}^{in\frac{2\pi k}{N}}\psi'^{\dagger}_u(n) \\
& - \mathrm{sgn}(k)\sum_n \mathrm{e}^{in\frac{2\pi k}{N}}\psi'^{\dagger}_d(n)\bigg] \\
& \times \prod_{k'=-\frac{N-1}{2}}^{\frac{N-1}{2}} \frac{1}{\sqrt{2N}}\bigg[\mathrm{sgn}(k')\mathrm{e}^{-i\frac{1}{2}\frac{2\pi k'}{N}}\sum_n \mathrm{e}^{-in\frac{2\pi k'}{N}}\psi'^{\dagger}_u(n) \\
& - \sum_n \mathrm{e}^{-in\frac{2\pi k'}{N}}\psi'^{\dagger}_d(n)\bigg]|0\rangle_F, \qquad (40)
\end{aligned}
$$

$$
\begin{aligned}
|q;-\rangle_F = \sqrt{2}\prod_{k'=-\frac{N-1}{2}}^{\frac{N-1}{2};k'\neq k} \frac{1}{\sqrt{2N}}\bigg[& \mathrm{sgn}(k')\mathrm{e}^{-i\frac{1}{2}\frac{2\pi k'}{N}}\sum_n \mathrm{e}^{-in\frac{2\pi k'}{N}}\psi'^{\dagger}_u(n) \\
& - \sum_n \mathrm{e}^{-in\frac{2\pi k'}{N}}\psi'^{\dagger}_d(n)\bigg]|0\rangle_F, \qquad (41)
\end{aligned}
$$

where $k = \frac{N}{2\pi}q$. It is worth noting that, in Eq. (41), we omit the potential negative sign, as quantum states that differ by a negative sign still describe the same physical system, and quantum states remain normalized.

V. THE CORRELATION FUNCTIONS AND RENORMALIZATION

As demonstrated in Eq. (16), the Hamiltonian can be divided into two distinct components: $H = H_B + H_F$. Notably, these components are entirely uncoupled, indicating that the vacuum state for the complete Hamiltonian (13) can be factored into a direct product of the ground state for the bosonic part and the ground state for the fermionic part: $|\Omega\rangle = |\Omega\rangle_F|\Omega\rangle_B$. Therefore, by utilizing the solution of the complete equation of motion for the bosonic field (24), we are ready to compute the two-point correlation function for the bosonic field:

$$
\langle\Omega|\phi_n(t_1)\phi_m(t_2)|\Omega\rangle = F^{-1}\sum_q \frac{1}{2L\omega_q}\mathrm{e}^{-i\omega_q(t_1-t_2)+i(n-m)aq}, \qquad (42)
$$

where $\omega_q = \sqrt{(\frac{2}{a}\sin\frac{aq}{2})^2 + (F^{-\frac{1}{2}}m)^2}$.

Since $q = \frac{2\pi k}{Na}$, where $k = -\frac{N-1}{2}, -\frac{N-1}{2}+1, -\frac{N-1}{2}+2, \ldots, \frac{N-1}{2}-1, \frac{N-1}{2}$, the difference between neighboring q values is $\Delta q = \frac{2\pi}{Na} = \frac{2\pi}{L}$. Consequently, we can express Eq. (42) in the following form:

$$
\langle\Omega|\phi_n(t_1)\phi_m(t_2)|\Omega\rangle = F^{-1}\sum_q \Delta q\frac{1}{4\pi\omega_q}\mathrm{e}^{-i\omega_q(t_1-t_2)+i(n-m)aq}. \qquad (43)
$$

To explore the continuum limit, our initial step involves keeping the lattice spacing a constant while letting N tend to infinity. As a result, Δq tends to zero, which implies that q becomes continuous. Consequently, the summation over q in Eq. (43) can be replaced by an integral with respect to q,

$$
\begin{aligned}
&\lim_{N\to\infty}\langle\Omega|\phi_n(t_1)\phi_m(t_2)|\Omega\rangle \\
&\quad = F^{-1}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}}\mathrm{d}q\frac{1}{4\pi\omega_q}\mathrm{e}^{-i\omega_q(t_1-t_2)+i(n-m)aq}. \qquad (44)
\end{aligned}
$$

In the continuum limit, the relationship between the continuous bosonic field operator $\phi(x,t)$ and the discrete field operator ϕ_n is given by $\phi_n(t) = \phi(x,t)|_{x=na}$. Considering two spatial coordinates $x_1 = na$ and $x_2 = ma$, with x_1 and x_2 fixed, the expression for the continuous two-point correlation function in the limit where the lattice spacing a tends to zero is

$$
\begin{aligned}
&\langle\Omega|\phi(x_1,t_1)\phi(x_2,t_2)|\Omega\rangle \\
&\quad = \lim_{a\to 0}\lim_{N\to\infty}\langle\Omega|\phi_n(t_1)\phi_m(t_2)|\Omega\rangle \\
&\quad = \lim_{a\to 0}F^{-1}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}}\mathrm{d}q\frac{1}{4\pi\omega_q}\mathrm{e}^{-i\omega_q(t_1-t_2)+i(x_1-x_2)q}. \qquad (45)
\end{aligned}
$$

In order to arrive at a more explicit form, we can introduce a deformation to the integral. To start, we separate the integral in Eq. (45) into two distinct parts,

$$
\begin{aligned}
&\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}}\mathrm{d}q\frac{1}{\omega_q}\mathrm{e}^{-i\omega_q(t_1-t_2)+i(x_1-x_2)q} \\
&\quad = \lim_{\epsilon\to 0}\int_0^{\frac{\pi}{a}}\mathrm{d}q\frac{1}{\omega_q}\mathrm{e}^{-i\omega_q(t_1-t_2)+i(x_1-x_2)q-\epsilon q} \\
&\qquad + \lim_{\epsilon\to 0}\int_{-\frac{\pi}{a}}^{0}\mathrm{d}q\frac{1}{\omega_q}\mathrm{e}^{-i\omega_q(t_1-t_2)+i(x_1-x_2)q+\epsilon q}. \qquad (46)
\end{aligned}
$$

In Eq. (46), we have introduced an ϵ suppression, which is a common technique in standard field theory. Additionally, we will make a reasonable assumption that, for the complete integral expression, the limits $\lim_{\epsilon\to 0}$ and $\lim_{a\to 0}$ can be interchanged. In Eq. (46), the last two integrals have identical structures, so we only need to focus on the first integral. Taking the limit $a\to 0$, we split it into two new integrals,

$$
\begin{aligned}
&\lim_{a\to 0}\int_0^{\frac{\pi}{a}}\mathrm{d}q\frac{1}{\omega_q}\mathrm{e}^{-i\omega_q(t_1-t_2)+i(x_1-x_2)q-\epsilon q} \\
&\quad = \lim_{a\to 0}\int_0^{\frac{\pi}{\sqrt{a}}}\mathrm{d}q\frac{1}{\omega_q}\mathrm{e}^{-i\omega_q(t_1-t_2)+i(x_1-x_2)q-\epsilon q} \\
&\qquad + \lim_{a\to 0}\int_{\frac{\pi}{\sqrt{a}}}^{\frac{\pi}{a}}\mathrm{d}q\frac{1}{\omega_q}\mathrm{e}^{-i\omega_q(t_1-t_2)+i(x_1-x_2)q-\epsilon q}. \qquad (47)
\end{aligned}
$$

Based on the expression for ω_q, it is easy to show that, for sufficiently small a (implying that $q > \pi/\sqrt{a}$ is large), we have $\omega_q > 1$. Hence, the last integral in Eq. (47) satisfies the following inequality:

$$\begin{aligned}
&\left|\lim_{a\to 0}\int_{\frac{\pi}{\sqrt{a}}}^{\frac{\pi}{a}} \mathrm{d}q\,\frac{1}{\omega_q}\,\mathrm{e}^{-i\omega_q(t_1-t_2)+i(x_1-x_2)q-\epsilon q}\right| \\
&\leq \lim_{a\to 0}\int_{\frac{\pi}{\sqrt{a}}}^{\frac{\pi}{a}} \mathrm{d}q\,\frac{1}{\omega_q}\left|\mathrm{e}^{-i\omega_q(t_1-t_2)+i(x_1-x_2)q}\right|\mathrm{e}^{-\epsilon q} \\
&\leq \lim_{a\to 0}\int_{\frac{\pi}{\sqrt{a}}}^{\frac{\pi}{a}} \mathrm{d}q\,\mathrm{e}^{-\epsilon q} = \lim_{a\to 0}\frac{\mathrm{e}^{-\epsilon\frac{\pi}{a}} - \mathrm{e}^{-\epsilon\frac{\pi}{\sqrt{a}}}}{-\epsilon} = 0,
\end{aligned} \tag{48}$$

which indicates that the last integral in Eq. (47) becomes negligible. Therefore, by substituting the dispersion relation for ω_q, Eq. (47) can be expressed as

$$\begin{aligned}
&\lim_{a\to 0}\int_0^{\frac{\pi}{a}} \mathrm{d}q\,\frac{1}{\omega_q}\,\mathrm{e}^{-i\omega_q(t_1-t_2)+i(x_1-x_2)q-\epsilon q} \\
&\quad = \int_0^{\infty} \mathrm{d}q\,\frac{1}{\sqrt{q^2+(F^{-\frac{1}{2}}m)^2}} \\
&\quad\quad \times \mathrm{e}^{-i\sqrt{q^2+(F^{-\frac{1}{2}}m)^2}(t_1-t_2)+i(x_1-x_2)q-\epsilon q}.
\end{aligned} \tag{49}$$

We can apply the same method to analyze the continuum limit of the last integral in (46). Therefore, according to (49), (46), and (45), we can derive the continuum limit of the two-point correlation functions for the bosonic field,

$$\begin{aligned}
&\langle\Omega|\phi(x_1,t_1)\phi(x_2,t_2)|\Omega\rangle \\
&\quad = \lim_{a\to 0}\lim_{N\to\infty}\langle\Omega|\phi_n(t_1)\phi_m(t_2)|\Omega\rangle \\
&\quad = F^{-1}\int_{-\infty}^{\infty}\mathrm{d}q\,\frac{1}{4\pi\sqrt{q^2+(F^{-\frac{1}{2}}m)^2}} \\
&\quad\quad \times \mathrm{e}^{-i(t_1-t_2)\sqrt{q^2+(F^{-\frac{1}{2}}m)^2}+i(x_1-x_2)q}.
\end{aligned} \tag{50}$$

Now, introducing the renormalized bosonic field $\phi_r \equiv F^{\frac{1}{2}}\phi$ and the renormalized mass $m_r \equiv F^{-\frac{1}{2}}m$, the continuum limit of the lattice correlation function (50) recovers the two-point correlation function of the original RS model (8).

Considering the relationship between Ψ and ψ as given in Eq. (14), $\langle\Omega|\Psi(n,t_1)\bar{\Psi}(m,t_2)|\Omega\rangle$ can be expressed in terms of ψ. For example, one component of the fermionic two-point correlation function can be written as

$$\begin{aligned}
\langle\Omega|\Psi_d(n,t_1)\bar{\Psi}_u(m,t_2)|\Omega\rangle &= \mathrm{e}^{-g^2 f(0)}\mathrm{e}^{g^2\langle\Omega|\phi_n(t_1)\phi_m(t_2)|\Omega\rangle} \\
&\quad \times\langle\Omega|\psi_d(n,t_1)\psi_d^\dagger(m,t_2)|\Omega\rangle.
\end{aligned} \tag{51}$$

Using Eqs. (30), (31), and (28), we can express the correlation function of ψ as

$$\begin{aligned}
&\langle\Omega|\psi_d(n,t_1)\psi_d^\dagger(m,t_2)|\Omega\rangle \\
&\quad = -\frac{1}{4N}\sum_{q\neq 0}\mathrm{sgn}(q)\mathrm{e}^{-iE_q(t_1-t_2)+i(n-m+\frac{1}{2})aq} \\
&\quad\quad -\frac{1}{4N}\sum_{q\neq 0}\mathrm{sgn}(q)\mathrm{e}^{-iE_q(t_1-t_2)+i(n-m-\frac{1}{2})aq} \\
&\quad\quad +\frac{1}{2N}\sum_{q\neq 0}\mathrm{e}^{-iE_q(t_1-t_2)+i(n-m)aq}+\frac{1}{2N}.
\end{aligned} \tag{52}$$

Using the techniques for handling the continuum limit of bosonic correlation functions, we can also obtain the continuum limit of Eq. (52) (keeping $x_1 = na$ and $x_2 = ma$ invariant),

$$\begin{aligned}
&\langle\Omega|\psi_{cd}(x_1,t_1)\psi_{cd}^\dagger(x_2,t_2)|\Omega\rangle \\
&\quad = \lim_{a\to 0}\frac{1}{a}\lim_{N\to\infty}\langle\Omega|\psi_d(n,t_1)\psi_d^\dagger(m,t_2)|\Omega\rangle \\
&\quad = -\frac{i}{2\pi}\frac{1}{(t_1-t_2)+(x_1-x_2)},
\end{aligned} \tag{53}$$

where the relationship between the discrete field $\psi(n,t)$ and the continuous fermionic field $\psi_c(x,t)$ is given by $\psi(n,t) = \sqrt{a}\psi_c(x,t)|_{x=na}$.

Here we introduce the renormalized fermionic field $\Psi(n,t)_R \equiv \mathrm{e}^{\frac{1}{2}g^2 f(0)}\Psi(n,t)$, then based on Eqs. (51) and (53), the continuum limit of the renormalized correlation function $\langle\Omega|\Psi_d(n,t_1)_R\bar{\Psi}_u(m,t_2)_R|\Omega\rangle$ can be expressed as

$$\begin{aligned}
&\langle\Omega|\Psi_{rd}(x_1,t_1)\bar{\Psi}_{ru}(x_2,t_2)|\Omega\rangle \\
&\quad = \lim_{a\to 0}\frac{1}{a}\lim_{N\to\infty}\langle\Omega|\Psi_d(n,t_1)_R\bar{\Psi}_u(m,t_2)_R|\Omega\rangle \\
&\quad = -\frac{i}{2\pi}\frac{1}{(t_1-t_2)+(x_1-x_2)}\mathrm{e}^{g^2\langle\Omega|\phi(x_1,t_1)\phi(x_2,t_2)|\Omega\rangle},
\end{aligned} \tag{54}$$

where we utilized the relation between the discrete renormalized fermionic field $\Psi(n,t)_R$ and the continuous renormalized fermionic field $\Psi_r(x,t)$, which is given by $\Psi(n,t)_R = \sqrt{a}\Psi_r(x,t)|_{x=na}$.

Using a similar method, we can calculate the other components. If we further define the renormalized coupling constant as $g_r \equiv F^{-\frac{1}{2}}g$ and combine it with the definition of the renormalized bosonic field, $\phi_r \equiv F^{\frac{1}{2}}\phi$, then the continuum limit of the two-point correlation function for the fermionic field is

$$\begin{aligned}
&\langle\Omega|\Psi_r(x_1,t_1)\bar{\Psi}_r(x_2,t_2)|\Omega\rangle \\
&\quad = -\frac{i}{2\pi}\frac{\gamma_\mu x^\mu}{x^2}\mathrm{e}^{g_r^2\langle\Omega|\phi_r(x_1,t_1)\phi_r(x_2,t_2)|\Omega\rangle},
\end{aligned} \tag{55}$$

where $x^\mu = (x^0, x^1) = (t_1 - t_2, x_1 - x_2)$. This is indeed the two-point fermion correlation function (9) in the original RS model.

It is worth noting that here we have defined $g_r \equiv F^{-\frac{1}{2}} g$ to match the two-point fermion correlation functions on the lattice with those in the original RS model. However, in practice, the most accurate definition of the renormalized coupling constant typically involves the three-point correlation functions corresponding to interaction vertices. So, it is essential to verify whether $g_r \equiv F^{-\frac{1}{2}} g$ also ensures that the continuum limit of the three-point correlation functions matches those of the original RS model.

In a similar vein, we can compute all components of $\langle\Omega|\phi_n(t_1)\Psi(m,t_2)\bar{\Psi}(0,0)|\Omega\rangle$ and its continuum limit. Combining the results from all components, we obtain the continuum limit of the lattice three-point correlation function corresponding to the interaction vertex as

$$\begin{aligned}\langle\Omega|\phi_r(x)\Psi_r(y)\bar{\Psi}_r(0)|\Omega\rangle = ig_r\gamma^5\langle\Omega|\Psi_r(y)\bar{\Psi}_r(0)|\Omega\rangle\\ \times\left[\langle\Omega|\phi_r(x)\phi_r(y)|\Omega\rangle\right.\\ \left.-\langle\Omega|\phi_r(x)\phi_r(0)|\Omega\rangle\right].\end{aligned} \tag{56}$$

Equation (56) corresponds precisely to the three-point correlation function in the original RS model associated with the interaction vertex, as given in Eq. (10). This also confirms that the renormalized coupling constant can indeed be defined as $g_r \equiv F^{-\frac{1}{2}} g$. It is necessary to clarify that the three-point correlation function does not contradict our previous discussion, wherein the interaction can be eliminated by field redefinition. This is because the fields involved are nonlinear functions of the free fields in Eq. (14), which are directly related to the particle content of the theory. Indeed, the same phenomenon would occur if an arbitrary nonlinear field transformation were applied in any free field theory.

In this section, we have computed the two- and three-point correlation functions for the RS model on the lattice. We discovered that their continuum limit converges to the correlation functions of the original model after undergoing renormalization. This observation demonstrates that the intricate Hamiltonian we constructed in Eq. (13) effectively reproduces the correct behavior of the RS model in the continuum limit. It is worth noting that, in both the continuum limit of lattice theory and the original RS model, the bosonic field's correlation functions expressed in terms of the bare field and bare mass do not diverge (this is not the case for fermionic field correlation functions). Therefore, the bare fields ϕ and ϕ_0 are well-defined quantities in both theories. We can directly equate ϕ to ϕ_0, implying that the continuum limit of the lattice theory's bare field is equal to the original RS model's bare field. Consequently, the connection between the lattice bare parameters (m, F, g) and the bare parameters of the original RS model (m_0, g_0) can be summarized as

$$F = 1 - \frac{g_0^2}{\pi}, \quad m = m_0, \quad g = g_0. \tag{57}$$

However, in the original RS model, the correlation functions of the bare fermionic field would exhibit divergences, which can also be observed from the computation of lattice correlation functions. To be more specific, considering the definition of the renormalized lattice fermionic field as $\Psi(n,t)_R = e^{\frac{1}{2}g^2 f(0)}\Psi(n,t)$ and the expression for $f(n)$ in Eq. (25), it is apparent that the field-strength renormalization constant $Z = e^{-g^2 f(0)}$ for the lattice fermionic field tends to zero in the continuum limit, i.e., $\lim_{a\to 0}\lim_{N\to\infty} Z = 0$. To further summarize the relationship between the lattice RS model's renormalization parameters and bare parameters, we have the following:

$$g_r \equiv F^{-\frac{1}{2}} g, \quad \phi_r \equiv F^{\frac{1}{2}}\phi, \quad \Psi(n)_R = e^{\frac{1}{2}g^2 f(0)}\Psi(n). \tag{58}$$

VI. EIGENSTATES AND FIELD MIXING

In this section, our attention is drawn to the vacuum state and the clothed particles (one-particle states) of the complete Hamiltonian (13). Our objective is to understand how the Hamiltonian's original bosonic component and fermionic component combine to give rise to the eigenstates of the full Hamiltonian. This mixing of the bosonic field and the fermionic field provides insights into the interactions among the fundamental degrees of freedom, shaping the spectrum of the Hamiltonian and, ultimately, giving rise to the observable degrees of freedom.

The previously defined representations, namely, $\{|\phi\rangle_B\}$ and $\{\prod\psi'^\dagger|0\rangle_F\}$, each represent a subspace of the entire system. The complete system encompasses both the fermionic and bosonic fields, and it can be described as a composite system. The basis for this composite system can be formed by taking the direct product of the basis states from the representation $\{\prod\psi'^\dagger|0\rangle_F\}$ [as given in Eq. (35)] and the basis states from the representation $\{|\phi\rangle_B\}$,

$$\psi'^\dagger_{\alpha_1}(n_1)\psi'^\dagger_{\alpha_2}(n_2)\psi'^\dagger_{\alpha_3}(n_3)\cdots\psi'^\dagger_{\alpha_s}(n_s)|0\rangle_F|\phi\rangle_B, \tag{59}$$

where

$$s = 1,2,\ldots,N, \quad \alpha_i = u,d, \quad n_i \neq n_j, \quad \forall\ \alpha_i = \alpha_j.$$

We will denote the representation formed by the basis (59) as $\{\prod\psi'^\dagger|0\rangle_F|\phi\rangle_B\}$.

A representation similar to $\{\prod\psi'^\dagger|0\rangle_F\}$ can also be defined for the fermionic field Ψ. We define the quantum state $|0\rangle$ as an eigenstate of the fermionic field Ψ with an eigenvalue of zero,

$$\Psi_u(n)|0\rangle = 0, \quad \Psi_d(n)|0\rangle = 0. \tag{60}$$

Following this, we can apply any number of fermionic field operators $\Psi^\dagger(n)$ to the state $|0\rangle$, yielding a series of quantum states

$$\Psi^\dagger_{\alpha_1}(n_1)\Psi^\dagger_{\alpha_2}(n_2)\Psi^\dagger_{\alpha_3}(n_3)\cdots\Psi^\dagger_{\alpha_s}(n_s)|0\rangle. \tag{61}$$

It can be easily verified that the quantum states (61) are normalized by the commutation relations between Ψ and $\Psi^\dagger$ as well as the properties (60). Using the states (61) as a basis, we can define a new representation, denoted as $\{\prod\Psi^\dagger|0\rangle\}$. As for the bosonic field ϕ in the original Hamiltonian (13), a representation similar to $\{|\phi\rangle_B\}$ can be defined. The eigenstates of ϕ, denoted as $|\phi\rangle$, satisfy the equation

$$\hat{\phi}|\phi\rangle = \phi|\phi\rangle. \tag{62}$$

The representation constructed using $|\phi\rangle$ as a basis is denoted as $\{|\phi\rangle\}$. It is worth noting that, even though $|\phi\rangle$ in Eq. (62) and $|\phi\rangle_B$ are both eigenstates of the bosonic field $\hat{\phi}$ with an eigenvalue of ϕ, they belong to different subsystems, and they are not the same quantum state. The direct product of the basis from the representation $\{\prod\Psi^\dagger|0\rangle\}$ with the basis from the representation $\{|\phi\rangle\}$ yields a basis for the entire Hilbert space,

$$\Psi^\dagger_{\alpha_1}(n_1)\Psi^\dagger_{\alpha_2}(n_2)\Psi^\dagger_{\alpha_3}(n_3)\cdots\Psi^\dagger_{\alpha_s}(n_s)|0\rangle|\phi\rangle. \tag{63}$$

The representation constructed using (63) as a basis is denoted as $\{\prod\Psi^\dagger|0\rangle|\phi\rangle\}$.

Let us now derive the relationship between the representation $\{\prod\psi'^\dagger|0\rangle_F|\phi\rangle_B\}$ and the representation $\{\prod\Psi^\dagger|0\rangle|\phi\rangle\}$. In other words, we will investigate the connection between the quantum states (59) and (63). To begin, let us analyze the most special states in both representations, namely, $|0\rangle_F|\phi\rangle_B$ and $|0\rangle|\phi\rangle$. Based on Eqs. (14) and (28), we can derive the relationship between the field operators ψ' and the original field operators Ψ,

$$\psi'_u(n) = \frac{1}{\sqrt{2}}\left[\mathrm{e}^{-ig\phi_n}\Psi_u(n) + \mathrm{e}^{ig\phi_n}\Psi_d(n)\right],$$
$$\psi'_d(n) = \frac{1}{\sqrt{2}}\left[-\mathrm{e}^{-ig\phi_n}\Psi_u(n) + \mathrm{e}^{ig\phi_n}\Psi_d(n)\right], \tag{64}$$

$$\Psi_u(n) = \frac{1}{\sqrt{2}}\mathrm{e}^{ig\phi_n}\left[\psi'_u(n) - \psi'_d(n)\right],$$
$$\Psi_d(n) = \frac{1}{\sqrt{2}}\mathrm{e}^{-ig\phi_n}\left[\psi'_u(n) + \psi'_d(n)\right]. \tag{65}$$

Because both $\psi'_u(n)$ and $\psi'_d(n)$ annihilate the state $|0\rangle_F|\phi\rangle_B$, utilizing the relation given by Eq. (65) we have

$$\Psi_u(n)|0\rangle_F|\phi\rangle_B = \Psi_d(n)|0\rangle_F|\phi\rangle_B = 0. \tag{66}$$

This demonstrates that the quantum state $|0\rangle_F|\phi\rangle_B$ is an eigenstate of the field operator Ψ with an eigenvalue of zero. Additionally, this state is also an eigenstate of the field operator $\hat{\phi}$ with an eigenvalue of ϕ. Similarly, the quantum state $|0\rangle|\phi\rangle$ is an eigenstate of field operator Ψ with a zero eigenvalue and is simultaneously an eigenstate of field operator $\hat{\phi}$ with an eigenvalue of ϕ. Therefore, we can conclude that

$$|0\rangle_F|\phi\rangle_B = |0\rangle|\phi\rangle. \tag{67}$$

It is worth noting the basis of $\{\prod\psi'^\dagger|0\rangle_F|\phi\rangle_B\}$ and $\{\prod\Psi^\dagger|0\rangle|\phi\rangle\}$ are constructed based on $|0\rangle_F|\phi\rangle_B$ and $|0\rangle|\phi\rangle$, respectively. Therefore, we can establish the relationship between these two basis sets using Eq. (67). To be precise, by substituting (64) and (67) into (59), we can derive the explicit expression for the transformation between these two distinct bases,

$$\begin{aligned}&\psi'^\dagger_{\alpha_1}(n_1)\psi'^\dagger_{\alpha_2}(n_2)\psi'^\dagger_{\alpha_3}(n_3)\cdots\psi'^\dagger_{\alpha_s}(n_s)|0\rangle_F|\phi\rangle_B\\ &\quad=\prod_{i=1}^{s}\frac{1}{\sqrt{2}}\left[(-1)^{f(\alpha_i)}\mathrm{e}^{ig\phi_{n_i}}\Psi^\dagger_u(n_i)+\mathrm{e}^{-ig\phi_{n_i}}\Psi^\dagger_d(n_i)\right]|0\rangle|\phi\rangle,\end{aligned} \tag{68}$$

where $f(u) = 0$ and $f(d) = 1$. Upon expanding the product in Eq. (68), we obtain a series of summations over quantum states (63). This indicates that we have effectively expressed the basis of $\{\prod\psi'^\dagger|0\rangle_F|\phi\rangle_B\}$ in terms of the basis of $\{\prod\Psi^\dagger|0\rangle|\phi\rangle\}$. With the relation given by Eq. (68), we can now express the vacuum state and clothed particles of the Hamiltonian (13) in the representation $\{\prod\Psi^\dagger|0\rangle|\phi\rangle\}$.

Using Eqs. (21) and (39), we can derive the representation of the vacuum state $|\Omega\rangle$ of the total Hamiltonian (13) in terms of the basis $\{\prod\psi'^\dagger|0\rangle_F|\phi\rangle_B\}$,

$$\begin{aligned}|\Omega\rangle &= |\Omega\rangle_F|\Omega\rangle_B = \sqrt{2}\mathcal{N}F^{\frac{1}{2}}\int \mathrm{d}\phi\,\mathrm{e}^{-\frac{F}{2}\sum_{n,m}\mathcal{E}_{nm}\phi_n\phi_m}\\ &\quad\times\prod_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}}\frac{1}{\sqrt{2N}}\left[\mathrm{sgn}(k)\mathrm{e}^{-i\frac{1}{2}\frac{2\pi k}{N}}\sum_n\mathrm{e}^{-in\frac{2\pi k}{N}}\psi'^\dagger_u(n)\right.\\ &\quad\left.-\sum_n\mathrm{e}^{-in\frac{2\pi k}{N}}\psi'^\dagger_d(n)\right]|0\rangle_F|\phi\rangle_B.\end{aligned} \tag{69}$$

Based on the transformation relations between the bases of the representations $\{\prod\psi'^\dagger|0\rangle_F|\phi\rangle_B\}$ and $\{\prod\Psi^\dagger|0\rangle|\phi\rangle\}$

given by Eq. (68), we can express the vacuum state (69) in the representation $\{\prod \Psi^\dagger|0\rangle|\phi\rangle\}$ as follows:

$$|\Omega\rangle = \sqrt{2}\mathcal{N}F^{\frac{1}{2}}\int \mathrm{d}\phi\, \mathrm{e}^{-\frac{F}{2}\sum_{n,m}\mathcal{E}_{nm}\phi_n\phi_m} \times \prod_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} \frac{1}{2\sqrt{N}}\Bigg[\left(\mathrm{sgn}(k)\mathrm{e}^{-i\frac{1}{2}\frac{2\pi k}{N}}+1\right) \times \sum_n \mathrm{e}^{-in\frac{2\pi k}{N}}\mathrm{e}^{ig\phi_n}\Psi_u^\dagger(n) + \left(\mathrm{sgn}(k)\mathrm{e}^{-i\frac{1}{2}\frac{2\pi k}{N}}-1\right)\sum_n \mathrm{e}^{-in\frac{2\pi k}{N}}\mathrm{e}^{-ig\phi_n}\Psi_d^\dagger(n)\Bigg]|0\rangle|\phi\rangle. \tag{70}$$

This is the physical vacuum state represented using the original field degrees of freedom in the Hamiltonian (13). Note that each fermionic field operator $\Psi^\dagger(n)$ is first attached by a value $\mathrm{e}^{ig\phi_n}$ related to the bosonic field eigenstate before being summed. This makes the vacuum state an entangled state, where fermionic and bosonic field degrees of freedom are entangled, exhibiting a mixing between the bosonic and fermionic fields, rather than a simple direct product of the bare fermion vacuum and bare boson vacuum.

Likewise, we can derive the one-boson state $|q;B\rangle$ in the representation $\{\prod \Psi^\dagger|0\rangle|\phi\rangle\}$ using (22), (39), and (68),

$$|q;B\rangle = 2\mathcal{N}F\sqrt{L\omega_q}\frac{1}{N}\int \mathrm{d}\phi\, \mathrm{e}^{-\frac{F}{2}\sum_{n,m}\mathcal{E}_{nm}\phi_n\phi_m}\left(\sum_n \mathrm{e}^{inaq}\phi_n\right) \times \prod_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} \frac{1}{2\sqrt{N}}\Bigg[\left(\mathrm{sgn}(k)\mathrm{e}^{-i\frac{1}{2}\frac{2\pi k}{N}}+1\right) \times \sum_n \mathrm{e}^{-in\frac{2\pi k}{N}}\mathrm{e}^{ig\phi_n}\Psi_u^\dagger(n) + \left(\mathrm{sgn}(k)\mathrm{e}^{-i\frac{1}{2}\frac{2\pi k}{N}}-1\right)\sum_n \mathrm{e}^{-in\frac{2\pi k}{N}}\mathrm{e}^{-ig\phi_n}\Psi_d^\dagger(n)\Bigg]|0\rangle|\phi\rangle. \tag{71}$$

This is the one-boson state represented using the original field degrees of freedom in the Hamiltonian (13). Once again, we can observe that the one-boson state also exhibits entanglement, mixing the fermionic and bosonic degrees of freedom. The only difference between the one-boson state and the vacuum state is the additional term in the one-boson state, which is given by $\sum_n \mathrm{e}^{inaq}\phi_n$. However, this term does not involve fermionic content. Therefore, it may be said that the entanglement between the fermionic and bosonic fields in the one-boson state arises from the entanglement in the vacuum state.

Similarly, we can apply a nearly identical approach to derive the one-fermion state in the basis of $\{\prod \Psi^\dagger|0\rangle|\phi\rangle\}$. By employing (21), (40), and (41), we can deduce the one-fermion state $|q;F+\rangle$ as well as the one-antifermion state $|q;F-\rangle$ of the total Hamiltonian (13),

$$|q;F+\rangle = \mathcal{N}F^{\frac{1}{2}}\int \mathrm{d}\phi\, \mathrm{e}^{-\frac{F}{2}\sum_{n,m}\mathcal{E}_{nm}\phi_n\phi_m}\frac{1}{\sqrt{N}}\Bigg[\left(\mathrm{e}^{i\frac{1}{2}\frac{2\pi k}{N}}+\mathrm{sgn}(k)\right)\sum_n \mathrm{e}^{in\frac{2\pi k}{N}}\mathrm{e}^{ig\phi_n}\Psi_u^\dagger(n) + \left(\mathrm{e}^{i\frac{1}{2}\frac{2\pi k}{N}}-\mathrm{sgn}(k)\right)\sum_n \mathrm{e}^{in\frac{2\pi k}{N}}\mathrm{e}^{-ig\phi_n}\Psi_d^\dagger(n)\Bigg]\prod_{k'=-\frac{N-1}{2}}^{\frac{N-1}{2}}\frac{1}{2\sqrt{N}}\Bigg[\left(\mathrm{sgn}(k')\mathrm{e}^{-i\frac{1}{2}\frac{2\pi k'}{N}}+1\right)\sum_n \mathrm{e}^{-in\frac{2\pi k'}{N}}\mathrm{e}^{ig\phi_n}\Psi_u^\dagger(n) + \left(\mathrm{sgn}(k')\mathrm{e}^{-i\frac{1}{2}\frac{2\pi k'}{N}}-1\right)\sum_n \mathrm{e}^{-in\frac{2\pi k'}{N}}\mathrm{e}^{-ig\phi_n}\Psi_d^\dagger(n)\Bigg]|0\rangle|\phi\rangle, \tag{72}$$

$$|q;F-\rangle = \sqrt{2}\mathcal{N}F^{\frac{1}{2}}\int \mathrm{d}\phi\, \mathrm{e}^{-\frac{F}{2}\sum_{n,m}\mathcal{E}_{nm}\phi_n\phi_m}\prod_{k'=-\frac{N-1}{2}}^{\frac{N-1}{2};k'\neq k}\frac{1}{2\sqrt{N}}\Bigg[\left(\mathrm{sgn}(k')\mathrm{e}^{-i\frac{1}{2}\frac{2\pi k'}{N}}+1\right)\sum_n \mathrm{e}^{-in\frac{2\pi k'}{N}}\mathrm{e}^{ig\phi_n}\Psi_u^\dagger(n) + \left(\mathrm{sgn}(k')\mathrm{e}^{-i\frac{1}{2}\frac{2\pi k'}{N}}-1\right)\sum_n \mathrm{e}^{-in\frac{2\pi k'}{N}}\mathrm{e}^{-ig\phi_n}\Psi_d^\dagger(n)\Bigg]|0\rangle|\phi\rangle, \tag{73}$$

where $k \equiv \frac{aN}{2\pi}q$. This is the one-fermion state represented using the original field degrees of freedom in the Hamiltonian (13). It also exhibits entanglement between fermionic and bosonic degrees of freedom. However, the structure of fermionic (antifermionic) one-particle states is vastly different from that of one-boson states. In addition, Eqs. (71)–(73) indicate that there is a significant distinction in the structure between clothed particles (real one-particle states) and bare particles. Because of the interaction between the fermionic field and bosonic field in the Hamiltonian, the excitation of the fermionic clothed particles involves both the fermionic field and the bosonic field [considering Eqs. (37), (38), and (64) together], rather than just the fermionic field alone. Although the boson creation operator is composed entirely of the bosonic field [considering Eqs. (14) and (19) together], due to the vacuum entanglement between bosonic and fermionic field degrees of freedom, bosonic clothed particles also exhibit a mixture of fermionic and bosonic content.

Furthermore, the representation $\{\prod \Psi^\dagger|0\rangle|\phi\rangle\}$ can also reveal the spatial entanglement structure of quantum states. As seen from (63), the basis in the representation $\{\prod \Psi^\dagger|0\rangle|\phi\rangle\}$ is formed by the direct product of the fermionic part $\Psi^\dagger_{\alpha_1}(n_1)\Psi^\dagger_{\alpha_2}(n_2)\Psi^\dagger_{\alpha_3}(n_3)\cdots\Psi^\dagger_{\alpha_s}(n_s)|0\rangle$ and the bosonic part $|\phi\rangle$. The quantum states of the bosonic part are eigenstates of the bosonic field operator $\hat{\phi}|\phi\rangle = \phi|\phi\rangle$. The field operators $\hat{\phi}$ at different points are independent of each other and satisfy the commutation relations $[\hat{\phi}_n, \hat{\phi}_m] = 0$. As a result, the eigenstate of $\hat{\phi}$ can be written in a direct product formulation as follows: $|\phi\rangle = |\phi_1\rangle_1|\phi_2\rangle_2|\phi_3\rangle_3\cdots = \prod_n|\phi_n\rangle_n$, where $|\phi\rangle_n$ is the eigenstate of $\hat{\phi}_n$ but in a Hilbert space constructed only for the lattice point n.

Similarly, due to the anticommutation of fermionic fields at different spatial points, one can define the following four quantum states at a specific spatial point n: $|00\rangle_n$, $|10\rangle_n \equiv \Psi^\dagger_u(n)|00\rangle_n$, $|01\rangle_n \equiv \Psi^\dagger_d(n)|00\rangle_n$, $|11\rangle_n \equiv \Psi^\dagger_u(n)\times \Psi^\dagger_d(n)|00\rangle_n$. Then, the quantum state $|0\rangle$ can be expressed as the direct product of quantum states at different spatial points: $|0\rangle = |00\rangle_0|00\rangle_1\cdots|00\rangle_{N-1}$. For quantum states of the form like (61), they can also be expressed as a direct product of quantum states at different spatial points,

$$\begin{aligned}&\Psi^\dagger_u(n_1)\Psi^\dagger_u(n_2)\Psi^\dagger_d(n_2)\cdots\Psi^\dagger_d(n_s)|0\rangle\\&=\cdots|00\rangle_{n_1-1}|10\rangle_{n_1}|00\rangle_{n_1+1}\cdots|00\rangle_{n_2-1}|11\rangle_{n_2}|00\rangle_{n_2+1}\cdots\\&\quad\times|00\rangle_{n_s-1}|01\rangle_{n_s}|00\rangle_{n_s+1}\cdots. \qquad (74)\end{aligned}$$

This signifies that the basis of the representation $\{\prod \Psi^\dagger|0\rangle|\phi\rangle\}$ can be expressed as a direct product of quantum states at different spatial points. Consequently, employing the representation $\{\prod \Psi^\dagger|0\rangle|\phi\rangle\}$ not only highlights the entanglement between fermionic and bosonic fields, as mentioned earlier, but also reveals the spatial entanglement structure of the vacuum state through (70), while (71)–(73) also illustrate the spatial entanglement structure of the clothed particles.

VII. CONCLUSIONS AND DISCUSSIONS

We have presented the lattice Hamiltonian of the RS model, diagonalized it, and subsequently derived lattice correlation functions, as well as the physical vacuum and clothed particles. The continuum limit of the lattice correlation functions matches the original RS model's correlation functions, affirming that the continuum limit of the lattice theory corresponds to the original RS model. In order to gain a more intuitive understanding of the complex Hamiltonian, we have analyzed the equations of motion for the lattice theory in the Appendixes. In Appendix A, we obtained the equations of motion for the bosonic field in the lattice RS model and compared them with those of the original RS model. Similarly, in Appendix B, we derived the equations of motion for the fermionic field in the lattice RS model and compared them to the original RS model. It is worth noting that the equations of motion for the lattice RS model share the same structure as those of the original RS model, with only some differences in the coefficients of regularization terms. These differences arise because the original RS model describes the infrared behavior of the lattice RS model. Even the ultraviolet behavior of the original RS model falls under the infrared behavior of the lattice model. Consequently, the coefficients of the regularization terms in the equations of motion for the original RS model differ slightly from those of the lattice model. However, this discrepancy does not imply that the continuum limit of the lattice theory is not the original RS model. As mentioned earlier, the continuum limit of the lattice correlation functions matches the original RS model's correlation functions, and the behavior exhibited by taking the continuum limit first and then letting the field spacing tend to zero aligns with that of the original RS model.

Creation and annihilation operators directly associated with the bare fields are referred to as "bare operators," denoted as a_p. One-particle states that are eigenstates of the Hamiltonian are called clothed particles, and the creation and annihilation operators that produce clothed particles from the physical vacuum are called clothed operators, denoted as α_p. Then, clothed operators α can be expressed in terms of bare operators a, with the specific "clothing transformation" given by $\alpha_p = W^\dagger a_p W$, where the transformation operator W is a function of all bare operators a and satisfies $W^\dagger W = WW^\dagger = 1$ [55]. It is worth noting that the transformation operators W induced by interactions such as $\mathcal{L}_I = -m_{e\mu}(\bar{\nu}_e\nu_\mu + \bar{\nu}_\mu\nu_e)$ and $L_I = -\lambda(\phi^\dagger_\alpha\phi_\beta + \phi^\dagger_\beta\phi_\alpha)$ have been studied in previous works [37,40].

Although we did not adopt the Fock representation in this paper, we can still convert the results of the paper into the Fock representation and express the clothed operators in terms of bare operators. In fact, the operators a_p in Eq. (19), d_p in Eq. (37), and b_p in Eq. (38) are already clothed operators. If we denote the corresponding bare operators as A_p, D_p, and B_p, then by combining Eqs. (19), (37), (38), and (14), we can express the clothed operators (a, d, b) in terms of bare operators (A, D, B), i.e., $x_p = f(A, D, B)$, where $x = a, d, b$. Since we are considering the interaction $\Delta\mathcal{L} = -g\partial_u\phi\bar{\Psi}\gamma^5\gamma^\mu\Psi$, which is more complex than the quadratic interactions mentioned above, expressing it in the form $x_p = W(A, D, B)^\dagger X_p W(A, D, B)$ (where $X = A$, D, B) would require further derivations and calculations. However, it can be anticipated that $W(A, D, B)$ will be highly complex and challenging to intuitively understand.

Therefore, in this paper we did not employ the conventional Fock representation but instead chose the representation defined by the basis (63), denoted as $\{\prod \Psi^\dagger|0\rangle|\phi\rangle\}$. In this representation, the specific form of the physical vacuum of the RS model is given by Eq. (70), the specific form of the one-boson state is given by Eq. (71), and the one-fermion states and one-antifermion states are given by Eqs. (72) and (73), respectively. It can be observed that both the physical vacuum and the clothed particles exhibit entanglement between the bosonic and fermionic fields.

In addition to the entanglement between the fermionic and bosonic fields, the basis vectors of the representation $\{\prod \Psi^\dagger|0\rangle|\phi\rangle\}$ can all be expressed as direct products of quantum states at different spatial points, allowing us to directly observe the spatial entanglement structure of the quantum states. The vacuum in classical theory is local, meaning that if space is divided into many parts, each part is still a vacuum. However, in quantum field theory, due to the nonlocality of quantum states and the entanglement between spatial points, strictly speaking, the vacuum state cannot be simply divided into two parts. Nevertheless, since the vacuum state corresponds to the classical vacuum, it should exhibit locality on large scales. Let us first ignore the entanglement between the bosonic and fermionic fields and focus solely on the wave function of the bosonic part in the vacuum state (70), which is given by $\int \mathrm{d}\phi \mathrm{e}^{-\frac{F}{2}\sum_{n,m}\mathcal{E}_{nm}\phi_n\phi_m}$. If $\mathcal{E}_{nm} \propto \delta_{nm}$, then there would be no entanglement between different spatial points in the bosonic vacuum state, and it would exhibit the same locality as the classical vacuum. However, in reality, $\mathcal{E}_{nm} = \frac{1}{N}\sum_q a\omega_q \mathrm{e}^{i(n-m)aq}$ is not proportional to δ_{nm}, indicating entanglement between different spatial points and preventing the vacuum from being arbitrarily divided into two parts. This seems contradictory to the locality of the classical vacuum. Nevertheless, it can be easily proven that $\mathcal{E}_{nm} \propto e^{-m_r|na-ma|} \to 0$ as $|na-ma| \to \infty$. This implies that the entanglement between points with large spatial separations becomes weak. Consequently, from a macroscopic perspective, the bosonic part of the vacuum state (70) does indeed exhibit locality similar to the classical vacuum.

However, for the fermionic part of the vacuum state (70), it is challenging to intuitively discern locality. Moreover, the complete quantum state exhibits entanglement between fermions and bosons. Therefore, we need a more quantitative analysis of this entanglement. Specifically, we can choose two regions, denoted as regions A and B, with their union referred to as region A + B. Since $\prod \Psi^\dagger|0\rangle|\phi\rangle$ is a representation based on real space, the entanglement entropy of regions A, B, and A + B can be computed in this representation. If the sum of the entanglement entropy of region A and the entanglement entropy of region B, minus the entanglement entropy of region A + B, decreases rapidly as the distance between the regions A and B increases, it confirms that the vacuum state exhibits locality similar to the classical vacuum from a macroscopic perspective.

In the future, in addition to computing the entanglement entropy of quantum states, we can consider introducing external sources to the RS model to make the system nonuniform and study the possible emergence of spatial cloud structures. We can also introduce a fermion mass term to the RS model and develop perturbation theory based on this work to compute quantum states. Essentially, this paper provides a solvable Hamiltonian containing a three-point interaction, from which Feynman rules can be derived in a well-defined manner, demonstrating how the bare parameters of the Hamiltonian evolve into various parameters of the lower-level Feynman diagram.

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APPENDIX A: THE EQUATION OF MOTION FOR THE BOSONIC FIELD

We denote the positive and negative frequency components of $\psi_u(n)$ as $\psi_u^+(n)$ and $\psi_u^-(n)$, respectively, and the positive and negative frequency components of $\psi_d(n)$ as $\psi_d^+(n)$ and $\psi_d^-(n)$. Based on (30), (31), and (28), we can further derive the following commutation relations:

$$\{(\psi_u^-(n))^\dagger, \psi_u^-(n+m)\} = \frac{1}{2}\left[\delta_{m,0} - i\frac{1}{2N}\cot\left[\left(m+\frac{1}{2}\right)\frac{\pi}{N}\right] - i\frac{1}{2N}\cot\left[\left(m-\frac{1}{2}\right)\frac{\pi}{N}\right]\right], \tag{A1}$$

$$\{(\psi_d^-(n))^\dagger, \psi_d^-(n+m)\} = \frac{1}{2}\left[\delta_{m,0} + i\frac{1}{2N}\cot\left[\left(m+\frac{1}{2}\right)\frac{\pi}{N}\right] + i\frac{1}{2N}\cot\left[\left(m-\frac{1}{2}\right)\frac{\pi}{N}\right]\right], \tag{A2}$$

$$\{(\psi_d^-(n))^\dagger, \psi_u^-(n+m)\} = \frac{1}{2}\left[i\frac{1}{2N}\cot\left[\left(m+\frac{1}{2}\right)\frac{\pi}{N}\right] - i\frac{1}{2N}\cot\left[\left(m-\frac{1}{2}\right)\frac{\pi}{N}\right] - \frac{1}{N}\right], \tag{A3}$$

$$\{(\psi_u^-(n))^\dagger, \psi_d^-(n+m)\} = \frac{1}{2}\left[i\frac{1}{2N}\cot\left[\left(m-\frac{1}{2}\right)\frac{\pi}{N}\right] - i\frac{1}{2N}\cot\left[\left(m+\frac{1}{2}\right)\frac{\pi}{N}\right] - \frac{1}{N}\right]. \tag{A4}$$

Let us further require that $\Psi_c = (\lim_{a\to 0}\lim_{N\to\infty} C^{-\frac{1}{2}})\Psi_0$, where $\Psi_c = \lim_{a\to 0}\lim_{N\to\infty}\frac{1}{\sqrt{a}}\Psi(n)$ and $C \equiv \mathrm{e}^{g^2[f(0)-f(1)]}$.

In this case, the continuous limit of $\Psi_0[n] \equiv C^{-\frac{1}{2}}\frac{1}{\sqrt{a}}\Psi(n)$ becomes the bare fermionic field Ψ_0 of the original RS model. Therefore, in order to compare the lattice equation of motion with those of the original RS model, we should express the equation of motion in terms of $\Psi_0[n]$. With the help of the commutation relations (A1)–(A4) and Eq. (58), we can derive the equation of motion for the bosonic field ϕ from the lattice Hamiltonian (13),

$$\begin{aligned}
&\partial^0\partial_0\phi_n - \frac{1}{a}\left(\frac{\phi_{n+1}-\phi_n}{a} - \frac{\phi_n-\phi_{n-1}}{a}\right) + m_0^2\phi_n \\
&= -g_0\frac{1}{2}\partial_0\left(\Psi_{0u}^\dagger[n+1]\Psi_{0u}[n] + \Psi_{0u}^\dagger[n]\Psi_{0u}[n+1] - \Psi_{0d}^\dagger[n+1]\Psi_{0d}[n] - \Psi_{0d}^\dagger[n]\Psi_{0d}[n+1]\right) \\
&\quad - g_0\frac{1}{2}\left(\frac{\Psi_{0u}^\dagger[n+1]\Psi_{0u}[n] - \Psi_{0u}^\dagger[n]\Psi_{0u}[n-1]}{a} + \frac{\Psi_{0u}^\dagger[n]\Psi_{0u}[n+1] - \Psi_{0u}^\dagger[n-1]\Psi_{0u}[n]}{a}\right. \\
&\quad \left. + \frac{\Psi_{0d}^\dagger[n+1]\Psi_{0d}[n] - \Psi_{0d}^\dagger[n]\Psi_{0d}[n-1]}{a} + \frac{\Psi_{0d}^\dagger[n]\Psi_{0d}[n+1] - \Psi_{0d}^\dagger[n-1]\Psi_{0d}[n]}{a}\right) \\
&\quad + \frac{g_0^2}{\pi}\partial^0\partial_0\phi_n + \frac{g_0^2}{3\pi}\frac{1}{a}\left(\frac{\phi_{n+1}-\phi_n}{a} - \frac{\phi_n-\phi_{n-1}}{a}\right) + O(a).
\end{aligned} \tag{A5}$$

To correspond to the lattice regularization established in the time-slicing framework, we set the regularization parameters of the original RS model to be at equal time intervals, i.e., $\epsilon^0 = 0$ and $\epsilon^1 = \epsilon$. In this case, the coefficient in front of the term $\partial_1\partial_1\phi_0$ in the motion equation of the original RS model, given by (3), is zero. By using the specific representation of the γ matrices provided in (2), we can observe that the only difference between the lattice bosonic field's motion equation (A5) and the continuum limit of the original RS model's Eq. (3) lies in the last term of (A5). In the lattice equation, this term is given by $\lim_{a\to 0}\frac{g_0^2}{3\pi}\frac{1}{a}(\frac{\phi_{0n+1}-\phi_{0n}}{a} - \frac{\phi_{0n}-\phi_{0n-1}}{a}) = \frac{g_0^2}{3\pi}\partial_1\partial_1\phi_0$, from which we can clearly see that the coefficient in front of the $\partial_1\partial_1\phi_0$ term is $\frac{g_0^2}{3\pi}$. In contrast, the corresponding term in the original RS model's Eq. (3) has a coefficient of 0.

Let us discuss why the continuum limit of the lattice motion equation (A5) includes an additional term $\frac{g_0^2}{3\pi}\partial_1\partial_1\phi_0$ compared to the original RS model's Eq. (3). Defining the variable $x \equiv na$, we can obtain the following result with the help of the commutation relations (A1)–(A4):

$$\begin{aligned}
C'\Psi_{cu}^\dagger(x)\Psi_{cu}(x+\epsilon) &= \lim_{a\to 0}\frac{1}{a}\lim_{N\to\infty} \mathrm{e}^{[g\phi_n^+, g\phi_n^-]-[g\phi_{n+m}^+, g\phi_n^-]} \\
&\quad\times \Psi_u^\dagger(n)\Psi_u(n+m) \\
&= \psi_{cu}^\dagger(x)\psi_{cu}(x+\epsilon) \\
&\quad + \frac{1}{2\pi}g\frac{\phi(x+\epsilon)-\phi(x)}{\epsilon},
\end{aligned} \tag{A6}$$

where C' is defined as

$$\begin{aligned}
C' &= \lim_{a\to 0}\lim_{N\to\infty} \mathrm{e}^{[g\phi_n^+, g\phi_n^-]-[g\phi_{n+m}^+, g\phi_n^-]} \\
&= \lim_{a\to 0}\lim_{N\to\infty} \mathrm{e}^{g^2[f(0)-f(m)]} = \lim_{a\to 0}\lim_{N\to\infty} \mathrm{e}^{g^2[f(0)-f(\frac{\epsilon}{a})]}.
\end{aligned} \tag{A7}$$

Utilizing (A6), we obtain the regularization for the field operator product of Ψ_c as

$$\begin{aligned}
\lim_{\epsilon\to 0} C'\Psi_{cu}^\dagger(x)\Psi_{cu}(x+\epsilon) &= \lim_{\epsilon\to 0}\psi_{cu}^\dagger(x)\psi_{cu}(x+\epsilon) \\
&\quad + \frac{1}{2\pi}g\lim_{\epsilon\to 0}\frac{\phi(x+\epsilon)-\phi(x)}{\epsilon}.
\end{aligned} \tag{A8}$$

Equation (A8) is precisely the axial-vector current formula in the original RS model [see Eq. (3.35) in [57]],

$$N[\bar{\Psi}_0(x)\gamma^5\gamma^\mu\Psi_0(x)] = j_{5f}^\mu(x) + \frac{g_0}{\pi}\partial^\mu\phi_0(x) \equiv j_5^\mu(x), \tag{A9}$$

where $\mu = 1$. The original RS model indeed used (A9) to derive the equations of motion (3), demonstrating that our lattice theory is consistent with the original RS model.

To further clarify the reason for the coefficient difference between the lattice motion equation and the original RS model motion equation, let us analyze the continuum limit

$$\begin{aligned}
\lim_{a\to 0}\frac{1}{a}\lim_{N\to\infty} C\Psi_u^\dagger(n)\Psi_u(n+1) &= \lim_{a\to 0}\frac{1}{a}\lim_{N\to\infty}\psi_u^\dagger(n)\psi_u(n+1) \\
&\quad + \frac{2}{3\pi}g\lim_{a\to 0}\lim_{N\to\infty}\frac{\phi_{n+1}-\phi_n}{a}.
\end{aligned} \tag{A10}$$

In Eq. (A10), the coefficient in front of the coupling constant g is $\frac{2}{3\pi}$, while in Eq. (A8), the coefficient in front of the coupling constant g is $\frac{1}{2\pi}$. The reason for this difference in coefficients between the two equations lies in the order of taking limits. To be more specific, in deriving Eq. (A8), we took three limits in total, denoted as $\lim_{\epsilon\to 0}\lim_{a\to 0}\lim_{N\to\infty}$. It is important to note that, in this sequence, we first let a tend to zero and then let ϵ tend to zero. However, if we require ϵ and a to simultaneously approach zero, i.e., $\lim_{\epsilon=a\to 0}\lim_{N\to\infty}$, we will obtain

Eq. (A10), and the coefficient in front of the coupling constant g will change from $\frac{1}{2\pi}$ to $\frac{2}{3\pi}$. This leads to an additional term in the continuum limit of the lattice motion equation (A5) compared to the original RS model's motion equation (3). This additional term is precisely the one mentioned repeatedly before, i.e., $\lim_{a\to 0}\frac{g_0^2}{3\pi}\frac{1}{a}\left(\frac{\phi_{0n+1}-\phi_{0n}}{a}-\frac{\phi_{0n}-\phi_{0n-1}}{a}\right)=\frac{g_0^2}{3\pi}\partial_1\partial_1\phi_0$.

Physically, this difference fundamentally originates from the lattice's ultraviolet behavior. After taking the continuum limit $\lim_{a\to 0}\lim_{N\to\infty}$ of the lattice theory, we obtain a continuous theory, which captures the infrared behavior of the lattice. Roughly speaking this is because, for any nonzero field separation ϵ, the distance between $\Psi^\dagger_{cu}(x)$ and $\Psi_{cu}(x+\epsilon)$ already encompasses infinitely many lattice sites in the continuum limit, and the lattice's ultraviolet behavior has been eliminated in this limit, leaving only the infrared behavior. In this context, even as the field separation ϵ tends to zero, it does not touch upon the lattice's ultraviolet behavior. However, if we set the field spacing $\epsilon = a$ while the continuum limit $\lim_{a\to 0}$ is not yet complete, it means that even later, no matter how small the lattice spacing becomes, the two fields will always be on adjacent lattice points. This implies that the lattice ultraviolet behavior continues to affect the system, even after taking the "continuum limit." The lattice theory's motion equation incorporates lattice ultraviolet behavior that the original RS model's motion equation lacks, which is why there are slight differences in their equations of motion. However, this does not imply that the continuum limit of the lattice theory is not the original RS model. By comparing (A8) and (A9), we observe that the lattice theory aligns with the continuous field theory of the RS model if we first take the continuum limit and subsequently let the interval ϵ vanish.

APPENDIX B: THE EQUATION OF MOTION FOR THE FERMIONIC FIELD

The equation of motion for Ψ can be derived from the lattice Hamiltonian (13),

$$\begin{aligned}
&i\gamma^0\partial_0\Psi(n)+i\gamma^1\frac{1}{2}\left\{\left[\Psi(n+1)-\Psi(n)\right]\frac{1}{a}\right.\\
&\quad\left.+\left[\Psi(n)-\Psi(n-1)\right]\frac{1}{a}\right\}\\
&=g\gamma^5\gamma^0\Psi(n)\partial_0\phi_{n-1}+g\gamma^5\gamma^1\Psi(n)(\phi_n-\phi_{n-1})\frac{1}{a}\\
&\quad-ig^2\frac{1}{F\pi}\gamma^5\frac{a\gamma_1}{-a^2}\Psi(n)+KO(a^{\frac{3}{2}}),
\end{aligned}\tag{B1}$$

where $K\equiv e^{-\frac{1}{2}g^2[\phi_n^+,\phi_n^-]}=e^{-\frac{1}{2}g^2 f(0)}$. Letting the lattice spacing $\epsilon = a$ in Eq. (6), and based on Eq. (58) and the definition of K, we find that $\Psi[n]_R\equiv\frac{1}{\sqrt{a}}e^{\frac{1}{2}g^2 f(0)}\Psi(n)=K^{-1}\frac{1}{\sqrt{a}}\Psi(n)$ has the continuum limit of the renormalized continuous fermionic field $\Psi_r(x)$. On the other hand, the renormalized lattice fermionic field given in Sec. V is $\Psi(n)_R=e^{\frac{1}{2}g^2 f(0)}\Psi(n)=K^{-1}\Psi(n)$, which exactly matches $\Psi[n]_R=\frac{1}{\sqrt{a}}\Psi(n)_R$. This implies that the fact that the continuum limit of $\Psi[n]_R$ is $\Psi_r(x)$ is consistent with the correspondence between lattice and continuous fermionic fields discussed in Sec. V. Based on the relation between renormalized parameters and bare parameters (58), we can rewrite Eq. (B1) as

$$\begin{aligned}
&i\gamma^0\partial_0\Psi[n]_R+i\gamma^1\frac{1}{2}\left\{\left[\Psi[n+1]_R-\Psi[n]_R\right]\frac{1}{a}\right.\\
&\quad\left.+\left[\Psi[n]_R-\Psi[n-1]_R\right]\frac{1}{a}\right\}\\
&=g_r\gamma^5\left[\gamma^0\Psi[n]_R\partial_0\phi_{n-1}+\gamma^1\Psi[n]_R(\phi_n-\phi_{n-1})\frac{1}{a}\right.\\
&\quad\left.-i\frac{g_r}{\pi}\frac{a\gamma_1}{-a^2}\Psi[n]_R\right]+O(a).
\end{aligned}\tag{B2}$$

Comparing the continuum limit of (B2) with the fermionic field equation of the original RS model (4) (with regularization parameters set as equal time intervals $\epsilon^0=0,\epsilon^1=\epsilon$), we notice that the two equations are nearly identical. The only difference lies in the coefficient of the last term: it is $\frac{g_r}{\pi}$ in Eq. (B2), whereas in Eq. (4), it is $\frac{g_r}{2\pi}$.

We have previously analyzed the reasons for the differences between the lattice bosonic field equation and the original RS model equation. Here, the same reasons lead to slight differences in the coefficients of the lattice fermionic field equation compared to the original RS model equation. Specifically, the mathematical origin of these differences lies in the distinct orders of taking limits. The correct limit to obtain the original RS Eq. (4) can be effectively thought of as $\lim_{\epsilon\to 0}\lim_{a\to 0}\lim_{N\to\infty}$, whereas the continuum limit of the lattice equation (B2) is $\lim_{\epsilon=a\to 0}\lim_{N\to\infty}$. From a physical perspective, the limit $\lim_{\epsilon=a\to 0}\lim_{N\to\infty}$ makes (B2) exhibit the lattice's ultraviolet behavior. Therefore, (B2) differs slightly from (4), which only contains lattice infrared behavior.

However, this does not imply that the continuum limit of the lattice theory is not the original RS model. It is worth noting that the alternative form of Eq. (B1) can be written as

$$\begin{aligned}
i\partial_0\Psi_u(n)=&-i\frac{1}{2}\left\{\left[\Psi_u(n+1)-\Psi_u(n)\right]\frac{1}{a}\right.\\
&\left.+\left[\Psi_u(n)-\Psi_u(n-1)\right]\frac{1}{a}\right\}\\
&-g\Psi_u(n)\partial_0\phi_n^+-g\partial_0\phi_n^-\Psi_u(n)\\
&-g(\phi_n^--\phi_{n-1}^-)\frac{1}{a}\Psi_u(n)\\
&-g\Psi_u(n)(\phi_n^+-\phi_{n-1}^+)\frac{1}{a}+KO(a^{\frac{3}{2}}).
\end{aligned}\tag{B3}$$

Equation (B3) does not directly manifest the lattice's ultraviolet behavior, and its continuum limit corresponds precisely to the equation of motion [i.e., Eq. (3.24) in Ref. [57]] in the original RS model.

APPENDIX C: COMPARISON OF THE RS MODEL AND THE SCHWINGER MODEL

Using $\gamma^5\gamma^\mu = \epsilon^{\mu\nu}\gamma_\nu$ and $A_\nu = \epsilon_{\mu\nu}\partial^\mu\phi$, the "interaction Lagrangian" of the RS model $-g\bar{\Psi}\gamma^5\gamma^\mu\Psi\partial_\mu\phi$ is equal to the one in the Schwinger model, i.e., $-g\bar{\Psi}\gamma^\mu\Psi A_\mu$. However, the RS model has fermionic particles, while the Schwinger model shows confinement and has no fermions in its spectrum.

A rough picture of confinement is that the potential energy between a positive and a negative charge is a linear potential in $1+1$ classical electrodynamics. Therefore, we first calculate the interaction force between particles in the classical RS model. During this process, we analyze the similarities and differences between the RS model and $1+1$ classical electrodynamics to understand how the differences in the bosonic part of the action affect the interaction force between a positive and a negative charge.

Since the fermionic part and the interaction part of the Lagrangian in the RS model are the same as those in the Schwinger model, the equations of motion for the fermions are also the same. Therefore, in the classical RS model, the force formula for particles can be obtained through the Lorentz force in $1+1$ electrodynamics with the help of $A_\nu = \epsilon_{\mu\nu}\partial^\mu\phi$. Specifically, the Lorentz force on a charge g is given by

$$\frac{dp}{dt} = gE = g(\partial_0 A_1 - \partial_1 A_0), \tag{C1}$$

where A and ϕ can be considered as external fields. Thus, using $A_\nu = \epsilon_{\mu\nu}\partial^\mu\phi$, we can obtain the force on charge g in the RS model as follows:

$$\frac{dp}{dt} = g\partial_\mu\partial^\mu\phi. \tag{C2}$$

Equation (C2) can also be regarded as another form of the Lorentz force in $1+1$ electrodynamics.

However, the bosonic part of the RS model Lagrangian is different from that of the Schwinger model, hence the corresponding equations of motion are also different. The bosonic field equation of motion for $1+1$ classical electrodynamics is as follows:

$$\partial_\mu F^{\mu\nu} = g\bar{\Psi}\gamma^\mu\Psi = gJ^\mu. \tag{C3}$$

Here, A and ϕ are not external fields, so the equation $A_\nu = \epsilon_{\mu\nu}\partial^\mu\phi$ cannot be directly used to obtain the bosonic field equation of motion for the RS model. Starting from the Lagrangian of the RS model, the equation of motion for the bosonic field can be derived as follows:

$$(\partial_\mu\partial^\mu + m^2)\phi = g\partial_\mu(\bar{\Psi}\gamma^5\gamma^\mu\Psi) = g\epsilon_{\mu\nu}\partial^\mu(\bar{\Psi}\gamma^\nu\Psi) = g\epsilon_{\mu\nu}\partial^\mu J^\nu. \tag{C4}$$

To more clearly compare Eqs. (C3) and (C4), we use $A_\nu = \epsilon_{\mu\nu}\partial^\mu\phi$ to express Eq. (C3) in terms of ϕ,

$$\partial_\mu\partial_\alpha\partial^\alpha\phi = g\epsilon_{\mu\nu}J^\nu. \tag{C5}$$

Equation (C5) can be further written as $\partial_\mu\partial^\mu\partial_\alpha\partial^\alpha\phi = g\epsilon_{\mu\nu}\partial^\mu J^\nu$, which looks very similar to Eq. (C4) in the case where $m=0$. Therefore, we will first discuss the case $m=0$. Now consider two point charges fixed in position. For the RS model, according to (C4), the scalar field generated by a point charge located at the origin satisfies $\partial_\mu\partial^\mu\phi = 0, x \neq 0$. Therefore, according to (C2), the force on another point charge is $\frac{dp}{dt} = g\partial_\mu\partial^\mu\phi = 0$. For $1+1$ classical electrodynamics, according to (C5), the scalar field generated by a point charge g located at the origin satisfies $\partial_\mu\partial^\mu\phi = \frac{1}{2}g, x > 0$. Furthermore, using (C2), the force on another charge $-g$ located at $x_1 > 0$ is $\frac{dp}{dt} = -\frac{1}{2}g^2$. Therefore, in $1+1$ classical electrodynamics, there is a linear potential between the two charges. Considering the effects of quantum field theory, "it is energetically favorable for a new pair to materialize from the vacuum when the separation is sufficiently great" (quoting Ref. [60]), which leads to confinement.

Although in the classical RS model with $m=0$, the force between point charges separated by a distance $r \neq 0$ is zero, the situation is different for $m \neq 0$. According to (C4), the scalar field produced by a point charge located at the origin is given by

$$\phi(x) = \begin{cases} \frac{1}{2}g e^{mx} & , x < 0, \\ -\frac{1}{2}g e^{-mx} & , x > 0. \end{cases} \tag{C6}$$

Further utilizing (C2), we find that another charge $-g$ located at $x_1 > 0$ experiences an exponentially decaying force given by $\frac{dp}{dt} = -\frac{1}{2}g^2 m^2 e^{-mx_1}$, indicating that the RS model does not exhibit confinement.

At the quantum level, the Hamiltonian formulation is the most direct method for studying eigenstates. As stated in the original paper on the RS model [57], "we note that on account of the derivative coupling in the Lagrangian we have $\mathcal{H}_\mathcal{I} \neq -\mathcal{L}_\mathcal{I}$," which seems to imply that the "interaction Hamiltonian" $\mathcal{H}_\mathcal{I}$ appears to be different although the "interaction Lagrangian" $\mathcal{L}_\mathcal{I}$ of the RS model and the Schwinger model appear to be the same. However, the Hamiltonian of the Schwinger model also depends on the chosen gauge, so further analysis is needed. To align as closely as possible with the form of the RS model Hamiltonian (13), we use the $A_0 = 0$ gauge Hamiltonian of the Schwinger model from Ref. [19],

$$H = \int_0^L \left(\frac{1}{2}E^2 - i\Psi^\dagger\gamma^0\gamma^1\partial_1\Psi + g\Psi^\dagger\gamma^0\gamma^1 A_1\Psi\right), \qquad \text{(C7)}$$

with $[A_1(x), E(y)] = i\delta(x-y)$ and the first-class constraint known as Gauss's law,

$$\partial_1 E = g\Psi^\dagger\Psi. \qquad \text{(C8)}$$

Comparing (C7) with the Hamiltonian of the RS model (13), it can be observed that the first two lines of (13) correspond to the second term of (C7), the third line of (13) corresponds to the first term of (C7), and the third term of (C7) corresponds to the fifth–seventh lines of (13). However, the fourth line of (13) is an additional term due to the derivative coupling of the RS model, and it is the absence of a similar term in (C7) that prevents the bosonic and fermionic parts from being separated in the same way as in the RS model. Specifically, similar to the fermionic field transformation $\psi(n) = \mathrm{e}^{-i\gamma^5 g\phi_n}\Psi(n)$ in (14), we can also perform the transformation $\psi(x) = \mathrm{e}^{ig\int_0^x dsA(s)}\Psi(x)$ for the fermionic field in the Schwinger model. This allows us to diagonalize the second and third terms of (C7), yielding

$$H = \int_0^L \left(\frac{1}{2}E^2 - i\psi^\dagger\gamma^0\gamma^1\partial_1\psi\right). \qquad \text{(C9)}$$

Formally, (C9) is very similar to (16), which features free fermion excitations. However, in (C9), E does not commute with the new fermionic field ψ. Similar to the transformation $\pi \to \pi'$ in (14), the bosonic field E also needs to undergo a transformation $E \to E'$ so that the new bosonic field E' commutes with the new fermionic field ψ. However, if the transformation $E \to E'$ is made, the Schwinger Hamiltonian (C9) obviously cannot be decomposed into bosonic and fermionic parts. The reason the RS model's Hamiltonian (13) can be decomposed into bosonic and fermionic parts after the transformation $\pi \to \pi'$ is precisely due to the sixth line of (13), which is caused by the derivative coupling and is not present in the Schwinger Hamiltonian. Therefore, although the interaction Lagrangian $\mathcal{L}_\mathcal{I}$ of the RS model and the Schwinger model appear to be the same, the Hamiltonian of the Schwinger model cannot be decomposed into a free fermion part in the same way as the RS model even when ignoring the Gauss constraint.

In fact, by using an unphysical indefinite metric field, we can derive a free fermion field from the Schwinger model through a transformation [60,61] similar to the RS model's transformation $\psi(n) = \mathrm{e}^{-i\gamma^5 g\phi_n}\Psi(n)$ in (14). However, this does not mean the Schwinger model has fermions in its spectrum, because unphysical negative-metric particles can obscure the propagation properties of the fermion field [60]. Alternatively, from the definition of the physical Hilbert space, one can also see that the Schwinger model has no fermion. Equation (2.32) in Ref. [61] presents a Hamiltonian decomposed into three free parts. Although the part concerning $\tilde{\varphi}$ appears to be a massless boson field, it is equivalent to the Hamiltonian of a massless free fermion field according to Eq. (17.98) in Ref. [62]. In this sense, it seems that the Schwinger model, like the RS model, has free fermions in its spectrum. However, the physical Hilbert space of the Schwinger model is defined by Eq. (2.10) in Ref. [61], which causes $\tilde{\varphi}$ to decouple from the physical spectrum.

[1] K. G. Wilson, Confinement of quarks, Phys. Rev. D **10**, 2445 (1974).

[2] J. B. Kogut and L. Susskind, Hamiltonian formulation of Wilson's lattice gauge theories, Phys. Rev. D **11**, 395 (1975).

[3] L. Susskind, Lattice fermions, Phys. Rev. D **16**, 3031 (1977).

[4] J. B. Kogut, A review of the lattice gauge theory approach to quantum chromodynamics, Rev. Mod. Phys. **55**, 775 (1983).

[5] J. B. Kogut, An introduction to lattice gauge theory and spin systems, Rev. Mod. Phys. **51**, 659 (1979).

[6] T. A. Degrand and C. DeTar, *Lattice Methods for Quantum Chromodynamics* (World Scientific, Singapore, 2006).

[7] C. Gattringer and C. Lang, *Quantum Chromodynamics on the Lattice: An Introductory Presentation* (Springer Science & Business Media, Heidelberg, 2009), Vol. 788.

[8] G. Magnifico, T. Felser, P. Silvi, and S. Montangero, Lattice quantum electrodynamics in $(3+1)$-dimensions at finite density with tensor networks, Nat. Commun. **12**, 3600 (2021).

[9] Z. Davoudi, I. Raychowdhury, and A. Shaw, Search for efficient formulations for Hamiltonian simulation of non-Abelian lattice gauge theories, Phys. Rev. D **104**, 074505 (2021).

[10] E. Zohar, Quantum simulation of lattice gauge theories in more than one space dimension—requirements, challenges and methods, Phil. Trans. A. Math. Phys. Eng. Sci. **380**, 20210069 (2021).

[11] M. Troyer and U.-J. Wiese, Computational complexity and fundamental limitations to fermionic quantum Monte Carlo simulations, Phys. Rev. Lett. **94**, 170201 (2005).

[12] A. L. Fitzpatrick and E. Katz, Snowmass white paper: Hamiltonian truncation, arXiv:2201.11696.

[13] A. Florio, D. Frenklakh, K. Ikeda, D. Kharzeev, V. Korepin, S. Shi, and K. Yu, Real-time nonperturbative dynamics of jet production in Schwinger model: Quantum entanglement and vacuum modification, Phys. Rev. Lett. **131**, 021902 (2023).

[14] C. W. Bauer, Z. Davoudi, A. B. Balantekin, T. Bhattacharya, M. Carena, W. A. De Jong *et al.*, Quantum simulation for high-energy physics, PRX Quantum **4**, 027001 (2023).

[15] J. Schwinger, Gauge invariance and mass. II, Phys. Rev. **128**, 2425 (1962).

[16] J. Lowenstein and J. Swieca, Quantum electrodynamics in two dimensions, Ann. Phys. (N.Y.) **68**, 172 (1971).

[17] N. Manton, The Schwinger model and its axial anomaly, Ann. Phys. (N.Y.) **159**, 220 (1985).

[18] J. E. Hetrick and Y. Hosotani, QED on a circle, Phys. Rev. D **38**, 2621 (1988).

[19] R. Link, Eigenstates of the Schwinger-model Hamiltonian, Phys. Rev. D **42**, 2103 (1990).

[20] S. Iso and H. Murayama, Hamiltonian formulation of the Schwinger model: Non-confinement and screening of the charge, Prog. Theor. Phys. **84**, 142 (1990).

[21] T. Banks, L. Susskind, and J. Kogut, Strong-coupling calculations of lattice gauge theories: $(1+1)$-dimensional exercises, Phys. Rev. D **13**, 1043 (1976).

[22] J. Ambjørn, J. Greensite, and C. Peterson, The axial anomaly and the lattice Dirac sea, Nucl. Phys. **B221**, 381 (1983).

[23] P. Nason, The lattice Schwinger model with SLAC fermions, Nucl. Phys. **B260**, 269 (1985).

[24] K. Melnikov and M. Weinstein, Lattice Schwinger model: Confinement, anomalies, chiral fermions, and all that, Phys. Rev. D **62**, 094504 (2000).

[25] T. Byrnes, P. Sriganesh, R. J. Bursill, and C. J. Hamer, Density matrix renormalization group approach to the massive Schwinger model, Phys. Rev. D **66**, 013002 (2002).

[26] M. C. Bañuls, K. Cichy, K. Jansen, and J. I. Cirac, The mass spectrum of the Schwinger model with matrix product states, J. High Energy Phys. 11 (2013) 158.

[27] F. Hebenstreit and J. Berges, Connecting real-time properties of the massless Schwinger model to the massive case, Phys. Rev. D **90**, 045034 (2014).

[28] B. Buyens, F. Verstraete, and K. Van Acoleyen, Hamiltonian simulation of the Schwinger model at finite temperature, Phys. Rev. D **94**, 085018 (2016).

[29] T. Pichler, M. Dalmonte, E. Rico, P. Zoller, and S. Montangero, Real-time dynamics in U(1) lattice gauge theories with tensor networks, Phys. Rev. X **6**, 011023 (2016).

[30] B. Buyens, S. Montangero, J. Haegeman, F. Verstraete, and K. Van Acoleyen, Finite-representation approximation of lattice gauge theories at the continuum limit with tensor networks, Phys. Rev. D **95**, 094509 (2017).

[31] K. Zapp and R. Orus, Tensor network simulation of QED on infinite lattices: Learning from $(1+1)$d, and prospects for $(2+1)$d, Phys. Rev. D **95**, 114508 (2017).

[32] J. C. Pinto Barros, M. Dalmonte, and A. Trombettoni, String tension and robustness of confinement properties in the Schwinger-Thirring model, Phys. Rev. D **100**, 036009 (2019).

[33] L. Funcke, K. Jansen, and S. Kühn, Topological vacuum structure of the Schwinger model with matrix product states, Phys. Rev. D **101**, 054507 (2020).

[34] N. Butt, S. Catterall, Y. Meurice, R. Sakai, and J. Unmuth-Yockey, Tensor network formulation of the massless Schwinger model with staggered fermions, Phys. Rev. D **101**, 094509 (2020).

[35] M. Rigobello, S. Notarnicola, G. Magnifico, and S. Montangero, Entanglement generation in $(1+1)$d QED scattering processes, Phys. Rev. D **104**, 114501 (2021).

[36] K. Ikeda, D. E. Kharzeev, and S. Shi, Nonlinear chiral magnetic waves, Phys. Rev. D **108**, 074001 (2023).

[37] M. Blasone and G. Vitiello, Quantum field theory of fermion mixing, Ann. Phys. (N.Y.) **244**, 283 (1995).

[38] M. Binger and C.-R. Ji, Quantum field theory of meson mixing, Phys. Rev. D **60**, 056005 (1999).

[39] M. Blasone, P. A. Henning, and G. Vitiello, The exact formula for neutrino oscillations, Phys. Lett. B **451**, 140 (1999).

[40] C.-R. Ji and Y. Mishchenko, Nonperturbative vacuum effect in the quantum field theory of meson mixing, Phys. Rev. D **64**, 076004 (2001).

[41] M. Blasone, A. Capolupo, O. Romei, and G. Vitiello, Quantum field theory of boson mixing, Phys. Rev. D **63**, 125015 (2001).

[42] C.-R. Ji and Y. Mishchenko, The general theory of quantum field mixing, Phys. Rev. D **65**, 096015 (2002).

[43] C. Giunti, Fock states of flavor neutrinos are unphysical, Eur. Phys. J. C **39**, 377 (2005).

[44] M. Blasone, F. Dell'Anno, S. De Siena, and F. Illuminati, A field-theoretical approach to entanglement in neutrino mixing and oscillations, Europhys. Lett. **106**, 30002 (2014).

[45] M. Blasone, P. Jizba, N. E. Mavromatos, and L. Smaldone, Dynamical generation of field mixing via flavor vacuum condensate, Phys. Rev. D **100**, 045027 (2019).

[46] M. Blasone, F. Illuminati, G. G. Luciano, and L. Petruzziello, Flavor vacuum entanglement in boson mixing, Phys. Rev. A **103**, 032434 (2021).

[47] L. Smaldone and G. Vitiello, Neutrino mixing and oscillations in quantum field theory: A comprehensive introduction, Universe **7**, 504 (2021).

[48] W. Pauli and M. Fierz, Zur theorie der emission langwelliger lichtquanten, Il Nuovo Cimento (1924–1942) **15**, 167 (1938).

[49] A. S. Blum and C. Joas, From dressed electrons to quasiparticles: The emergence of emergent entities in quantum field theory, Stud. Hist. Philos. Sci. Part B **53**, 1 (2016).

[50] V. Chung, Infrared divergence in quantum electrodynamics, Phys. Rev. **140**, B1110 (1965).

[51] P. P. Kulish and L. D. Faddeev, Asymptotic conditions and infrared divergences in quantum electrodynamics, Theor. Math. Phys. **4**, 745 (1970).

[52] P. Duch, Infrared problem in perturbative quantum field theory, Rev. Math. Phys. **33**, 2150032 (2021).

[53] W. Dybalski, From Faddeev–Kulish to LSZ. Towards a non-perturbative description of colliding electrons, Nucl. Phys. **B925**, 455 (2017).

[54] O. Greenberg and S. Schweber, Clothed particle operators in simple models of quantum field theory, Il Nuovo Cimento **8**, 378 (1958).

[55] A. Shebeko and M. Shirokov, Unitary transformations in quantum field theory and bound states, Phys. Part. Nucl. **32**, 15 (2001).

[56] Y. Kostylenko and A. Shebeko, Clothed particle representation in quantum field theory: Fermion mass renormalization due to vector boson exchange, Phys. Rev. D **108**, 125019 (2023).

[57] K. Rothe and I. Stamatescu, Study of a two-dimensional model with axial-current-pseudoscalar derivative interaction, Ann. Phys. (N.Y.) **95**, 202 (1975).

[58] H. J. Borchers, On structure of the algebra of field operators, Nuovo Cimento **24**, 214 (1962).

[59] L. Martinovič, Solvable models in the conventional and light-front field theory: Recent progress, Few Body Syst. **55**, 527 (2014).

[60] A. Casher, J. Kogut, and L. Susskind, Vacuum polarization and the quark-parton puzzle, Phys. Rev. Lett. **31**, 792 (1973).

[61] L. V. Belvedere, J. Swieca, K. Rothe, and B. Schroer, Generalized two-dimensional Abelian gauge theories and confinement, Nucl. Phys. **B153**, 112 (1979).

[62] R. Shankar, *Quantum Field Theory and Condensed Matter: An Introduction* (Cambridge University Press, Cambridge, England, 2017).