Thermodynamic stability in relativistic viscous and spin hydrodynamics

Xiang Ren,[1] Chen Yang,[1] Dong-Lin Wang,[1,*] and Shi Pu[1,2,†]
[1]*Department of Modern Physics, University of Science and Technology of China, Anhui 230026, China*
[2]*Southern Center for Nuclear-Science Theory (SCNT), Institute of Modern Physics, Chinese Academy of Sciences, Huizhou 516000, Guangdong Province, China*

(Received 11 May 2024; accepted 10 July 2024; published 8 August 2024)

We have applied thermodynamic stability analysis to derive the stability and causality conditions for conventional relativistic viscous hydrodynamics and spin hydrodynamics. We obtain the thermodynamic stability conditions for second-order relativistic hydrodynamics with shear and bulk viscous tensors, finding them identical to those derived from linear mode analysis. We then derive the thermodynamic stability conditions for minimal causal extended second-order spin hydrodynamics in canonical form, both with and without viscous tensors. Without viscous tensors, the constraints from thermodynamic stability exactly match those from linear mode analysis. In the presence of viscous tensors, the thermodynamic stability imposes more stringent constraints than those obtained from linear mode analysis. Our results suggest that conditions derived from thermodynamic stability analysis can guarantee both causality and stability in linear mode analysis.

DOI: 10.1103/PhysRevD.110.034010

I. INTRODUCTION

In relativistic heavy ion collisions, two nuclei are accelerated to speeds close to that of light, collide with each other, and generate a hot and dense matter known as the quark-gluon plasma (QGP) [1–4]. The evolution of the QGP is well described by relativistic hydrodynamics. Relativistic hydrodynamics serves as a macroscopic effective theory for relativistic many-body systems in the long-wavelength and low-frequency limit. The main equations of relativistic hydrodynamics are the conservation equations for the energy-momentum tensor and other conserved currents in the gradient expansion, e.g., the Israel-Stewart theory [5,6], the extended Baier-Romatschke-Son-Starinets-Stephanov theory [7], the Denicol-Niemi-Molnar-Rischke theory [8], and the more recently established Bemfica-Disconzi-Noronha-Kovtun theory [9–13]. For additional studies and developments, we refer the reader to the recent review papers [14,15] and the references therein.

In the early stages of noncentral collisions, the nuclei possess a huge initial orbital angular momentum, on the order of $10^7\hbar$. This initial orbital angular momentum is transferred to the spin polarization of quarks and subsequently to the final-state particles through spin-orbital coupling. This mechanism leads to the spin polarization of Λ and $\bar{\Lambda}$ hyperons and the spin alignment of vector mesons [16–18]. The STAR collaboration has observed both the global and local polarization of Λ and $\bar{\Lambda}$ hyperons [19,20], as well as the spin alignment of ϕ and $K^{0,*}$ mesons [21].

On the theoretical side, the global polarization can be well described by various phenomenological models [22–34] through the combination of the modified Cooper-Frye formula [35,36] with hydrodynamic simulations under the assumption that the system is close to global equilibrium. To understand local polarization, effects beyond global equilibrium, such as shear-induced polarization [37–44], spin Hall effects [34,45,46], weak magnetic fields induced polarization [47] and the corrections due to the interactions between quarks and back ground fields [48], need to be considered. Although hydrodynamic simulations can qualitatively describe local polarization as functions of azimuthal angle, understanding the dependence on centrality and transverse momentum remains challenging [20,49,50]. Therefore, it is necessary to consider the evolution of spin during collisions. Recently established spin hydrodynamics, which integrates the total angular momentum conservation equation with conventional relativistic hydrodynamic equations, has been developed from various theoretical frameworks, such as from effective action [51,52], entropy principle [53–63], kinetic theory [64–80], holography [81,82], and quantum statistics [83]. For recent reviews on this topic, see Refs. [84–86].

As a fundamental requirement, both conventional relativistic hydrodynamics and spin hydrodynamics must exhibit causality and stability. In pioneering works [87,88], linear mode analysis was implemented to study the causality and stability of hydrodynamic systems. Through linear mode analysis, the causality and stability conditions for various

*Contact author: donglinwang@mail.ustc.edu.cn
†Contact author: shipu@ustc.edu.cn

types of hydrodynamics are derived [87–101]. These conditions establish inequalities that constrain the range of transport coefficients. Recently, it was found that the conventional causality criterion [102] used in linear mode analysis is insufficient to guarantee causality. Consequently, several studies [103–108] have proposed new causality criteria that also explore the deep connection between causality and stability [13,107,109].

Very recently, Ref. [96] has systematically studied the causality and stability for the minimal extended second-order spin hydrodynamics in the linear mode analysis. Later, Ref. [101] also investigates the impact of other second-order terms. It was revealed that the system appears to be unstable at finite wavelengths, even though it satisfies asymptomatic stability conditions derived for both large and small wavelengths [96]. To address this issue, it is essential to explore the stability of spin hydrodynamics through an alternative approach.

In this work, we apply thermodynamic stability analysis [110–113], which is grounded in the second law of thermodynamics and the principle of maximizing total entropy in equilibrium states [114], to spin hydrodynamics. We will derive stability conditions from this thermodynamic stability analysis and compare them with those obtained through linear mode analysis.

The structure of this paper is organized as follows: In Sec. II, we briefly review thermodynamic stability analysis. Next, we apply this analysis to conventional relativistic viscous hydrodynamics as a test case in Sec. III. In Sec. IV, we analyze the thermodynamic stability conditions for spin hydrodynamics and compare the results with those obtained from linear mode analysis. We conclude with a summary in Sec. V.

Throughout this work, we choose the metric $g_{\mu\nu} = \mathrm{diag}\{+,-,-,-\}$ and define the projector $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ with u^μ being the fluid velocity. For an arbitrary tensor $A^{\mu\nu}$, we introduce the notations $A^{(\mu\nu)} = \frac{1}{2}(A^{\mu\nu} + A^{\nu\mu})$, $A^{[\mu\nu]} = \frac{1}{2}(A^{\mu\nu} - A^{\nu\mu})$, and $A^{\langle\mu\nu\rangle} \equiv \frac{1}{2}[\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\beta}\Delta^{\nu\alpha}]A_{\alpha\beta} - \frac{1}{3}\Delta^{\mu\nu}(\Delta^{\alpha\beta}A_{\alpha\beta})$.

II. BRIEF INTRODUCTION TO THE THERMODYNAMIC STABILITY

In this section, we briefly review the main idea in Ref. [113]. Consider an isolated system near thermodynamic equilibrium, consisting of a fluid connected to a sufficiently large heat-particle bath. According to the second law of thermodynamics, the entropy of the entire system, S, must not decrease, i.e., the variation of entropy ΔS follows:

$$\Delta S = \Delta S_F + \Delta S_B \geq 0, \tag{1}$$

where $S_{F,B}$ stand for the entropy for fluid and bath, respectively. Equation (1) is the original condition for the thermodynamic stability.

Now, let us consider conserved quantities $\mathcal{Q}^a$ and their thermodynamic conjugates α^a in the system, where $a = 1, 2, \ldots$ label different conserved quantities. For example, if the total number is conserved, then $\mathcal{Q}$ and α correspond to the total number and μ/T, respectively, with μ and T being the chemical potential and temperature. While, if the total energy is conserved, $\mathcal{Q}$ and α are total energy and $-1/T$, respectively. Then, the variation of entropy can be expressed as

$$dS = -\sum_a \alpha^a d\mathcal{Q}^a. \tag{2}$$

The $\mathcal{Q}^a$ can be divided as the part for fluid $\mathcal{Q}^a_F$ and the one for the bath $\mathcal{Q}^a_B$, with the following relationship:

$$d\mathcal{Q}^a_B = -d\mathcal{Q}^a_F. \tag{3}$$

Then the variation of total entropy becomes

$$\Delta S = \Delta S_F + \sum_a \alpha^a_B \Delta\mathcal{Q}^a_F \geq 0. \tag{4}$$

If defining

$$\Psi \equiv S_F + \sum_a \alpha^a_B \mathcal{Q}^a_F, \tag{5}$$

then Eq. (4) implies that the function Ψ should be maximized in the equilibrium state.

One can also define the information current E^μ as

$$E^\mu \equiv -\delta s^\mu_F - \sum_a \alpha^a_F \delta J^{a,\mu}_F, \tag{6}$$

where s^μ_F is the entropy current of the fluid and $J^{a,\mu}_F$ is the conserved current associated with $\mathcal{Q}^a_F$, and the symbol δ denotes the small perturbations from the thermodynamic equilibrium state.

Given that the whole system is near thermodynamic equilibrium and the heat-particle bath is sufficiently large, we can assume that the chemical potential and temperature in the fluid are equal to those in the bath, i.e., α^a_F for the fluid is approximately equal to α^a_B in the bath. Under this assumption, $\alpha^a_F \delta J^{a,\mu}_F$ can be simplified to $\delta(\alpha^a J^{a,\mu}_F)$, where we do not distinguish between α^a_F in the fluid and α^a_B in the bath. Consequently, Eq. (4) can be further written as

$$E \equiv \int d\Sigma E^\mu n_\mu \geq 0, \tag{7}$$

holds for an arbitrary spacelike three-dimensional surface Σ and its timelike and future-directed normal unit vector n^μ.

If the thermodynamic equilibrium state is unique, i.e., determined solely by the thermodynamic variables, then from Eq. (7) and the definition of E^μ in Eq. (6), the information current E^μ must satisfy the following conditions:

(i) $E^\mu n_\mu \geq 0$ for any n^μ with $n_0 > 0, n^\mu n_\mu = 1$,

(ii) $E^\mu n_\mu = 0$ if and only if all perturbations are zero,

(iii) $\partial_\mu E^\mu \leq 0$. (8)

As a remark, the conditions in Eq. (8) can be treated as criteria of thermodynamic stability [113]. It has also been found that these criteria in Eq. (8) can guarantee the causality of the system [113]. Moreover, when all these conditions are satisfied in one inertial frame of reference, the thermodynamic stability conditions in Eq. (1) are assured across all inertial frames of Refs. [107,109]. These criteria provide us with a novel tool for analyzing the stability and causality of the system.

III. THERMODYNAMIC STABILITY OF THE SECOND ORDER VISCOUS HYDRODYNAMICS

In this section, we implement the thermodynamic stability criteria (8) to the relativistic second order viscous hydrodynamics in the gradient expansion. It can be considered as an example to show the connection between the constraints from the thermodynamic stability and conventional linear mode analysis. For convenience, we focus on the quantities for fluid and omit all the lower index F from now on.

The energy momentum conservation equation reads

$$\partial_\mu T^{\mu\nu} = 0, \tag{9}$$

and the energy momentum tensor in the Landau or energy frame is given by [115]

$$T^{\mu\nu} = (e+P)u^\mu u^\nu - P g^{\mu\nu} + \pi^{\mu\nu} - \Pi\Delta^{\mu\nu}, \tag{10}$$

where $e, P, \pi^{\mu\nu}, \Pi$ are energy density, pressure, shear viscous tensor, and bulk pressure, respectively. Note that the net baryon number density of the QGP produced in relativistic heavy ion collisions is negligible [116]. For simplicity, the following discussions are limited in the cases where (baryon) currents vanish.

In order to compare the constraints from thermodynamic stability and linear mode analysis, we choose the minimal extension of second order viscous hydrodynamics [89]. The corresponding entropy current is given by

$$s^\mu = s u^\mu - Q^\mu + \mathcal{O}(\partial^3), \tag{11}$$

where Q^μ stands for the possible corrections from the second order. Following Refs. [5,6], we take

$$Q^\mu = \frac{1}{2}\frac{u^\mu}{T}(\chi_\Pi \Pi^2 + \chi_\pi \pi^{\rho\sigma}\pi_{\rho\sigma}), \tag{12}$$

as an example. Then the entropy principle $\partial_\mu s^\mu \geq 0$ gives the constitutive equations for $\pi^{\mu\nu}$ and Π as below,

$$\begin{aligned}
&\tau_\Pi (u\cdot\partial)\Pi + \Pi \\
&\quad = -\zeta\left[\partial_\mu u^\mu + \frac{1}{2}\chi_\Pi T \partial_\rho (u^\rho/T)\Pi\right], \\
&\tau_\pi \Delta^{\alpha<\mu}\Delta^{\nu>\beta}(u\cdot\partial)\pi_{\alpha\beta} + \pi^{\mu\nu} \\
&\quad = 2\eta\left[\partial^{\langle\mu}u^{\nu\rangle} - \frac{1}{2}\chi_\pi T\partial_\rho(u^\rho/T)\pi^{\mu\nu}\right],
\end{aligned} \tag{13}$$

where

$$\zeta, \eta > 0, \tag{14}$$

and

$$\tau_\Pi = \zeta\chi_\Pi, \qquad \tau_\pi = 2\eta\chi_\pi. \tag{15}$$

For the more comprehensive discussions on the second order theories, we refer to Refs. [5,6]. Later, we will compare our results from thermodynamic stability with those from linear mode analysis [90,91]. The terms proportional to χ_Π, χ_π on the right-hand side of Eq. (13) do not appear in the constitutive equations in Refs. [90,91], but these terms will not contribute to the causality and stability conditions in linear mode analysis.

Next, we choose local rest frame $u^\mu = (1, \mathbf{0})$ and assume the fluid reaches the thermodynamic equilibrium state, in which Π and $\pi^{\mu\nu}$ are zero. For the macroscopic variables $\varphi = (e, u^\mu, \Pi, \pi^{\mu\nu})$, we consider the perturbations near the thermodynamic equilibrium, $\delta\varphi$. We can expand the system in the power series of $\delta\varphi$. By using the following relationship,

$$\begin{aligned}
u^\mu \delta u_\mu &= -\frac{1}{2}\delta u^\mu \delta u_\mu, \\
u_\mu \delta\pi^{\mu\nu} &= -\delta u_\mu \delta\pi^{\mu\nu}, \\
\delta\pi^\mu_\mu &= 0,
\end{aligned} \tag{16}$$

we find

$$\begin{aligned}
\delta u^i, \delta\pi^{ij} &\sim \mathcal{O}(\delta), \\
\delta u^0, \delta\pi^{0i} &\sim \mathcal{O}(\delta^2), \\
\delta\pi^{00} &\sim \mathcal{O}(\delta^3).
\end{aligned} \tag{17}$$

With the help of Eq. (11), the information current is then given by [112,113]

$$
\begin{aligned}
E^\mu &= -\delta s^\mu + \frac{u_\nu}{T}\delta T^{\mu\nu}, \\
&= u^\mu\left(\frac{\delta e}{T} - \delta s\right) + \frac{1}{T}\delta u^\mu \delta P + \frac{1}{2}\frac{u^\mu}{T}(\chi_\pi \delta\pi^{\alpha\beta}\delta\pi_{\alpha\beta} + \chi_\Pi \delta\Pi\delta\Pi) \\
&\quad - \frac{1}{2T}(e+P)u^\mu \delta u_\nu \delta u^\nu - \frac{1}{T}\delta u_\nu \delta\pi^{\mu\nu} + \frac{1}{T}\delta u^\mu \delta\Pi + \mathcal{O}(\delta^3).
\end{aligned} \tag{18}
$$

By using the thermodynamic relations,

$$
ds = \frac{1}{T}de, \qquad dP = s dT, \tag{19}
$$

we find that

$$
\begin{aligned}
\delta s &= \frac{1}{T}\delta e + \frac{1}{2}\frac{\partial^2 s}{\partial e^2}(\delta e)^2 + \mathcal{O}(\delta^3), \\
&= \frac{1}{T}\delta e - \frac{1}{2T}\frac{\delta P}{e+P}\delta e + \mathcal{O}(\delta^3), \\
&= \frac{1}{T}\delta e - \frac{1}{2T}\frac{c_s^2}{e+P}(\delta e)^2 + \mathcal{O}(\delta^3),
\end{aligned} \tag{20}
$$

where c_s^2 is the speed of sound. Then, E^μ can be further simplified,

$$
\begin{aligned}
E^\mu &= \frac{1}{2T}\frac{u^\mu c_s^2}{e+P}(\delta e)^2 + \frac{c_s^2}{T}\delta u^\mu \delta e - \frac{1}{2T}(e+P)u^\mu \delta u_\nu \delta u^\nu \\
&\quad - \frac{1}{T}\delta u_\nu \delta\pi^{\mu\nu} + \frac{1}{T}\delta u^\mu \delta\Pi \\
&\quad + \frac{1}{2T}u^\mu(\chi_\pi \delta\pi^{\alpha\beta}\delta\pi_{\alpha\beta} + \chi_\Pi \delta\Pi\delta\Pi).
\end{aligned} \tag{21}
$$

Let us now impose the three conditions (8) on E^μ. From the definition (6), we have $\partial_\mu E^\mu = -\partial_\mu \delta s^\mu$, so that the condition (iii) in Eq. (8) leads to the inequality (14), which is consistent with the requirement from the conventional entropy principle. To analyze the constraints from the conditions (i) and (ii) in Eq. (8), we introduce an arbitrary timelike future-directed vector n^μ with $n_0 > 0, n^\mu n_\mu = 1$. After some tedious and straightforward calculations, we obtain

$$
\begin{aligned}
\frac{2n_0 T E^\mu n_\mu}{e+P} &= \frac{n_0^2\tau_\pi}{\eta(e+P)}\sum_{i<j}\left[\delta\pi^{ij} - \frac{1}{n_0\chi_\pi}n_{(i}\delta u_{j)}\right]^2 + \frac{n_0^2\tau_\pi}{\eta(e+P)}\left[\delta\pi^{11} + \frac{1}{2}\delta\pi^{22} + \frac{1}{2n_0\chi_\pi}(n_3\delta u_3 - n_1\delta u_1)\right]^2 \\
&\quad + \frac{3n_0^2\tau_\pi}{4\eta(e+P)}\left[\delta\pi^{22} + \frac{1}{3n_0\chi_\pi}(n_3\delta u_3 + n_1\delta u_1 - 2n_2\delta u_2)\right]^2 + \sum_{i=1}^{5} a_i(\delta A_i)^2,
\end{aligned} \tag{22}
$$

where the exact expressions for a_i and δA_i can be found in Appendix B. Imposing the conditions (i) and (ii) in Eq. (8) leads to[1]

$$
\begin{gathered}
c_s^2, \tau_\pi, \tau_\Pi > 0, \\
1 - c_s^2 - \frac{4\eta}{3\tau_\pi(e+P)} - \frac{\zeta}{\tau_\Pi(e+P)} > 0,
\end{gathered} \tag{23}
$$

which are exactly the same as those derived from linear mode analysis in the previous literature [90,91,118].

In general, if the baryon or other conserved current is considered, e.g., $j^\mu = n u^\mu + \nu^\mu$ with n and ν^μ being number density and diffusive current, the independent fields become $\varphi = (e, u^\mu, \Pi, \pi^{\mu\nu}, n, \nu^\mu)$ [119]. In these cases, the thermodynamic relations (19), constitutive relations (11)–(13), and information current (18) will be modified. More constraints for thermodynamic stability would occur and the final constraints become different with Eq. (23). For the general analysis including baryon currents, one can refer to Refs. [112,113,117,119].

[1]In this work, we assume $e + P > 0$, while Ref. [117] also explores cases where $e + P < 0$. Additionally, we note that the treatment of $\delta\pi^{\mu\nu}$ in Eq. (22) is different with Eq. (C14) of Ref. [117], since the number of independent components of $\delta\pi^{\mu\nu}$ is 5.

IV. THERMODYNAMIC STABILITY OF SPIN HYDRODYNAMICS

In this section, we implement the thermodynamic stability criteria (8) to the spin hydrodynamics. First, let us briefly review the spin hydrodynamics in the canonical form. Besides the energy momentum conservation, we also have the conservation equations for the total angular momentum, i.e.,

$$
\begin{aligned}
\partial_\lambda J^{\lambda\mu\nu} &= 0, \\
\partial_\mu \Theta^{\mu\nu} &= 0,
\end{aligned} \tag{24}
$$

where $J^{\lambda\mu\nu}$ and $\Theta^{\mu\nu}$ are the total angular momentum tensor and energy momentum tensor in canonical form, respectively. The constitutive equations of $J^{\lambda\mu\nu}$ and $\Theta^{\mu\nu}$ are

$$\Theta^{\mu\nu} = (e+P)u^\mu u^\nu - Pg^{\mu\nu} + 2q^{[\mu}u^{\nu]} + \phi^{\mu\nu} + \pi^{\mu\nu} - \Pi\Delta^{\mu\nu},$$
$$J^{\lambda\mu\nu} = x^\mu\Theta^{\lambda\nu} - x^\nu\Theta^{\lambda\mu} + \Sigma^{\lambda\mu\nu}, \tag{25}$$

where $q^\mu, \phi^{\mu\nu}$ are related to the spin and $\Sigma^{\lambda\mu\nu}$ is the spin tensor. In the following, we will limit our considerations to the cases where (baryon) currents vanish and, therefore, the terms for (baryon) number density do not contribute to constitutive relations and thermodynamic relations.

Inserting Eq. (25) into Eq. (24) yields

$$\partial_\lambda\Sigma^{\lambda\mu\nu} = -2\Theta^{[\mu\nu]}. \tag{26}$$

The spin tensor $\Sigma^{\lambda\mu\nu}$ is usually decomposed as [53,54]

$$\Sigma^{\lambda\mu\nu} = u^\lambda S^{\mu\nu} + \sigma^{\lambda\mu\nu}, \tag{27}$$

where $S^{\mu\nu}$ is named as spin density and $\sigma^{\lambda\mu\nu}$ is perpendicular to the fluid velocity. We follow Ref. [54] to consider the power counting of the spin tensor,

$$S^{\mu\nu} \sim \mathcal{O}(\partial^0), \qquad \sigma^{\lambda\mu\nu} \sim \mathcal{O}(\partial^1). \tag{28}$$

Analogy to charge chemical potential, one can introduce the spin chemical potential $\omega^{\mu\nu}$, which modifies the thermodynamic relations in the presence of spin density [53,54],

$$e + P = Ts + \omega_{\mu\nu}S^{\mu\nu},$$
$$de = Tds + \omega_{\mu\nu}dS^{\mu\nu},$$
$$dP = sdT + S^{\mu\nu}d\omega_{\mu\nu}. \tag{29}$$

The entropy current in Eq. (11) can also be extended as

$$s^\mu = su^\mu + \frac{1}{T}q^\mu - Q^\mu. \tag{30}$$

The complete second order terms for the entropy current is complicated, see e.g., Ref. [63]. For simplicity, we write down the Q^μ analogy to Eq. (12),

$$Q^\mu = \frac{1}{2T}u^\mu(\chi_q q^\nu q_\nu + \chi_\phi\phi^{\alpha\beta}\phi_{\alpha\beta} + \chi_\Pi\Pi^2 + \chi_\pi\pi^{\alpha\beta}\pi_{\alpha\beta}). \tag{31}$$

From the second law of thermodynamics, we can get

$$\tau_q\Delta^{\mu\nu}(u\cdot\partial)q_\nu + q^\mu = \lambda\left[u^\rho\partial_\rho u^\mu - T\Delta^{\mu\nu}\partial_\nu\frac{1}{T} - 4\omega^{\mu\nu}u_\nu + \frac{1}{2}\chi_q T\partial_\rho\left(\frac{u^\rho}{T}\right)q_\nu\right],$$
$$\tau_\phi\Delta^{\mu\alpha}\Delta^{\nu\beta}(u\cdot\partial)\phi_{\alpha\beta} + \phi^{\mu\nu} = 2\gamma_s\Delta^{\mu\alpha}\Delta^{\nu\beta}\left[\partial_{[\alpha}u_{\beta]} + 2\omega_{\alpha\beta} - \frac{1}{2}\chi_\phi T\partial_\rho\left(\frac{u^\rho}{T}\right)\phi_{\alpha\beta}\right], \tag{32}$$

with the transport coefficients,

$$\tau_q = -\lambda\chi_q, \qquad \tau_\phi = 2\chi_\phi\gamma_s, \qquad \lambda,\gamma_s > 0. \tag{33}$$

The equation for $\pi^{\mu\nu}$ and Π are the same as Eq. (13). We notice that the terms proportional to $\chi_q\chi_\phi$ on the right-hand side of Eq. (32) differs with the constitutive equations for q^μ and $\phi^{\mu\nu}$ in the minimal causal extended second order theory in Ref. [96]. However, these new terms proportional to χ_q,χ_ϕ will not contribute to the causality and stability conditions in linear mode analysis.

A. Information current for spin hydrodynamics

Considering the small perturbations around thermodynamic equilibrium $\varphi \to \varphi + \delta\varphi$, where $\varphi = (e, u^\mu, \Pi, \pi^{\mu\nu}, S^{\mu\nu}, q^\mu, \phi^{\mu\nu})$, we can construct the information current E^μ for spin hydrodynamics. According to the definition of E^μ in Eq. (6), we next consider the conserved currents.

We note that different with Eq. (18), $u_\nu\delta\Theta^{\mu\nu}/T$ is no longer a conserved current in spin hydrodynamics due to the nonvanishing antisymmetric part of $\delta\Theta^{\mu\nu}$. Recalling that u_μ/T is a killing vector in thermodynamic equilibrium state, i.e., $\partial_{(\mu}(u_{\nu)}/T) = 0$, leading to the general solutions for u_μ/T as [120]

$$u_\mu/T = b_\mu + \varpi_{\mu\nu}x^\nu, \tag{34}$$

where b_μ and $\varpi_{\mu\nu} = -\varpi_{\nu\mu}$ are space-time independent, and $\varpi_{\mu\nu}$ is named as the thermal vorticity in spin hydrodynamics in the global equilibrium [53,54]. Then, we find

$$\partial_\mu\left(\frac{u_\nu}{T}\delta\Theta^{\mu\nu}\right) = -\varpi_{\mu\nu}\delta\Theta^{[\mu\nu]}, \tag{35}$$

indicating that $u_\nu\delta\Theta^{\mu\nu}/T$ is not a conserved current. According to Eq. (26), we notice that $\partial_\mu\delta\Sigma^{\mu\rho\sigma} = -2\delta\Theta^{[\mu\nu]}$, and then construct a new conserved current $u_\nu\delta\Theta^{\mu\nu}/T - \frac{1}{2}\varpi_{\rho\sigma}\delta\Sigma^{\mu\rho\sigma}$, which can also be written as

$$\frac{u_\nu}{T}\delta\Theta^{\mu\nu} - \frac{1}{2}\varpi_{\rho\sigma}\delta\Sigma^{\mu\rho\sigma} = b_\nu\delta\Theta^{\mu\nu} - \frac{1}{2}\varpi_{\rho\sigma}\delta J^{\mu\rho\sigma}. \tag{36}$$

The $b_\nu\delta\Theta^{\mu\nu}$ corresponds to energy and momentum conservation. The $-\frac{1}{2}\varpi_{\rho\sigma}\delta J^{\mu\rho\sigma}$ comes from total angular momentum conservation. Interestingly, from Eq. (36), the thermal vorticity $\varpi_{\mu\nu}$ plays a role like the chemical potential corresponding to the total angular momentum. Numerous studies [53,54,58,120–122] prove that the thermal vorticity in the global equilibrium are proportional to spin chemical potential,

$$\varpi_{\rho\sigma} = \frac{2\omega_{\rho\sigma}}{T}. \tag{37}$$

The independent currents in spin hydrodynamics are $u_\nu \delta\Theta^{\mu\nu}/T - \frac{1}{2}\varpi_{\rho\sigma}\delta\Sigma^{\mu\rho\sigma}$ and $\varpi_{\rho\sigma}\delta J^{\mu\rho\sigma}$. Recalling the definition (6), we assume

$$E^\mu = -\delta s^\mu + m_1\left(\frac{u_\nu}{T}\delta\Theta^{\mu\nu} - \frac{1}{2}\varpi_{\rho\sigma}\delta\Sigma^{\mu\rho\sigma}\right) + m_2\varpi_{\rho\sigma}\delta J^{\mu\rho\sigma}, \tag{38}$$

with two constants $m_{1,2}$. Since the leading order of E^μ is $\mathcal{O}(\delta^2)$ [112,113], Eq. (38) implies that

$$\delta s^\mu = m_1\left(\frac{u_\nu}{T}\delta\Theta^{\mu\nu} - \frac{1}{2}\varpi_{\rho\sigma}\delta\Sigma^{\mu\rho\sigma}\right) + m_2\varpi_{\rho\sigma}\delta J^{\mu\rho\sigma}, \tag{39}$$

holds at order $\mathcal{O}(\delta)$. By contracting u_μ on both sides of Eq. (39), we derive

$$\delta s = \frac{m_1}{T}(\delta e - \omega_{\rho\sigma}\delta S^{\rho\sigma}) + \frac{2m_2}{T}\omega_{\rho\sigma}(2x^\rho u_\mu \delta\Theta^{\mu\sigma} + \delta S^{\rho\sigma}), \tag{40}$$

where the identity (37) is used. Comparison of Eq. (40) with the thermodynamic relations (29) yields $m_1 = 1$ and $m_2 = 0$, resulting in

$$E^\mu = -\delta s^\mu + \frac{u_\nu}{T}\delta\Theta^{\mu\nu} - \frac{1}{T}\omega_{\rho\sigma}\delta\Sigma^{\mu\rho\sigma}. \tag{41}$$

Following the same strategy as in Sec. III, we will choose the rest frame of the fluid without rotation and assume the irrotational system reaches the thermodynamic equilibrium,

$$\{q^\mu, \phi^{\mu\nu}, \omega^{\mu\nu}, S^{\mu\nu}\} = 0. \tag{42}$$

The perturbation δs in Eq. (20) becomes

$$\delta s = \frac{1}{T}\delta e - \frac{1}{2T}\frac{c_s^2}{e+P}(\delta e)^2 - \frac{1}{2T}\delta\omega_{\alpha\beta}\delta S^{\alpha\beta} + O(\delta^3). \tag{43}$$

With the above results and Eqs. (25), (29), and (30), the information current can be expressed as

$$\begin{aligned} E^\mu &= -\delta s^\mu + \frac{1}{T}u_\nu\delta\Theta^{\mu\nu} - \frac{1}{T}\omega_{\rho\sigma}\delta\Sigma^{\mu\rho\sigma}, \\ &= \frac{1}{2T}\frac{c_s^2}{e+P}(\delta e)^2 u^\mu + \frac{c_s^2}{T}\delta e\delta u^\mu + \frac{c_s^2}{T(e+P)}\delta e\delta q^\mu + \frac{1}{2T}\delta\omega_{\alpha\beta}\delta S^{\alpha\beta}u^\mu \\ &\quad - \frac{1}{2T}(e+P)u^\mu\delta u_\nu\delta u^\nu + \frac{1}{T}\delta u_\nu\delta q^\nu u^\mu - \frac{1}{T}\delta u_\nu\delta\phi^{\mu\nu} - \frac{1}{T}\delta u_\nu\delta\pi^{\mu\nu} + \frac{1}{T}\delta\Pi\delta u^\mu \\ &\quad + \frac{1}{2T}u^\mu(\chi_q\delta q^\nu\delta q_\nu + \chi_\phi\delta\phi^{\alpha\beta}\delta\phi_{\alpha\beta} + \chi_\Pi\delta\Pi\delta\Pi + \chi_\pi\delta\pi^{\alpha\beta}\delta\pi_{\alpha\beta}), \end{aligned} \tag{44}$$

where we have used

$$\begin{aligned} u_\mu\delta q^\mu &= -\delta u_\mu\delta q^\mu, \\ u_\nu\delta\phi^{\mu\nu} &= -\delta u_\nu\delta\phi^{\mu\nu}. \end{aligned} \tag{45}$$

As a cross-check, we derive Eq. (44) by using a different approach shown in Appendix A.

Again, let us take $u^\mu = (1, 0)$. For arbitrary n^μ with $n_0 > 0$ and $n^\mu n_\mu = 1$, we can get

$$\begin{aligned} \frac{2n_0 T E^\mu n_\mu}{e+P} &= \frac{n_0^2\tau_\pi}{\eta(e+P)}\sum_{i<j}\left[\delta\pi^{ij} - \frac{1}{n_0\chi_\pi}n_{(i}\delta u_{j)}\right]^2 + \frac{n_0^2\tau_\pi}{\eta(e+P)}\left[\delta\pi^{11} + \frac{1}{2}\delta\pi^{22} + \frac{1}{2n_0\chi_\pi}(n_3\delta u_3 - n_1\delta u_1)\right]^2 \\ &\quad + \frac{3n_0^2\tau_\pi}{4\eta(e+P)}\left[\delta\pi^{22} + \frac{1}{3n_0\chi_\pi}(n_3\delta u_3 + n_1\delta u_1 - 2n_2\delta u_2)\right]^2 + \frac{n_0^2\tau_q}{\lambda(e+P)}\sum_i\left[\delta q^i - \frac{1}{n_0\chi_q}\left(\frac{c_s^2}{e+P}\delta e n_i + n_0\delta u_i\right)\right]^2 \\ &\quad + \frac{n_0^2\tau_\phi}{\gamma_s(e+P)}\sum_{i<j}\left(\delta\phi^{ij} - \frac{1}{n_0\chi_\phi}n_{[i}\delta u_{j]}\right)^2 + \frac{n_0^2}{e+P}\delta\omega_{\alpha\beta}\delta S^{\alpha\beta} + \sum_{i=6}^{10}a_i(\delta A_i)^2 + O(\delta^3), \end{aligned} \tag{46}$$

where the expressions for a_i and δA_i are presented in Appendix B. Next, we analyze the thermodynamic stability in two cases: with and without viscous tensors, $\pi^{\mu\nu}$ and $\Pi\Delta^{\mu\nu}$. The main reason is as follows. In the previous study by some of us [96], we find that there exist zero modes in the linear mode analysis for the spin hydrodynamics with vanishing viscous tensors. Such zero modes disappear once we turn on the finite viscous tensors. It is questionable whether the spin hydrodynamics can be stable and causal with vanishing viscous tensors. Therefore, it is necessary to study the thermodynamic stability with and without viscous tensors separately.

B. Case I: With vanishing viscous tensors

By simply setting $\delta\pi^{\mu\nu}$ and $\delta\Pi$ to zero in Eq. (46), we find that the sufficient and necessary conditions for thermodynamic stability (8) are

$$\begin{aligned} c_s^2, \gamma_s, \lambda, \tau_\phi, \tau_q, \delta\omega_{\alpha\beta}\delta S^{\alpha\beta} &> 0, \\ 1 - \frac{\lambda}{\tau_q(e+P)} - \frac{\gamma_s}{\tau_\phi(e+P)} &> 0, \\ 1 - c_s^2 - \frac{(3c_s^2+1)\lambda}{\tau_q(e+P)} &> 0. \end{aligned} \tag{47}$$

The last two inequalities can be rewritten as

$$\begin{aligned} 0 < \frac{2\gamma'\tau_q}{(2\tau_q - \lambda')\tau_\phi} &< 1, \\ 0 < \frac{c_s^2(2\tau_q + 3\lambda')}{2\tau_q - \lambda'} &< 1, \end{aligned} \tag{48}$$

where

$$\lambda' = \frac{2\lambda}{e+P}, \qquad \gamma' = \frac{\gamma_s}{e+P}. \tag{49}$$

We find that the conditions (48) are exactly the same as the causality conditions derived by linear mode analysis [96].

The stability conditions from linear mode analysis are given by [96]

$$\begin{aligned} c_s^2, \gamma_s, \lambda, \tau_\phi, \tau_q, \chi_s, -\chi_b &> 0, \\ 2\tau_q - \lambda' &> 0, \\ \chi_e^{0i} &= 0, \end{aligned} \tag{50}$$

where $\chi_e^{\mu\nu}$ and χ_b, χ_s are the spin susceptibilities with respect to e and S^{0i}, S^{ij}, i.e.,

$$\begin{aligned} \delta\omega^{0i} &= \chi_e^{0i}\delta e + \chi_b \delta S^{0i}, \\ \delta\omega^{ij} &= \chi_e^{ij}\delta e + \chi_s \delta S^{ij}. \end{aligned} \tag{51}$$

The inequality $2\tau_q > \lambda'$ can be directly derived from the thermodynamic stability conditions (47).

With the parametrization (51), the inequalities $\chi_b < 0$ and $\chi_s > 0$ are necessary conditions for $\delta\omega_{\alpha\beta}\delta S^{\alpha\beta} > 0$ in Eq. (47). However, $\chi_e^{0i} = 0$ does not arise immediately from the thermodynamic stability conditions. In fact, the spin susceptibility $\chi_e^{\mu\nu}$ introduced in Eq. (51) is a high order correction in our setup. Let us consider the equations of state,

$$e = e(T, \omega^{\mu\nu}), \qquad S^{\mu\nu} = S^{\mu\nu}(T, \omega^{\mu\nu}). \tag{52}$$

For simplicity, let us focus on S^{xy} and ω^{xy}, and assume other components of $S^{\mu\nu}$ and $\omega^{\mu\nu}$ are vanishing. Since the $\omega^{xy} \sim \mathcal{O}(\partial^1)$ is the quantum correction to the thermodynamic variables, the equations of state can be expressed as power series of ω^{xy} based on symmetry considerations,[2]

$$\begin{pmatrix} \delta e \\ \delta S^{xy} \end{pmatrix} = \begin{pmatrix} a_{11}T^3 & a_{12}\omega^{xy}T^2 \\ a_{21}\omega^{xy}T & a_{22}T^2 \end{pmatrix} \begin{pmatrix} \delta T \\ \delta\omega^{xy} \end{pmatrix} + \mathcal{O}(\omega_{xy}^2\delta\omega^{xy}, \omega_{xy}^2\delta T), \tag{53}$$

where a_{ij} are dimensionless constants and a_{11}, $a_{22} \neq 0$. The inverse of Eq. (53) gives

$$\begin{pmatrix} \delta T \\ \delta\omega^{xy} \end{pmatrix} = \frac{1}{a_{11}a_{22}T^4} \begin{pmatrix} a_{22}T & -a_{12}\omega^{xy}T \\ -a_{21}\omega^{xy} & a_{11}T^2 \end{pmatrix} \begin{pmatrix} \delta e \\ \delta S^{xy} \end{pmatrix} + \mathcal{O}(\omega_{xy}^2\delta e, \omega_{xy}^2\delta S^{\mu\nu}). \tag{54}$$

We find that $\chi_e^{xy} \propto \omega^{xy}$. When the system reaches irrotational equilibrium state shown in Eq. (42), $\chi_e^{xy} \propto \delta\omega^{xy}$, therefore $\chi_e^{xy}\delta e \sim \mathcal{O}(\delta^2)$ are high order corrections. While $\chi_s \sim 1/(a_{22}T^2) \sim \mathcal{O}(\delta^0)$ can survive. Hence, the condition $\chi_e^{\mu\nu} = \mathcal{O}(\delta)$ does not arise from stability demand but rather from our choice of an irrotational background.

Taking the parameterization (51) with $\chi_e^{\mu\nu} = \mathcal{O}(\delta)$, the inequalities $\chi_b < 0$ and $\chi_s > 0$ now become equivalent to $\delta\omega_{\alpha\beta}\delta S^{\alpha\beta} > 0$. Consequently, in the case of $\Pi, \pi^{\mu\nu} = 0$, the thermodynamic stability conditions align with the stability and causality conditions derived from linear mode analysis in Ref. [96]. It also indicates that the zero modes in the dispersion relations appeared in linear mode analysis [96] will not lead to instabilities.

C. Case II: With finite viscous tensors

Let us consider the full form of E^μ shown in Eq. (44). Imposing the thermodynamic stability conditions (8) yields

[2]Here, we assume the absence of characteristic or external tensors. In other words, the system is considered "isotropic." Clearly, this assumption implies $\chi_e^{\mu\nu} \sim \omega^{\mu\nu}$ by considering the antisymmetric tensor structure of $\chi_e^{\mu\nu}$. If this assumption does not hold, then $\chi_e^{\mu\nu}$ may be nonzero even if $\omega^{\mu\nu} = 0$.

$$c_s^2, \lambda, \gamma_s, \eta, \zeta, \tau_q, \tau_\phi, \tau_\pi, \tau_\Pi, -\chi_b, \chi_s > 0, \tag{55}$$

$$1 - \frac{\lambda'}{2\tau_q} - \frac{4\gamma_\perp}{3\tau_\pi} - \frac{1}{3\tau_\Pi}(3\gamma_\parallel - 4\gamma_\perp) > 0, \tag{56}$$

$$1 - \frac{\lambda'}{2\tau_q} - \frac{\gamma_\perp}{\tau_\pi} - \frac{\gamma'}{\tau_\phi} > 0, \tag{57}$$

$$1 - c_s^2 - \frac{(1+3c_s^2)\lambda'}{2\tau_q} - \frac{(2\tau_q - c_s^2\lambda')[4\gamma_\perp\tau_\Pi + \tau_\pi(3\gamma_\parallel - 4\gamma_\perp)]}{6\tau_q\tau_\pi\tau_\Pi} > 0, \tag{58}$$

$$2 - c_s^2 - \frac{(2+3c_s^2)\lambda'}{2\tau_q} - \frac{4\gamma_\perp\tau_\Pi + \tau_\pi(3\gamma_\parallel - 4\gamma_\perp)}{3\tau_\pi\tau_\Pi} > 0, \tag{59}$$

where we have used the parametrization (51) and the shorthand notations (49) and

$$\gamma_\perp = \frac{\eta}{e+P}, \qquad \gamma_\parallel = \frac{\frac{4}{3}\eta + \zeta}{e+P}. \tag{60}$$

We now compare these conditions (55)–(59) to those derived from linear mode analysis [96]. The causality conditions in linear mode analysis are given by

$$0 < \frac{2\tau_q(\gamma'\tau_\pi + \gamma_\perp\tau_\phi)}{(2\tau_q - \lambda')\tau_\pi\tau_\phi} < 1, \tag{61}$$

$$0 < \frac{b_1^{1/2} \pm (b_1 - b_2)^{1/2}}{6(2\tau_q - \lambda')\tau_\pi\tau_\Pi} < 1, \tag{62}$$

where $b_{1,2}$ are defined as

$$b_1^{1/2} = 8\gamma_\perp\tau_q\tau_\Pi + \tau_\pi[2\tau_q(3\gamma_\parallel - 4\gamma_\perp) + 3\tau_\Pi c_s^2(3\lambda' + 2\tau_q)],$$
$$b_2 = 12c_s^2\lambda'(2\tau_q - \lambda')\tau_\pi\tau_\Pi[\tau_\pi(3\gamma_\parallel - 4\gamma_\perp) + 4\gamma_\perp\tau_\Pi]. \tag{63}$$

It is straightforward to show that the inequality (61) can be derived from inequalities (55), (57). Similarly, one can derive (62) by using inequalities (55), (59). We then conclude that the causality in linear mode analysis is ensured by thermodynamic stability conditions.

The stability conditions derived by linear mode analysis are [96]

$$c_s^2, \lambda, \gamma_s, \eta, \zeta, \tau_q, \tau_\phi, \tau_\pi, \tau_\Pi, -\chi_b, \chi_s > 0, \tag{64}$$

$$2\tau_q - \lambda' > 0, \tag{65}$$

$$b_1 > b_2 > 0, \tag{66}$$

$$\frac{c_2}{c_3} > 0, \tag{67}$$

where the definitions of $c_{2,3}$ are presented in Appendix C. After performing the calculations detailed in Appendix D, we show that the inequalities (66), (67) can be derived from (64), (65). Consequently, the independent stability conditions in linear mode analysis reduce to Eqs. (64) and (65). It is worth noting that the inequality (64) aligns precisely with inequality (55) under the parameterization (51), while inequality (65) can be derived from either inequality (56) or (57).

Our results reveal that the stability and causality conditions derived in linear mode analysis can indeed be derived from thermodynamic stability conditions. However, the reverse does not hold in the current case. For instance, the inequality (56) cannot be derived from the causality and stability conditions identified in linear mode analysis. Therefore, unlike the scenarios discussed in Secs. III and IV B, the thermodynamic stability conditions for spin hydrodynamics involving nonvanishing components q^μ, $\phi^{\mu\nu}$, Π, and $\pi^{\mu\nu}$ are more stringent than those derived from linear mode analysis.

Let us discuss the above observation. A dissipative process is called real or on shell if it satisfies the equations of motion, otherwise, it is called virtual or off shell. Linear mode analysis solely considers real processes, whereas thermodynamic stability analysis encompasses both real and virtual processes [113,123]. If there are no virtual processes, meaning all forms of perturbations are allowed, then the conditions derived from thermodynamic stability analysis and linear mode analysis coincide, as the cases in Secs. III and IV B. However, in the presence of virtual processes, additional conditions emerge from thermodynamic stability analysis and are invisible in linear mode analysis. Consequently, the thermodynamic stability are more stringent compared to linear-mode stability. This implies that the thermodynamic stability analysis for spin hydrodynamics with viscous tensors may involve virtual processes that are not allowed by linearized hydrodynamic equations. A systematic verification of this statement is left for our future work.

In Ref. [96], it was found that the conditions derived from linear mode analysis might be necessary but are not sufficient to ensure stability. In contrast, the thermodynamic stability criteria (8) are both necessary and sufficient for ensuring stability. The reasoning is as follows.

Clearly, the thermodynamic stability criteria (8) are necessary to uphold the fundamental laws of stability, specifically the second law of thermodynamics and the principle of maximizing total entropy in the equilibrium state. On the other hand, the functional $E[\delta\varphi]$ defined in Eq. (7) is positive definite and nonincreasing in time when the criteria (8) are fulfilled. Then $E[\delta\varphi]$ can be interpreted as a Lyapunov functional, which is sufficient to guarantee the stability of the corresponding linearized hydrodynamic equations [113,124,125]. Therefore, we argue that the unstable modes identified in Ref. [96] would disappear if we adopt the conditions from thermodynamic stability (55)–(59). A rigorous proof of this assertion will require more general discussions on the structure of linearized hydrodynamic equations and will be presented elsewhere.

V. CONCLUSION

In this work, we have applied thermodynamic stability analysis to derive the stability and causality conditions for conventional relativistic viscous hydrodynamics and spin hydrodynamics.

As a test, we first derived the thermodynamic stability conditions in Eq. (23) for second-order relativistic viscous hydrodynamics without (baryon) currents and heat currents. We found that these conditions are consistent with those derived from linear mode analysis in Refs. [90,91,118].

We next studied the thermodynamic stability of minimal causal extended second-order spin hydrodynamics in canonical form, both with and without viscous tensors. In the absence of viscous tensors, the constraints derived from thermodynamic stability analysis exactly match those obtained from linear mode analysis. This indicates that the zero modes found in the linear mode analysis will not affect the causality and stability of the spin hydrodynamics in this case.

As another important observation, we also note that the inequality $\delta\omega_{\alpha\beta}\delta S^{\alpha\beta} > 0$ in Eq. (47) can be satisfied by adopting physical equations of state. The spin susceptibilities with respect to energy density, $\chi_e^{\mu\nu}$, are found to be $\sim\mathcal{O}(\delta)$ and therefore can be neglected in the current setup. This finding could help us understand the unstable modes identified in Ref. [96] when the asymptotic stability conditions are met in the linear modes analysis.

We then derive the thermodynamic stability conditions in Eqs. (55)–(59) for spin hydrodynamics in the presence of viscous tensors. These conditions are consistent with the causality conditions derived from linear mode analysis and are more stringent than the stability conditions found in linear mode analysis. Our studies suggest that the conditions derived from thermodynamic stability analysis can guarantee both causality and stability in linear mode analysis.

In the current studies, we have only considered irrotational spin hydrodynamics. The inclusion of a rotating background will affect the analysis, as noted in Ref. [98], and should be studied systematically in future work.

ACKNOWLEDGMENTS

We thank Lorenzo Gavassino for explaining the differences between thermodynamic stability analysis and linear mode analysis and Shokri for fruitful discussions on the information current for spin hydrodynamics. This work is supported in part by the National Key Research and Development Program of China under Contract No. 2022YFA1605500, by the Chinese Academy of Sciences (CAS) under Grant No. YSBR-088 and by National Nature Science Foundation of China (NSFC) under Grants No. 12075235 and No. 12135011.

APPENDIX A: ANOTHER APPROACH TO DERIVE THE INFORMATION CURRENT FOR SPIN HYDRODYNAMICS

Here we employ the method used in Ref. [119] (see also the Supplemental Material of Ref. [113]) to derive the information current (44) for spin hydrodynamics. This method is based on the fact that the function Ψ, defined in Eq. (5), should be maximized in the equilibrium state. We now introduce θ to characterize a smooth one-parameter family of solutions to hydrodynamic equations, where only $\theta = 0$ corresponds to the equilibrium state. Then $\Psi = \Psi(\theta)$ is a function of θ. Since Ψ is maximized in the equilibrium state, we have

$$\dot{\Psi}(0) = 0, \qquad \ddot{\Psi}(0) \leq 0, \tag{A1}$$

where the dot represents the derivative with respect to θ. Given an arbitrary three-dimensional spacelike Cauchy surface Σ with the future-directed and timelike normal unit vector n^μ, we can express Ψ as $\Psi = \int_\Sigma d\Sigma n_\mu \psi^\mu$, with the current $\psi^\mu = \psi^\mu(\theta)$ given by

$$\psi^\mu = s^\mu + \sum_a \alpha^a J^{a,\mu}. \tag{A2}$$

Due to arbitrariness of the Cauchy surface Σ, Eq. (A1) implies that

$$\dot{\psi}^\mu(0) = 0, \tag{A3}$$

and $\ddot{\psi}^\mu(0)$ is past directed and nonspacelike. For small θ, the information current E^μ can be derived through [113,119]

$$E^\mu = -\frac{1}{2}\theta^2 \ddot{\psi}^\mu(0). \tag{A4}$$

To calculate the information current E^μ using Eq. (A4), let us first construct the current ψ^μ. According to the discussion in Sec. IV A, there are two independent conserved currents,

$$\kappa_\nu \Theta^{\mu\nu} + \frac{1}{2}\partial_{[\rho}\kappa_{\sigma]}\Sigma^{\mu\rho\sigma}, \qquad \xi_{\rho\sigma}J^{\mu\rho\sigma}, \tag{A5}$$

where κ^μ is a killing vector and $\xi_{\rho\sigma}$ is an antisymmetric constant tensor. The general form for ψ^μ is

$$\psi^\mu = s^\mu - \kappa_\nu \Theta^{\mu\nu} - \frac{1}{2}\partial_{[\rho}\kappa_{\sigma]}\Sigma^{\mu\rho\sigma} - \xi_{\rho\sigma}J^{\mu\rho\sigma}. \tag{A6}$$

By introducing another killing vector

$$\beta_\nu = \kappa_\nu + 2\xi_{\rho\nu}x^\rho, \tag{A7}$$

the expression (A6) can be equivalently written as

$$\psi^\mu = s^\mu - \beta_\nu \Theta^{\mu\nu} - \frac{1}{2}\partial_{[\rho}\beta_{\sigma]}\Sigma^{\mu\rho\sigma}. \tag{A8}$$

Substituting the constitutive equations (25) into it, we obtain

$$\psi^\mu = \left[s - (e+P+\Pi)\beta_\nu u^\nu - \frac{1}{2}\partial_{[\rho}\beta_{\sigma]}S^{\rho\sigma} + q^\nu\beta_\nu - \mathcal{K}\right]u^\mu + (P+\Pi)\beta^\mu - q^\mu\left(\beta_\nu u^\nu - \frac{1}{T}\right) - (\phi^{\mu\nu} + \pi^{\mu\nu})\beta_\nu, \tag{A9}$$

where

$$\mathcal{K} = \frac{1}{2T}(\chi_q q^\nu q_\nu + \chi_\phi \phi^{\alpha\beta}\phi_{\alpha\beta} + \chi_\Pi \Pi^2 + \chi_\pi \pi^{\alpha\beta}\pi_{\alpha\beta}). \tag{A10}$$

The next step is to impose the constraint (A3) on ψ^μ. We find

$$\begin{aligned}\dot\psi^\mu = &\left[\dot s - (\dot e + \dot P + \dot\Pi)\beta_\nu u^\nu - (e+P+\Pi)\beta_\nu \dot u^\nu - \frac{1}{2}\partial_{[\rho}\beta_{\sigma]}\dot S^{\rho\sigma} + \dot q^\nu\beta_\nu - \dot{\mathcal{K}}\right]u^\mu \\ &+ \left[s - (e+P+\Pi)\beta_\nu u^\nu - \frac{1}{2}\partial_{[\rho}\beta_{\sigma]}S^{\rho\sigma} + q^\nu\beta_\nu - \mathcal{K}\right]\dot u^\mu \\ &+ (\dot P + \dot\Pi)\beta^\mu - \dot q^\mu\left(\beta_\nu u^\nu - \frac{1}{T}\right) - q^\mu\left(\beta_\nu \dot u^\nu + \frac{1}{T^2}\dot T\right) - (\dot\phi^{\mu\nu} + \dot\pi^{\mu\nu})\beta_\nu. \end{aligned} \tag{A11}$$

Note that here u^μ and $\dot u^\mu$ are independent, and this is true for other variables. The constraint (A3) demands

$$\frac{u_\nu}{T} = \beta_\nu, \qquad \frac{2}{T}\omega_{\rho\sigma} = -\partial_{[\rho}\beta_{\sigma]}, \qquad \Pi, q^\mu, \phi^{\mu\nu}, \pi^{\mu\nu} = 0, \tag{A12}$$

in the equilibrium state. These conditions are exactly the same as those from entropy current analysis [53,54].

With the equilibrium conditions (A12), we can get

$$\begin{aligned} u_\nu \ddot u^\nu = \dot u_\nu \dot u^\nu, \qquad u_\nu \dot q^\nu = 0, \qquad u_\nu \ddot q^\nu = -2\dot u_\nu \dot q^\nu, \\ \ddot\phi^{\mu\nu}u_\nu = -2\dot\phi^{\mu\nu}\dot u_\nu, \qquad \ddot\pi^{\mu\nu}u_\nu = -2\dot\pi^{\mu\nu}\dot u_\nu. \end{aligned} \tag{A13}$$

The thermodynamic relations (29) give

$$\ddot e = T\ddot s + \omega_{\rho\sigma}\ddot S^{\rho\sigma} + \dot T\,\dot s + \dot\omega_{\rho\sigma}\dot S^{\rho\sigma}. \tag{A14}$$

With the help of these identities (A13), (A14), we derive

$$\begin{aligned}\ddot\psi^\mu(0) = &-\left[\frac{1}{T}\dot T\,\dot s + \frac{1}{T}\dot\omega_{\rho\sigma}\dot S^{\rho\sigma} - \frac{1}{T}(e+P)\dot u_\nu \dot u^\nu + \frac{2}{T}\dot u_\nu \dot q^\nu\right]u^\mu - \frac{1}{T}(\chi_q \dot q^\nu \dot q_\nu + \chi_\phi \dot\phi^{\alpha\beta}\dot\phi_{\alpha\beta} + \chi_\Pi \dot\Pi^2 + \chi_\pi \dot\pi^{\alpha\beta}\dot\pi_{\alpha\beta})u^\mu \\ &- \frac{2}{T}(\dot P + \dot\Pi)\dot u^\mu - \frac{2}{T^2}\dot q^\mu \dot T + \frac{2}{T}\dot\phi^{\mu\nu}\dot u_\nu + \frac{2}{T}\dot\pi^{\mu\nu}\dot u_\nu. \end{aligned} \tag{A15}$$

Notice that, for small θ, the quantity $\theta\dot\varphi$ represent the small perturbation around the equilibrium state, i.e.,

$$\delta\varphi = \theta\dot\varphi, \tag{A16}$$

where φ stands for the hydrodynamic variables T, s, Π, u^μ, q^μ, etc. Hence, the information current from Eq. (A4) can be expressed as

$$
\begin{aligned}
E^\mu =& \frac{1}{2}\left[\frac{1}{T}\delta T\delta s + \frac{1}{T}\delta\omega_{\rho\sigma}\delta S^{\rho\sigma} - \frac{1}{T}(e+P)\delta u_\nu \delta u^\nu + \frac{2}{T}\delta u_\nu \delta q^\nu\right]u^\mu \\
&+ \frac{1}{2T}(\chi_q \delta q^\nu \delta q_\nu + \chi_\phi \delta\phi^{\alpha\beta}\delta\phi_{\alpha\beta} + \chi_\Pi \delta\Pi\delta\Pi + \chi_\pi \delta\pi^{\alpha\beta}\delta\pi_{\alpha\beta})u^\mu \\
&+ \frac{1}{T}(\delta P + \delta\Pi)\delta u^\mu + \frac{1}{T^2}\delta q^\mu \delta T - \frac{1}{T}\delta\phi^{\mu\nu}\delta u_\nu - \frac{1}{T}\delta\pi^{\mu\nu}\delta u_\nu + \mathcal{O}(\delta^3).
\end{aligned}
\tag{A17}
$$

The formula (A17) works for both rotational and irrotational background.

In an irrotational background where $\omega^{\mu\nu}, S^{\mu\nu} = 0$, we have

$$
\begin{aligned}
\delta s &= \frac{1}{T}\delta e + \mathcal{O}(\delta^2), \\
\delta P &= c_s^2 \delta e + \mathcal{O}(\delta^2), \\
\delta T &= \frac{c_s^2 T}{e+P}\delta e + \mathcal{O}(\delta^2).
\end{aligned}
\tag{A18}
$$

Plugging Eq. (A18) into Eq. (A17), we obtain the same information current as Eq. (44).

APPENDIX B: EXPRESSIONS FOR a_i AND δA_i IN EQS. (22), (46)

Here, we present the expressions for a_i and δA_i in Eqs. (22), (46),

$$
\begin{aligned}
a_1 &= a_6 = \zeta^{-1}\tau_\Pi(e+P), \\
a_2 &= \frac{1}{[1 + C_1 n_1^2 + C_2(n_2^2 + n_3^2)]}, \\
a_3 &= \frac{1 + C_2(n_1^2 + n_2^2 + n_3^2)}{[1 + C_1(n_1^2 + n_2^2) + C_2 n_3^2][1 + C_1 n_1^2 + C_2(n_2^2 + n_3^2)]}, \\
a_4 &= \frac{1 + C_2(n_1^2 + n_2^2 + n_3^2)}{[1 + C_1(n_1^2 + n_2^2) + C_2 n_3^2][1 + C_1(n_1^2 + n_2^2 + n_3^2)]}, \\
a_5 &= \frac{1 + (C_1 - c_s^2)(n_1^2 + n_2^2 + n_3^2)}{1 + C_1(n_1^2 + n_2^2 + n_3^2)}, \\
a_7 &= \frac{1}{C_3 + C_4 n_1^2 + C_5(n_2^2 + n_3^2)}, \\
a_8 &= \frac{C_3 + C_5(n_1^2 + n_2^2 + n_3^2)}{[C_3 + C_4(n_1^2 + n_2^2) + C_5 n_3^2][C_3 + C_4 n_1^2 + C_5(n_2^2 + n_3^2)]}, \\
a_9 &= \frac{C_3 + C_5(n_1^2 + n_2^2 + n_3^2)}{[C_3 + C_4(n_1^2 + n_2^2) + C_5 n_3^2][C_3 + C_4(n_1^2 + n_2^2 + n_3^2)]}, \\
a_{10} &= \frac{\{C_4 - c_s^2[(C_3 - 2)^2 - (C_3 - 1)C_4]\}(n_1^2 + n_2^2 + n_3^2)^2}{n_0^2[C_3 + C_4(n_1^2 + n_2^2 + n_3^2)]} + \frac{[C_3 + C_4 + (3C_3 - 4)c_s^2](n_1^2 + n_2^2 + n_3^2) + C_3}{n_0^2[C_3 + C_4(n_1^2 + n_2^2 + n_3^2)]},
\end{aligned}
\tag{B1}
$$

and

$$\delta A_1 = \delta A_6 = \frac{n_0}{e+P}\delta\Pi - \frac{\zeta}{\tau_\Pi(e+P)}(n_1\delta u_1 + n_2\delta u_2 + n_3\delta u_3),$$
$$\delta A_2 = [1 + C_1 n_1^2 + C_2(n_2^2+n_3^2)]\delta u_1 + (C_1 - C_2)n_1(n_2\delta u_2 + n_3\delta u_3) - \frac{c_s^2 n_0 n_1}{e+P}\delta e,$$
$$\delta A_3 = [1 + C_1(n_1^2+n_2^2) + C_2 n_3^2]\delta u_2 + (C_1 - C_2)n_2 n_3\delta u_3 - \frac{c_s^2 n_0 n_2}{e+P}\delta e,$$
$$\delta A_4 = [1 + C_1(n_1^2+n_2^2+n_3^2)]\delta u_3 - \frac{c_s^2 n_0 n_3}{e+P}\delta e,$$
$$\delta A_5 = \delta A_{10} = \frac{c_s n_0}{e+P}\delta e,$$
$$\delta A_7 = [C_3 + C_4 n_1^2 + C_5(n_2^2+n_3^2)]\delta u_1 + (C_4 - C_5)n_1(n_2\delta u_2 + n_3\delta u_3) + \frac{(C_3-2)c_s^2 n_1 n_0}{e+P}\delta e,$$
$$\delta A_8 = [C_3 + C_4(n_1^2+n_2^2) + C_5 n_3^2]\delta u_2 + (C_4 - C_5)n_2 n_3\delta u_3 + \frac{(C_3-2)c_s^2 n_2 n_0}{e+P}\delta e,$$
$$\delta A_9 = [C_3 + C_4(n_1^2+n_2^2+n_3^2)]\delta u_3 + \frac{(C_3-2)c_s^2 n_3 n_0}{e+P}\delta e, \tag{B2}$$

where we have defined

$$C_1 = 1 - \frac{4\eta}{3\tau_\pi(e+P)} - \frac{\zeta}{\tau_\Pi(e+P)},$$
$$C_2 = 1 - \frac{\eta}{\tau_\pi(e+P)},$$
$$C_3 = 1 - \frac{\lambda}{\tau_q(e+P)},$$
$$C_4 = 1 - \frac{\lambda}{\tau_q(e+P)} - \frac{4\eta}{3\tau_\pi(e+P)} - \frac{\zeta}{\tau_\Pi(e+P)},$$
$$C_5 = 1 - \frac{\lambda}{\tau_q(e+P)} - \frac{\eta}{\tau_\pi(e+P)} - \frac{\gamma_s}{\tau_\phi(e+P)}. \tag{B3}$$

APPENDIX C: EXPRESSIONS FOR $c_{2,3}$ IN INEQUALITY (67)

The expressions for $c_{2,3}$ in the inequality (67) are given by

$$c_1 = \sqrt{\frac{b_1^{1/2} \pm (b_1 - b_2)^{1/2}}{6(2\tau_q - \lambda')\tau_\pi\tau_\Pi}}, \quad \text{or} \quad -\sqrt{\frac{b_1^{1/2} \pm (b_1 - b_2)^{1/2}}{6(2\tau_q - \lambda')\tau_\pi\tau_\Pi}},$$
$$c_2 = -3c_1^4[2\tau_\pi\tau_\Pi + (2\tau_q - \lambda')(\tau_\pi + \tau_\Pi)] + c_1^2\{6\gamma_\parallel\tau_q + (6\gamma_\parallel - 8\gamma_\perp)\tau_\pi + 8\gamma_\perp\tau_\Pi + 3c_s^2[2\tau_\pi\tau_\Pi + (3\lambda' + 2\tau_q)(\tau_\pi + \tau_\Pi)]\} - 3c_s^2\gamma_\parallel\lambda',$$
$$c_3 = -2c_s^2\lambda'[(3\gamma_\parallel - 4\gamma_\perp)\tau_\pi + 4\gamma_\perp\tau_\Pi] - 18c_1^4(2\tau_q - \lambda')\tau_\pi\tau_\Pi + 4c_1^2[3c_s^2(3\lambda' + 2\tau_q)\tau_\pi\tau_\Pi + 2(3\gamma_\parallel - 4\gamma_\perp)\tau_q\tau_\pi + 8\gamma_\perp\tau_q\tau_\Pi]. \tag{C1}$$

Note that here we have set $\chi_e^{\mu\nu} = 0$, but the corresponding formulas in Ref. [96] contain nonzero $\chi_e^{\mu\nu}$.

APPENDIX D: DERIVE INEQUALITIES (66), (67) FROM (64), (65)

In this appendix, we will show that the inequalities (66), (67) can be derived from (64), (65). In the following calculations, we adopt the notations (49), (60), in which we have

$$3\gamma_{\parallel} - 4\gamma_{\perp} > 0. \tag{D1}$$

The inequalities (65), (D1) will be frequently used.

For the inequality (66), we note that

$$\begin{aligned} b_2 &= 12c_s^2\lambda'(2\tau_q - \lambda')\tau_\pi\tau_\Pi[\tau_\pi(3\gamma_{\parallel} - 4\gamma_{\perp}) + 4\gamma_{\perp}\tau_\Pi], \\ b_1 - b_2 &= 9(3\lambda' + 2\tau_q)^2\tau_\pi^2\tau_\Pi^2 c_s^4 + 4\tau_q^2[(3\gamma_{\parallel} - 4\gamma_{\perp})\tau_\pi + 4\gamma_{\perp}\tau_\Pi]^2 \\ &\quad + 12(\lambda'^2 + \lambda'\tau_q + 2\tau_q^2)\tau_\pi\tau_\Pi[(3\gamma_{\parallel} - 4\gamma_{\perp})\tau_\pi + 4\gamma_{\perp}\tau_\Pi]c_s^2. \end{aligned} \tag{D2}$$

Using (64), (65), (D1), we find that $b_2 > 0$ and $b_1 - b_2 > 0$, proving the inequality (66).

To show the inequality (67), it is equivalent to show $c_2c_3 > 0$. Straightforward calculation gives

$$c_2c_3 = f_0 \pm f_1(b_1 - b_2)^{1/2}, \tag{D3}$$

where

$$\begin{aligned} f_0 &= \frac{1}{9\tau_\pi^3\tau_\Pi^3(2\tau_q - \lambda')^3} f_0^{(1)} f_0^{(2)}, \\ f_1 &= \frac{1}{9\tau_\pi^3\tau_\Pi^3(2\tau_q - \lambda')^3}[f_1^{(0)} + c_s^2 f_1^{(2)} + c_s^4 f_1^{(4)} + c_s^6 f_1^{(6)}], \end{aligned} \tag{D4}$$

with

$$\begin{aligned} f_0^{(1)} &= 16\tau_\pi\gamma_{\perp}\tau_\Pi[3c_s^2\tau_\Pi(\lambda'^2 + \tau_q\lambda' + 2\tau_q^2) + 2(3\gamma_{\parallel} - 4\gamma_{\perp})\tau_q^2] + 9c_s^4\tau_\pi^2\tau_\Pi^2(3\lambda' + 2\tau_q)^2 \\ &\quad + 12c_s^2\tau_\pi^2\tau_\Pi(3\gamma_{\parallel} - 4\gamma_{\perp})(\lambda'^2 + \tau_q\lambda' + 2\tau_q^2) + 4\tau_\pi^2\tau_q^2(3\gamma_{\parallel} - 4\gamma_{\perp})^2 + 64\gamma_{\perp}^2\tau_\Pi^2\tau_q^2, \\ f_0^{(2)} &= 72c_s^4\lambda'\tau_\pi^3\tau_\Pi^3(3\lambda' + 2\tau_q) + 64\gamma_{\perp}^2\tau_\Pi^3\tau_q(\tau_\pi\lambda' + \tau_q\lambda' + 2\tau_q^2) + 12c_s^2\tau_\pi\tau_\Pi^3\gamma_{\perp}(2\tau_q + \lambda')(4\tau_q^2 - \lambda'^2 + 8\lambda\tau_\pi) \\ &\quad + 4(3\gamma_{\parallel} - 4\gamma_{\perp})^2\tau_\pi^3\tau_q[\lambda'\tau_\Pi + \tau_q(2\tau_q - \lambda')] + (3\gamma_{\parallel} - 4\gamma_{\perp})\tau_\pi\tau_\Pi\{3c_s^2\tau_\pi^2\tau_\Pi(2\tau_q + \lambda')(4\tau_q^2 - \lambda'^2 + 8\lambda'\tau_\Pi) \\ &\quad + 16\gamma_{\perp}\tau_\pi\tau_q[2\lambda'\tau_\Pi + \tau_q(2\tau_q - \lambda')] + 16\gamma_{\perp}\tau_\Pi\tau_q^2(2\tau_q - \lambda')\}, \\ f_1^{(0)} &= 8\tau_q^2[4\gamma_{\perp}(\tau_\Pi - \tau_\pi) + 3\tau_\pi\gamma_{\parallel}]^2\{4\gamma_{\perp}\tau_\Pi^2[\lambda'\tau_\pi + \tau_q(2\tau_q - \lambda')] + \lambda'\tau_\pi^2\tau_\Pi(3\gamma_{\parallel} - 4\gamma_{\perp}) + \tau_\pi^2\tau_q(3\gamma_{\parallel} - 4\gamma_{\perp})(2\tau_q - \lambda')\}, \\ f_1^{(2)} &= 6\tau_\pi\tau_\Pi[\tau_\pi(3\gamma_{\parallel} - 4\gamma_{\perp}) + 4\gamma_{\perp}\tau_\Pi]\{\tau_\pi^2(3\gamma_{\parallel} - 4\gamma_{\perp})[2\lambda'\tau_\Pi(\lambda'^2 + 5\tau_q\lambda' + 10\tau_q^2) + 4\tau_q^3(\lambda' + 4\tau_q) - 3\lambda'^3\tau_q] \\ &\quad + 4\gamma_{\perp}\tau_\Pi^2[2\lambda'\tau_\pi(\lambda'^2 + 5\tau_q\lambda' + 10\tau_q^2) + 4\tau_q^3(\lambda' + 4\tau_q) - 3\lambda'^3\tau_q]\}, \\ f_1^{(4)} &= 72[(3\gamma_{\parallel} - 4\gamma_{\perp})\tau_\pi + 4\tau_\Pi\gamma_{\perp}]\tau_\pi^3\tau_\Pi^3\lambda'(5\lambda'^2 + 10\lambda'\tau_q + 8\tau_q^2) \\ &\quad + 9[(3\gamma_{\parallel} - 4\gamma_{\perp})\tau_\pi^2 + 4\tau_\Pi^2\gamma_{\perp}]\tau_\pi^2\tau_\Pi^2(2\tau_q - \lambda')(2\tau_q + \lambda')^2(3\lambda' + 2\tau_q), \\ f_1^{(6)} &= 216\lambda'\tau_\pi^4\tau_\Pi^4(3\lambda' + 2\tau_q)^2. \end{aligned} \tag{D5}$$

From the inequalities (64), (65), (D1), we have

$$f_0 > 0. \tag{D6}$$

Next we calculate

$$f_0^2 - f_1^2(b_1 - b_2) = (g_0 + g_2c_s^2 + g_4c_s^2)G, \tag{D7}$$

where

$$
\begin{aligned}
G &= \frac{4\lambda'^2 c_s^4}{9\tau_\pi^3\tau_\Pi^3(2\tau_q-\lambda')^3}[\tau_\pi(3\gamma_\parallel-4\gamma_\perp)+4\tau_\Pi\gamma_\perp]\{\tau_\Pi^2[48\tau_\pi c_s^2\gamma_\perp(\lambda^2+\tau_q(\lambda+2\tau_q))+9\tau_\pi^2 c_s^4(3\lambda+2\tau_q)^2+64\gamma_\perp^2\tau_q^2] \\
&\quad +4\tau_\pi\tau_\Pi(3\gamma_\parallel-4\gamma_\perp)[3\tau_\pi c_s^2(\lambda^2+\tau_q(\lambda+2\tau_q))+8\gamma_\perp\tau_q^2]+4\tau_\pi^2\tau_q^2(3\gamma_\parallel-4\gamma_\perp)^2\}, \\
g_0 &= 4\{4\gamma_\perp\tau_\Pi^2[\lambda'\tau_\pi+\tau_q(2\tau_q-\lambda')]+\tau_\pi^2[\tau_q(2\tau_q-\lambda')+\lambda'\tau_\Pi](3\gamma_\parallel-4\gamma_\perp)\}[4\gamma_\perp\tau_\Pi+\tau_\pi(3\gamma_\parallel-4\gamma_\perp)]^2, \\
g_2 &= 24\tau_\pi^3(3\gamma_\parallel-4\gamma_\perp)\gamma_\perp\tau_\Pi^2[4\lambda'^2+16\lambda'\tau_\Pi+(2\tau_q-\lambda')^2]+96\tau_\pi\gamma_\perp^2\tau_\Pi^4[4\lambda'^2+(2\tau_q-\lambda')^2]+48\gamma_\perp^2\tau_\Pi^4(2\tau_q+\lambda')^2(2\tau_q-\lambda') \\
&\quad +768\lambda'\gamma_\perp^2\tau_\pi^2\tau_\Pi^4+24\gamma_\perp\tau_\pi^2\tau_\Pi^3(3\gamma_\parallel-4\gamma_\perp)[4\lambda'^2+(2\tau_q-\lambda')^2]+24\gamma_\perp\tau_\pi^2\tau_\Pi^2(3\gamma_\parallel-4\gamma_\perp)(2\tau_q+\lambda')^2(2\tau_q-\lambda') \\
&\quad +6\tau_\pi^4\tau_\Pi(3\gamma_\parallel-4\gamma_\perp)^2[4\lambda'^2+8\lambda'\tau_\Pi+(2\tau_q-\lambda')^2]+3\tau_\pi^4(3\gamma_\parallel-4\gamma_\perp)^2(2\tau_q+\lambda')^2(2\tau_q-\lambda'), \\
g_4 &= 72\lambda'\tau_\pi^2\tau_\Pi^2\{4\gamma_\perp\tau_\Pi^2(3\lambda'+2\tau_q+2\tau_\pi)+\tau_\pi^2(3\gamma_\parallel-4\gamma_\perp)[(3\lambda'+2\tau_q)+2\tau_\Pi]\}.
\end{aligned} \tag{D8}
$$

Again, we can find from the inequalities (64), (65), (D1) that

$$
G, g_0, g_2, g_4 > 0, \tag{D9}
$$

which leads to

$$
f_0^2 - f_1^2(b_1 - b_2) > 0. \tag{D10}
$$

Combing the results (D6) and (D10), we obtain

$$
c_2 c_3 = f_0 \pm f_1(b_1 - b_2)^{1/2} > 0, \tag{D11}
$$

or the equivalent form, $c_2/c_3 > 0$, i.e., the inequality (67).

[1] I. Arsene *et al.* (BRAHMS Collaboration), Nucl. Phys. **A757**, 1 (2005).
[2] K. Adcox *et al.* (PHENIX Collaboration), Nucl. Phys. **A757**, 184 (2005).
[3] J. Adams *et al.* (STAR), Nucl. Phys. **A757**, 102 (2005).
[4] K. Aamodt *et al.* (ALICE Collaboration), J. Instrum. **3**, S08002 (2008).
[5] W. Israel and J. M. Stewart, Ann. Phys. (N.Y.) **118**, 341 (1979).
[6] W. Israel and J. M. Stewart, Proc. R. Soc. A **365**, 43 (1979).
[7] R. Baier, P. Romatschke, D. T. Son, A. O. Starinets, and M. A. Stephanov, J. High Energy Phys. 04 (2008) 100.
[8] G. S. Denicol, H. Niemi, E. Molnar, and D. H. Rischke, Phys. Rev. D **85**, 114047 (2012); **91**, 039902(E) (2015).
[9] F. S. Bemfica, M. M. Disconzi, and J. Noronha, Phys. Rev. D **98**, 104064 (2018).
[10] P. Kovtun, J. High Energy Phys. 10 (2019) 034.
[11] F. S. Bemfica, M. M. Disconzi, and J. Noronha, Phys. Rev. D **100**, 104020 (2019); **105**, 069902(E) (2022).
[12] R. E. Hoult and P. Kovtun, J. High Energy Phys. 06 (2020) 067.
[13] F. S. Bemfica, M. M. Disconzi, and J. Noronha, Phys. Rev. X **12**, 021044 (2022).
[14] L. Gavassino and M. Antonelli, Front. Astron. Space Sci. **8**, 686344 (2021).
[15] G. S. Rocha, D. Wagner, G. S. Denicol, J. Noronha, and D. H. Rischke, Entropy **26**, 189 (2024).
[16] Z.-T. Liang and X.-N. Wang, Phys. Rev. Lett. **94**, 102301 (2005); **96**, 039901(E) (2006).
[17] Z.-T. Liang and X.-N. Wang, Phys. Lett. B **629**, 20 (2005).
[18] J.-H. Gao, S.-W. Chen, W.-t. Deng, Z.-T. Liang, Q. Wang, and X.-N. Wang, Phys. Rev. C **77**, 044902 (2008).
[19] L. Adamczyk *et al.* (STAR Collaboration), Nature (London) **548**, 62 (2017).
[20] J. Adam *et al.* (STAR Collaboration), Phys. Rev. Lett. **123**, 132301 (2019).
[21] M. S. Abdallah *et al.* (STAR Collaboration), Nature (London) **614**, 244 (2023).
[22] F. Becattini, F. Piccinini, and J. Rizzo, Phys. Rev. C **77**, 024906 (2008).
[23] I. Karpenko and F. Becattini, Eur. Phys. J. C **77**, 213 (2017).
[24] Y. Xie, D. Wang, and L. P. Csernai, Phys. Rev. C **95**, 031901 (2017).
[25] H. Li, L.-G. Pang, Q. Wang, and X.-L. Xia, Phys. Rev. C **96**, 054908 (2017).
[26] Y. Sun and C. M. Ko, Phys. Rev. C **96**, 024906 (2017).
[27] S. Shi, K. Li, and J. Liao, Phys. Lett. B **788**, 409 (2019).
[28] D.-X. Wei, W.-T. Deng, and X.-G. Huang, Phys. Rev. C **99**, 014905 (2019).
[29] X.-L. Xia, H. Li, Z.-B. Tang, and Q. Wang, Phys. Rev. C **98**, 024905 (2018).

[30] O. Vitiuk, L. V. Bravina, and E. E. Zabrodin, Phys. Lett. B **803**, 135298 (2020).
[31] B. Fu, K. Xu, X.-G. Huang, and H. Song, Phys. Rev. C **103**, 024903 (2021).
[32] S. Ryu, V. Jupic, and C. Shen, Phys. Rev. C **104**, 054908 (2021).
[33] A. Lei, D. Wang, D.-M. Zhou, B.-H. Sa, and L. P. Csernai, Phys. Rev. C **104**, 054903 (2021).
[34] X.-Y. Wu, C. Yi, G.-Y. Qin, and S. Pu, Phys. Rev. C **105**, 064909 (2022).
[35] F. Becattini, V. Chandra, L. Del Zanna, and E. Grossi, Ann. Phys. (Amsterdam) **338**, 32 (2013).
[36] R.-h. Fang, L.-g. Pang, Q. Wang, and X.-n. Wang, Phys. Rev. C **94**, 024904 (2016).
[37] S. Y. F. Liu and Y. Yin, J. High Energy Phys. 07 (2021) 188.
[38] Y.-C. Liu and X.-G. Huang, Sci. China Phys. Mech. Astron. **65**, 272011 (2022).
[39] B. Fu, S. Y. F. Liu, L. Pang, H. Song, and Y. Yin, Phys. Rev. Lett. **127**, 142301 (2021).
[40] F. Becattini, M. Buzzegoli, and A. Palermo, Phys. Lett. B **820**, 136519 (2021).
[41] C. Yi, S. Pu, and D.-L. Yang, Phys. Rev. C **104**, 064901 (2021).
[42] C. Yi, S. Pu, J.-H. Gao, and D.-L. Yang, Phys. Rev. C **105**, 044911 (2022).
[43] C. Yi, X.-Y. Wu, D.-L. Yang, J.-H. Gao, S. Pu, and G.-Y. Qin, Phys. Rev. C **109**, L011901 (2024).
[44] X.-Y. Wu, C. Yi, G.-Y. Qin, and S. Pu, EPJ Web Conf. **296**, 05008 (2024).
[45] S. Y. F. Liu and Y. Yin, Phys. Rev. D **104**, 054043 (2021).
[46] B. Fu, L. Pang, H. Song, and Y. Yin, arXiv:2201.12970.
[47] J.-A. Sun and L. Yan, arXiv:2401.07458.
[48] S. Fang, S. Pu, and D.-L. Yang, Phys. Rev. D **109**, 034034 (2024).
[49] S. Acharya *et al.* (ALICE Collaboration), Phys. Rev. Lett. **128**, 172005 (2022).
[50] STAR Collaboration, Phys. Rev. Lett. **131**, 202301 (2023).
[51] D. Montenegro, L. Tinti, and G. Torrieri, Phys. Rev. D **96**, 056012 (2017); **96**, 079901(A) (2017).
[52] D. Montenegro, L. Tinti, and G. Torrieri, Phys. Rev. D **96**, 076016 (2017).
[53] K. Hattori, M. Hongo, X.-G. Huang, M. Matsuo, and H. Taya, Phys. Lett. B **795**, 100 (2019).
[54] K. Fukushima and S. Pu, Phys. Lett. B **817**, 136346 (2021).
[55] S. Li, M. A. Stephanov, and H.-U. Yee, Phys. Rev. Lett. **127**, 082302 (2021).
[56] A. D. Gallegos, U. Gürsoy, and A. Yarom, SciPost Phys. **11**, 041 (2021).
[57] D. She, A. Huang, D. Hou, and J. Liao, Sci. Bull. **67**, 2265 (2022).
[58] M. Hongo, X.-G. Huang, M. Kaminski, M. Stephanov, and H.-U. Yee, J. High Energy Phys. 11 (2021) 150.
[59] D.-L. Wang, S. Fang, and S. Pu, Phys. Rev. D **104**, 114043 (2021).
[60] D.-L. Wang, X.-Q. Xie, S. Fang, and S. Pu, Phys. Rev. D **105**, 114050 (2022).
[61] Z. Cao, K. Hattori, M. Hongo, X.-G. Huang, and H. Taya, Prog. Theor. Exp. Phys. **2022**, 071D01 (2022).
[62] J. Hu, Phys. Rev. C **107**, 024915 (2023).
[63] R. Biswas, A. Daher, A. Das, W. Florkowski, and R. Ryblewski, Phys. Rev. D **108**, 014024 (2023).
[64] W. Florkowski, B. Friman, A. Jaiswal, and E. Speranza, Phys. Rev. C **97**, 041901 (2018).
[65] W. Florkowski, B. Friman, A. Jaiswal, R. Ryblewski, and E. Speranza, Phys. Rev. D **97**, 116017 (2018).
[66] W. Florkowski, E. Speranza, and F. Becattini, Acta Phys. Pol. B **49**, 1409 (2018).
[67] N. Weickgenannt, X.-L. Sheng, E. Speranza, Q. Wang, and D. H. Rischke, Phys. Rev. D **100**, 056018 (2019).
[68] S. Bhadury, W. Florkowski, A. Jaiswal, A. Kumar, and R. Ryblewski, Phys. Lett. B **814**, 136096 (2021).
[69] N. Weickgenannt, E. Speranza, X.-l. Sheng, Q. Wang, and D. H. Rischke, Phys. Rev. Lett. **127**, 052301 (2021).
[70] S. Shi, C. Gale, and S. Jeon, Phys. Rev. C **103**, 044906 (2021).
[71] E. Speranza and N. Weickgenannt, Eur. Phys. J. A **57**, 155 (2021).
[72] S. Bhadury, W. Florkowski, A. Jaiswal, A. Kumar, and R. Ryblewski, Phys. Rev. D **103**, 014030 (2021).
[73] R. Singh, G. Sophys, and R. Ryblewski, Phys. Rev. D **103**, 074024 (2021).
[74] H.-H. Peng, J.-J. Zhang, X.-L. Sheng, and Q. Wang, Chin. Phys. Lett. **38**, 116701 (2021).
[75] X.-L. Sheng, N. Weickgenannt, E. Speranza, D. H. Rischke, and Q. Wang, Phys. Rev. D **104**, 016029 (2021).
[76] J. Hu, Phys. Rev. D **105**, 076009 (2022).
[77] N. Weickgenannt, D. Wagner, E. Speranza, and D. H. Rischke, Phys. Rev. D **106**, 096014 (2022).
[78] N. Weickgenannt, D. Wagner, and E. Speranza, Phys. Rev. D **105**, 116026 (2022).
[79] N. Weickgenannt, D. Wagner, E. Speranza, and D. H. Rischke, Phys. Rev. D **106**, L091901 (2022).
[80] D. Wagner, M. Shokri, and D. H. Rischke, arXiv:2405.00533.
[81] A. D. Gallegos and U. Gürsoy, J. High Energy Phys. 11 (2020) 151.
[82] M. Garbiso and M. Kaminski, J. High Energy Phys. 12 (2020) 112.
[83] F. Becattini, A. Daher, and X.-L. Sheng, Phys. Lett. B **850**, 138533 (2024).
[84] Y. Hidaka, S. Pu, Q. Wang, and D.-L. Yang, Prog. Part. Nucl. Phys. **127**, 103989 (2022).
[85] P. Shi and H. Xu-Guang, Acta Phys. Sin. **72**, 071202 (2023).
[86] F. Becattini *et al.*, arXiv:2402.04540.
[87] W. A. Hiscock and L. Lindblom, Phys. Rev. D **31**, 725 (1985).
[88] W. A. Hiscock and L. Lindblom, Phys. Rev. D **35**, 3723 (1987).
[89] T. Koide, G. S. Denicol, P. Mota, and T. Kodama, Phys. Rev. C **75**, 034909 (2007).
[90] G. S. Denicol, T. Kodama, T. Koide, and P. Mota, J. Phys. G **35**, 115102 (2008).
[91] S. Pu, T. Koide, and D. H. Rischke, Phys. Rev. D **81**, 114039 (2010).
[92] C. V. Brito and G. S. Denicol, Phys. Rev. D **102**, 116009 (2020).
[93] C. V. Brito and G. S. Denicol, Phys. Rev. D **105**, 096026 (2022).

[94] G. Sarwar, M. Hasanujjaman, J. R. Bhatt, H. Mishra, and J.-e. Alam, Phys. Rev. D **107**, 054031 (2023).
[95] A. Daher, A. Das, and R. Ryblewski, Phys. Rev. D **107**, 054043 (2023).
[96] X.-Q. Xie, D.-L. Wang, C. Yang, and S. Pu, Phys. Rev. D **108**, 094031 (2023).
[97] N. Weickgenannt, Phys. Rev. D **108**, 076011 (2023).
[98] M. Shokri and D. H. Rischke, Phys. Rev. D **108**, 096029 (2023).
[99] C. V. P. de Brito, G. S. Rocha, and G. S. Denicol, arXiv: 2311.07272.
[100] Z. Fang, K. Hattori, and J. Hu, arXiv:2402.18601.
[101] A. Daher, W. Florkowski, R. Ryblewski, and F. Taghinavaz, Phys. Rev. D **109**, 114001 (2024).
[102] E. Krotscheck and W. Kundt, Commun. Math. Phys. **60**, 171 (1978).
[103] M. P. Heller, A. Serantes, M. Spaliński, and B. Withers, Phys. Rev. Lett. **130**, 261601 (2023).
[104] L. Gavassino, Phys. Lett. B **840**, 137854 (2023).
[105] M. P. Heller, A. Serantes, M. Spaliński, and B. Withers, arXiv:2305.07703.
[106] L. Gavassino, M. M. Disconzi, and J. Noronha, Phys. Rev. Lett. **132**, 162301 (2024).
[107] D.-L. Wang and S. Pu, Phys. Rev. D **109**, L031504 (2024).
[108] R. E. Hoult and P. Kovtun, Phys. Rev. D **109**, 046018 (2024).
[109] L. Gavassino, Phys. Rev. X **12**, 041001 (2022).
[110] W. A. Hiscock and L. Lindblom, Ann. Phys. (N.Y.) **151**, 466 (1983).
[111] T. S. Olson, Ann. Phys. (N.Y.) **199**, 18 (1990).
[112] L. Gavassino, Classical Quantum Gravity **38**, 21LT02 (2021).
[113] L. Gavassino, M. Antonelli, and B. Haskell, Phys. Rev. Lett. **128**, 010606 (2022).
[114] L. D. Landau and E. M. Lifshitz, *Statistical Physics, Part 1*, Course of Theoretical Physics Vol. 5 (Butterworth-Heinemann, Oxford, 1980).
[115] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, Course of Theoretical Physics Vol. 6 (Pergamon, New York, 1987).
[116] K. Yagi, T. Hatsuda, and Y. Miake, *Quark-Gluon Plasma: From Big Bang to Little Bang* (Cambridge University Press, Cambridge, England, 2005).
[117] D. Almaalol, T. Dore, and J. Noronha-Hostler, arXiv: 2209.11210.
[118] S. Pu, Relativistic fluid dynamics in heavy ion collisions, Ph.D. thesis, Hefei, CUST, 2011.
[119] L. Gavassino and M. Shokri, Phys. Rev. D **108**, 096010 (2023).
[120] F. Becattini, Phys. Rev. Lett. **108**, 244502 (2012).
[121] F. Becattini, L. Bucciantini, E. Grossi, and L. Tinti, Eur. Phys. J. C **75**, 191 (2015).
[122] F. Becattini, W. Florkowski, and E. Speranza, Phys. Lett. B **789**, 419 (2019).
[123] L. Gavassino, N. Mullins, and M. Hippert, Phys. Rev. D **109**, 125002 (2024).
[124] J. LaSalle and S. Lefschetz, *Stability by Liapunov's Direct Method: With Applications*, Mathematics in Science and Engineering Vol. 4 (Academic Press, New York, 1961).
[125] L. Gavassino, M. M. Disconzi, and J. Noronha, Phys. Rev. Lett. **132**, 222302 (2024).