

Kaon superfluidity in the early Universe

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(Received 20 May 2024; accepted 26 June 2024; published 5 August 2024)

Previously, it was found that pion superfluidity could be realized in the quantum chromodynamics (QCD) epoch of the early Universe, when lepton flavor asymmetry $|l_e + l_\mu|$ is large enough to generate a charge chemical potential $|\mu_Q|$ larger than vacuum pion mass. By following the same logic, kaon superfluidity might also be possible when $|l_e + l_\mu|$ is so large that $|\mu_Q|$ becomes larger than vacuum kaon mass. Such a possibility is checked by adopting Ginzburg-Landau approximation within the three-flavor Polyakov–Nambu–Jona-Lasinio model. Consider the case with full chemical balance, though kaon superfluidity could be stable compared to the chiral phases with only σ condensations, it would get killed by the more favored homogeneous pion superfluidity. If we introduce mismatch between s and d quarks, kaon superfluidity would require so large s quark density that such a state is impossible in the early Universe.

DOI: [10.1103/PhysRevD.110.034004](https://doi.org/10.1103/PhysRevD.110.034004)

I. INTRODUCTION

One important mission of nuclear physics is to explore possible phases of quantum chromodynamics (QCD) systems under different circumstances. Usually, neutron stars are believed to be in the low energy regime of QCD with hadrons the basic degrees of freedom, and many relevant phases had been proposed for them, such as neutron Cooper pairing [1], pasta structure [2,3], pion condensation [4–6], and kaon condensation [7], see also the review [8]. On the other hand, relativistic heavy ion collisions (HICs) are expected to be in the high energy regime of QCD with quarks and gluons the basic degrees of freedom, and the transition between quark gluon plasma (QGP) and hadron phases were widely studied [9,10]. However, the line between QGP and hadron phases is never very clear due to two facts: no sign of ordered phase transition was ever found in HICs [11–14] and quarkyonic matter was supposed to be possible in neutron stars [15–19].

The QCD epoch of the early Universe is another playground of high energy nuclear physics. Previously, primordial magnetic field was assumed to be generated in the electroweak epoch which then leaves relic in recent galaxies according to scaling law [20–22], and lepton flavor asymmetry was taken as a free parameter there since the early lepton flavors could not be constrained by recent observations for they are not conserved [23,24]. It turns out

that pion superfluidity could be favored for large enough lepton flavor asymmetry [23–25] and the QCD transition becomes of first order after taking the magnetic effect into account [26]. The first-order transition could generate gravitational wave directly in the QCD epoch, and the European Pulsar Timing Array group considered it to be a possible origin of their updated observations [27]. Inspired from the studies on neutron stars [4–7] and pure isospin matter [28–31], if the lepton flavor asymmetry is so large that the chemical potential of K^\pm is larger than their mass, kaon condensation can possibly be realized in a similar way as pions in the early Universe [23–26]. But it is also possible that pion superfluidity would kill kaon superfluidity at large charge density, in a similar way to the killing of rho superconductivity at finite isospin density [32]. Indeed, in the case of full chemical balance, no sign of kaon superfluidity was found for however large isospin chemical potential in the three-flavor chiral perturbation theory with mesons the basic degrees of freedom [33].

This work is devoted to exploring the possibility of kaon superfluidity in the early Universe by adopting the three-flavor Polyakov–Nambu–Jona-Lasinio (PNJL) model with quarks and gluons the basic degrees of freedom, the same as QCD. Note that gluons are important for realistic study as they contribute significantly to thermodynamic quantities at large temperature, such as entropy. If, due to its internal structure, kaon mass is greatly reduced to be comparable to pion mass in the medium, as that happens in neutron stars [7], kaon superfluidity could probably coexist with pion superfluidity. The paper is organized as follows. In Sec. II, formalisms are developed for chiral phases in Sec. II A and pion/kaon superfluidity in Sec. II B, respectively, for the general case with a primordial magnetic field. Specifically, the part for

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kaon condensation is developed by applying Ginzburg-Landau approximation based on the background of pion superfluidity. Within the subsections, the nontrivial sectors of strong interaction are presented in detail in Secs. II A 1 and II B 1, and the trivial sectors of electroweak interaction are simply summarized in Secs. II A 2 and II B 2. In Sec. III, numerical results are demonstrated for the case without primordial magnetic field. Finally, a brief summary is given in Sec. IV.

II. THE THREE-FLAVOR PNJL MODEL

In this section, we adopt the three-flavor PNJL model [34–36] for the QCD sector and develop the formalism for a finite primordial magnetic field.

A. Chiral phases in the magnetic field

Usually, chiral symmetry breaking and restoration are related to the expectation values of scalar field condensates. Below, we specifically refer to such phases as *chiral phases* to distinguish from pion or kaon superfluidity phase by following the previous convention [26]. Due to the Meissner effect, the chiral phases and pion/kaon superfluidity should be treated separately in an external magnetic field H , and Gibbs free energy must be adopted to determine the ground state [37]. As magnetization is small in chiral phases [26], we would directly use H instead of magnetic induction intensity B in the following.

1. The strong interaction sector

In a primordial magnetic field, the Lagrangian of the PNJL model can be modified from the previous one [26] to

$$\begin{aligned} \mathcal{L}_{\text{PNJL}} = & -\frac{H^2}{2} + \bar{\psi} \left[i\mathcal{D} - i\gamma^4 \left(ig\mathcal{A}^4 + Q_q\mu_Q + \frac{\mu_B}{3} \right) - m_0 \right] \psi \\ & + G \sum_{a=0}^8 [(\bar{\psi}\lambda^a\psi)^2 + (\bar{\psi}i\gamma_5\lambda^a\psi)^2] + \mathcal{L}_{\text{tH}} - V(L) \end{aligned} \quad (1)$$

by adopting the covariant derivative $D_\mu = \partial_\mu + iQ_q eA_\mu$. Here, the field variables are defined as the following: $\psi = (u, d, s)^T$ is the three-flavor quark field, H is the magnetic field with A_μ the corresponding vector potential, and the Polyakov loop is $L = \frac{1}{N_c} \text{tr} e^{ig \int dx_4 \mathcal{A}^4}$ with $\mathcal{A}^4 = A^{4c} T^c / 2$ the non-Abelian gauge field. For the quarks, the current mass and electric charge number matrices are, respectively,

$$m_0 \equiv \text{diag}(m_{0u}, m_{0d}, m_{0s}),$$

$$Q_q \equiv \text{diag}(q_u, q_d, q_s) = \frac{1}{3} \text{diag}(2, -1, -1); \quad (2)$$

and the interaction vertices $\lambda^i (i = 1, \dots, 8)$ are Gell-Mann matrices in flavor space and $\lambda^0 = \sqrt{2/3} \mathbb{1}_3$. For later use, the 't Hooft term, $\mathcal{L}_{\text{tH}} \equiv -K \sum_{t=\pm} \text{Det} \bar{\psi} \Gamma^t \psi$, can be rewritten as

$$\mathcal{L}_{\text{tH}} = -\frac{K}{2} \sum_{t=\pm} \epsilon_{ijk} \epsilon_{imn} (\bar{\psi}^i \Gamma^t \psi^j) (\bar{\psi}^j \Gamma^t \psi^m) (\bar{\psi}^k \Gamma^t \psi^n) \quad (3)$$

with $\Gamma^\pm = \mathbb{1}_4 \pm \gamma_5$ for right- and left-handed channels, respectively. Here, one should note the correspondences between 1, 2, 3 and u, d, s and the Einstein summation convention for the flavor indices i, j, k, m, n . The pure gluon potential could be obtained by fitting to the lattice QCD data [34]:

$$\begin{aligned} \frac{V(L)}{T^4} = & -\frac{1}{2} \left(3.51 - \frac{2.47}{\tilde{T}} + \frac{15.2}{\tilde{T}^2} \right) L^2 - \frac{1.75}{\tilde{T}^3} \\ & \times \ln [1 - 6L^2 + 8L^3 - 3L^4], \end{aligned} \quad (4)$$

where $\tilde{T} \equiv T/T_0$ is the reduced temperature with $T_0 = 0.27 \text{ GeV}$.

In the chiral phases, we only consider chiral condensates $\sigma_i \equiv \langle \bar{\psi}^i \psi^i \rangle$ with i flavor index, then the 't Hooft term \mathcal{L}_{tH} can be effectively reduced to four fermion interaction terms in Hartree approximation [35]:

$$\begin{aligned} \mathcal{L}_{\text{tH}}^4 = & -\frac{K}{2} \sum_{s=\pm} \epsilon_{ijk} \epsilon_{imn} \langle \bar{\psi}^i \Gamma^s \psi^j \rangle (\bar{\psi}^j \Gamma^s \psi^m) (\bar{\psi}^k \Gamma^s \psi^n) \\ = & -\frac{K}{6} \left\{ 2 \sum_{f=u,d,s} \sigma_f (\bar{\psi} \lambda^0 \psi)^2 - 3\sigma_s \sum_{i=1}^3 (\bar{\psi} \lambda^i \psi)^2 - 3\sigma_d \sum_{i=4}^5 (\bar{\psi} \lambda^i \psi)^2 - 3\sigma_u \sum_{i=6}^7 (\bar{\psi} \lambda^i \psi)^2 + (\sigma_s - 2\sigma_u - 2\sigma_d) (\bar{\psi} \lambda^8 \psi)^2 \right. \\ & \left. + \sqrt{2} (2\sigma_s - \sigma_u - \sigma_d) (\bar{\psi} \lambda^0 \psi) (\bar{\psi} \lambda^8 \psi) - \sqrt{6} (\sigma_u - \sigma_d) (\bar{\psi} \lambda^3 \psi) (\bar{\psi} \lambda^0 \psi - \sqrt{2} \bar{\psi} \lambda^8 \psi) \right\} - (\lambda^a \rightarrow i\lambda^a \gamma^5) \end{aligned} \quad (5)$$

with ϵ_{ijk} the Levi-Civita symbol. As a consequence, the Lagrangian becomes the one with only four fermion effective interactions,

$$\mathcal{L}_{\text{PNJL}}^4 = -\frac{H^2}{2} - V(L) + \bar{\psi} \left[i\mathcal{D} - i\gamma^4 \left(ig\mathcal{A}^4 + Q_q\mu_Q + \frac{\mu_B}{3} \right) - m_0 \right] \psi + \sum_{a,b=0}^8 [G_{ab}^- (\bar{\psi} \lambda^a \psi) (\bar{\psi} \lambda^b \psi) + G_{ab}^+ (\bar{\psi} i\gamma_5 \lambda^a \psi) (\bar{\psi} i\gamma_5 \lambda^b \psi)], \quad (6)$$

where the nonvanishing elements of the symmetric coupling matrices G^\pm are given by [35]

$$\begin{aligned} G_{00}^\mp &= G \mp \frac{K}{3} \sum_{f=u,d,s} \sigma_f, & G_{11}^\mp &= G_{22}^\mp = G_{33}^\mp = G \pm \frac{K}{2} \sigma_s, & G_{44}^\mp &= G_{55}^\mp = G \pm \frac{K}{2} \sigma_d, & G_{66}^\mp &= G_{77}^\mp = G \pm \frac{K}{2} \sigma_u, \\ G_{88}^\mp &= G \mp \frac{K}{6} (\sigma_s - 2\sigma_u - 2\sigma_d), & G_{08}^\mp &= \mp \frac{\sqrt{2}K}{12} (2\sigma_s - \sigma_u - \sigma_d), & G_{38}^\mp &= -\sqrt{2}G_{03}^\mp = \mp \frac{\sqrt{3}K}{6} (\sigma_u - \sigma_d). \end{aligned} \quad (7)$$

By contracting a pair of field and conjugate field further in $\mathcal{L}_{\text{th}}^4$, we find

$$\mathcal{L}_{\text{th}}^2 = - \sum_{s=\pm}^{i(\neq j \neq k)} K \langle \bar{\psi}^j \Gamma^s \psi^j \rangle \langle \bar{\psi}^k \Gamma^s \psi^k \rangle [\bar{\psi}^i \Gamma^s \psi^i] = -2K \sigma_j \sigma_k \bar{\psi}^i \psi^i (i \neq j \neq k, j < k), \quad (8)$$

and we could obtain the total dynamical quark masses as

$$m_i = m_{0i} - 4G\sigma_i + 2K\sigma_j\sigma_k \quad (9)$$

after taking the initial four quark interactions into account. Then, the gap equations directly follow the definitions of chiral condensates, that is,

$$\sigma_i \equiv \langle \bar{\psi}^i \psi^i \rangle = -\frac{i}{V_4} \text{Tr } \mathcal{S}_i, \quad (10)$$

where the effective quark propagators are given by [38]

$$\begin{aligned} \hat{\mathcal{S}}_i(k) &= i \int ds \exp \left\{ -i(m_i^2 + k_4^2 + k_3^2)s - i \frac{\tan(q_i e H s)}{q_i e H} (k_1^2 + k_2^2) \right\} [m_i - \gamma^4 k_4 - \gamma^3 k_3 - \gamma^2 (k_2 + \tan(q_i e H s) k_1) \\ &\quad - \gamma^1 (k_1 - \tan(q_i e H s) k_2)] [1 + \gamma^1 \gamma^2 \tan(q_i e H s)] \end{aligned} \quad (11)$$

for the case with a constant magnetic field.

By adopting vacuum regularization, the explicit forms of the gap equations are [38]

$$\begin{aligned} -\sigma_f &= N_c \frac{m_f^3}{2\pi^2} \left[\tilde{\Lambda}_f \left(1 + \tilde{\Lambda}_f^2 \right)^{\frac{1}{2}} - \ln \left(\tilde{\Lambda}_f + \left(1 + \tilde{\Lambda}_f^2 \right)^{\frac{1}{2}} \right) \right] + N_c \frac{m_f}{4\pi^2} \int_0^\infty \frac{ds}{s^2} e^{-m_f^2 s} \left(\frac{q_f e H s}{\tanh(q_f e H s)} - 1 \right) \\ &\quad - 6 \sum_{u=\pm} \frac{|q_f e H|}{2\pi} \sum_{n=0}^\infty \alpha_n \int_{-\infty}^\infty \frac{dk_3}{2\pi} \frac{m_f}{E_f^n} F_f^u(E_f^n, L, T, \mu_Q, \mu_B), \end{aligned} \quad (12)$$

where the reduced cutoff $\tilde{\Lambda}_f = \Lambda/m_f$, Landau level factor $\alpha_n = 1 - \delta_{n0}/2$, particle energy $E_f^n(k_3, m_f) = (2n|q_f e H| + k_3^2 + m_f^2)^{1/2}$, and the fermion distribution function

$$F_f^u(E_f^n, L, T, \mu_Q, \mu_B) \equiv \frac{L e^{-\frac{1}{T}(E_f^n - u(q_f \mu_Q + \frac{\mu_B}{3}))} + 2L e^{-\frac{2}{T}(E_f^n - u(q_f \mu_Q + \frac{\mu_B}{3}))} + e^{-\frac{3}{T}(E_f^n - u(q_f \mu_Q + \frac{\mu_B}{3}))}}{1 + 3L e^{-\frac{1}{T}(E_f^n - u(q_f \mu_Q + \frac{\mu_B}{3}))} + 3L e^{-\frac{2}{T}(E_f^n - u(q_f \mu_Q + \frac{\mu_B}{3}))} + e^{-\frac{3}{T}(E_f^n - u(q_f \mu_Q + \frac{\mu_B}{3}))}}.$$

Eventually, the quark part of thermodynamic potential can be consistently obtained as [39]

$$\begin{aligned} \Omega_q(H) &= 2G \sum_{f=u,d,s} \sigma_f^2 - 4K \prod_{f=u,d,s} \sigma_f - N_c \sum_{f=u,d,s} \left\{ \frac{m_f^4}{8\pi^2} \left[\tilde{\Lambda}_f \left(1 + 2\tilde{\Lambda}_f^2 \right) \left(1 + \tilde{\Lambda}_f^2 \right)^{\frac{1}{2}} - \ln \left(\tilde{\Lambda}_f + \left(1 + \tilde{\Lambda}_f^2 \right)^{\frac{1}{2}} \right) \right] \right. \\ &\quad - \frac{1}{8\pi^2} \int_0^\infty \frac{ds}{s^3} \left(e^{-m_f^2 s} - e^{-m_f^2 s} \right) \left(\frac{q_f e H s}{\tanh(q_f e H s)} - 1 \right) - \frac{1}{8\pi^2} \int_0^\infty \frac{ds}{s^3} e^{-m_f^2 s} \left(\frac{q_f e H s}{\tanh(q_f e H s)} - 1 - \frac{1}{3} (q_f e H s)^2 \right) \\ &\quad \left. + 2T \sum_{u=\pm} \frac{|q_f e H|}{2\pi} \sum_{n=0}^\infty \alpha_n \int_{-\infty}^\infty \frac{dk_3}{2\pi} K_f^u(E_f^n, L, T, \mu_Q, \mu_B) \right\}, \end{aligned} \quad (13)$$

with

$$K_f^u(E_f^n, L, T, \mu_Q, \mu_B) = \frac{1}{N_c} \ln \left[1 + 3Le^{-\frac{1}{T}(E_f^n - t(q_f \mu_Q + \frac{\mu_B}{3}))} + 3Le^{-\frac{2}{T}(E_f^n - t(q_f \mu_Q + \frac{\mu_B}{3}))} + e^{-\frac{3}{T}(E_f^n - t(q_f \mu_Q + \frac{\mu_B}{3}))} \right].$$

And the Gibbs free energy for the QCD sector is $\Omega_\chi^M = -\frac{H^2}{2} + V(L) + \Omega_q(H)$ [26].

With the free energy at hand, the gap equation for L can be given by $\partial_L \Omega_\chi^M = 0$ as

$$T^3 \left[-\left(3.51 - \frac{2.47}{\tilde{T}} + \frac{15.2}{\tilde{T}^2} \right) L + \frac{1.75}{\tilde{T}^3} \frac{12L(1-L)^2}{1-6L^2+8L^3-3L^4} \right] = 6 \sum_{f=u,d,s} \sum_{u=\pm} \frac{|q_f e H|}{2\pi} \sum_{n=0}^{\infty} \alpha_n \int_{-\infty}^{\infty} \frac{dk_3}{2\pi} \\ \times \frac{e^{-\frac{1}{T}(E_f^n - u(q_f \mu_Q + \frac{\mu_B}{3}))} + e^{-\frac{2}{T}(E_f^n - u(q_f \mu_Q + \frac{\mu_B}{3}))}}{1 + 3Le^{-\frac{1}{T}(E_f^n - u(q_f \mu_Q + \frac{\mu_B}{3}))} + 3Le^{-\frac{2}{T}(E_f^n - u(q_f \mu_Q + \frac{\mu_B}{3}))} + e^{-\frac{3}{T}(E_f^n - u(q_f \mu_Q + \frac{\mu_B}{3}))}}, \quad (14)$$

and the entropy, electric charge, and baryon densities follow the well-known thermodynamic relations as

$$s_\chi^M = 2N_c \sum_{f=u,d,s} \sum_{u=\pm} \frac{|q_f e H|}{2\pi} \sum_{n=0}^{\infty} \alpha_n \int_{-\infty}^{\infty} \frac{dk_3}{2\pi} \left[K_f^u(E_f^n, L, T, \mu_Q, \mu_B) + \frac{1}{T} \left(E_f^n - u \left(q_f \mu_Q + \frac{\mu_B}{3} \right) \right) F_f^u(E_f^n, L, T, \mu_Q, \mu_B) \right] \\ + T^3 \left\{ \frac{1}{2} \left(4 \times 3.51 - 3 \times \frac{2.47}{\tilde{T}} + 2 \times \frac{15.2}{\tilde{T}^2} \right) L^2 + \frac{1.75}{\tilde{T}^3} \ln[1 - 6L^2 + 8L^3 - 3L^4] \right\}, \quad (15)$$

$$n_Q^{q,M} = 2N_c \sum_{f=u,d,s} \sum_{u=\pm} \frac{|q_f e H|}{2\pi} \sum_{n=0}^{\infty} \alpha_n \int_{-\infty}^{\infty} \frac{dk_3}{2\pi} u q_f F_f^u(E_f^n, L, T, \mu_Q, \mu_B), \quad (16)$$

$$n_B^M = 2 \sum_{f=u,d,s} \sum_{u=\pm} \frac{|q_f e H|}{2\pi} \sum_{n=0}^{\infty} \alpha_n \int_{-\infty}^{\infty} \frac{dk_3}{2\pi} u F_f^u(E_f^n, L, T, \mu_Q, \mu_B). \quad (17)$$

2. The electroweak interaction sector

In free gas approximation, the thermodynamic potentials for the quantum electroweak dynamics (QEWD) sector can be simply given by [40,41]

$$\Omega_\gamma = 2T \int \frac{d^3k}{(2\pi)^3} \log(1 - e^{-k/T}), \quad (18)$$

$$\Omega_l^M = \sum_{i=e,\mu,\tau} \left\{ -T \sum_{u=\pm} \int \frac{d^3k}{(2\pi)^3} \log \left[1 + e^{-(k - u\mu_i)/T} \right] + \frac{1}{8\pi^2} \int_0^\infty \frac{ds}{s^3} e^{-m_i^2 s} \left[\frac{eHs}{\tanh(eHs)} - 1 - \frac{1}{3}(eHs)^2 \right] \right. \\ \left. - 2T \sum_{u=\pm} \frac{|eH|}{2\pi} \sum_{n=0}^{\infty} \alpha_n \int_{-\infty}^{\infty} \frac{dk_3}{2\pi} \log \left[1 + e^{-(\epsilon_i^n(k_3, eH) - u(-\mu_Q + \mu_i))/T} \right] \right\}, \quad (19)$$

where $\epsilon_i^n(k_3, eH) = (k_3^2 + 2n|eH| + m_i^2)^{1/2}$ and the degeneracy is one for (anti-)neutrinos due to their definite chiralities. Then, the corresponding entropy, electric charge and lepton flavor densities directly as

$$s_\gamma = 2 \int \frac{d^3k}{(2\pi)^3} \left[-\log(1 - e^{-k/T}) + \frac{k/T}{e^{k/T} - 1} \right], \quad (20)$$

$$s_l^M = \sum_{i=e,\mu,\tau} \left\{ \int \frac{d^3k}{(2\pi)^3} \left\{ \log \left[1 + e^{-(k - u\mu_i)/T} \right] + \frac{(k - u\mu_i)/T}{1 + e^{(k - u\mu_i)/T}} \right\} + 2 \sum_{u=\pm} \frac{|eH|}{2\pi} \sum_{n=0}^{\infty} \alpha_n \int_{-\infty}^{\infty} \frac{dk_3}{2\pi} \right. \\ \left. \times \left\{ \log \left[1 + e^{-(\epsilon_i^n(k_3, eH) - u(-\mu_Q + \mu_i))/T} \right] + \frac{(\epsilon_i^n(k_3, eH) - u(-\mu_Q + \mu_i))/T}{1 + e^{(\epsilon_i^n(k_3, eH) - u(-\mu_Q + \mu_i))/T}} \right\} \right\}, \quad (21)$$

$$n_Q^{l,M} = 2T \sum_{u=\pm} \frac{|eH|}{2\pi} \sum_{n=0}^{\infty} \alpha_n \int_{-\infty}^{\infty} \frac{dk_3}{2\pi} \frac{-u}{1 + e^{(\epsilon_1^n(k_3, eH) - u(-\mu_Q + \mu_l))/T}}, \quad (22)$$

$$n_i^M = T \sum_{u=\pm} \int \frac{d^3k}{(2\pi)^3} \frac{u}{1 + e^{(k - u\mu_i)/T}} + 2T \sum_{u=\pm} \frac{|eH|}{2\pi} \sum_{n=0}^{\infty} \alpha_n \int_{-\infty}^{\infty} \frac{dk_3}{2\pi} \frac{u}{1 + e^{(\epsilon_1^n(k_3, eH) - u(-\mu_Q + \mu_i))/T}}. \quad (23)$$

So, in the chiral phases, the total thermodynamic potential, entropy, electric charge and lepton densities are, respectively,

$$\begin{aligned} \Omega_M &= \Omega_\gamma + \Omega_1^M + \Omega_\chi^M, & s_M &= s_\gamma + s_1^M + s_\chi^M, \\ n_Q^M &= n_Q^{l,M} + n_Q^{q,M}, & n_1^M &= \sum_{i=e,\mu,\tau} n_i^M \end{aligned} \quad (24)$$

after combining the QEWD and QCD sectors together. According to the conventions [23,24], the following reduced quantities are frequently utilized

$$b^M = n_B^M/s_M, \quad l^M = n_l^M/s_M, \quad l_i^M = n_i^M/s_M. \quad (25)$$

B. The superconducting pion or kaon superfluidity

Since a homogeneous pion or kaon superfluid is also a type-I electric superconductor, the external magnetic field will be entirely expelled from the bulk due to the Meissner effect [37]. Then, the formalism is as if no external magnetic field exists, see that for the pion superfluidity in Ref. [26]. In this section, we would devote more effort to deriving the thermodynamic potential for kaon condensate based on the pion superfluidity.

1. The strong interaction sector

Without magnetic field in the bulk, the Lagrangian is given by [34–36]

$$\begin{aligned} \mathcal{L}_{\text{PNJL}} &= -V(L) + \bar{\psi} \left[i\not{\partial} - i\gamma^4 \left(igA^4 + Q_q\mu_Q + \frac{\mu_B}{3} \right) - m_0 \right] \psi \\ &+ G \sum_{a=0}^8 [(\bar{\psi}\lambda^a\psi)^2 + (\bar{\psi}i\gamma_5\lambda^a\psi)^2] + \mathcal{L}_{\text{tH}}. \end{aligned} \quad (26)$$

For the pion or kaon superfluidity, we choose the following scalar and charged pseudoscalar condensates to be nonzero

$$\begin{aligned} \sigma_f &= \langle \bar{\psi}_f \psi_f \rangle, & \Delta_\pi &= \langle \bar{u} i \gamma^5 d \rangle, & \Delta_\pi^* &= \langle \bar{d} i \gamma^5 u \rangle, \\ \Delta_K &= \langle \bar{u} i \gamma^5 s \rangle, & \Delta_K^* &= \langle \bar{s} i \gamma^5 u \rangle. \end{aligned} \quad (27)$$

Without loss of generality, we set $\Delta_\pi = \Delta_\pi^*$ and $\Delta_K = \Delta_K^*$ in the following. To facilitate the study, we would like first to reduce \mathcal{L}_{tH} to an effective form with four-fermion interactions at most. By applying the Hartree approximation

to contract a pair of quark and antiquark in each six-fermion interaction term [26,35], we immediately find

$$\begin{aligned} \mathcal{L}_{\text{tH}}^4 &= -K \{ \epsilon_{ijk} \epsilon_{lmn} \sigma_i (\bar{\psi}^j \psi^m \bar{\psi}^k \psi^n - \bar{\psi}^j i \gamma^5 \psi^m \bar{\psi}^k i \gamma^5 \psi^n) \\ &+ 2\Delta_\pi [\bar{s} s (\bar{u} i \gamma^5 d + \bar{d} i \gamma^5 u - \Delta_\pi) + \bar{s} i \gamma^5 s (\bar{u} d + \bar{d} u)] \\ &+ 2\Delta_K [\bar{d} d (\bar{u} i \gamma^5 s + \bar{s} i \gamma^5 u - \Delta_K) + \bar{d} i \gamma^5 d (\bar{u} s + \bar{s} u)] \}, \end{aligned} \quad (28)$$

where the second and third terms in the brace are induced by π^\pm and K^\pm condensations, respectively. Armed with the reduced Lagrangian density

$$\begin{aligned} \mathcal{L}_{\text{PNJL}} &= -V(L, L) \\ &+ \bar{\psi} \left[i\not{\partial} - i\gamma^4 \left(igA^4 + Q_q\mu_Q + \frac{\mu_B}{3} \right) - m_0 \right] \psi \\ &+ G \sum_{a=0}^8 [(\bar{\psi}\lambda^a\psi)^2 + (\bar{\psi}i\gamma_5\lambda^a\psi)^2] + \mathcal{L}_{\text{tH}}^4, \end{aligned} \quad (29)$$

the left calculations can just follow that of two-flavor case in principle.

By contracting quark and antiquark pairs once more in the interaction terms of Eq. (29), we obtain the quark bilinear as

$$\begin{aligned} \mathcal{L}_{\text{PNJL}}^2 &= \bar{\psi} \left[i\not{\partial} - i\gamma^4 \left(igA^4 + Q_q\mu_Q + \frac{\mu_B}{3} \right) - m \right. \\ &\left. - i\gamma^5 \left(\lambda^1 \Pi + \lambda^4 \mathcal{K} \right) \right] \psi, \end{aligned} \quad (30)$$

where the scalar and pseudoscalar masses are, respectively,

$$\begin{aligned} m_i &= m_{0i} - 4G\sigma_i + 2K(\sigma_j\sigma_k + \Delta_\pi^2\delta_{i3} + \Delta_K^2\delta_{i2}), \\ \Pi &= (-4G + 2K\sigma_3)\Delta_\pi, \quad \mathcal{K} = (-4G + 2K\sigma_2)\Delta_K \end{aligned} \quad (31)$$

with $i \neq j \neq k$. Due to the non-diagonal nature of the pseudoscalar masses Π and \mathcal{K} , the full propagator of ψ cannot be further reduced and it is hard to derive the eigenenergy analytically. Since $m_\pi \ll m_K$ in the vacuum and $\mu_d = \mu_s$, we expect $\Delta_K \ll \Delta_\pi$ in the superfluid phase, so it is a good approximation to neglect Δ_K for u and d quarks and assume $m_u = m_d \equiv m_l$. Eventually, to give consistent result on the kaon vacuum mass, we would adopt

the following approximate scalar and pseudoscalar masses for further study

$$\begin{aligned} m_l &= m_{0l} - 4G\sigma_l + 2K\sigma_l\sigma_s, \\ m_s &= m_{0s} - 4G\sigma_s + 2K(\sigma_l\sigma_s + \Delta_\pi^2), \\ \Pi &= (-4G + 2K\sigma_s)\Delta_\pi, \quad \mathcal{K} = (-4G + 2K\sigma_l)\Delta_K, \end{aligned} \quad (32)$$

where m_l , m_s , and Π are exactly the same as those without kaon condensation [25]. However, even with those simplifications, it is not possible to work out a simple analytic form for the bilinear thermodynamic potential from (30), that is,

$$\begin{aligned} \Omega_{bl} &= -\frac{1}{V_4} \text{Tr} \ln \left[k - i\gamma^4 \left(igA^4 + Q_q\mu_Q + \frac{\mu_B}{3} \right) - m \right. \\ &\quad \left. - i\gamma^5 (\lambda^1 \Pi + \lambda^4 \mathcal{K}) \right] \end{aligned} \quad (33)$$

in energy momentum space.

Inspired from the fact that only one local minimum of thermodynamic potential is involved with respect to Δ_π in the pion superfluidity [26], the same fact should be true with respect to Δ_K in the kaon superfluidity. Hence, Δ_K is expected to change smoothly within pion superfluidity, and we could take Ginzburg-Landau approximation to expand Ω_{bl} in orders of \mathcal{K} and calculate to a convergent order. Then, the thermodynamic potential can be separated into two parts, $\Omega_{bl} = \Omega'_{\pi s} + \Omega'_K$, where

$$\Omega'_{\pi s} \equiv -\frac{1}{V_4} \text{Tr} \ln S_{\pi s}^{-1}, \quad (34)$$

$$\begin{aligned} \Omega'_K &\equiv -\frac{1}{V_4} \text{Tr} \ln (1 - S_{\pi s} i\gamma^5 \lambda^4 \mathcal{K}) \\ &= \frac{1}{V_4} \sum_{n=1} \frac{1}{n} \text{Tr} (S_{\pi s} i\gamma^5 \lambda^4 \mathcal{K})^n \end{aligned} \quad (35)$$

are the ones with pion and kaon condensations, respectively. Here, $S_{\pi s}$ is the three-flavor fermion propagator in the pion superfluidity with its inverse defined by [25,26]

$$S_{\pi s}^{-1} \equiv k - i\gamma^4 \left(igA^4 + Q_q\mu_Q + \frac{\mu_B}{3} \right) - m - i\gamma^5 \lambda^1 \Pi. \quad (36)$$

Only u and d quarks couple with each other in $S_{\pi s}^{-1}$, so the propagator of s quark can be directly given by

$$S_s = \left[k - i\gamma^4 \left(igA^4 + Q_s\mu_Q + \frac{\mu_B}{3} \right) - m_s \right]^{-1}. \quad (37)$$

For the u and d quark parts, the matrix elements of $S_{\pi s}$ had been worked out to be [29]

$$S_{uu} = \sum_{t=\pm} \frac{\tilde{k}_0 + t\epsilon_l(\mathbf{k}) - \frac{\mu_Q}{2}}{\tilde{k}_0^2 - (E_{\mathbf{k}}^{-t})^2} \Lambda_t \gamma^0, \quad (38)$$

$$S_{dd} = \sum_{t=\pm} \frac{\tilde{k}_0 + t\epsilon_l(\mathbf{k}) + \frac{\mu_Q}{2}}{\tilde{k}_0^2 - (E_{\mathbf{k}}^t)^2} \Lambda_t \gamma^0, \quad (39)$$

$$S_{ud} = \sum_{t=\pm} \frac{-i\Pi}{\tilde{k}_0^2 - (E_{\mathbf{k}}^{-t})^2} \Lambda_t \gamma^5, \quad (40)$$

$$S_{du} = \sum_{t=\pm} \frac{-i\Pi}{\tilde{k}_0^2 - (E_{\mathbf{k}}^t)^2} \Lambda_t \gamma^5 \quad (41)$$

with the help of massive energy projectors

$$\Lambda_{\pm} = \frac{1}{2} \left(1 \pm \gamma^0 \frac{\boldsymbol{\gamma} \cdot \mathbf{k} + m_l}{\epsilon_l(\mathbf{k})} \right). \quad (42)$$

Here, we have defined $\tilde{k}_0 \equiv k_0 + igA^4 + \frac{\mu_Q + 2\mu_B}{6}$ and $E_{\mathbf{k}}^t = \sqrt{[\epsilon_l(k) + t\frac{\mu_Q}{2}]^2 + \Pi^2}$ with $\epsilon_l(k) = \sqrt{k^2 + m_l^2}$. Note the orthogonality: $\Lambda_{t_1} \Lambda_{t_2} = \Lambda_{t_1} \delta_{t_1, t_2}$.

The explicit form of the thermodynamic potential had been worked out for the pion superfluidity in our previous works [25,26], refer to Eq. (66). We now focus on deriving the explicit expression for the kaon part, Ω'_K . As the s quark propagator is diagonal in $S_{\pi s}$, it is easy to check that odd terms of \mathcal{K} do not contribute in Ω'_K , thus we have

$$\Omega'_K = \frac{1}{V_4} \sum_{n=1} \frac{1}{n} \text{Tr} (S_s i\gamma^5 S_{uu} i\gamma^5)^n \mathcal{K}^{2n}. \quad (43)$$

If we define an effective s -quark propagator

$$S_{5s5} \equiv i\gamma^5 S_s i\gamma^5 = \gamma^0 \sum_{u=\pm} \frac{\Lambda_u^s}{\tilde{k}_{0s} + u\epsilon_s(\mathbf{k})} \quad (44)$$

with $\tilde{k}_{0s} = \tilde{k}_0 - \frac{\mu_Q}{2}$ and $\Lambda_u^s = \frac{1}{2} (1 + u\gamma^0 \frac{\boldsymbol{\gamma} \cdot \mathbf{k} + m_s}{\epsilon_s(\mathbf{k})})$, the kaon part can be further reduced to

$$\Omega'_K = \sum_{n=1} \frac{\beta'_{2n}}{n} \mathcal{K}^{2n}, \quad \beta'_{2n} \equiv \frac{1}{V_4} \text{Tr} (S_{uu} S_{5s5})^n. \quad (45)$$

By substituting the explicit expressions of S_{uu} and S_{5s5} from Eqs. (38) and (44), respectively, the expansion coefficient β'_{2n} can be calculated to any given order in principle. The quark propagators are diagonal in color space, so the trace over Dirac space is the most tough part in the calculation.

As we can see, the basic element in the trace terms is

$$S_{uu} S_{5s5} = \sum_{t, u=\pm} \frac{[\tilde{k}_{0s} + t\epsilon_l(\mathbf{k})] \Lambda_t \Lambda_u^s}{[\tilde{k}_0^2 - (E_{\mathbf{k}}^{-t})^2][\tilde{k}_{0s} + u\epsilon_s(\mathbf{k})]}. \quad (46)$$

To help evaluate higher order coefficients, it is useful to check the commuting relation between Λ_t and Λ_u^s and we find

$$\Lambda_u^s \Lambda_t = \Lambda_t \Lambda_u^s + \delta\Lambda_{u,t}, \quad \delta\Lambda_{u,t} \equiv \frac{1}{2} u t \frac{\delta m}{\epsilon_1 \epsilon_s} \gamma \cdot \mathbf{k} \quad (47)$$

with $\delta m \equiv m_s - m_1$. The term $\delta\Lambda_{u,t}$ bears the following properties:

$$\Lambda_t^{(s)} \delta\Lambda_{u,t} = \delta\Lambda_{u,t} \Lambda_{-t}^{(s)}, \quad (48)$$

$$\delta\Lambda_{u,t} \delta\Lambda_{u',t'} = -\frac{1}{4} u t u' t' f(k, m_1, m_s) \quad (49)$$

with $f(k, m_1, m_s) \equiv (\frac{k\delta m}{\epsilon_1 \epsilon_s})^2$. Now, we can evaluate the trace over Dirac space order by order by utilizing the properties of $\Lambda_t^{(s)}$ and recursions, that is,

$$\alpha_{t,u} \equiv \text{Tr}_D \Lambda_t \Lambda_u^s = \text{Tr}_D \Lambda_u^s \Lambda_t = 1 + tu \frac{\mathbf{k}^2 + m_s m_1}{\epsilon_1 \epsilon_s}; \quad (50)$$

$$\begin{aligned} \alpha_{t_1, u_1; t_2, u_2} &\equiv \text{Tr}_D \Lambda_{t_1} \Lambda_{u_1}^s \Lambda_{t_2} \Lambda_{u_2}^s = \text{Tr}_D \Lambda_{t_1} (\Lambda_{t_2} \Lambda_{u_1}^s + \delta\Lambda_{u_1, t_2}) \Lambda_{u_2}^s = \delta_{t_1, t_2} \delta_{u_1, u_2} \alpha_{t_1, u_1} + \delta^2 \alpha_{t_1, u_1; t_2, u_2}, \\ \delta^2 \alpha_{t_1, u_1; t_2, u_2} &\equiv \text{Tr}_D \Lambda_{t_1} \delta\Lambda_{u_1, t_2} \Lambda_{u_2}^s = -\frac{1}{2} u_1 t_1 u_2 t_2 f(k, m_1, m_s); \end{aligned} \quad (51)$$

$$\begin{aligned} \alpha_{t_1, u_1; t_3, u_3} &\equiv \text{Tr}_D \Lambda_{t_1} \Lambda_{u_1}^s \Lambda_{t_2} \Lambda_{u_2}^s \Lambda_{t_3} \Lambda_{u_3}^s = \delta_{t_1, t_2} \delta_{u_1, u_2} \text{Tr}_D \Lambda_{t_1} \Lambda_{u_1}^s \Lambda_{t_3} \Lambda_{u_3}^s + \text{Tr}_D \Lambda_{t_1} \delta\Lambda_{u_1, t_2} \Lambda_{u_2}^s \Lambda_{t_3} \Lambda_{u_3}^s \\ &= \delta_{t_1, t_2} \delta_{u_1, u_2} \alpha_{t_1, u_1; t_3, u_3} + \delta_{t_1, -t_3} \delta_{u_2, u_3} \delta^2 \alpha_{t_1, u_1; t_2, u_2} + \text{Tr}_D \Lambda_{t_1} \delta\Lambda_{u_1, t_2} \delta\Lambda_{u_2, t_3} \Lambda_{u_3}^s \\ &= \delta_{t_1, t_2} \delta_{u_1, u_2} \alpha_{t_1, u_1; t_3, u_3} + \delta_{t_1, -t_3} \delta_{u_2, u_3} \delta^2 \alpha_{t_1, u_1; t_2, u_2} + \frac{1}{2} \alpha_{t_1, u_3} \delta^2 \alpha_{t_2, u_1; t_3, u_2}; \end{aligned} \quad (52)$$

$$\begin{aligned} \alpha_{t_1, u_1; t_4, u_4} &\equiv \text{Tr}_D \Lambda_{t_1} \Lambda_{u_1}^s \Lambda_{t_2} \Lambda_{u_2}^s \Lambda_{t_3} \Lambda_{u_3}^s \Lambda_{t_4} \Lambda_{u_4}^s = \delta_{t_1, t_2} \delta_{u_1, u_2} \alpha_{t_2, u_2; t_4, u_4} + \text{Tr}_D \Lambda_{t_1} \delta\Lambda_{u_1, t_2} \Lambda_{u_2}^s \Lambda_{t_3} \Lambda_{u_3}^s \Lambda_{t_4} \Lambda_{u_4}^s \\ &= \delta_{t_1, t_2} \delta_{u_1, u_2} \alpha_{t_2, u_2; t_4, u_4} + \delta_{t_3, t_4} \delta_{u_3, u_4} \left(\delta_{t_1, -t_3} \delta_{u_2, u_3} \delta^2 \alpha_{t_1, u_1; t_2, u_2} + \frac{1}{2} \alpha_{t_1, u_3} \delta^2 \alpha_{t_2, u_1; t_3, u_2} \right) \\ &\quad + \frac{1}{2} \alpha_{t_1, -u_2; -t_3, u_4} \delta^2 \alpha_{t_2, u_1; t_4, u_3}. \end{aligned} \quad (53)$$

We notice that the coefficients are all even functions of δm , and the leading nontrivial dependence is of quadratic except $\alpha_{t,u}$. Then, the expansion coefficients for the kaon part of thermodynamic potential follow as

$$\beta'_{2n} \equiv \frac{1}{V_4} \sum_k \sum_{q_j} \sum_{t_1, u_2; t_n, u_n = \pm} \alpha_{t_1, u_1; t_n, u_n} \prod_{i=1, \dots, n} \frac{[\tilde{k}_{0s} + t_i \epsilon_1(\mathbf{k})]}{[\tilde{k}_0^2 - (E_{\mathbf{k}}^{-t_i})^2][\tilde{k}_{0s} + u_i \epsilon_s(\mathbf{k})]}, \quad (54)$$

where q_j ($j = 1, 2, 3$) are the color charges of quarks and the conventions $\alpha_{t_1, u_1; t_1, u_1} = \alpha_{t_1, u_1}$ and $\alpha_{t_1, u_1; t_2, u_2} = \alpha_{t_1, u_1; t_2, u_2}$ should be understood. At finite temperature and chemical potential, we should work in Euclidean space by shifting k_0 to $i\omega_m$ in β'_{2n} and complete the summation over the Matsubara frequency $\omega_m \equiv (2m+1)\pi T$ in order to facilitate numerical calculations, see the Appendix for more details.

By the way, if we neglect δm dependence in $\alpha_{t_1, u_1; t_n, u_n}$, only the terms with $t_1 = \dots = t_n$ and $u_1 = \dots = u_n$, ($n = 1, 2, \dots$) are nonzero and we simply have

$$\alpha_{t_1, u_1; t_n, u_n} = \alpha_{t_1, u_1} \prod_{i=1}^n \delta_{t_1, t_i} \delta_{u_1, u_i}. \quad (55)$$

Then, the coefficients β'_{2n} would be greatly simplified to

$$\beta'_{2n} = \frac{1}{V_4} \sum_k \sum_{q_j} \sum_{t, u = \pm} \alpha_{t, u} \left(\frac{\tilde{k}_{0s} + t \epsilon_1(\mathbf{k})}{[\tilde{k}_0^2 - (E_{\mathbf{k}}^{-t})^2][\tilde{k}_{0s} + u \epsilon_s(\mathbf{k})]} \right)^n. \quad (56)$$

Note that β_2 in Eq. (56) is exactly the same as that in Eq. (54) as $\alpha_{t,u}$ does not depend on δm . With this approximation, the summation over n can even be carried out analytically in Ω'_K to arrive at

$$\begin{aligned} \Omega'_K &= \frac{-1}{V_4} \sum_k \sum_{q_j} \sum_{t, u = \pm} \alpha_{t, u} \ln \\ &\quad \times \left\{ 1 - \frac{[\tilde{k}_{0s} + t \epsilon_1(\mathbf{k})] \mathcal{K}^2}{[\tilde{k}_0^2 - (E_{\mathbf{k}}^{-t})^2][\tilde{k}_{0s} + u \epsilon_s(\mathbf{k})]} \right\}. \end{aligned} \quad (57)$$

We have checked numerically that such an approximation is very bad when $m_s \gg m_1$, so we will stick to the exact form (54) in the following.

Eventually, the coupled gap equations follow directly from the definitions of condensates

$$\begin{aligned}\sigma_s &\equiv \langle \bar{s}s \rangle = \frac{\partial \Omega'_{\pi s}}{\partial m_s}, & 2\sigma_1 &\equiv \langle \bar{u}u \rangle + \langle \bar{d}d \rangle = \frac{\partial \Omega'_{\pi s}}{\partial m_1}, \\ 2\Delta_\pi &\equiv \langle \bar{u}i\gamma^5 d \rangle + \langle \bar{d}i\gamma^5 u \rangle = \frac{\partial \Omega'_{\pi s}}{\partial \Pi}, \\ 2\Delta_K &\equiv \langle \bar{u}i\gamma^5 s \rangle + \langle \bar{s}i\gamma^5 u \rangle = \frac{\partial \Omega'_K}{\partial \mathcal{K}}\end{aligned}\quad (58)$$

and the minimal condition $\partial_L[V(L, L) + \Omega_{\pi s}] = 0$ as [25]

$$\sigma_s = -2N_c \int^\Lambda \frac{d^3k}{(2\pi)^3} \frac{m_s}{\epsilon_s(k)} + 2N_c \int \frac{d^3k}{(2\pi)^3} \frac{m_s}{\epsilon_s(k)} \sum_{u=\pm} dV_1\left(L, u, \epsilon_s(k), \frac{-\mu_Q + \mu_B}{3}\right), \quad (59)$$

$$2\sigma_1 = -2N_c \int^\Lambda \frac{d^3k}{(2\pi)^3} \sum_{t=\pm} \frac{m_1}{\epsilon_1(k)} \frac{\epsilon_1(k) + t\frac{\mu_Q}{2}}{E_1^t(k)} + 2N_c \int \frac{d^3k}{(2\pi)^3} \sum_{t,u=\pm} \frac{m_1}{\epsilon_1(k)} \frac{\epsilon_1(k) + t\frac{\mu_Q}{2}}{E_1^t(k)} dV_1\left(L, u, E_1^t(k), \frac{\mu_Q + 2\mu_B}{6}\right), \quad (60)$$

$$2\Delta_\pi = -2N_c \int^\Lambda \frac{d^3k}{(2\pi)^3} \sum_{t=\pm} \frac{\Pi}{E_1^t(k)} + 2N_c \int \frac{d^3k}{(2\pi)^3} \sum_{t,u=\pm} \frac{\Pi}{E_1^t(k)} dV_1\left(L, u, E_1^t(k), \frac{\mu_Q + 2\mu_B}{6}\right), \quad (61)$$

$$2\Delta_K = 2 \sum_{n=1} \beta'_{2n} \mathcal{K}^{2n-1}, \quad (62)$$

$$\begin{aligned}T^4 \left[-\left(3.51 - \frac{2.47}{\tilde{T}} + \frac{15.2}{\tilde{T}^2}\right)L + \frac{1.75}{\tilde{T}^3} \frac{12L(1-L)^2}{1-6L^2+8L^3-3L^4} \right] &= 6T \int \frac{d^3k}{(2\pi)^3} \sum_{u=\pm} \left[\sum_{t=\pm} dV_2\left(L, u, E_1^t(k), \frac{\mu_Q + 2\mu_B}{6}\right) \right. \\ &\quad \left. + dV_2\left(L, u, \epsilon_s(k), \frac{-\mu_Q + \mu_B}{3}\right) \right].\end{aligned}\quad (63)$$

Note that $\Delta_\pi = 0$ and $\Delta_K = 0$ are trivial solutions of Eqs. (61) and (62), respectively. As pion superfluidity is usually more favored than kaon superfluidity in this study, Δ_π (or Π) and Δ_K (or \mathcal{K}) are actually true order parameters for I_3 [28] and I_8 flavor symmetries, respectively. The total self-consistent thermodynamic potential can be found to be

$$\Omega_{\pi\text{KSF}} = V(L, L) + \Omega'_{\pi s} + \Omega_K + 2G(\sigma_s^2 + 2\sigma_1^2 + 2\Delta_\pi^2) - 4K(\sigma_1^2 + \Delta_\pi^2)\sigma_s \quad (64)$$

by utilizing the definitions of condensates and their relations to scalar and pseudoscalar masses, refer to Eqs. (32) and (58). Here, the pion bilinear [25,26] and full kaon parts of thermodynamic potential are, respectively,

$$\Omega'_{\pi s} = -2N_c \int^\Lambda \frac{d^3k}{(2\pi)^3} \left[\sum_{t=\pm} E_{\mathbf{k}}^t + \epsilon_s(k) \right] - 2T \int \frac{d^3k}{(2\pi)^3} \sum_{u=\pm} \left[\sum_{t=\pm} Fl\left(L, u, E_{\mathbf{k}}^t, \frac{\mu_Q + 2\mu_B}{6}\right) + Fl\left(L, u, \epsilon_s(k), \frac{-\mu_Q + \mu_B}{3}\right) \right], \quad (65)$$

$$\left\{ Fl(L, u, x, y) = \log \left[1 + 3Le^{-\frac{1}{T}(x-uy)} + 3Le^{-\frac{2}{T}(x-uy)} + e^{-\frac{3}{T}(x-uy)} \right] \right\}, \quad (66)$$

$$\Omega_K \equiv \Omega'_K + (4G - 2K\sigma_1)\Delta_K^2 = \sum_{n=1} \frac{\beta_{2n}}{n} \mathcal{K}^{2n}, \quad \beta_{2n} = \beta'_{2n} + \frac{\delta_{n,1}}{4G - 2K\sigma_1}. \quad (67)$$

The contribution of Ω_K is usually small compared to that of $\Omega'_{\pi s}$ but is important for the evaluation of Δ_K . Following that, the entropy, electric charge and baryon densities of the system can be approximately evaluated by simply neglecting Ω_K and we have [25]

$$\begin{aligned}
s_{\pi\text{KSF}} = & 2 \int \frac{d^3k}{(2\pi)^3} \sum_{t,u=\pm} \left[Fl \left(L, u, E_1^t(k), \frac{\mu_Q + 2\mu_B}{6} \right) + \frac{3 \left(E_1^t(k) - u \frac{\mu_Q + 2\mu_B}{6} \right)}{T} dV_1 \left(L, u, E_1^t(k), \frac{\mu_Q + 2\mu_B}{6} \right) \right] \\
& + 2 \int \frac{d^3k}{(2\pi)^3} \sum_{u=\pm} \left[Fl \left(L, u, \epsilon_s(k), \frac{-\mu_Q + \mu_B}{3} \right) + \frac{3 \left(E_1^t(k) - u \frac{-\mu_Q + \mu_B}{3} \right)}{T} dV_1 \left(L, u, \epsilon_s(k), \frac{-\mu_Q + \mu_B}{3} \right) \right] \\
& + T^3 \left\{ \frac{1}{2} \left(4 \times 3.51 - 3 \times \frac{2.47}{\tilde{T}} + 2 \times \frac{15.2}{\tilde{T}^2} \right) L^2 + \frac{1.75}{\tilde{T}^3} \ln [1 - 6L^2 + 8L^3 - 3L^4] \right\}, \quad (68)
\end{aligned}$$

$$\begin{aligned}
n_Q^{\pi\text{KSF}} = & N_c \int^\Lambda \frac{d^3k}{(2\pi)^3} \sum_{t=\pm} t \frac{\epsilon_1(k) + t \frac{\mu_Q}{2}}{E_1^t(k)} - 3 \int \frac{d^3k}{(2\pi)^3} \sum_{t,u=\pm} t \frac{\epsilon_1(k) + t \frac{\mu_Q}{2}}{E_1^t(k)} dV_1 \left(L, u, E_1^t(k), \frac{\mu_Q + 2\mu_B}{6} \right) \\
& + \int \frac{d^3k}{(2\pi)^3} \sum_{t,u=\pm} u dV_1 \left(L, u, E_1^t(k), \frac{\mu_Q + 2\mu_B}{6} \right) - 2 \int \frac{d^3k}{(2\pi)^3} \sum_{t,u=\pm} u dV_1 \left(L, u, \epsilon_s(k), \frac{-\mu_Q + \mu_B}{3} \right), \quad (69)
\end{aligned}$$

$$n_B^{\pi\text{KSF}} = 2 \int \frac{d^3k}{(2\pi)^3} \sum_{t,u=\pm} u dV_1 \left(L, u, E_1^t(k), \frac{\mu_Q + 2\mu_B}{6} \right) + 2 \int \frac{d^3k}{(2\pi)^3} \sum_{t,u=\pm} u dV_1 \left(L, u, \epsilon_s(k), \frac{-\mu_Q + \mu_B}{3} \right). \quad (70)$$

2. The electroweak interaction sector

In free gas approximation, the thermodynamic potentials for the QEWD sector are [40]

$$\Omega_\gamma = 2T \int \frac{d^3k}{(2\pi)^3} \log(1 - e^{-k/T}), \quad (71)$$

$$\begin{aligned}
\Omega_l = & -T \sum_{u=\pm}^{i=e,\mu,\tau} \int \frac{d^3k}{(2\pi)^3} \left\{ 2 \log \left[1 + e^{-(\epsilon_i(k) - u(-\mu_Q + \mu_i))/T} \right] \right. \\
& \left. + \log \left[1 + e^{-(k - u\mu_i)/T} \right] \right\}, \quad (72)
\end{aligned}$$

and the corresponding entropy, electric charge and lepton flavor densities are, respectively,

$$s_\gamma = 2 \int \frac{d^3k}{(2\pi)^3} \left[-\log(1 - e^{-k/T}) + \frac{k/T}{e^{k/T} - 1} \right], \quad (73)$$

$$\begin{aligned}
s_l = & \sum_{u=\pm}^{i=e,\mu,\tau} \int \frac{d^3k}{(2\pi)^3} \left\{ 2 \log \left[1 + e^{-(\epsilon_i(k) - u(-\mu_Q + \mu_i))/T} \right] \right. \\
& + \log \left[1 + e^{-(k - u\mu_i)/T} \right] + \frac{2(\epsilon_i(k) - u(-\mu_Q + \mu_i))/T}{1 + e^{(\epsilon_i(k) - u(-\mu_Q + \mu_i))/T}} \\
& \left. + \frac{(k - u\mu_i)/T}{1 + e^{(k - u\mu_i)/T}} \right\}, \quad (74)
\end{aligned}$$

$$n_Q^l = 2T \sum_{u=\pm}^{i=e,\mu,\tau} \int \frac{d^3k}{(2\pi)^3} \frac{-u}{1 + e^{(\epsilon_i(k) - u(-\mu_Q + \mu_i))/T}}, \quad (75)$$

$$\begin{aligned}
n_i = & -\frac{\partial \Omega_l}{\partial \mu_i} = T \sum_{u=\pm} \int \frac{d^3k}{(2\pi)^3} \left[\frac{2u}{1 + e^{(\epsilon_i(k) - u(-\mu_Q + \mu_i))/T}} \right. \\
& \left. + \frac{u}{1 + e^{(k - u\mu_i)/T}} \right], \quad i = e, \mu, \tau. \quad (76)
\end{aligned}$$

So, in the pion or kaon superfluidity, the total thermodynamic potential, entropy, electric charge, and lepton densities are, respectively,

$$\begin{aligned}
\Omega = & \Omega_\gamma + \Omega_l + \Omega_{\pi\text{KSF}}, \quad s = s_\gamma + s_l + s_{\pi\text{KSF}}, \\
n_Q = & n_Q^l + n_Q^{\pi\text{KSF}}, \quad n_l = \sum_{i=e,\mu,\tau} n_i \quad (77)
\end{aligned}$$

after combining the QEWD and QCD sectors together. According to the conventions, we define the following reduced quantities:

$$b = n_B^{\pi\text{KSF}}/s, \quad l = n_l/s, \quad l_i = n_i/s. \quad (78)$$

III. NUMERICAL RESULTS FOR THE CASE WITHOUT MAGNETIC FIELD

To carry out numerical calculations, we fix the electron and muon masses from the Particle Data Group as $m_e = 0.53$ MeV and $m_\mu = 113$ MeV and suppress the contribution of heavy τ lepton. The model parameters for the strong interaction sector are fixed as [42,43]

$$\begin{aligned}
m_{0l} = & 5.5 \text{ MeV}, \quad m_{0s} = 140.7 \text{ MeV}, \quad \Lambda = 602.3 \text{ MeV}, \\
G\Lambda^2 = & 1.835, \quad K\Lambda^5 = 12.36 \quad (79)
\end{aligned}$$

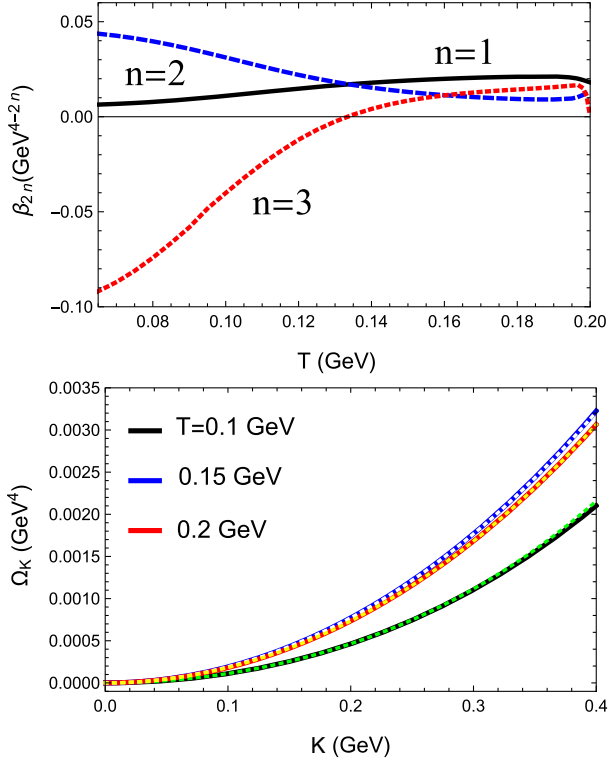


FIG. 1. Upper: the coefficients β_{2n} ($n = 1, 2, 3$) as functions of temperature T for lepton flavor asymmetry $l_e + l_\mu = -0.3$, lower: the corresponding kaon part of thermodynamic potential Ω_K as a function of K for $T = 0.1, 0.15$, and 0.2 GeV with solid lines to $o(K^6)$ and dotted lines to $o(K^4)$.

by fitting to the observations

$$\begin{aligned} m_\pi &= 135 \text{ MeV}, & m_K &= 497.7 \text{ MeV}, \\ m_{\eta'} &= 957.8 \text{ MeV}, & f_\pi &= 92.4 \text{ MeV}. \end{aligned} \quad (80)$$

As the total lepton and charge densities are well constrained in the early Universe, the densities of e, μ and τ leptons can be determined separately once we fix the flavor asymmetry between e and μ leptons. Then, the flavor asymmetry between e and τ leptons or μ and τ leptons can also be uniquely fixed. In the following, we will take e and μ leptons to present the flavor asymmetry without loss of generality. In this work, we only consider the case without primordial magnetic field, and simply set $l = -0.012$ and $l_e = 0$ as the results would not be affected qualitatively [25]. As we have mentioned, in the case of full chemical balance with $\mu_d = \mu_s$, we expect $|K| \leq |\Pi|$ since π^\pm mass is much smaller than K^\pm mass. So we mainly focus on the pion superfluidity regime and check if kaon superfluidity could coexist with that in the following.

For the lepton flavor asymmetry $l_e + l_\mu = -0.3$, the coefficients β_{2n} ($n = 1, 2, 3$) are illustrated as functions of temperature in the upper panel of Fig. 1, and the corresponding kaon part of thermodynamic potential Ω_K

is shown for different temperatures in the lower panel of Fig. 1. Note: so many terms are involved in the full evaluation of β_6 that we only consider β_6^V for demonstration here and in the following; the largest value of Π is about 0.36 GeV in this case, so the range $K \in [0, 0.4]$ GeV is large enough to cover all possible minima of Ω_K . According to the upper panel, the signs of β_2 and β_4 keep positive up to $T_c = 0.21$ GeV, while the sign of β_6 changes from negative to positive with increasing temperature. For all the chosen temperatures, it is easy to check from the lower panel that the quartic approximation up to $o(K^4)$ is very precise compared to that up to $o(K^6)$. Since there is no extra minimum of Ω_K in the considered range except at $K = 0$, the kaon superfluidity is not favored in this case. The medium part β_6^M would become important with increasing temperature but should be at most of the same order as β_6^V , therefore the quartic approximation is still very good and the quantitative results remain. In that sense, though the expressions of β_{2n} are very complicated for $n \geq 3$, we can safely stick to the quartic Ginzburg-Landau approximation.

In Fig. 1, the charge chemical potential μ_Q is never found to satisfy $|\mu_Q| > m_K^V (\sim 0.5 \text{ GeV})$, so we are not surprised to find that kaon superfluidity is not favored. How about the case when we achieve $|\mu_Q| > m_K^V$ by increasing $|l_e + l_\mu|$? For a fixed temperature, $T = 0.06$ GeV, the coefficients β_{2n} ($n = 1, 2, 3$) are illustrated as functions of lepton flavor asymmetry $l_e + l_\mu$ in the upper panel of Fig. 2, and the corresponding Ω_K is shown for different $l_e + l_\mu$ in the lower panel of Fig. 2. Similar to Fig. 1, the quartic approximation up to $o(K^4)$ is very precise compared to that up to $o(K^6)$ according to the lower panel. According to the upper panel, β_2 decreases with increasing $|l_e + l_\mu|$ and becomes negative when $|\mu_Q| > m_K^V$ in the chiral phases; however, β_2 increases with increasing $|l_e + l_\mu|$ and keeps positive in the pion superfluidity. As β_4 is positive for both cases, we conclude that kaon superfluidity is possible in the background of chiral phases but would get killed by pion superfluidity. Combine the results in Figs. 1 and 2, both β_2 and β_4 keep positive with increasing T or $|l_e + l_\mu|$, thus kaon superfluidity is not possible to coexist with pion superfluidity in the case of chemical balance.

According to Ref. [30], the pion superfluidity could be in Sarma phase for $m_\pi^V < |\mu_Q| < 0.23$ GeV and in Larkin-Ovchinnikov-Fudde-Ferrell (LOFF) phase for $|\mu_Q| > 0.23$ GeV at small baryon density and zero temperature. Besides, by following the discussions on two-flavor color superconductor [44], the LO phase with an antipodal-wave form $\Pi(z) = \Pi \cos(qz)$ is expected to be favored in the mismatch range $\delta\mu \equiv (|\mu_Q + 2\mu_B|/6) \in (0.7, 0.8)\Pi_0$, where Π_0 is the pion condensate at $\delta\mu = 0$. So, $\Pi(z)$ could vanish at spatial points $z_n = (2n + 1)\frac{\pi}{2}q^{-1}$, $n = 0, \pm 1, \dots$ in the LOFF phase and kaon condensation would be nonzero there if $|\mu_Q| > m_K$, see the discussions on the

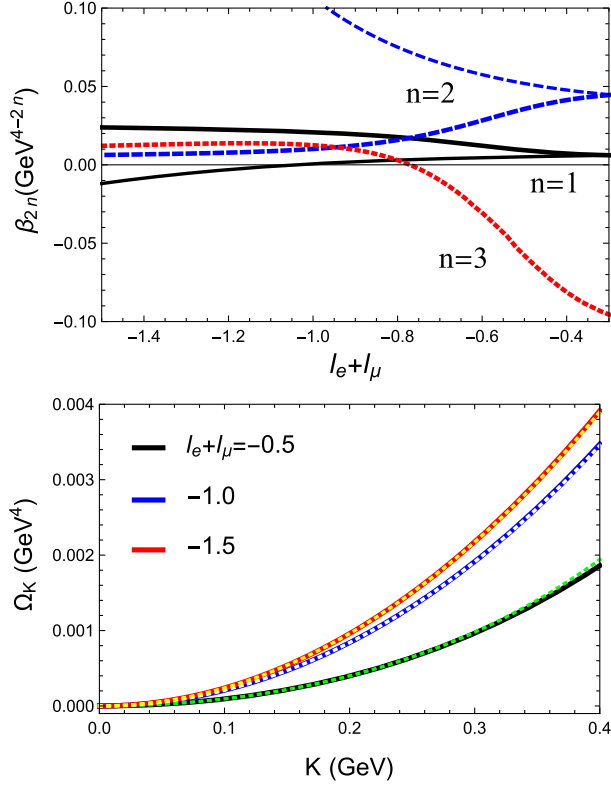


FIG. 2. Upper: the coefficients β_{2n} ($n = 1, 2, 3$) as functions of lepton flavor asymmetry $l_e + l_\mu$ at temperature $T = 0.06$ GeV in pion superfluidity (thick lines) and chiral phases (thin lines), lower: the corresponding kaon part of thermodynamic potential Ω_K as a function of K in the pion superfluid phase for $l_e + l_\mu = -0.5, -1$ and -1.5 with solid lines to $\mathcal{O}(K^6)$ and dotted lines to $\mathcal{O}(K^4)$.

upper panel of Fig. 2. By choosing a large enough lepton flavor asymmetry, $l_e + l_\mu = -1.5$, the mismatch and pion condensate are shown together as functions of μ_Q in Fig. 3 by evolving the temperature in the range (15,64) MeV.

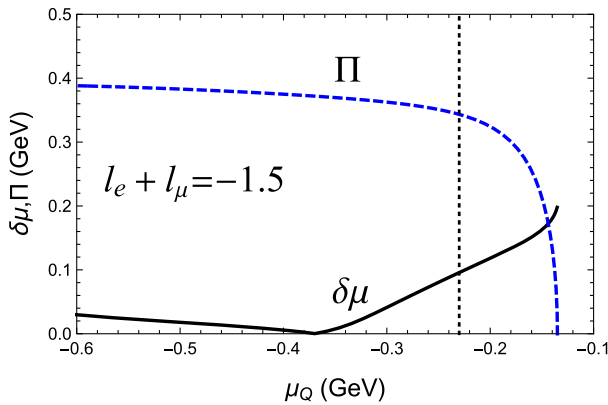


FIG. 3. For the lepton flavor asymmetry $l_e + l_\mu = -1.5$, the mismatch $\delta\mu$ and pion condensate Π as functions of μ_Q by evolving the temperature in the range (15,64) MeV. The vertical line corresponds to $\mu_Q = -0.23$ GeV.

As $\Pi \leq \Pi_0$ for a given μ_Q , we never find $\delta\mu$ to be in the range $(0.7, 0.8)\Pi_0$ for $|\mu_Q| > 0.23$ GeV, hence the LOFF phase cannot be realized and the inhomogeneous kaon superfluidity is disfavored.

In order to check how hard can kaon superfluidity be realized in the early Universe, we introduce the strangeness chemical potential μ_s to s quark and then the full chemical potential changes to $\mu_s = (-\mu_Q + \mu_B)/3 + \mu_s$. As we can tell, μ_s now plays a role of mismatch between s and d quarks, so s quark density increases with μ_s and d quark density would get suppressed since the total baryon density is small. Eventually, it can be realized that kaon superfluidity, instead of pion superfluidity, is the ground state of the system. For $T = 0.06$ GeV and $l_e + l_\mu = -1.2$, the kaon part of thermodynamic potential Ω_K and s quark density n_s are illustrated as functions of μ_s in Fig. 4. We can easily identify a first-order transition from pion superfluidity to chiral phases around $\mu_s = 0.45$ GeV in the upper panel, but the latter is unstable to kaon superfluidity as $|\mu_Q| > m_K$ there. According to the lower panel, n_s is required to be three orders larger than n_d in order to realize kaon superfluidity. If the original mismatch between s and d quarks is generated in the electroweak epoch ($10^{-11}s$ after big bang), it is impossible to obtain such a large mismatch in the QCD epoch ($10^{-6}s$ after big bang) since the lifetime of s quark is $10^{-8}s$.

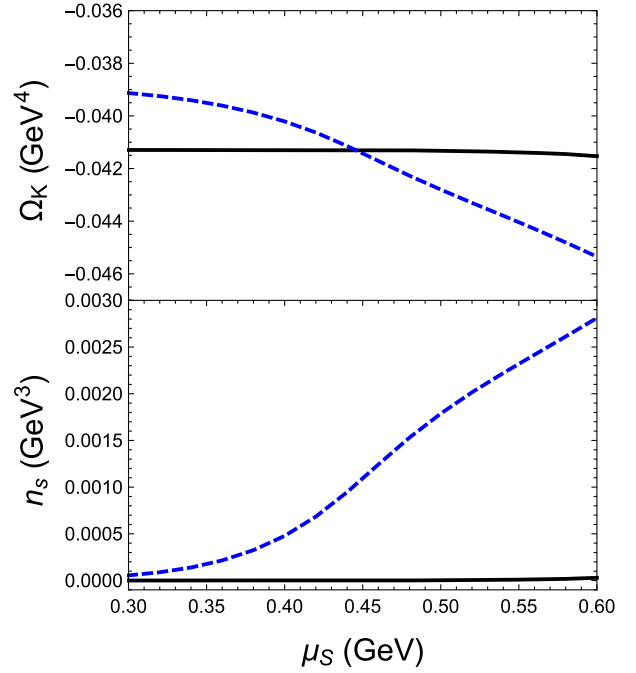


FIG. 4. For the temperature $T = 0.06$ GeV and lepton flavor asymmetry $l_e + l_\mu = -1.2$, the kaon part of thermodynamic potential Ω_K (upper panel) and s quark density n_s (lower panel) as functions of strangeness chemical potential μ_s in the pion superfluidity (black solid lines) and chiral phases (blue dashed lines).

IV. SUMMARY

In this work, the possibility of kaon superfluidity is checked for the QCD epoch of the early Universe by considering the case with a vanishing primordial magnetic field and a large lepton flavor asymmetry. We work within the three-flavor Polyakov–Nambu–Jona-Lasinio model and adopt the Ginzburg–Landau approximation to develop a formalism for the study of kaon condensation within pion superfluidity. Actually, numerical calculations show that quartic Ginzburg–Landau approximation is precise enough to determine the value of kaon condensates. In the case of full chemical balances among elementary particles, kaon superfluidity could be stable compared to the chiral phases with only σ condensations, but it would get killed by the true ground state—homogeneous pion superfluidity. As baryon density is small in the early Universe, we can only find Sarma gapless phase but not LOFF phase for the pion superfluidity at low temperature, so inhomogeneous kaon superfluidity seems impossible either. If we relax the

chemical balance between s and d quark by introducing the strangeness chemical potential μ_s , kaon superfluidity could be theoretically realized by increasing μ_s whence pion superfluidity becomes disfavored. But the corresponding s quark density is so larger than d quark density that we do not think that is realistic whence the lifetime of s quark is self-consistently taken into account. By the way, a primordial magnetic field is not expected to cure the issue since s and d quarks respond the same to it. In a word, kaon superfluidity is impossible in the early Universe.

ACKNOWLEDGMENTS

G. C. is supported by the Natural Science Foundation of Guangdong Province with Grant No. 2024A1515011225.

APPENDIX: EXPANSION COEFFICIENTS

At finite temperature and chemical potential, the coefficient β'_{2n} in Eq. (54) becomes

$$\beta'_{2n} \equiv - \int \frac{d^3\mathbf{k}}{(2\pi)^3} T \sum_m \sum_{q_j} \sum_{t_1, u_2; t_n, u_n = \pm} \alpha_{t_1, u_1; t_n, u_n} \prod_{i=1, \dots, n} \frac{\left[i\tilde{\omega}_{m,j} - \frac{\mu_Q}{2} + t_i \epsilon_l(\mathbf{k}) \right]}{\left[\tilde{\omega}_{m,j}^2 + (E_{\mathbf{k}}^{-t_i})^2 \right] \left[i\tilde{\omega}_{m,j} - \frac{\mu_Q}{2} + u_i \epsilon_s(\mathbf{k}) \right]}, \quad (\text{A1})$$

where $i\tilde{\omega}_{m,j} = i\omega_m + iq_j T + \frac{\mu_Q + 2\mu_B}{6}$ with the fermion Matsubara frequency $\omega_m \equiv (2m+1)\pi T$. To facilitate numerical calculations, we are going to complete the summations over ω_m and q_j analytically, that is, explicitly evaluating the general term

$$\xi_{2n} \equiv -T \sum_m \sum_{q_j} \prod_{i=1, \dots, n} \frac{\left[i\tilde{\omega}_{m,j} - \frac{\mu_Q}{2} + t_i \epsilon_l(\mathbf{k}) \right]}{\left[\tilde{\omega}_{m,j}^2 + (E_{\mathbf{k}}^{-t_i})^2 \right] \left[i\tilde{\omega}_{m,j} - \frac{\mu_Q}{2} + u_i \epsilon_s(\mathbf{k}) \right]}. \quad (\text{A2})$$

By introducing contour integral around the imaginary axis, the summation over ω_m can be alternatively presented as

$$\xi_{2n} \equiv \left(\int_{-i\infty-\varepsilon}^{i\infty+\varepsilon} + \int_{i\infty+\varepsilon}^{-i\infty-\varepsilon} \right) \frac{dk_0}{4\pi i} \tanh \frac{k_0}{2T} \sum_{q_j} \prod_{i=1, \dots, n} \frac{\left[\tilde{k}_{0j} - \frac{\mu_Q}{2} + t_i \epsilon_l(\mathbf{k}) \right]}{\left[\tilde{k}_{0j}^2 - (E_{\mathbf{k}}^{-t_i})^2 \right] \left[\tilde{k}_{0j} - \frac{\mu_Q}{2} + u_i \epsilon_s(\mathbf{k}) \right]} \quad (\text{A3})$$

with $\tilde{k}_{0j} = k_0 + iq_j T + \frac{\mu_Q + 2\mu_B}{6}$. Then, by including the vanishing integral over lines with $|k_0| = \infty$, the contour integral can be separated into two clockwise integral loops for $\Re(k_0) > 0$ and $\Re(k_0) < 0$. To carry out the complex variable integral, it is convenient to rewrite the coefficient as

$$\xi_{2n} \equiv \left(\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} + \int_{i\infty+\varepsilon}^{-i\infty-\varepsilon} \right) \frac{dk_0}{4\pi i} \sum_{q_j} \frac{\left[\tilde{k}_{0j} - \frac{\mu_Q}{2} + \epsilon_l(\mathbf{k}) \right]^{\frac{n+T_n}{2}} \left[\tilde{k}_{0j} - \frac{\mu_Q}{2} - \epsilon_l(\mathbf{k}) \right]^{\frac{n-T_n}{2}} \tanh \frac{k_0}{2T}}{\left[\tilde{k}_{0j}^2 - (E_{\mathbf{k}}^-)^2 \right]^{\frac{n+T_n}{2}} \left[\tilde{k}_{0j}^2 - (E_{\mathbf{k}}^+)^2 \right]^{\frac{n-T_n}{2}} \left[\tilde{k}_{0j} - \frac{\mu_Q}{2} + \epsilon_s(\mathbf{k}) \right]^{\frac{n+U_n}{2}} \left[\tilde{k}_{0j} - \frac{\mu_Q}{2} - \epsilon_s(\mathbf{k}) \right]^{\frac{n-U_n}{2}}} \quad (\text{A4})$$

by collecting all the terms with same t_i or u_i together. Here, we have defined $T_n \equiv \sum_{i=1, \dots, n} t_i$ and $U_n \equiv \sum_{i=1, \dots, n} u_i$ which are both in the range $[-n, n]$. Then, $\frac{n \pm T_n}{2}, \frac{n \pm U_n}{2} \in [0, n]$.

Eventually, we apply the theorem of complex variable integral to find

$$\begin{aligned} \xi_{2n} \equiv & -\sum_{v=\pm} \sum_{q_j} \left\{ \frac{\partial_x^{\frac{n+T_n}{2}-1}}{\left(\frac{n+T_n}{2}-1\right)!} \frac{[x - \frac{\mu_Q}{2} + \epsilon_l(\mathbf{k})]^{\frac{n+T_n}{2}} [x - \frac{\mu_Q}{2} - \epsilon_l(\mathbf{k})]^{\frac{n-T_n}{2}} \tanh \frac{x - (iq_j T + \frac{\mu_Q + 2\mu_B}{6})}{2T}}{2(x + vE_{\mathbf{k}}^-)^{\frac{n+T_n}{2}} [x^2 - (E_{\mathbf{k}}^+)^2]^{\frac{n-T_n}{2}} [x - \frac{\mu_Q}{2} + \epsilon_s(\mathbf{k})]^{\frac{n+U_n}{2}} [x - \frac{\mu_Q}{2} - \epsilon_s(\mathbf{k})]^{\frac{n-U_n}{2}}} \Big|_{x=vE_{\mathbf{k}}^-} \right. \\ & + \frac{\partial_x^{\frac{n-T_n}{2}-1}}{\left(\frac{n-T_n}{2}-1\right)!} \frac{[x - \frac{\mu_Q}{2} + \epsilon_l(\mathbf{k})]^{\frac{n+T_n}{2}} [x - \frac{\mu_Q}{2} - \epsilon_l(\mathbf{k})]^{\frac{n-T_n}{2}} \tanh \frac{x - (iq_j T + \frac{\mu_Q + 2\mu_B}{6})}{2T}}{2(x + vE_{\mathbf{k}}^+)^{\frac{n+T_n}{2}} [x^2 - (E_{\mathbf{k}}^-)^2]^{\frac{n-T_n}{2}} [x - \frac{\mu_Q}{2} + \epsilon_s(\mathbf{k})]^{\frac{n+U_n}{2}} [x - \frac{\mu_Q}{2} - \epsilon_s(\mathbf{k})]^{\frac{n-U_n}{2}}} \Big|_{x=vE_{\mathbf{k}}^+} \\ & + \frac{\partial_x^{\frac{n+U_n}{2}-1}}{\left(\frac{n+U_n}{2}-1\right)!} \frac{[x + \epsilon_l(\mathbf{k})]^{\frac{n+T_n}{2}} [x - \epsilon_l(\mathbf{k})]^{\frac{n-T_n}{2}} \tanh \frac{x - (iq_j T + \frac{-\mu_Q + \mu_B}{3})}{2T}}{4[(x + \frac{\mu_Q}{2})^2 - (E_{\mathbf{k}}^-)^2]^{\frac{n-T_n}{2}} [(x + \frac{\mu_Q}{2})^2 - (E_{\mathbf{k}}^+)^2]^{\frac{n-T_n}{2}} [x - \epsilon_s(\mathbf{k})]^{\frac{n-U_n}{2}}} \Big|_{x=-\epsilon_s(\mathbf{k})} \\ & \left. + \frac{\partial_x^{\frac{n-U_n}{2}-1}}{\left(\frac{n-U_n}{2}-1\right)!} \frac{[x + \epsilon_l(\mathbf{k})]^{\frac{n+T_n}{2}} [x - \epsilon_l(\mathbf{k})]^{\frac{n-T_n}{2}} \tanh \frac{x - (iq_j T + \frac{-\mu_Q + \mu_B}{3})}{2T}}{4[(x + \frac{\mu_Q}{2})^2 - (E_{\mathbf{k}}^-)^2]^{\frac{n-T_n}{2}} [(x + \frac{\mu_Q}{2})^2 - (E_{\mathbf{k}}^+)^2]^{\frac{n-T_n}{2}} [x + \epsilon_s(\mathbf{k})]^{\frac{n+U_n}{2}}} \Big|_{x=\epsilon_s(\mathbf{k})} \right\}, \quad (\text{A5}) \end{aligned}$$

where the conventions $\partial_x^0/0! = 1$ and $\partial_x^{-1}/(-1)! = 0$ should be understood. The summation over q_j can be completed ahead by utilizing the following property

$$\begin{aligned} \sum_{q_j} \tanh \frac{x - iq_j T}{2T} &= \sum_{q_j} \tanh \frac{x - iq_j T}{2T} = \sum_{q_j} \frac{1 - e^{-x/T + iq_j}}{1 + e^{-x/T + iq_j}} = 3 + 2T \partial_x \sum_{q_j} \ln \left(1 + e^{-x/T + iq_j} \right) \\ &= 3 + 2T \partial_x \ln \left(1 + 3Le^{\frac{-x}{T}} + 3L^* e^{\frac{-2x}{T}} + e^{\frac{-3x}{T}} \right) = 3 - 6g(x, T, L, L^*) = -3 + 6g^*(-x, T, L, L^*), \\ g(x, T, L, L^*) &\equiv \frac{Le^{\frac{-x}{T}} + 2L^* e^{\frac{-2x}{T}} + e^{\frac{-3x}{T}}}{1 + 3Le^{\frac{-x}{T}} + 3L^* e^{\frac{-2x}{T}} + e^{\frac{-3x}{T}}} \quad (\text{A6}) \end{aligned}$$

with $\lim_{x \rightarrow \infty} g(x, T, L, L^*) = \lim_{x \rightarrow -\infty} g^*(-x, T, L, L^*) = 0$. By following the assumption $L = L^*$, we have $g(x, T, L, L) = g^*(x, T, L, L)$ and then the function be simply rewritten as $g(x, T, L)$ instead. With the help of $g(x, T, L)$, the coefficient can be separated into vacuum and medium parts, that is, $\xi_{2n} \equiv \xi_{2n}^V + \xi_{2n}^M$ with

$$\begin{aligned} \xi_{2n}^V \equiv & -N_c \sum_{v=\pm} \left\{ \frac{\partial_x^{\frac{n+T_n}{2}-1}}{\left(\frac{n+T_n}{2}-1\right)!} \frac{v[x - \frac{\mu_Q}{2} + \epsilon_l(\mathbf{k})]^{\frac{n+T_n}{2}} [x - \frac{\mu_Q}{2} - \epsilon_l(\mathbf{k})]^{\frac{n-T_n}{2}}}{2(x + vE_{\mathbf{k}}^-)^{\frac{n+T_n}{2}} [x^2 - (E_{\mathbf{k}}^+)^2]^{\frac{n-T_n}{2}} [x - \frac{\mu_Q}{2} + \epsilon_s(\mathbf{k})]^{\frac{n+U_n}{2}} [x - \frac{\mu_Q}{2} - \epsilon_s(\mathbf{k})]^{\frac{n-U_n}{2}}} \Big|_{x=vE_{\mathbf{k}}^-} \right. \\ & + \frac{\partial_x^{\frac{n-T_n}{2}-1}}{\left(\frac{n-T_n}{2}-1\right)!} \frac{v[x - \frac{\mu_Q}{2} + \epsilon_l(\mathbf{k})]^{\frac{n+T_n}{2}} [x - \frac{\mu_Q}{2} - \epsilon_l(\mathbf{k})]^{\frac{n-T_n}{2}}}{2(x + vE_{\mathbf{k}}^+)^{\frac{n+T_n}{2}} [x^2 - (E_{\mathbf{k}}^-)^2]^{\frac{n-T_n}{2}} [x - \frac{\mu_Q}{2} + \epsilon_s(\mathbf{k})]^{\frac{n+U_n}{2}} [x - \frac{\mu_Q}{2} - \epsilon_s(\mathbf{k})]^{\frac{n-U_n}{2}}} \Big|_{x=vE_{\mathbf{k}}^+} \\ & - \frac{\partial_x^{\frac{n+U_n}{2}-1}}{\left(\frac{n+U_n}{2}-1\right)!} \frac{[x + \epsilon_l(\mathbf{k})]^{\frac{n+T_n}{2}} [x - \epsilon_l(\mathbf{k})]^{\frac{n-T_n}{2}}}{4[(x + \frac{\mu_Q}{2})^2 - (E_{\mathbf{k}}^-)^2]^{\frac{n-T_n}{2}} [(x + \frac{\mu_Q}{2})^2 - (E_{\mathbf{k}}^+)^2]^{\frac{n-T_n}{2}} [x - \epsilon_s(\mathbf{k})]^{\frac{n-U_n}{2}}} \Big|_{x=-\epsilon_s(\mathbf{k})} \\ & \left. + \frac{\partial_x^{\frac{n-U_n}{2}-1}}{\left(\frac{n-U_n}{2}-1\right)!} \frac{[x + \epsilon_l(\mathbf{k})]^{\frac{n+T_n}{2}} [x - \epsilon_l(\mathbf{k})]^{\frac{n-T_n}{2}}}{4[(x + \frac{\mu_Q}{2})^2 - (E_{\mathbf{k}}^-)^2]^{\frac{n-T_n}{2}} [(x + \frac{\mu_Q}{2})^2 - (E_{\mathbf{k}}^+)^2]^{\frac{n-T_n}{2}} [x + \epsilon_s(\mathbf{k})]^{\frac{n+U_n}{2}}} \Big|_{x=\epsilon_s(\mathbf{k})} \right\}, \quad (\text{A7}) \end{aligned}$$

$$\begin{aligned}
\xi_{2n}^M \equiv & 6 \sum_{v=\pm} \left\{ \frac{\partial_x^{\frac{n+T_n}{2}-1}}{\left(\frac{n+T_n}{2}-1\right)!} \frac{v[x - \frac{\mu_Q}{2} + \epsilon_l(\mathbf{k})]^{\frac{n+T_n}{2}} [x - \frac{\mu_Q}{2} - \epsilon_l(\mathbf{k})]^{\frac{n-T_n}{2}} g\left(vx - v\frac{\mu_Q+2\mu_B}{6}, T, L\right)}{2(x + vE_{\mathbf{k}}^{\pm})^{\frac{n+T_n}{2}} [x^2 - (E_{\mathbf{k}}^{\pm})^2]^{\frac{n-T_n}{2}} [x - \frac{\mu_Q}{2} + \epsilon_s(\mathbf{k})]^{\frac{n+U_n}{2}} [x - \frac{\mu_Q}{2} - \epsilon_s(\mathbf{k})]^{\frac{n-U_n}{2}}} \Big|_{x=vE_{\mathbf{k}}^-} \right. \\
& + \frac{\partial_x^{\frac{n-T_n}{2}-1}}{\left(\frac{n-T_n}{2}-1\right)!} \frac{v[x - \frac{\mu_Q}{2} + \epsilon_l(\mathbf{k})]^{\frac{n+T_n}{2}} [x - \frac{\mu_Q}{2} - \epsilon_l(\mathbf{k})]^{\frac{n-T_n}{2}} g\left(vx - v\frac{\mu_Q+2\mu_B}{6}, T, L\right)}{2(x + vE_{\mathbf{k}}^+)^{\frac{n+T_n}{2}} [x^2 - (E_{\mathbf{k}}^+)^2]^{\frac{n-T_n}{2}} [x - \frac{\mu_Q}{2} + \epsilon_s(\mathbf{k})]^{\frac{n+U_n}{2}} [x - \frac{\mu_Q}{2} - \epsilon_s(\mathbf{k})]^{\frac{n-U_n}{2}}} \Big|_{x=vE_{\mathbf{k}}^+} \\
& - \frac{\partial_x^{\frac{n+U_n}{2}-1}}{\left(\frac{n+U_n}{2}-1\right)!} \frac{[x + \epsilon_l(\mathbf{k})]^{\frac{n+T_n}{2}} [x - \epsilon_l(\mathbf{k})]^{\frac{n-T_n}{2}} g\left(-x + \frac{-\mu_Q+\mu_B}{3}, T, L\right)}{4[(x + \frac{\mu_Q}{2})^2 - (E_{\mathbf{k}}^-)^2]^{\frac{n-T_n}{2}} [(x + \frac{\mu_Q}{2})^2 - (E_{\mathbf{k}}^+)^2]^{\frac{n-T_n}{2}} [x - \epsilon_s(\mathbf{k})]^{\frac{n-U_n}{2}}} \Big|_{x=-\epsilon_s(\mathbf{k})} \\
& \left. + \frac{\partial_x^{\frac{n-U_n}{2}-1}}{\left(\frac{n-U_n}{2}-1\right)!} \frac{[x + \epsilon_l(\mathbf{k})]^{\frac{n+T_n}{2}} [x - \epsilon_l(\mathbf{k})]^{\frac{n-T_n}{2}} g\left(x - \frac{-\mu_Q+\mu_B}{3}, T, L\right)}{4[(x + \frac{\mu_Q}{2})^2 - (E_{\mathbf{k}}^-)^2]^{\frac{n-T_n}{2}} [(x + \frac{\mu_Q}{2})^2 - (E_{\mathbf{k}}^+)^2]^{\frac{n-T_n}{2}} [x + \epsilon_s(\mathbf{k})]^{\frac{n-U_n}{2}}} \Big|_{x=\epsilon_s(\mathbf{k})} \right\}. \quad (\text{A8})
\end{aligned}$$

Correspondingly, the coefficient β'_{2n} can be separated into vacuum and medium parts, that is, $\beta'_{2n} \equiv \beta_{2n}^V + \beta_{2n}^M$ with

$$\beta_{2n}^V \equiv \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{t_1, u_2; t_n, u_n=\pm} \alpha_{t_1, u_1; t_n, u_n} \xi_{2n}^V, \quad (\text{A9})$$

$$\beta_{2n}^M \equiv \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{t_1, u_2; t_n, u_n=\pm} \alpha_{t_1, u_1; t_n, u_n} \xi_{2n}^M. \quad (\text{A10})$$

Keep in mind that $\xi_{2n}^{V/M}$ depends on both t_i and u_i . Usually, β_{2n}^M is convergent as $\lim_{k \rightarrow \infty} k^3 \xi_{2n}^M = 0$, while β_{2n}^V is divergent and can be regularized by three-momentum cutoff.

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