Effective mass and symmetry breaking in the Ishibashi-Kawai-Kitazawa-Tsuchiya matrix model from compactification

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The Ishibashi-Kawai-Kitazawa-Tsuchiya (IKKT) model is a promising candidate for a nonperturbative description of type IIB superstring theory. It is known from analytic approaches and numerical simulations that the IKKT matrix model with a mass term admits interesting cosmological solutions. However, this mass term is often introduced by hand and serves as a regulator in the theory. In the present paper, we show that an effective mass matrix can arise naturally in the IKKT model by imposing a toroidal compactification where the space-time fermions acquire antiperiodic boundary conditions. When six spatial dimensions are chosen to be compact, the effective mass matrix breaks the SO(1,9) space-time symmetry of the IKKT model to $\mathrm{SO}(1,3)\times\mathrm{SO}(6)$. This paves the way for space-time solutions of the IKKT model where SO(1,9) symmetry is naturally broken to $\mathrm{SO}(1,3)\times\mathrm{SO}(6)$.

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I. INTRODUCTION

Superstring theory is a promising candidate for a self-consistent unified theory of quantum gravity. An interesting feature of the theory is that the dimensionality of space-time is not arbitrary, but comes from the consistency of the theory. Specifically, the theory is only consistently defined in ten space-time dimensions. For this theory to describe our world, one must impose that six out of the nine spatial dimensions are compactified. This can be done in many ways, resulting in a vast landscape of effective descriptions of string theory in four dimensions. In addition to the four-dimensional vacua, there exist other ways to consistently compactify string theory to an arbitrary number of dimensions, which results in vacua that are not four dimensional. Clarifying why four-dimensional vacua are preferred in the theory remains an open question, which to this day does not have an answer in perturbative string theory.

Another area where perturbative string theory lacks predictive power is in the context of cosmology. While string theory can be used to make predictions about the Universe at late times, the cosmic singularity is not resolved generally in perturbative string theory [1–4]. Therefore, to study the very early Universe and explain the dimensionality of our world, we definitely need a nonperturbative description.

There have been many proposals for a nonperturbative description of string theory, most of them relying on matrix models [5–7]. Among these theories, the Ishibashi-Kawai-Kitazawa-Tsuchiya (IKKT) model [7], a nonperturbative description of type IIB superstring theory, stands out as a natural choice to explain the birth of the Universe. This model is built around the action

$$S_{\mathrm{IKKT}} = -\frac{1}{4g^2}\mathrm{Tr}[A^M, A^N]^2 - \frac{1}{2g^2}\mathrm{Tr}\bar{\psi}\Gamma^M[A_M, \psi], \quad (1)$$

where large bosonic matrices A^M's encode information about space-time, and large fermionic matrix ψ's are added to preserve supersymmetry. In the action above, g is a gauge coupling that is related to the string scale l_s via $g \sim l_s^2$, and the indices are contracted using the Minkowski metric in the mostly minus sign convention $\eta_{\mathrm{MN}} = \mathrm{diag}(1, -1, -1, \ldots, -1)$. Given the causal structure of the space, A^0 encodes information about time and A^i encodes information about space, where $i \in \{1, \ldots, 9\}$ labels the nine space dimensions.

Over the years, there have been many attempts to find solutions of the IKKT model that correspond to an emergent four-dimensional universe. The first steps toward finding these solutions were done by analyzing the model using the Gaussian expansion method [8,9] to investigate symmetry breaking in the theory. Using this method, it was shown that the SO(10) symmetry of the Euclidean version of this model can be spontaneously

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broken to SO(3) [10–13].[1] Other studies based on the complex Langevin method have also shown consistent results [14,15], and a recent analysis in the context of the Banks-Fischler-Shenker-Susskind (BFSS) model has a lso shown progress in this direction [16]. Similarly, Monte Carlo simulations of the Lorentzian model showed that an expanding $(1+3)$-dimensional space-time can emerge from an SO(1,9) symmetric state of the model after a critical time [17–20]. To achieve this result, approximations were made to avoid the sign problem of the Lorentzian theory. However, further studies have shown that these approximations are no longer valid when the four space-time dimensions emerge [21]. Since then, the Lorentzian model has been studied without this approximation using the complex Langevin method, where the emergence of four space-time dimensions remains a topic of study [22–27].

In earlier numerical analyses of the Lorentzian IKKT model, it was found in [17] that an important feature seems to be required to obtain the emergence of $(1+3)$ dimensions. When $(1+3)$ dimensions become large, certain bounds used to regulate theory,

$$\frac{1}{N}\mathrm{Tr}(A_0)^2 \leq \kappa \frac{1}{N}\mathrm{Tr}(A_i)^2, \tag{2}$$

$$\frac{1}{N}\mathrm{Tr}(A_i)^2 \leq \kappa L^2, \tag{3}$$

become saturated. Saturating the constraints above is equivalent to adding the following piece to the IKKT action:

$$S_{\mathrm{const}} = \frac{\tilde{\lambda}}{2}\mathrm{Tr}(A_0^2 - \kappa L^2) - \frac{\lambda}{2}\mathrm{Tr}(A_i^2 - L^2), \tag{4}$$

where λ and $\tilde{\lambda}$ are Lagrange multipliers. Minimizing the IKKT action in the presence of the constraint above, it was shown analytically [28] and numerically [29] that various cosmological solutions of the equation of motions can be found. An important point to notice is that adding the constraint piece in Eq. (4) to the IKKT action is equivalent to adding a mass term to the theory, which may or may not be Lorentz invariant depending on the choice of λ and $\tilde{\lambda}$. Hence, adding a mass term to the theory can lead to interesting cosmological solutions. In fact, various analyses of the Lorentzian IKKT model with a mass term have been done before, in which case it was shown that cosmological solutions can also be found [30–33].[2]

Since an effective mass term arises as a possible explanation for the emergence of cosmological solutions, it seems natural to ask what conditions are necessary for a mass term to naturally appear in the theory and what causes the symmetry of space-time to break from SO(1,9) to $\mathrm{SO}(1,3)\times\mathrm{SO}(6)$. In the present paper, we explore this question by studying compactifications of the IKKT model. We find that if six spatial dimensions are compactified in a way that supersymmetry is broken, space-time fermions are quenched and the IKKT model action develops an effective mass matrix that breaks the SO(1,9) symmetry of the model to $\mathrm{SO}(1,3)\times\mathrm{SO}(6)$. This leads the way for solutions of the IKKT model where the SO(1,9) symmetry of space-time is naturally broken to $\mathrm{SO}(1,3)\times\mathrm{SO}(6)$.

A. Outline

To obtain the mass matrix, we will proceed as follows. In Sec. II, we will Wick rotate the Lorentzian IKKT model by imposing the change of variables $A^0 \to iA^0$ and $\Gamma^i \to i\Gamma^i$ to obtain the Euclidean IKKT model action

$$S_{\mathrm{IKKT}} = -\frac{1}{4g^2}\mathrm{Tr}[A^M, A^N]^2 - \frac{i}{2g^2}\mathrm{Tr}\bar{\psi}\Gamma^M[A_M, \psi]. \tag{5}$$

This transformation to Euclidean space will be done to simplify computations. Then, we will give the theory boundary conditions similar to the Scherk-Schwarz torus [42]. Namely, we will compactify the theory on a six-dimensional torus where supersymmetry is broken by imposing that the space-time fermions ψ acquire antiperiodic boundary conditions. As a result, the IKKT model action under compactification will become equivalent to a six-dimensional Yang-Mills theory with the following action:

$$S_C = \frac{L^6}{2g_{\mathrm{eff}}^2}\int d\sigma^6 \mathrm{Tr}\bigg(\frac{1}{2}F_{ab}F^{ab} + D_a A_\mu D^a A^\mu - \frac{1}{2}[A^\mu, A^\nu]^2 + \bar{\psi}\Gamma^a D_a \psi - i\bar{\psi}\Gamma^\mu[A_\mu, \psi]\bigg), \tag{6}$$

where we have substituted the mode expansion

$$A^M = \sum_{n^a \in \mathbb{Z}^6} A^M(n^a) e^{i2\pi L n^a \sigma^a}, \qquad \psi = \sum_{r^a \in \mathbb{Z}^6 + 1/2} \psi(r^a) e^{i2\pi L r^a \sigma^a}. \tag{7}$$

Here, $\mu \in \{0, \ldots, 3\}$ labels the noncompact directions, $a \in \{4, 5, \ldots, 9\}$ labels the compact directions, $g_{\mathrm{eff}}^2 = g^2/N$ is an effective gauge coupling, and N is a large

[1]Earlier results [10–12] showed that SO(10) is broken to SO(4), while a more recent and careful study [13] suggests that SO(10) is broken to SO(3) instead. The relationship between this SO(3) symmetry and our four-dimensional universe is still under study.

[2]See [34–38] for other deformations of the IKKT model which admit cosmological solutions and [39–41] for recent progress in the study of cosmological solutions in the IKKT model.

integer that we will introduce later. In this six-dimensional Yang-Mills theory, the zero modes describe noncompact degrees of freedom, and the nonzero models describe interactions between these noncompact degrees of freedom. Hence, by integrating out the nonzero modes in the theory, one can obtain a Wilsonian effective action for the noncompact degrees of freedom in the theory. In Sec. III, we will compute this Wilsonian effective action from the expression

$$S^0_{\mathrm{eff}} = -\ln\left(\prod'_{n^a\in\mathbb{Z}^6}\prod_{r^b\in\mathbb{Z}^6+1/2}\int \mathcal{D}A^M(n^a)\mathcal{D}\psi(r^b)e^{-S_E}\right). \tag{8}$$

Here, $\prod'$ means that we are not integrating over the zero modes $n^a = 0$ of the theory. This computation will be done in the decompactification limit $L \gg g_{\mathrm{eff}}^{1/2}$, where perturbation theory is valid and we expect to obtain a result close to the IKKT action without the compactification constraint [Eq. (1)]. Carrying out the computation to leading order in perturbation theory and Wick rotating back to Lorentzian space, we will find that the effective action takes the form

$$S^0_{\mathrm{eff}} = -\frac{1}{4g^2_{\mathrm{eff}}}\mathrm{Tr}[A^M(0), A^N(0)]^2 + \frac{1}{2}M^2_{MN}\mathrm{Tr}(A^M(0)A^N(0))^2 + \cdots, \tag{9}$$

where the mass matrix

$$M^2_{MN} = \begin{bmatrix} \eta_{\mu\nu}M_4^2 & 0 \\ 0 & \eta_{ab}M_6^2 \end{bmatrix} \tag{10}$$

arises as a first-order correction which breaks SO(1,9) symmetry to $\mathrm{SO}(1,3)\times\mathrm{SO}(6)$. In the expression above, the masses M_4^2 and M_6^2 take the values

$$M_4^2 = 16(S_{F_1} - S_{B_1})\frac{NM}{L^2}, \tag{11}$$

$$M_6^2 = \frac{32}{3}(S_{F_1} - S_{B_1})\frac{NM}{L^2}, \tag{12}$$

where the constants S_{B_1} and S_{F_1} are determined by the following sums:

$$S_{B_1} = \sum'_{n^a\in\mathbb{Z}^6}\frac{1}{(2\pi n^a)^2}, \qquad S_{F_1} = \sum_{r^a\in\mathbb{Z}^6+1/2}\frac{1}{(2\pi r^a)^2}. \tag{13}$$

The sums S_{B_1} are S_{F_1} are divergent in the large n^a and r^a limit. However, the difference between these two sums is finite and takes the value $S_{F_1} - S_{B_1} \approx 0.0397887$ when evaluated numerically.

The reason why we obtain Eq. (9) and not Eq. (1) in the decompactification limit is because of broken supersymmetry. Since the fermions have antiperiodic boundary conditions, the fermionic zero modes are projected away in the mode expansion. Hence, the fermionic sector does not enter the zero-mode effective action. We are left with the bosonic part of the IKKT action and a mass matrix coming from integrating out interactions between the zero-mode degrees of freedom in the theory. If supersymmetry is restored by imposing that fermions have periodic boundary conditions, r^a becomes summed over $\mathbb{Z}^6$ instead of $\mathbb{Z}^6 + 1/2$ in the sum S_{F_1}. In this case, the masses M_4^2 and M_6^2 vanish since $S_{B_1} = S_{B_2}$, the fermions acquire a zero-mode term, and we obtain the IKKT model action [Eq. (1)] with an effective gauge coupling g_{eff}.

II. COMPACTIFICATION OF THE IKKT MODEL

Compactifying a matrix model presents a different challenge than compactifying a field theory. For one, there are no free parameters in the matrix model that we can choose to be compact. Hence, we must impose conditions on the matrices themselves. To overcome this challenge, we will make use of the method of mirror images, which was first brought forward by Washington Taylor in the context of D-brane mechanics [43]. This method proved successful to explain graviton scattering under toroidal compactification of the BFSS model [44] and has recently been used to explain three graviton amplitudes [45] and soft theorems [46] in this same model.

This method builds on the fact that toroidal compactification is equivalent to duplicating a fundamental region of the target space an infinite number of times along said direction. For example, let us suppose we wish to compactify the real line $x \in \mathbb{R}$ on a circle S^1 of radius R. One option would be to confine the real line to an interval $x \in [0, 2\pi R[$ where we impose periodic boundary conditions. Another would be to invoke the fact that periodic boundary conditions are equivalent to duplicating the interval $[0, 2\pi R[$ an infinite number of times along the real line. In other words, each point on the real line can be associated with a point a distance $x \to x + 2\pi R$ away from this point. The mathematical term for this operation is called going to the universal cover of the circle.

The same procedure can be applied to matrix models to impose a compactification. Since the matrix model describes a target space, we can impose that the target space contains duplicated objects in the direction we want to compactify in an attempt to replicate the effects of a compact space. To see how this is done in the context of the IKKT model, let us first Wick rotate the Lorentzian IKKT model to Euclidean space by imposing the change of variables $A^0 \to iA^0$ and $\Gamma^i \to i\Gamma^i$ in the Lorentzian IKKT model action. We obtain

$$S_{\mathrm{IKKT}} = -\frac{1}{4g^2}\mathrm{Tr}[A^M, A^N]^2 - \frac{i}{2g^2}\mathrm{Tr}\bar{\psi}\Gamma^M[A_M, \psi]. \quad (14)$$

As previously mentioned, we will be interested in configurations of the IKKT model where six spatial directions A^a are compact and where fermions acquire antiperiodic boundary conditions. Such compactifications were first studied by Scherk and Schwarz [42] in the context of supergravity as a way to break supersymmetry in the lower-dimensional theory. Similar studies were also done by Rohm in string theory [47] and by Banks and Motl in the context of matrix models [48]. In the present case, we will generalize the one-dimensional case in [49], where these boundary conditions were used to obtain a thermal state of the IKKT model, to the case where six dimensions are compactified. To do this, we will invoke the existence of unitary operators U^a, which generate a translation in the A^a direction of the target space. In addition, we impose that these operators commute with each other,

$$U^a U^b = U^b U^a, \quad (15)$$

so that translations in different compact directions can be made independently of each other. Following our previous discussion, compactifying the target space on a six-dimensional torus where fermions acquire antiperiodic boundary conditions should be equivalent to imposing the conditions

$$(U^b)^{-1} A^\mu U^b = A^\mu, \quad (16)$$

$$(U^b)^{-1} A^a U^b = A^a + 2\pi L \delta_{ab}, \quad (17)$$

$$(U^b)^{-1} \psi U^b = -\psi, \quad (18)$$

where L is the torus radius. Here, μ labels the noncompact space-time directions and a labels the compact space directions. To solve the constraint equation above, we will use an approach similar to the one in [50] and assume that the Hilbert space that the A's and ψ's act on has the tensor product form

$$X = Y \otimes Z, \quad (19)$$

where X is a $M \times M$ matrix that will remain invariant under the translation, and Z is a $N \times N$ matrix associated with the Hilbert space the translations act on. We will then invoke that U^a takes the following form:

$$U^a = \mathbb{I}_M \otimes e^{-i2\pi q^a} e^{-ip^a}, \quad (20)$$

where $\mathbb{I}_M$ is the M-dimensional identity operator and q^a and p^b are operators that satisfy the Heisenberg algebra $[q^a, p^b] = i\delta_{ab}$. With the form above, the unitary operator U^a satisfies $(U^a)^{-1} q^a U^a = q^a + 1$ and generates a shift from q^a to $q^a + 1$. The extra factor of $e^{-i2\pi q^a}$ does not affect this shift. However, it will play a role in achieving the antiperiodic boundary conditions for the fermions. Next, we will note that a matrix of the form

$$B = \sum_{n^a} B(n^a) \otimes e^{in^a p^a}, \quad (21)$$

satisfies $(U^a)^{-1} B U^a = B$ if n^a is an integer, and $(U^a)^{-1} B U^a = -B$ if n^a if a half-integer. Consequently, it is possible to solve the constraint equations by imposing that the matrices A^μ, A^a, and ψ take the following form:

$$A^\mu = \sum_{n^b \in \mathbb{Z}^6} A^\mu(n^b) \otimes e^{in^b p^b}, \quad (22)$$

$$A^a = \sum_{n^b \in \mathbb{Z}^6} A^a(n^b) \otimes e^{in^b p^b} + 2\pi L \mathbb{I}_M \otimes q^a, \quad (23)$$

$$\psi = \sum_{r^b \in \mathbb{Z}^6 + 1/2} \psi(r^b) \otimes e^{ir^b p^b}. \quad (24)$$

In the expressions above, n^b and r^b are summed over N integers and half-integers, respectively, where N is taken to be large but finite. It is possible to show that, when written in the $|q^a\rangle$ basis, the matrices above take the block Toeplitz form depicted in Fig. 1. In this block Toeplitz form, the diagonal blocks describe the distribution of objects within an interval $[0, 2\pi L[$ and their interactions. The off-diagonal blocks, on their side, describe interactions between the duplicated fundamental regions. Substituting the matrices above in the IKKT model action and using the identities

$$[q^a, e^{inp^b}] = -n e^{inp^b} \delta_{ab}, \quad \mathrm{Tr} e^{i(n \pm m)p^b} = N\delta(n \pm m), \quad (25)$$

we obtain the momentum space representation of the Yang-Mills action

$$S_C = \frac{L^6}{2g_{\mathrm{eff}}^2} \int d\sigma^6 \mathrm{Tr}\bigg(\frac{1}{2} F_{ab} F^{ab} + D_a A_\mu D^a A^\mu - \frac{1}{2}[A^\mu, A^\nu]^2 + \bar{\psi}\Gamma^a D_a \psi - i\bar{\psi}\Gamma^\mu [A_\mu, \psi]\bigg), \quad (26)$$

where $g_{\mathrm{eff}}^2 = g^2/N$ is an effective gauge coupling and A^M and ψ are expanded using the mode decomposition

$$A^M = \sum_{n^a \in \mathbb{Z}^6} A^M(n^a) e^{i2\pi L n^a \sigma^a},$$
$$\psi = \sum_{r^a \in \mathbb{Z}^6 + 1/2} \psi(r^a) e^{i2\pi L r^a \sigma^a}. \quad (27)$$

$$A^\mu = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & A^\mu(0) & A^\mu(1) & A^\mu(2) & \cdots \\ \cdots & A^\mu(-1) & A^\mu(0) & A^\mu(1) & \cdots \\ \cdots & A^\mu(-2) & A^\mu(-1) & A^\mu(0) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

$$A^a = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & A^a(0) - 2\pi L & A^a(1) & A^a(2) & \cdots \\ \cdots & A^a(-1) & A^a(0) & A^a(1) & \cdots \\ \cdots & A^a(-2) & A^a(-1) & A^a(0) + 2\pi L & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

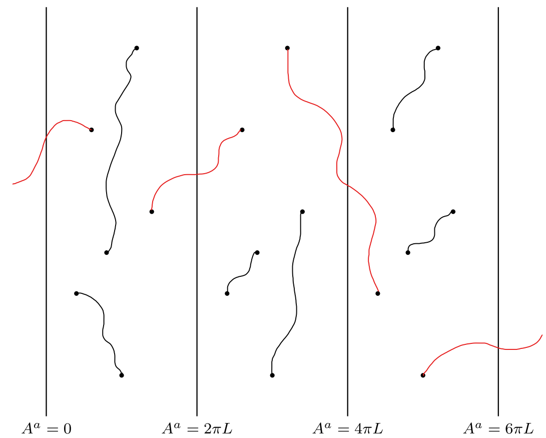

FIG. 1. Left: the diagonal blocks (black) describe the distribution of objects and their interactions in the duplicated regions, and the off-diagonal blocks (red) describe interactions between the duplicated regions. Right: sketch of the duplicated regions along a compact direction A^a. The line between the black dots depicts interactions inside (black) and across duplicated regions (red).

Here, σ takes values inside the interval $[0, L^{-1}[$. Moreover, A^M and ψ respectively satisfy periodic and antiperiodic boundary conditions. In the mode expansion above, the zero modes are related to the distribution of objects and their interactions in the fundamental regions, and the nonzero modes are associated with interactions between fundamental regions [51]. In the decompactification limit $L \gg g_{\rm eff}^{1/2}$, one should expect the fundamental regions to be far away from each other. In this case, interactions will be suppressed, and we should obtain a theory that is approximately described by the dynamics of the zero modes of the theory. This can be seen by looking at the mode expansion

$$S_C = -\frac{1}{4g_{\rm eff}^2}{\rm Tr}[A^M(0), A^N(0)]^2 + \frac{1}{2g_{\rm eff}^2}\sum_{n^a\in\mathbb{Z}^6}(2\pi L n^a)^2 {\rm Tr}(A^M(-n^a)A^M(n^a)) + \frac{1}{2g_{\rm eff}^2}\sum_{r^a\in\mathbb{Z}^6+1/2}(2\pi L r^a i){\rm Tr}(\bar\psi(r^a)\Gamma^a\psi(r^a)) + \cdots,$$

of the compact IKKT action. In the $L \gg g_{\rm eff}^{1/2}$ limit, the nonzero winding modes $\omega_{n^a} = 2\pi L n^a$ and $\omega_{r^a} = 2\pi L r^a$ associated with the second and third term become heavy, and interactions become suppressed in the path integral. As a result, we expect the compact IKKT action to be effectively described by the zero modes of the system. This means we should recover the bosonic IKKT model action

$$S_C = -\frac{1}{4g_{\rm eff}^2}{\rm Tr}[A^M(0), A^N(0)]^2 + \cdots \tag{28}$$

and possible corrections coming from interactions between the fundamental regions. The fermions, in this case, do not contribute since their zero modes are projected away by the antiperiodic boundary conditions. As the radius of compactification L decreases, one should expect that interactions become important, leading to more corrections to Eq. (28). In the following sections, we will derive the leading corrections to Eq. (28) by evaluating a Wilsonian effective action for the zero modes $A^M(0)$ of the theory. In the limit where $L \gg g_{\rm eff}^{1/2}$, we will see that the effective action of the zero modes acquires a mass matrix as a leading-order correction, leading to symmetry breaking in the theory.

III. WILSONIAN EFFECTIVE ACTION

To compute an effective action for the zero modes of the theory, which describes the noncompact degrees of freedom, we will adopt a Wilsonian approach. This approach will consist of integrating out the nonzero modes in the path integral to obtain an action that depends exclusively on the zero modes of the theory.

To see how this can be done, let us remind ourselves that a Wilsonian effective action can be used to find an effective description of the low energy modes of a theory by integrating out high energy modes above a cutoff Λ. For example, let us consider the action $S[\Phi]$ related to a scalar field Φ. To obtain the low energy effective action for some long wavelength modes Φ_L, we can split the scalar field $\Phi = \Phi_L + \Phi_S$ into the contributions from Φ_L and the short wavelength component Φ_S. Then, the contribution of the short wavelength modes Φ_S can be integrated out in the partition function in the following way:

$$Z = \int \mathcal{D}\Phi e^{-S[\Phi]} \tag{29}$$

$$= \int \mathcal{D}\Phi_L \left(\int \mathcal{D}\Phi_S e^{S[\Phi_L+\Phi_S]} \right) \tag{30}$$

$$= \int \mathcal{D}\Phi_L e^{-S_{\text{eff}}[\Phi_L]} \tag{31}$$

to obtain a Wilsonian effective action $S_{\text{eff}}[\Phi_L]$ of the short wavelength component Φ_L. This Wilsonian effective action can be then computed from the expression

$$S_{\text{eff}}[\Phi_L] = -\ln \left(\int \mathcal{D}\Phi_S e^{S[\Phi_L+\Phi_S]} \right). \tag{32}$$

In the present case, we want to obtain an effective action of the zero modes $A^M(0)$ of the theory. This means that, in the Wilsonian sense, we must integrate out all the nonzero modes $A^M(n^a)$ for $n^a \neq 0$ and $\psi(r^a)$ in the path integral. To do this, we can split $A^M = A^M(0) + \sum'_{n^a\in\mathbb{Z}^6} A^M(n^a)e^{in^a\sigma^a}$ into the zero-mode component $A^M(0)$ and the nonzero-mode component $\sum'_{n^a\in\mathbb{Z}^6} A^M(n^a)e^{in^a\sigma^a}$. Here, $\sum'$ means that we do not sum over the zero modes $n^a = 0$. We will then integrate out the nonzero modes in the partition function in same way as for our scalar field example. For the compact IKKT action [Eq. (26)], this gives us

$$Z = \prod_{n^a r^b \in \mathbb{Z}^6} \int \mathcal{D}A^M(n^a)\mathcal{D}\psi(r^b)e^{-S_C} \tag{33}$$

$$= \int \mathcal{D}A^M(0) \left(\prod_{r^b\in\mathbb{Z}^6+1/2} \int \mathcal{D}A^M(n^a)\mathcal{D}\psi(r^b)e^{-S_C} \right) \tag{34}$$

$$= \int \mathcal{D}A^M(0)e^{-S^0_{\text{eff}}}, \tag{35}$$

where

$$S^0_{\text{eff}} = -\ln \left(\prod'_{n^a\in\mathbb{Z}^6} \prod_{r^b\in\mathbb{Z}^6+1/2} \int \mathcal{D}A^M(n^a)\mathcal{D}\psi(r^b)e^{-S_C} \right) \tag{36}$$

can be identified as the zero-mode effective action. Here again, we remind the reader that $\prod'$ means we integrate over all the modes $n^a \in \mathbb{Z}^6$ except the zero modes $n^a = 0$ of the theory. This means that S^0_{eff} will depend exclusively on the zero modes $A^M(0)$ that have not been integrated over. The goal of the next sections will be to compute the quantity above. This will be done using standard perturbative methods.

A. Choice of γ matrix representation and gauge fixing

As a first step toward computing Eq. (36), we will choose a convenient representation for the γ matrices that reflects the fact that $SO(10)$ symmetry is broken to $SO(4) \times SO(6)$ by our choice of compactification. We will do this in a way to preserves the Majorana and Weyl conditions

$$\Gamma_{11}\psi = \psi, \qquad \bar{\psi} = \psi^T C_{10} \tag{37}$$

which the fermions must satisfy for the theory to be supersymmetric. Here, Γ_{11} and C_{10} are respectively the chirality operator and the charge conjugation operator in 10 dimensions. In the present case, we will use the representation introduced in [52] and consider Γ matrices of the form

$$\Gamma^a = \tilde{\Gamma}^a \otimes 1, \qquad \Gamma^\mu = \tilde{\Gamma}_7 \otimes \gamma^\mu, \tag{38}$$

where $\tilde{\Gamma}^a$ are SO(6) γ matrices, $\tilde{\Gamma}_7$ is the chirality operator for these matrices, and γ^μ are SO(4) γ matrices (see [53] for other convenient representations). We will further require that the SO(4) γ matrices are in the Weyl representation

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \tag{39}$$

where σ^μ and $\bar{\sigma}^\mu$ are Pauli 4-vectors which satisfy

$$\bar{\sigma}_0 = \sigma_0 = 1, \qquad \bar{\sigma}_i = -\sigma_i, \qquad \{\sigma_i, \sigma_j\} = -2\delta_{ij}. \tag{40}$$

In this representation, the chirality and charge conjugation operator for the ten-dimensional Γ matrices take the form

$$\Gamma_{11} = \tilde{\Gamma}_7 \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad C_{10} = C_6 \otimes \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}. \tag{41}$$

Therefore, the Majorana and Weyl conditions reduce to

$$\psi = \begin{pmatrix} \psi^A_+ \\ \psi^A_- \end{pmatrix}, \qquad \tilde{\Gamma}_7\psi^A_\pm = \pm\psi^A_\pm, \qquad \psi^A_\pm = \pm\epsilon^{AB}C_6(\bar{\psi}^A_\pm)^T, \tag{42}$$

where $A = 1, 2$. Given our choice of γ matrices, the compact IKKT action takes the form

$$S_C = \frac{L^6}{2g_{\text{eff}}^2}\int d\sigma^6 \text{Tr}\left(\frac{1}{2}F_{ab}F^{ab} + D_a A_\mu D^a A^\mu - \frac{1}{2}[A^\mu, A^\nu]^2 + \frac{1}{2}\bar{\psi}_+^A \tilde{\Gamma}^a \partial_a \psi_+^A + \frac{1}{2}\bar{\psi}_-^A \tilde{\Gamma}^a \partial_a \psi_-^A \right. \tag{43}$$

$$\left. - \frac{i}{2}\bar{\psi}_+^A \tilde{\Gamma}^a [A_a, \psi_+^A] - \frac{i}{2}\bar{\psi}_-^A \tilde{\Gamma}^a [A_a, \psi_-^A] + \frac{i}{2}\bar{\psi}_+^A (\sigma^\mu)^{AB}[A_\mu, \psi_-^B] - \frac{i}{2}\bar{\psi}_-^A (\bar{\sigma}^\mu)^{AB}[A_\mu, \psi_+^B]\right). \tag{44}$$

In addition to our choice of γ matrices, we will choose to work in the Lorenz gauge $\partial_a A^a = 0$. This choice can be imposed by adding the ghost term

$$S_{gh} = \frac{L^6}{g_{\text{eff}}^2}\int d\sigma^6 \text{Tr}(\partial^a \bar{c} D_a c) \tag{45}$$

to the compact IKKT action.

B. Mode expansion

Next, we will decompose the compact IKKT action into its different Fourier modes and separate the zero mode and the nonzero mode of the action. To do this, we will first separate the compact IKKT action $S_c = S_{\text{kin}} + S_{\text{int}}$ in a kinetic part

$$S_{\text{kin}} = L^6 \int d\sigma^6 \text{Tr}\left(\frac{1}{2}\partial_a A_N \partial^a A^N + \frac{1}{2}\bar{\psi}_+^A \tilde{\Gamma}^a \partial_a \psi_+^A + \frac{1}{2}\bar{\psi}_-^A \tilde{\Gamma}^a \partial_a \psi_-^A + \partial_a \bar{c}\partial^a c\right) \tag{46}$$

and an interaction part

$$S_{\text{int}} = L^6 \int d\sigma^6 \text{Tr}\left(-i\partial_a A_N [A^a, A^N] - \frac{1}{4}[A^M, A^N]^2 - \frac{i}{2}\bar{\psi}_+^A \tilde{\Gamma}^a [A_a, \psi_+^A] \right. \tag{47}$$

$$\left. - \frac{i}{2}\bar{\psi}_-^A \tilde{\Gamma}^a [A_a, \psi_-^A] + \frac{i}{2}\bar{\psi}_+^A (\sigma^\mu)^{AB}[A_\mu, \psi_-^B] - \frac{i}{2}\bar{\psi}_-^A (\bar{\sigma}^\mu)^{AB}[A_\mu, \psi_+^B] - i\partial_a \bar{c}[A^a, c]\right). \tag{48}$$

Then, we will rescale the gauge fields, the fermions, and the ghosts to make them dimensionless using the change of variable

$$A^M \to \lambda L A^M, \qquad \psi_+ \to \lambda L^{3/2}\psi_+, \qquad \psi_- \to \lambda L^{3/2}\psi_-, \qquad c \to \lambda L c. \tag{49}$$

Here, λ is a dimensionless parameter defined by $\lambda^2 \equiv g_{\text{eff}}^2/L^4$, which will play a role later in the perturbative expansion of Eq. (36). Finally, we will substitute the mode expansion

$$A^M = \sum_{n^a \in \mathbb{Z}^6} A^M(n^a) e^{i2\pi L n^a \sigma^a}, \qquad \psi = \sum_{r^a \in \mathbb{Z}^6 + 1/2} \psi(r^a) e^{i2\pi L r^a \sigma^a}, \qquad c = \sum_{n^a \in \mathbb{Z}^6} c(n^a) e^{i2\pi L n^a \sigma^a}, \tag{50}$$

in $S_C = S_{\text{kin}} + S_{\text{int}}$. After this substitution, the compact IKKT action can be written in the form $S_C = S_0 + S'_{\text{kin}} + S'_{\text{int}}$, where

$$S_0 = -\frac{\lambda^2}{4}\text{Tr}[A^M(0), A^N(0)]^2 \tag{51}$$

is the zero-mode part of the action, and

$$S'_{\text{kin}} = \frac{1}{2}\sum_{n^a \in \mathbb{Z}^6}{}' (2\pi n^a)^2 \text{Tr}(A_M(n^a) A^M(-n^a)) + \frac{1}{2}\sum_{r^a \in \mathbb{Z}^6 + 1/2} (2\pi r^a i)\bar{\psi}_+^A(r^a)\tilde{\Gamma}^a \psi_+^A(r^a) \tag{52}$$

$$+ \frac{1}{2}\sum_{r^a \in \mathbb{Z}^6 + 1/2} (2\pi r^a i)\bar{\psi}_-^A(r^a)\tilde{\Gamma}^a \psi_-^A(r^a) + \sum_{n^a \in \mathbb{Z}^6}{}' (2\pi n^a)^2 \text{Tr}(\bar{c}(n^a) c(n^a)) \tag{53}$$

is the kinetic part where the zero modes, which do not contribute, are not summed over. The final term, corresponding to the interaction part where the zero modes have been removed, takes the form

$$S'_{\text{int}} = \sum_{i=1}^{5} V_i, \tag{54}$$

where the V_i's are given by

$$V_1 = -\frac{\lambda^2}{4} \sideset{}{'}\sum_{n^a m^a l^a \in \mathbb{Z}^6} \mathrm{Tr}([A^M(-n^a - m^a - l^a), A^N(n^a)][A_M(m^a), A^N(l^a)]), \tag{55}$$

$$V_2 = -\lambda \sideset{}{'}\sum_{n^a m^a \in \mathbb{Z}^6} 2\pi(n^a + m^a)\mathrm{Tr}(A_M(-n^a - m^a)[A^a(n^a), A^M(m^a)]), \tag{56}$$

$$V_3 = -\frac{i}{2}\lambda \sideset{}{'}\sum_{r^a \in \mathbb{Z}^6+1/2, n^a \in \mathbb{Z}^6} \mathrm{Tr}(\bar{\psi}^A_+(r^a + n^a)\tilde{\Gamma}^b[A_b(n^a), \psi^A_+(r^a)]), \tag{57}$$

$$V_4 = -\frac{i}{2}\lambda \sideset{}{'}\sum_{r^a \in \mathbb{Z}^6+1/2, n^a \in \mathbb{Z}^6} \mathrm{Tr}(\bar{\psi}^A_-(r^a + n^a)\tilde{\Gamma}^b[A_b(n^a), \psi^A_-(r^a)]), \tag{58}$$

$$V_5 = \frac{i}{2}\lambda \sideset{}{'}\sum_{r^a \in \mathbb{Z}^6+1/2, n^a \in \mathbb{Z}^6} \mathrm{Tr}(\bar{\psi}^A_+(r^a + n^a)(\sigma^\mu)^{AB}[A_\mu(n^a), \psi^B_-(r^a)]), \tag{59}$$

$$V_6 = -\frac{i}{2}\lambda \sideset{}{'}\sum_{r^a \in \mathbb{Z}^6+1/2, n^a \in \mathbb{Z}^6} \mathrm{Tr}(\bar{\psi}^A_-(r^a + n^a)(\bar{\sigma}^\mu)^{AB}[A_\mu(n^a), \psi^B_+(r^a)]), \tag{60}$$

$$V_7 = -\lambda \sideset{}{'}\sum_{n^a \in \mathbb{Z}^6+1/2, m^a \in \mathbb{Z}^6} 2\pi(n^a + m^b)\mathrm{Tr}(\bar{c}(n^a + m^a)[A_a(n^a), c(m^a)]). \tag{61}$$

C. Zero mode effective action

We are now in a position to evaluate the Wilsonian effective action for the zero modes of the theory. Before taking on the task of evaluating Eq. (36), let us pause and notice that the only free parameter in Eqs. (51)–(61) is the dimensionless quantity λ. In the computation that follows, λ will play the role of expansion parameter. Since S_0 is an $\mathcal{O}(\lambda^2)$ quantity, we will only be concerned with corrections to S_0 that contribute at $\mathcal{O}(\lambda^2)$ order, neglecting the higher-order corrections. This approximation is valid when $\lambda \ll 1$ or, in other words, when $L \gg g_{\text{eff}}^{1/2}$. In the IKKT model, g^2 is related to the string scale l_s via $g^2 \sim l_s^4$. Hence, our approximation will be valid when the compactification radius L is much larger than the string length l_s.

To evaluate (36), we will first substitute $S_E = S_0 + S'_{\text{kin}} + S'_{\text{int}}$ in our definition for the zero-mode effective action of the theory [Eq. (36)]. We obtain

$$\begin{aligned} S^0_{\text{eff}} &= -\ln\left(\sideset{}{'}\prod_{n^a \in \mathbb{Z}^6} \prod_{r^b \in \mathbb{Z}^6+1/2} \int \mathcal{D}A^M(n^a)\mathcal{D}\psi(r^b)e^{-S_C} \right) \\ &= -\ln\left(e^{-S_0} \sideset{}{'}\prod_{n^a \in \mathbb{Z}^6} \prod_{r^b \in \mathbb{Z}^6+1/2} \right. \\ &\quad \left. \times \int \mathcal{D}A^M(n^a)\mathcal{D}\psi(r^b)e^{-S'_{\text{kin}} - S'_{\text{int}}} \right) \\ &= S_0 - \ln Z_{\text{kin}} - \ln\langle e^{-S'_{\text{int}}} \rangle, \end{aligned}$$

where we have defined

$$Z_{\rm kin} = \prod'_{n^a\in\mathbb{Z}^6}\prod_{r^b\in\mathbb{Z}^6+1/2}\mathcal{D}A^M(n^a)\mathcal{D}\psi(r^b)e^{-S'_{\rm kin}},$$
$$\langle . \rangle = \frac{1}{Z_{\rm kin}}\prod'_{n^a\in\mathbb{Z}^6}\prod_{r^b\in\mathbb{Z}^6+1/2}\int\mathcal{D}A^M(n^a)\mathcal{D}\psi(r^b).e^{-S'_{\rm kin}}. \tag{62}$$

As expected, the first term lets us recover the bosonic part of the IKKT action. The second term, on its side, does not depend on $A^M(0)$ and is nondynamical. For this reason, we will simply ignore it. Finally, we have the term $-\ln\langle e^{-S_{\rm int}}\rangle$ which is dynamical and will bring correction to the bosonic IKKT action. This term can be evaluated perturbatively by expanding it in the form

$$\begin{aligned}-\ln\langle e^{-S_{\rm int}}\rangle &= \left\langle S_{\rm int} - \frac{1}{2}S_{\rm int}^2 + \cdots\right\rangle_c \\ &= \langle V_1\rangle_c - \frac{1}{2}\langle V_2^2\rangle_c - \frac{1}{2}\langle V_3^2\rangle_c - \frac{1}{2}\langle V_4^2\rangle_c \\ &\quad - \langle V_5V_6\rangle_c - \frac{1}{2}\langle V_7^2\rangle_c + \cdots,\end{aligned}$$

where $\langle . \rangle_c$ denotes the fact that only connected diagrams contribute to the expectation value. In the expression above, we have only kept the terms that contribute to leading order [$\mathcal{O}(\lambda^2)$]. All other contributions from the vertex terms [Eqs. (55)–(61)] either vanish or contribute at next to leading [$\mathcal{O}(\lambda^4)$] order or at a higher order in the expansion parameter λ. To evaluate the quantities above, it is useful to write down the two-point functions

$$\langle A^M(n^a)A^N(m^a)\rangle = \frac{\delta_{MN}\delta_{n^a+m^a,0}}{(2\pi n^a)^2}, \tag{63}$$

$$\langle\bar{\psi}^A_{+\alpha}(r^a)\psi^B_{+\beta}(s^a)\rangle = -i\frac{2\pi r^a\tilde{\Gamma}^a_{\alpha\beta}\delta_{AB}\delta_{r^a,s^a}}{(2\pi r^a)^2}, \tag{64}$$

$$\langle\bar{\psi}^A_{-\alpha}(r^a)\psi^B_{-\beta}(s^a)\rangle = -i\frac{2\pi r^a\tilde{\Gamma}^a_{\alpha\beta}\delta_{AB}\delta_{r^a,s^a}}{(2\pi r^a)^2}, \tag{65}$$

$$\langle\bar{c}(n^a)c(m^a)\rangle = \frac{\delta_{n^a,m^b}}{(2\pi n^a)^2}, \tag{66}$$

for the gauge fields, the fermions, and the ghosts.[3] Using the two-point functions above, we find

[3]In Eqs. (63)–(66), we did not write down the matrix indices to avoid cluttering the notation. Here, the two-point functions of any two matrices A_{ab} and B_{cd} should take the form $\langle A_{ab}B_{cd}\rangle \sim \delta_{ad}\delta_{bc}$, where a, b, c, and d are matrix indices.

$$\begin{aligned}\langle V_1\rangle_c &= 9\lambda^2 MS_{B_1}\mathrm{Tr}(A^N(0))^2,\\ \langle V_2^2\rangle_c &= 2\lambda^2 M((17S_{B_2}+S_{B_1})\mathrm{Tr}(A_0^a)^2 + S_{B_1}\mathrm{Tr}(A^\mu(0))^2),\\ \langle V_3^2\rangle_c &= -8\lambda^2 M(2S_{F_2}-S_{F_1})\mathrm{Tr}(A^a(0))^2,\\ \langle V_4^2\rangle_c &= -8\lambda^2 M(2S_{F_2}-S_{F_1})\mathrm{Tr}(A^a(0))^2,\\ \langle V_5V_6\rangle_c &= 8\lambda^2 MS_{F_1}\mathrm{Tr}(A^\mu(0))^2,\\ \langle V_7^2\rangle_c &= -2\lambda^2 MS_{B_2}\mathrm{Tr}(A^a(0))^2,\end{aligned}$$

where S_{B_1}, S_{B_2}, S_{F_1}, and S_{F_2} are defined as follows:

$$S_{B_1} = \sum'_{n^a\in\mathbb{Z}^6}\frac{1}{(2\pi n^a)^2}, \qquad S_{F_1} = \sum_{r^a\in\mathbb{Z}^6+1/2}\frac{1}{(2\pi r^a)^2}, \tag{67}$$

$$S_{B_2} = \sum'_{n^a\in\mathbb{Z}^6}\frac{(2\pi n^1)^2}{(2\pi n^a)^4}, \qquad S_{F_2} = \sum_{r^a\in\mathbb{Z}^6+1/2}\frac{(2\pi r^1)^2}{(2\pi r^a)^4}. \tag{68}$$

Adding each term in the expansion, we find

$$\begin{aligned}-\ln\langle e^{iS_{\rm int}}\rangle &= -8(S_{F_1}-S_{B_1})\lambda^2 M\mathrm{Tr}(A^\mu(0))^2\\ &\quad - 8(S_{F_1}-S_{B_1}-2(S_{F_2}-S_{B_2}))\lambda^2 M\mathrm{Tr}(A^a(0))^2\\ &\quad + \mathcal{O}(\lambda^4).\end{aligned} \tag{69}$$

The expression above can be simplified by noting that $S_{B_1} = 6S_{B_2}$ and $S_{F_1} = 6S_{F_2}$. Adding the correction terms to S_0, the zero-mode effective action at $\mathcal{O}(\lambda^2)$ takes the form

$$\begin{aligned}S^0_{\rm eff} = \Big(&-\frac{1}{4}\mathrm{Tr}[A^M(0),A^N(0)]^2 - 8(S_{F_1}-S_{B_1})M\mathrm{Tr}(A^\mu(0))^2\\ &-\frac{16}{3}(S_{F_1}-S_{B_1})\mathrm{Tr}(A^a(0))^2\Big)\lambda^2 + \mathcal{O}(\lambda^4).\end{aligned} \tag{70}$$

Hence, we find that, at leading order, the corrections to S_0 take the form of two mass terms: one associated with the noncompact directions A^μ and one associated with the compact directions A^a. As expected, these corrections break the $SO(10)$ symmetry of the target space to $SO(3)\times SO(6)$. This is to be expected since by choosing to compactify six spatial dimensions, we are picking six special directions in space. The zero-mode effective action at leading order in perturbation theory reflects this fact.

The expression above can be more neatly written after undoing our previous change of variable via $A^M \to \lambda^{-1}L^{-1}A^M$. In passing, we will also go back to Lorentzian signature by imposing $A^0 \to -iA^0$. In this case, the effective action takes the form

$$\begin{aligned}S^0_{\rm eff} &= -\frac{1}{4g^2_{\rm eff}}\mathrm{Tr}[A^M(0),A^N(0)]^2\\ &\quad + \frac{1}{2}M^2_{MN}\mathrm{Tr}(A^M(0)A^N(0))^2 + \cdots,\end{aligned} \tag{71}$$

where we have defined a mass matrix

$$M^2_{MN} = \begin{bmatrix} \eta_{\mu\nu} M_4^2 & 0 \\ 0 & \eta_{ab} M_6^2 \end{bmatrix}, \tag{72}$$

which includes two mass terms

$$M_4^2 = 16(S_{F_1} - S_{B_1}) \frac{NM}{L^2}, \tag{73}$$

$$M_6^2 = \frac{32}{3}(S_{F_1} - S_{B_1}) \frac{NM}{L^2}. \tag{74}$$

In the expression above, the sums S_{B1} and S_{F1} individually diverge in the limit where N is large. However, it is possible to isolate the divergence in these sums by rewriting them as an integral and using Poisson resummation. What we find is rather interesting. It turns out that S_{B1} and S_{F_1} have the same divergent piece which is canceled by the difference $S_{F_1} - S_{B_1}$. It is then possible to evaluate the difference numerically, which gives $S_{F_1} - S_{B_1} \approx 0.0397887$. A detailed derivation of this result can be found in the Appendix.

Notice that the breaking of supersymmetry plays a crucial role in obtaining nonvanishing masses M_4^2 and M_6^2. If supersymmetry is restored by imposing that fermions have periodic boundary conditions, then r^a becomes summed over $\mathbb{Z}^6$ instead of $\mathbb{Z}^6 + 1/2$, and the masses vanish since $S_{B_1} = S_{B_2}$. Moreover, the fermions are indeed projected away by the antiperiodic boundary conditions, as expected. When supersymmetry is restored, the fermions have zero-mode terms that will appear at leading order in perturbation theory, and we recover the noncompact IKKT model action [Eq. (1)] with an effective gauge coupling g_{eff}.

This quenching of the fermions in the decompactification limit is something we fully expect from this system given the field theory interpretation of the compact matrix models. In the literature, compact matrix theory actions of the form (26) are often defined as a Yang-Mills theory compactified on a dual torus of radius $L' = 1/(2\pi L)$ [43]. This is why, in action (26), the parameter σ takes values in the interval $[0, L^{-1}]$ instead of $[0, 2\pi L]$ as one would expect. From the field theory point of view, the fermions become quenched when the radius L' of the dual torus goes to zero, as one would expect for fermions with antiperiodic boundary conditions. In the matrix model, this compactification limit is in fact a decompactification limit. Hence, we obtain a quenching of the fermions in the large volume limit instead.

The mass term correction, on its side, arises at leading order when integrating out the nonzero modes of the theory. This means that, in the decompactification limit, one cannot ignore residual interactions between duplicated regions. This potentially implies that interactions between regions are long ranged and cannot be ignored even at large distances. A consequence of this phenomenon seems to be the breaking of gauge invariance in the fundamental regions. Since interactions between regions cannot be ignored, the theory develops an effective potential that takes the form of a mass term. This mass term, which impacts the distribution of objects and their interactions in the fundamental regions, also breaks the gauge invariance of the theory.[4] One may view this as being problematic since, naively, it should be expected that gauge invariance is preserved in the decompactification limit. This intuition comes from the fact that, in the decompactification limit, we should recover the same theory we started with, along with the same symmetries. However, we should remind ourselves that this is not the case when compactifying matrix theories. Instead of recovering the initial system, we recover a large N number of copies of the initial system, as reflected by the overall factor of N in the equation which is absorbed in the effective coupling $g^2_{\mathrm{eff}} = g^2/N$. These copies come from the fact that we have duplicated a fundamental region N times along the compact directions. Since we do not recover the same system we started with, it is possible that some symmetries of the original system are not preserved. In the present case, we find that gauge symmetry in the fundamental regions is dependent on the structure of the interaction between them. If supersymmetry is preserved, interactions vanish and gauge symmetry is preserved. If supersymmetry is broken, the gauge symmetry is broken.

It is worth noting that compactifying a matrix theory on a higher-dimensional torus can lead to some issues. For example, in the BFSS matrix model, decoupling breaks down when the theory is compactified on T^k where $k > 5$ (see [54] for more detail). However, this problem only arises when the compactification radius L is taken to be small and the system starts behaving like a dual quantum field theory. In the present case, the compactification radius is taken to be large, and the obtained system is closer to the IKKT model than a dual quantum field theory. Hence, we do not expect this issue to arise here.[5]

IV. CONCLUSION AND DISCUSSION

In this paper, we compactified the IKKT matrix model on a six-dimensional torus where the space-time fermions acquire antiperiodic boundary conditions, and we found that the Wilsonian effective action for the noncompact degrees of freedom in the theory acquires an effective mass term which breaks the SO(1,9) symmetry of the IKKT model to $\mathrm{SO}(1,3) \times \mathrm{SO}(6)$. This mass matrix arises as a result of broken supersymmetry. If supersymmetry is

[4]The IKKT model action is invariant under the gauge variations $\delta A^M = i[A^M, \alpha]$ and $\delta\psi = i[\psi, \alpha]$, where α is an arbitrary matrix. Including a mass term in the theory breaks this symmetry.

[5]We thank Savdeep Sethi for bringing this point to our attention.

restored, the conventional IKKT action [Eq. (1)] is recovered.

It would be interesting to see if the equations of motion of the effective action we have found have interesting cosmological solutions. Given that the SO(1,9) space-time symmetry of the IKKT model is broken to $\mathrm{SO}(1,3) \times \mathrm{SO}(6)$, one may expect there exist solutions where three space dimensions expand, and the six others stay small. In this case, it might be possible that a supersymmetry-breaking compactification is responsible for the emergence of three large space dimensions in recent numerical simulations of the IKKT model.

Assuming interesting cosmological solutions exist, it might be possible to use them to test recent predictions in matrix cosmology, one of them being the scale invariance of cosmological perturbations [49,55] (see for [56] a summary of progress and challenges in these scenarios). Another avenue of research would be to test a recent space-time metric proposal in the IKKT matrix model [57] using these solutions or to repeat our analysis in the BFSS matrix model. In this case, one may find a possible connection with cosmological scenarios found in nonsupersymmetric string theories [58].

Another exciting perspective is that higher-order correction to the Wilsonian effective action allows for fuzzy de Sitter space solutions [59–62]. For example, fuzzy dS_4 is described by four "Pauli-Lubanski" vectors that act as Casimir operators of the SO(1,4) group. Since these operators are built out of Lorentz generators of the SO (1,4) group, they satisfy well-known commutation relations. It would be interesting to see if these commutation relations are solutions of the IKKT model under compactification when higher-order corrections are considered.

Finally, it is worthwhile to mention that effective mass terms have been found in matrix models before, notably in the following work [63,64]. However, in this case, the analysis was done for bosonic $(1+\mathrm{D})$- and $(2+\mathrm{D})$-dimensional Yang-Mills theories where all but one or two of the space-time matrices are integrated out. $(1+\mathrm{D})$- and $(2+\mathrm{D})$-dimensional Yang-Mills theories can be viewed as a $(1+\mathrm{D})$- and $(2+\mathrm{D})$-dimensional IKKT model where one or two dimensions are compactified on a torus. Hence, our analysis can be viewed as a special case of this work in which we consider a $(6+4)$-dimensional Yang-Mills theory where fermions are included, supersymmetry is broken, and all bosonic matrices remain in the effective action. Contrary to [63,64], we have restrained ourselves to the limit where the compactification radius L is large but finite. It would be interesting to see if phase transitions appear as we decrease the compactification radius.

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APPENDIX: EPSTEIN SERIES REGULARIZATION

When deriving the zero-mode effective action of the IKKT model, we encountered the sums S_{B1} and S_{F1}. These sums involve Epstein and Epstein-Hurwitz series that are divergent in the limit where $N \to \infty$. In the present section, we show that the divergent part of these sums can be isolated by introducing a regulator in the sums. When this is done, we find that S_{B1} and S_{F1} have the same divergent part, which is canceled in the difference $S_{F1} - S_{B1}$. Evaluating the difference numerically, we find $S_{F1} - S_{B1} \approx 0.0397887$.

1. Bosonic sum

We will first start by treating the divergence of the bosonic sum

$$S_{B_1} = \sum_{\vec{n}\in\mathbb{Z}^6}{}' \frac{1}{|2\pi\vec{n}|^2}. \tag{A1}$$

Here, we will use the vector notation $n^a = \vec{n}$ for simplicity. This sum involves the Epstein series

$$E_B = \sum_{\vec{n}\in\mathbb{Z}^d}{}' \frac{1}{|\vec{n}|^2}, \tag{A2}$$

which diverges when $d/2 > 1$. To treat the divergence, we will modify the sum to include a UV regulator. Let us consider the expression

$$\sum_{\vec{n}\in\mathbb{Z}^d}{}' \frac{e^{-\alpha^2|\vec{n}|^2}}{|\vec{n}|^2}. \tag{A3}$$

Here, α^2 plays the role of cutoff which truncates the modes above $N \sim \alpha^{-1}$ out of the sum, hence taming the divergence. In the limit where $\alpha^2 \to 0$, all modes contribute to the sum and the expression above reduces to E_B. Equation (A3) can rewritten in integral form using the property

$$\frac{1}{|\vec{n}|^2} = \int_0^\infty dt e^{-t|\vec{n}|^2}. \tag{A4}$$

We obtain

$$\sum_{\vec{n}\in\mathbb{Z}^d}{}' \frac{e^{-\alpha^2|\vec{n}|^2}}{|\vec{n}|^2} = \int_0^\infty dt \sum_{\vec{n}\in\mathbb{Z}^d}{}' e^{-(t+\alpha^2)|\vec{n}|^2} \tag{A5}$$

$$= \pi \int_{\alpha^2/\pi}^\infty dt(\theta^d(t) - 1), \tag{A6}$$

where we made use of the function

$$\theta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi t n^2}. \tag{A7}$$

Since $\theta(t) \sim t^{-1/2}$ when $t \to 0$, the integrand in Eq. (A6) diverges in the limit when the regulator α^2 goes to zero. To deal with the divergent part of the integral, we will rewrite the part of the integral in the interval $t \in [\alpha^2/\pi, 1]$ by making use of the property

$$\theta(t) = \frac{1}{t^{1/2}}\theta(1/t), \tag{A8}$$

which can be derived using Poisson's resummation formula. Substituting the expression above in (A6), we obtain

$$\int_{\alpha^2/\pi}^1 dt(\theta^d(t) - 1) = \int_1^{\pi/\alpha^2} dt t^{d/2-1}(\theta^d(t) - 1) - \frac{d}{d-2} + \alpha^2 + \frac{2}{d-2}\left(\frac{\pi}{\alpha^2}\right)^{d/2-1}. \tag{A9}$$

In the expression above, the integral is finite for all values of d. Hence, for $d/2 > 1$, the only divergent piece when $\alpha^2 \to 0$ comes from the last term which is inversely proportional to α^2. Piecing everything together and letting α^2 go to zero, we obtain

$$E_B = \pi\left(\int_1^\infty dt(1 + t^{d/2-1})(\theta^d(t) - 1) - \frac{d}{d-2} + \frac{2}{d-2}\left(\frac{\pi}{\alpha^2}\right)^{d/2-1}\right). \tag{A10}$$

When $d = 6$, which is the case we are interested in, substituting the value of the E_B in S_{B_1} gives us

$$S_{B_1} = \frac{1}{4\pi}\left(\int_1^\infty dt(1 + t^2)(\theta^6(t) - 1) - \frac{3}{2} + \frac{1}{2}\left(\frac{\pi}{\alpha^2}\right)^2\right). \tag{A11}$$

2. Fermionic sum

Finally, we will evaluate the fermionic sum

$$S_{F_1} = \sum_{\vec{n}\in\mathbb{Z}^6+1/2} \frac{1}{|2\pi\vec{n}|^2}, \tag{A12}$$

where the vector notation $n^a = \vec{n}$ is used for simplicity. In this case, we will be interested in the Epstein-Hurwitz series

$$E_F = \sum_{\vec{n}\in\mathbb{Z}^d} \frac{1}{|\vec{n} + a|^2}, \tag{A13}$$

when $a \neq 0$. Here again, we will modify the sum to include a regulator α^2, which truncates the modes above $N \sim \alpha^{-1}$ out of the sum. In the present case, the expression of interest will be

$$\sum_{\vec{n}\in\mathbb{Z}^d} \frac{e^{-\alpha^2|\vec{n}+a|^2}}{|\vec{n} + a|^2}, \tag{A14}$$

which reduces to E_F when α^2 goes to zero. Making use of Eq. (A4), the sum above can be rewritten as an integral. We obtain

$$\sum_{\vec{n}\in\mathbb{Z}^d} \frac{e^{-\alpha^2|\vec{n}+a|^2}}{|\vec{n} + a|^2} = \int_0^\infty dt \sum_{\vec{n}\in\mathbb{Z}^d} e^{-(t+\alpha^2)|\vec{n}+a|^2} \tag{A15}$$

$$= \pi \int_{\alpha^2/\pi}^\infty dt \theta^d(t|a), \tag{A16}$$

where we defined the function

$$\theta(t|a) = \sum_{n=-\infty}^{\infty} e^{-\pi t(n+a)^2}. \tag{A17}$$

Just like $\theta(t)$, the function above can be approximated as $\theta(t|a) \sim t^{-1/2}$ when $t \to 0$, so the integrand in Eq. (A16) diverges in the limit when the regulator α^2 goes to zero. To treat this divergence, we will rewrite the divergent part of the integral by making use of the property

$$\theta(t|a) = \frac{e^{-\pi a^2 t}}{t^{1/2}}\theta(1/t|iat), \tag{A18}$$

which can be derived using Poisson's resummation formula. In this case, the divergent part of the integral can be written as

$$\int_{\alpha^2/\pi}^1 dt\theta^d(t|a) = \int_1^{\pi/\alpha^2} dt t^{d/2-1}(e^{-\pi d a^2/t}\theta^d(t|ia/t) - 1) - \frac{2}{d-2} + \frac{2}{d-2}\left(\frac{\pi}{\alpha^2}\right)^{d/2-1}. \tag{A19}$$

The integral on the right-hand side of the expression above is convergent for all values of d. Hence, when $d/2 > 1$, the only divergent piece comes from the last term which is inversely proportional to α^2. Piecing everything together and letting α^2 go to zero, we obtain

$$E_F = \pi\left(\int_1^\infty dt\theta^d(t|a) + \int_1^\infty dt t^{d/2-1}(e^{-\pi d a^2/t}\theta^d(t|ia/t) - 1) - \frac{2}{d-2} + \frac{2}{d-2}\left(\frac{\pi}{\alpha^2}\right)^{d/2-1}\right). \tag{A20}$$

Letting $d = 6$ and $a = 1/2$, we can finally evaluate S_{F_1} by substituting E_F in Eq. (A12). We obtain

$$S_{F_1} = \frac{1}{4\pi}\left(\int_1^\infty dt\theta^6(t|1/2) + \int_1^\infty dt t^2(e^{-\frac{3\pi}{2t}}\theta^6(t|i(2t)^{-1}) - 1) - \frac{1}{2} + \frac{1}{2}\left(\frac{\pi}{\alpha^2}\right)^2\right). \tag{A21}$$

As we can see, the divergent piece in S_{F1} is the same one that we obtained for S_{B1}. This is because in the $t \to 0$ limit, the θ function in the integrand (A16) behaves as $\theta(t|a) \sim t^{-1/2}$ independent of a. Consequently, integrating $\theta(t|a)^d$ in the vicinity of $t \to 0$ yields the same divergent piece regardless of if a takes the value zero (in the bosonic case) or 1/2 (in the fermionic case). This means that subtracting S_{B_1} from S_{F_1} should give a finite value, which can be obtained by carrying out each integral in S_{B_1} and S_{F_1}. Carrying out the integrals numerically, we obtain

$$S_{F1} - S_{B_1} = 0.0397887. \tag{A22}$$

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