All OPEs invariant under the infinite symmetry algebra for gluons on the celestial sphere

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S algebra is an infinite-dimensional Lie algebra, which is known to be the symmetry algebra of some gauge theories. It is a "colored version" of the $w_{1+\infty}$. In this paper, we write down all possible S-invariant (celestial) operator product expansions between two positive-helicity outgoing gluons and also find the Knizhnik-Zamolodchikov type null states for these theories. Our analysis hints at the existence of an infinite number of S-invariant gauge theories which include the (tree-level) maximally helicity-violating sector and the self-dual Yang-Mills theory.

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I. INTRODUCTION

The S matrix is an important observable in any quantum field theory in asymptotically flat spacetime. In fact, in quantum theory of gravity the S-matrix is argued to be the only observable. Therefore, any holographic dual theory has to compute the S-matrix elements or scattering amplitudes in the bulk. Celestial holography is an attempt in this direction [1–3]. The realization that the soft theorems of gauge theories and gravity are the Ward identities for different asymptotic symmetries [4–33] has led to important insights into the study of scattering amplitudes rewritten as correlators of a conformal field theory (CFT) on the two-dimensional celestial sphere. This CFT is commonly known as the celestial conformal field theory (CCFT). The map from scattering amplitudes to the correlators of CCFT is done via the Mellin transformation. Usually, scattering amplitudes are written in the momentum basis. The job of the Mellin transformation is to change the momentum eigenbasis into the boost eigenbasis [34–37]. The isomorphism between the global conformal group in two dimensions and the Lorentz group in four dimensions is at the heart of this basis change.

The operator product expansion (OPE) in CCFT corresponds to the collinear limit in the bulk and it plays a very important role in the study of the dual theory [14,17,19,23,24,38–53]. In a previous paper [38], we have studied the $w_{1+\infty}$ invariant OPEs in theories of gravity. Our analysis showed that there are an infinite number of theories on the celestial sphere which are $w_{1+\infty}$ invariant. By deriving the OPE from graviton scattering amplitudes we have explicitly shown in [53] that the self dual gravity is one example of this infinite family.

In this paper, we perform a similar analysis for gluons. In the case of gluons, the infinite symmetry algebra is know as the S algebra [17,18]. We write down all possible S-invariant OPE structures between two positive-helicity outgoing gluons. We find that there is a (discrete) infinite number of such structures and, presumably, each one corresponds to an S-invariant theory of gluons in the bulk. However, a more explicit Lagrangian description of these theories is not known to us.

There is an important difference between the analyses of $w_{1+\infty}$ and S-invariant theories, which we want to point out. S algebra does not contain the Poincaré generators. Therefore, the consistent OPEs need not be Poincaré invariant. However, in this paper we make sure that all of the OPEs are (conformal) Lorentz invariant, and this plays an important role. This is along the lines of [46–48,54].

We start with a brief review of the soft gluon symmetry algebra known as the S algebra in Sec. II. In Sec. III, the

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general structure of the OPE between two positive-helicity outgoing gluons on the celestial sphere is discussed. We argue that the null states of the maximally helicity-violating (MHV) sector can be used to write down the general OPE. In Sec. IV, we write down the null states that appear at $\mathcal{O}(z^0\bar{z}^0)$ of the gluon-gluon OPE in the MHV sector. These are not the complete set of null states that the MHV sector has at $\mathcal{O}(z^0\bar{z}^0)$; there are more of them. We talk about them in Sec. VIII, where we discuss the Knizhnik-Zamolodchikov (KZ)-type null states. Section V explicitly shows how to organize the OPE at every order. For simplicity, we focus on the $\mathcal{O}(z^0\bar{z}^0)$ terms in the OPE. We also discuss the transformation properties of MHV null states under the S algebra in this section, which are required to organize the OPE. Section VI shows the invariance of the $\mathcal{O}(z^0\bar{z}^0)$ OPE under the S algebra. In Sec. VII, we argue that an infinite number theories can exist on the celestial sphere. We conclude with a discussion of the results found in this paper and some future directions in Sec. X.

II. THE S ALGEBRA

We start by describing the soft symmetry algebra, which follows from the universal singular terms in the OPE between two positive-helicity outgoing gluons [17,18]. Let $O^{a,+}_{\Delta}(z,\bar{z})$ denote a positive-helicity outgoing gluon conformal primary operator of dimension Δ at the point $(z,\bar{z})$ on the celestial sphere. The universal singular terms in the OPE are given by

$$O^{a,+}_{\Delta_1}(z_1,\bar{z}_1)O^{b,+}_{\Delta_2}(z_2,\bar{z}_2) \sim \frac{-if^{abc}}{z_{12}}\sum_{n=0}^{\infty}B(\Delta_1-1+n,\Delta_2-1) \times \frac{\bar{z}_{12}^n}{n!}\bar{\partial}^n O^{c,+}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2). \quad (2.1)$$

The interesting fact about these singular OPE coefficients is that they allow us to define an infinite tower of conformally soft [39,55–60] gluons [17,18] by

$$R^{k,a}(z,\bar{z}) \coloneqq \lim_{\Delta\to k}(\Delta-k)O^{a,+}_{\Delta}(z,\bar{z}), \quad k=1,0,-1,\cdots. \quad (2.2)$$

Now it follows from the structure of the OPE (2.1) that one can introduce the following truncated mode expansion:

$$R^{k,a}(z,\bar{z}) = \sum_{n=\frac{k-1}{2}}^{\frac{1-k}{2}}\frac{R^{k,a}_n(z)}{\bar{z}^{n+\frac{k-1}{2}}}, \quad (2.3)$$

where the expansion coefficients $R^{k,a}_n(z)$ are the conserved holomorphic currents. For fixed values of k and a, there are $(2-k)$ such currents and they transform in the $(2-k)$-dimensional representation of the $\overline{SL_2(R)}$.

The holomorphic currents $R^{k,a}_n(z)$ can be further mode expanded as

$$R^{k,a}_n(z) = \sum_{\alpha\in\mathbb{Z}-\frac{k+1}{2}}\frac{R^{k,a}_{\alpha,n}}{z^{\alpha+\frac{k+1}{2}}}. \quad (2.4)$$

The algebra of these modes can be obtained from the singular terms (2.1) of the RR OPE and is given by [17]

$$[R^{k,a}_{\alpha,n},R^{l,b}_{\alpha',n'}] = -if^{abc}\frac{\left(\frac{1-k}{2}-n+\frac{1-l}{2}-n'\right)!}{\left(\frac{1-k}{2}-n\right)!\left(\frac{1-l}{2}-n'\right)!} \times\frac{\left(\frac{1-k}{2}+n+\frac{1-l}{2}+n'\right)!}{\left(\frac{1-k}{2}+n\right)!\left(\frac{1-l}{2}+n'\right)!}R^{k+l-1,c}_{\alpha+\alpha',n+n'}. \quad (2.5)$$

Now one can define the following generators [18]:

$$S^{q,a}_{\alpha,m} = (q-m-1)!(q+m-1)!R^{3-2q,a}_{\alpha,m}, \quad (2.6)$$

in terms of which the algebra (2.5) simplifies to

$$[S^{p,a}_{\alpha,m},S^{q,b}_{\beta,n}] = -if^{abc}S^{p+q-1,c}_{\alpha+\beta,m+n}. \quad (2.7)$$

This infinite-dimensional algebra of the conformally soft gluons, known as the S algebra, plays a central role in this paper.[1]

In this paper, we want to classify all possible OPEs between two positive-helicity outgoing gluons that are invariant under the S algebra. The strategy we adopt here is similar to that in the gravity case [38] but the details are very different. For example, in the gravity case, the soft symmetry algebra, which is isomorphic to the $w_{1+\infty}$, contains all four global space-time translations. But this is not the case for the S algebra, and so there are S-invariant theories that are not space-time translationally invariant [46–48,54]. However, we preserve Lorentz invariance because it translates into conformal invariance on the celestial sphere and the structure of the S algebra depends on that.

III. GENERAL STRUCTURE OF THE OPE BETWEEN TWO POSITIVE-HELICITY OUTGOING GLUONS

We can write the general structure of the OPE between two positive-helicity gluons invariant under the S algebra as

[1] In this paper, we write the OPE in terms of the descendants of the R algebra (2.5). However, we continue to refer to (2.5) as the S algebra.

$$O^{a,+}_{\Delta_1}(z_1,\bar{z}_1)O^{b,+}_{\Delta_2}(z_2,\bar{z}_2) = \frac{-if^{abc}}{z_{12}}\sum_{n=0}^{\infty}B(\Delta_1-1+n,\Delta_2-1)\frac{\bar{z}_{12}^n}{n!}\bar{\partial}^n O^{c,+}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2) + \sum_{p,q=0}^{\infty} z_{12}^p\bar{z}_{12}^q\sum_{k=1}^{\tilde{n}_{p,q}}\tilde{C}^k_{p,q}(\Delta_1,\Delta_2)\tilde{O}^{ab}_{k,p,q}(\Delta_1,\Delta_2,z_2,\bar{z}_2), \tag{3.1}$$

where in the second line we added the S algebra descendants of a positive-helicity gluon. The sum over k could be finite or infinite, depending on the theory. Our goal is to determine the descendants $\tilde{O}^{ab}_{k,p,q}$ and the OPE coefficients $\tilde{C}^k_{p,q}$ in a general S-invariant theory.

In the gravity case [38], we found that any w-invariant OPE can be written in terms of the MHV OPE and the MHV null states. We have also checked by detailed calculation that this structure holds in the self-dual gravity theory [53], which is w invariant. The same reasoning also holds for the S algebra and gluons. We summarize the argument below.

Since the S algebra is universal, i.e., the *same*[2] algebra holds in any S-invariant theory, it is reasonable to assume that there is a master OPE that holds in *all* S-invariant theories. Let us now consider the gluon-gluon OPE in the (tree-level) MHV sector of the pure Yang-Mills theory. Since the MHV sector is S invariant, the master OPE, when inserted into an MHV gluon scattering amplitude, should reproduce the known MHV sector OPE. Therefore, one can write

$$\text{Master OPE} = \text{MHV-sector OPE} + R, \tag{3.2}$$

where R should *vanish* inside an MHV scattering amplitude. This is possible only if R is a linear combination of MHV *null states*. Now, since the MHV-sector OPE already contains the universal singular terms (2.1) of the gluon-gluon OPE, R consists only of nonsingular terms. Thus, we can write

$$O^{a,+}_{\Delta_1}(z_1,\bar{z}_1)O^{b,+}_{\Delta_2}(z_2,\bar{z}_2)|_{\text{Any Theory}} = O^{a,+}_{\Delta_1}(z_1,\bar{z}_1)O^{b,+}_{\Delta_2}(z_2,\bar{z}_2)|_{\text{MHV}} + \sum_{p,q=0}^{\infty} z_{12}^p\bar{z}_{12}^q\sum_{k=1}^{\tilde{n}_{p,q}}\tilde{C}^k_{p,q}(\Delta_1,\Delta_2)M^{ab}_{k,p,q}(\Delta_1,\Delta_2,z_2,\bar{z}_2), \tag{3.3}$$

where $M^{ab}_{k,p,q}$ are the MHV null states. So when "any theory" is taken to be the MHV sector, $M^{ab}_{k,p,q}$ vanishes and we get back the MHV sector OPE by construction.

We now describe the MHV null states, which are of interest to us. In this paper, we apply this general procedure to write down the OPE at $\mathcal{O}(z_{12}^0\bar{z}_{12}^0)$.

IV. NULL STATES IN THE MHV SECTOR

The general null state at order $z_{12}^0\bar{z}_{12}^0$ is given by[3]

$$\Psi^{ab}_j(\Delta) = R^{-j,a}_{\frac{j-1}{2},\frac{j+1}{2}}\mathcal{O}^{b,+}_{\Delta+j} - \frac{(-1)^j j}{\Gamma(j+2)}\frac{\Gamma(\Delta+j-1)}{\Gamma(\Delta-2)}R^{1,a}_{-1,0}\mathcal{O}^{b,+}_{\Delta-1} - \frac{(-1)^j}{\Gamma(j+1)}\frac{\Gamma(\Delta+j-1)}{\Gamma(\Delta-1)}R^{0,a}_{-\frac{1}{2},\frac{1}{2}}\mathcal{O}^{b,+}_{\Delta}. \tag{4.1}$$

Here we have ignored the (p,q) index and have simply written M^{ab}_k instead of $M^{ab}_{k,0,0}$ for the order $z_{12}^0\bar{z}_{12}^0$ MHV null states.

Now it turns out that the basis of null states

$$M^{ab}_k(\Delta) = \sum_{i=1}^{k}\frac{1}{\Gamma(k-i+1)}\frac{\Gamma(\Delta+k-1)}{\Gamma(\Delta+i-1)}\Psi^{ab}_i(\Delta) \tag{4.2}$$

is more convenient because they transform nicely under the S algebra. We will discuss their transformation law in the next section.

We conclude this section by defining the antisymmetric part of the null states $M^{ab}_k(\Delta)$ as

$$M^a_k(\Delta) = f^{abc}M^{bc}_k. \tag{4.3}$$

V. ORGANIZING THE OPE AT EVERY ORDER

Since the MHV sector is S invariant, the MHV null states must form a representation of the S algebra. In other words, every generator of the S algebra must map any MHV null state to another MHV null state. Our analysis shows that this representation is reducible and different S-invariant theories correspond to different irreducible components of this representation. So our first job is to study the action of the S algebra generators on the MHV null states. This is facilitated by the following observation.

In Sec. II, we discussed that the S algebra is generated by an infinite number of holomorphic soft currents $\{R^{k,a}_p(z)\}$,[4] where $k = 1, 0, -1, -2, \ldots$ is the dimension Δ of the soft operator and $\frac{k-1}{2} \le p \le \frac{1-k}{2}$. For a fixed k, the soft currents $R^{k,a}_{\frac{1-k}{2}}, \ldots, R^{k,a}_{\frac{k-1}{2}}$ transform in a $(2-k)$-dimensional representation of the $\overline{sl_2(R)}$.[5] This can be seen from the following commutation relations:

[2]For example, this is not true in the conventional $2-D$ CFTs because different CFTs have different Virasoro central charges and thus different conformal symmetry algebras.

[3]These null states can be obtained by taking soft limits of the gluon-gluon MHV OPE [15]. The relevant terms in the gluon-gluon OPE in the MHV sector, which gives rise to these null states, are given in (6.2).

[4]Here, $-p$ is the antiholomorphic weight of the current.

[5]Note that we are assuming the theory to be (conformal) Lorentz invariant.

$$[H^0_{0,-1}, R^{k,a}_{m,p}] = \frac{1}{2}(2p+k-3)R^{k,a}_{m,p-1} \quad \text{for } p > \frac{k-1}{2},$$
$$[H^0_{0,-1}, R^{k,a}_{m,\frac{k-1}{2}}] = 0,$$
$$[H^0_{0,0}, R^{k,a}_{m,p}] = -2pR^{k,a}_{m,p},$$
$$[H^0_{0,1}, R^{k,a}_{m,p}] = \frac{1}{2}(2p-k+3)R^{k,a}_{m,p+1} \quad \text{for } p < -\frac{k-1}{2},$$
$$[H^0_{0,1}, R^{k,a}_{m,-\frac{k-1}{2}}] = 0. \tag{5.1}$$

Now let us consider the currents $R^{1,a}_0, R^{0,a}_{\frac{1}{2}}, R^{-1,a}_1, \cdots$ with the lowest $\overline{sl_2(R)}$ weights. Starting from $R^{1,a}_0$, all of the currents in this family can be obtained by applying the global subleading soft gluon operator $R^{0,b}_{\frac{1}{2},\frac{1}{2}}$ (Fig. 1). This can be seen from the following commutation relations:

$$[R^{0,a}_{\frac{1}{2},\frac{1}{2}}, R^{k,b}_{m,\frac{1-k}{2}}] = -if^{abc}(2-k)R^{k-1,c}_{m+\frac{1}{2},\frac{2-k}{2}}. \tag{5.2}$$

Equations (5.1) and (5.2) show that we can write any generator of the S algebra as a sum of products of the generators $(R^{1,a}_{n,0}, R^{0,a}_{\frac{1}{2},\frac{1}{2}}, H^0_{0,0}, H^0_{0,\pm1})$. Therefore, in order to study the action of the S algebra generators on the MHV null states, we just need to focus on this finite number of generators.

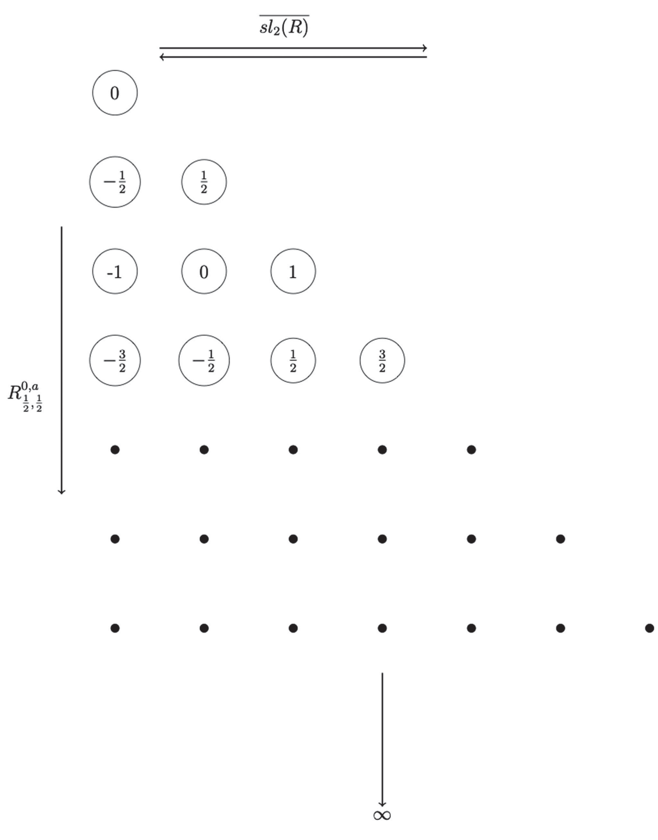

FIG. 1. Soft gluon currents arranged in representations of $\overline{sl_2(R)}$. The $\overline{sl_2(R)}$ generators move the currents horizontally in both directions, whereas the global subleading soft gluon symmetry generator $R^{0,a}_{\frac{1}{2},\frac{1}{2}}$ moves the currents vertically downward.

A. Transformation properties of the null states under $\overline{sl_2(R)}$ algebra

Using the action of different generators of $\overline{sl_2(R)}$ algebra on the gluon primary operators and the commutation relations (5.1), it is easy to show that

$$H^0_{0,1}\Psi^{ab}_k(\Delta) = 0. \tag{5.3}$$

Thus, (4.2) implies that

$$H^0_{0,1}M^{ab}_k(\Delta) = 0. \tag{5.4}$$

Therefore, the null states M^{ab}_k are $\overline{sl_2(R)}$ primaries.

B. Transformation properties of the null states under the leading soft gluon current algebra

One can easily check that under the leading soft gluon current algebra, the null states (4.2) transform as[6]

$$R^{1,a}_{0,0}M^{bc}_k(\Delta) = -if^{abd}M^{dc}_k(\Delta) - if^{acd}M^{bd}_k(\Delta)$$
$$R^{1,a}_{n,0}M^{bc}_k(\Delta) = 0, \qquad n > 0. \tag{5.5}$$

Therefore, the null states M^{ab}_k are the leading soft gluon current algebra primaries.

C. Transformation properties of the null states under subleading soft gluon operator $R^{0,a}_{\frac{1}{2},\frac{1}{2}}$

This is perhaps the most important transformation property because it mixes the null states M^{ab}_k with different k values. The action of $R^{0,a}_{\frac{1}{2},\frac{1}{2}}$ on $M^{bc}_k(\Delta)$ is given by

$$R^{0,a}_{\frac{1}{2},\frac{1}{2}}M^{bc}_k(\Delta) = -(k+2)if^{abx}M^{xc}_{k+1}(\Delta-1) + (\Delta+k-2) \times [if^{acx}M^{bx}_k(\Delta-1) + if^{abx}M^{xc}_k(\Delta-1)]. \tag{5.6}$$

Now let us consider the set of null states

$$M^{bc}_k(\Delta), \quad k = 1, 2, \ldots, n. \tag{5.7}$$

From (5.6), we can see that if we set

$$M^{ab}_{k+1}(\Delta) = 0, k \geq n \geq 0, \tag{5.8}$$

then the set (5.7) is closed under the action of $R^{0,a}_{\frac{1}{2},\frac{1}{2}}$. Moreover, it follows from (5.6) that the infinite set of

[6]We have used the action of S-algebra generators on the primary operators and on the Ψ-null states which are given in Appendices A and B.

equations (5.8) is also invariant under the action of $R^{0,a}_{\frac{1}{2},\frac{1}{2}}$ because the index k mixes only with $k' \geq k$. Therefore, the truncation (5.8) is S-algebra invariant and we can get an S-invariant OPE if we keep only the finite set (5.7). Let us emphasize that the integer n is in no way restricted by the S invariance.

VI. $\mathcal{O}(z^0_{12}\bar{z}^0_{12})$ OPE AND ITS INVARIANCE UNDER THE S ALGEBRA

Let us now consider the $\mathcal{O}(1)$ terms in the OPE when we keep only the finite set of MHV null states (5.7). In particular, we show that the $\mathcal{O}(1)$ terms in the OPE with the following coefficients are S invariant:

$$\begin{aligned}\mathcal{O}^{a,+}_{\Delta_1}(z,\bar{z})\mathcal{O}^{b,+}_{\Delta_2}(0,0)|_{\mathcal{O}(1)} &= \mathcal{O}^{a,+}_{\Delta_1}(z,\bar{z})\mathcal{O}^{b,+}_{\Delta_2}(0,0)|_{\mathrm{MHV\ OPE\ at}\ \mathcal{O}(1)} \\ &\quad + \sum_{k=1}^{n} B(\Delta_1+k,\Delta_2-1)M^{ab}_k(\Delta_1+\Delta_2) \end{aligned} \tag{6.1}$$

where $\mathcal{O}^{a,+}_{\Delta_1}(z,\bar{z})\mathcal{O}^{b,+}_{\Delta_2}(0,0)|_{\mathrm{MHV\ OPE\ at}\ \mathcal{O}(1)}$ is given by [15,42]

$$\begin{aligned}\mathcal{O}^{a,+}_{\Delta_1}(z,\bar{z})\mathcal{O}^{b,+}_{\Delta_2}(0,0)|_{\mathrm{MHV\ OPE\ at}\ \mathcal{O}(1)} &= B(\Delta_1-1,\Delta_2-1)\Big[\Delta_1 R^{1,a}_{-1,0}\mathcal{O}^{b,+}_{\Delta_1+\Delta_2-1}(0,0) \\ &\quad + \frac{\Delta_1-1}{\Delta_1+\Delta_2-2}R^{0,a}_{-\frac{1}{2},\frac{1}{2}}\mathcal{O}^{b,+}_{\Delta_1+\Delta_2}(0,0)\Big]. \end{aligned} \tag{6.2}$$

Let us first apply $R^{0,a}_{\frac{1}{2},\frac{1}{2}}$ to the OPE (6.1). After some straightforward algebra, we get

$$\begin{aligned}&R^{0,x}_{\frac{1}{2},\frac{1}{2}}(\mathcal{O}^{a,+}_{\Delta_1}(z,\bar{z})\mathcal{O}^{b,+}_{\Delta_2}(0,0)|_{\mathcal{O}(1)}) \\ &-R^{0,x}_{\frac{1}{2},\frac{1}{2}}\Big[\mathcal{O}^{a,+}_{\Delta_1}(z,\bar{z})\mathcal{O}^{b,+}_{\Delta_2}(0,0)|_{\mathrm{MHV\ OPE\ at}\ \mathcal{O}(1)} \\ &+\sum_{k=1}^{n}B(\Delta_1+k,\Delta_2-1)M^{ab}_k(\Delta_1+\Delta_2)\Big] \\ &= if^{xay}(n+2)B(\Delta_1+n,\Delta_2-1)M^{yb}_{n+1}(\Delta_1+\Delta_2-1). \end{aligned} \tag{6.3}$$

Now, we have argued in the previous section that if the $\mathcal{O}(1)$ OPE of an S-invariant theory truncates at $k=n$, then $M^{ab}_{n+1}(\Delta)$ will be a null state of that theory. Thus, we can set the rhs of (6.3) to 0, and hence (6.1) is invariant under the action of $R^{0,a}_{\frac{1}{2},\frac{1}{2}}$. Using (5.5), one can also verify that (6.1) is invariant under the actions of $R^{1,a}_{n,0}$.

In [15], it was shown that the OPE in the MHV sector is invariant under the action of $H^0_{0,1}$. We can also see from (5.3) that the null states $M^{ab}_k(\Delta)$ are annihilated by $H^0_{0,1}$. Thus, the OPE (6.1) is also invariant under $H^0_{0,1}$. Hence, we conclude that the truncated OPE (6.1) is invariant under the S algebra.

VII. INFINITE FAMILY OF S-INVARIANT THEORIES

In Sec. V C, we showed that the set of equations

$$M^{ab}_{k+1} = 0, k \geq n \geq 0 \tag{7.1}$$

is S invariant. Thus, at $\mathcal{O}(z^0\bar{z}^0)$ we can truncate the OPE (3.3) at an arbitrary n in an S-invariant way, that is, S invariance does not fix the value of the integer n. Hence, we can get a discrete infinite family of S-invariant OPEs for different choices of the integer n. Each of these consistent OPEs corresponds to an S-invariant theory of gluons. But, at present, we do not know the Lagrangian description of these theories except perhaps the self-dual Yang-Mills theory.

VIII. KNIZHNIK-ZAMOLODCHIKOV TYPE NULL STATES

KZ-type null states contain descendants of the holomorphic translation generator L_{-1} on the celestial sphere. They can be obtained algebraically by determining the relevant primary descendant, but in our case we can bypass this tedious procedure if we use the OPE commutativity,

$$\mathcal{O}^{a,+}_{\Delta_1}(z_1,\bar{z}_1)\mathcal{O}^{b,+}_{\Delta_2}(z_2,\bar{z}_2) = \mathcal{O}^{b,+}_{\Delta_2}(z_2,\bar{z}_2)\mathcal{O}^{a,+}_{\Delta_1}(z_1,\bar{z}_1). \tag{8.1}$$

The reason behind this is that the $\mathcal{O}(z^0_{12}\bar{z}^0_{12})$ terms of the OPE, as written in (6.1), are not manifestly symmetric under the exchange (8.1). Therefore, OPE commutativity imposes nontrivial constraints on the OPE coefficients and one such constraint is essentially the KZ equation. The process can be further simplified if we make the operator $\mathcal{O}^{b,+}_{\Delta_2}(z_2,\bar{z}_2)$ leading soft. The leading soft operator is defined using equation (2.2) with $k=1$. Now a straightforward calculation gives the KZ-type null state

$$K^a(\Delta) = \xi^a(\Delta) - i\sum_{k=1}^{n} M^a_k(\Delta+1) = 0 \tag{8.2}$$

where

$$\begin{aligned}\xi^a(\Delta) &= C_A L_{-1}\mathcal{O}^{a,+}_{\Delta} - (\Delta+1)R^{1,b}_{-1,0}R^{1,b}_{0,0}\mathcal{O}^{a,+}_{\Delta} \\ &\quad - R^{0,b}_{-\frac{1}{2},\frac{1}{2}}R^{1,b}_{0,0}\mathcal{O}^{a,+}_{\Delta+1} \end{aligned} \tag{8.3}$$

is the KZ-type null state in the MHV sector [15] and $M^a_k(\Delta)$ is the antisymmetric part of the null state $M^{ab}_k(\Delta)$ defined in (4.3). We have also used the identity $f^{abx}f^{aby} = C_A\delta^{xy}$ in deriving the KZ-type null-state equation (8.2).

Another null-state equation involving the descendant $L_{-1}\mathcal{O}^{a,+}_{\Delta}$ can be obtained from (8.1) in a similar way by taking the subleading conformal soft limit $\Delta_2 \to 0$. It is given by

$$(\Delta-1)\xi^a(\Delta) - \sum_{k=1}^{n}(\Delta+k)M_k^a(\Delta+1) = 0. \quad (8.4)$$

Now, by multiplying Eq. (8.2) by $(\Delta-1)$ and then subtracting it from (8.4), we get the following (current algebra) null state:

$$\chi_n^{1,a}(\Delta) = \sum_{k=1}^{n}(k+1)M_k^a(\Delta) = 0. \quad (8.5)$$

One can continue this procedure and get other current algebra null states by taking conformal soft limits $\Delta_2 \to k, k \leq -1$. We can denote them by $\{\chi_n^1(\Delta), \chi_n^2(\Delta), \cdots\}$. However, it can be shown that after a finite number of iterations this procedure stops due to the truncation (5.8).

A. S invariance of the KZ-type null state

In this section, we show that the KZ-type null state (8.2) is S invariant.

First of all, the states $M_k^{ab}(\Delta)$ and, as a result, $M_k^a(\Delta) = f^{abc}M_k^{bc}(\Delta)$ are annihilated by $H_{0,1}^0$. Therefore, the state $K^a(\Delta)$ is a primary of $\overline{sl_2(R)}$ because the KZ-type null state (8.3) in the MHV sector is annihilated [15] by $H_{0,1}^0$.

Similarly, one can show after some algebra that the following relation holds:

$$\begin{aligned} R^{0,c}_{\frac{1}{2},\frac{1}{2}}K^a(\Delta) &= (\Delta-2)if^{cax}K^x(\Delta-1) \\ &\quad - (n+2)f^{cbx}f^{aby}M_{n+1}^{xy}(\Delta) + f^{cax}\chi_n^{1,x}(\Delta) \\ &\quad + f^{cby}f^{bax}\left[\sum_{k=1}^{n}(k+1)M_k^{yx}(\Delta) + 2E^{yx}(\Delta)\right] \\ &\quad + f^{cab}f^{byx}E^{yx}(\Delta), \end{aligned} \quad (8.6)$$

where

$$E^{yx}(\Delta) = (\Delta-2)R^{1,y}_{-1,0}\mathcal{O}^{x,+}_{\Delta-1} + R^{0,y}_{-\frac{1}{2},\frac{1}{2}}\mathcal{O}^{x,+}_{\Delta}. \quad (8.7)$$

Now we know that, in a theory in which the $\mathcal{O}(z_{12}^0\bar{z}_{12}^0)$ OPE truncates at $k=n$, i.e., (5.8) holds, both $M_{n+1}^{bc}(\Delta)$ and $\chi_n^{1,a}(\Delta)$ are null states. Thus, we can set them to 0 and get

$$\begin{aligned} R^{0,c}_{\frac{1}{2},\frac{1}{2}}K^a(\Delta) &= (\Delta-2)if^{cax}K^x(\Delta-1) \\ &\quad + f^{cby}f^{bax}\left[\sum_{k=1}^{n}(k+1)M_k^{yx}(\Delta) + 2E^{yx}(\Delta)\right] \\ &\quad + f^{cab}f^{byx}E^{yx}(\Delta). \end{aligned} \quad (8.8)$$

In Appendix C we show that the second and the third terms on the rhs of (8.8) are actually zero. Taking this into account, we get

$$R^{0,c}_{\frac{1}{2},\frac{1}{2}}K^a(\Delta) = (\Delta-2)if^{cax}K^x(\Delta-1). \quad (8.9)$$

Thus, we see that $R^{0,c}_{\frac{1}{2},\frac{1}{2}}$ maps the KZ-type null state $K^a(\Delta)$ to a linear combination of other null states in the theory. Hence, the null state equation

$$K^a(\Delta) = 0 \quad (8.10)$$

is S invariant.

IX. EXAMPLE: CELESTIAL OPE IN SELF-DUAL YANG-MILLS THEORY

We now consider the example of self-dual Yang-Mills (SDYM) theory, which is known to be S invariant. In particular, we write the $\mathcal{O}(1)$ terms explicitly and show that it can be written completely in terms of MHV OPE and MHV null states. The color-dressed SDYM amplitude is given by [61,62]

$$\mathcal{A}^{(1)}_{n\mathrm{SDYM}} = \sum_{\sigma\in S_{n-1}/R} c^{a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)}}A^{(1)}_{n\mathrm{SDYM}}(\sigma(1)\sigma(2)\cdots\sigma(n)), \quad (9.1)$$

where the sum is over noncyclic permutations, modulo reflection of the list σ. The cyclic n-gluon color factors are given by

$$c^{a_1a_2\cdots a_n} = f^{b_1a_1b_2}f^{b_2a_2b_3}\cdots f^{b_na_nb_1}, \quad (9.2)$$

and $A^{(1)}_{\mathrm{SDYM}}(\sigma(1)\sigma(2)\cdots\sigma(n))$ are the color ordered amplitudes, given by [61,62]

$$A^{(1)}_{n\mathrm{SDYM}}(123\cdots n) = M_n \sum_{1\leq i_1<i_2<i_3<i_4\leq n}\frac{\langle i_1i_2\rangle[i_2i_3]\langle i_3i_4\rangle[i_4i_1]}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}, \quad (9.3)$$

where M_n is a numerical normalization constant. Here we are working in the split signature $(-,+,-,+)$ and in this signature the null momentum of a massless particle is parametrized as

$$p_i = \epsilon_i\omega_i(1+z_i\bar{z}_i, z_i+\bar{z}_i, z_i-\bar{z}_i, 1-z_i\bar{z}_i), \quad (9.4)$$

with $\epsilon_i = +1(-1)$ for outgoing (incoming) particles. $(z_i, \bar{z}_i)$ are the coordinates on the celestial torus. In our notation, the angle and square brackets are given by

$$\langle ij\rangle = -2\epsilon_i\epsilon_j\sqrt{\omega_i\omega_j}z_{ij}, \qquad [ij] = 2\sqrt{\omega_i\omega_j}gz_{ij}. \quad (9.5)$$

Here we are interested in the five-point color dressed amplitudes only. For $n=5$, the sum in (9.3) will give 20 terms. Let us start with the following term [63]:

$$A^{(1)}_{5\,\mathrm{SDYM}}(12345) = \frac{s_{12}s_{23} + s_{45}s_{51} + s_{25}s_{45} + s_{25}s_{14} + \langle 24\rangle[14]\langle 15\rangle[52]}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}, \tag{9.6}$$

where $s_{ij} = (p_i + p_j)^2$. The Mellin transformation for five-point color-ordered amplitudes is given by

$$\mathcal{M}_{5\,\mathrm{SDYM}}(1^+_{\Delta_1}, 2^+_{\Delta_2}, \cdots 5^+_{\Delta_5}) = \left(\prod_{k=1}^{5}\int_0^\infty d\omega_k \omega_k^{\Delta_k-1}\right) A^{(1)}_{5\,\mathrm{SDYM}}(12345)\delta^{(4)}\left(\sum_{k=1}^{5} p_k\right). \tag{9.7}$$

We are interested in the OPE limit $4 \to 5$. For our purpose, the following parametrization of the five-point momentum-conserving delta function will be convenient [14]:

$$\delta^{(4)}\left(\sum_{i=1}^{5}\epsilon_i\omega_i q_i\right) = \frac{1}{4\omega_p}\delta(\omega_1 - \omega_1^*)\delta(\omega_2 - \omega_2^*)\delta(\omega_3 - \omega_3^*) \\ \times \delta\left(x - \bar{x} - tz_{45}\left(\frac{x}{z_{35}} - \frac{\bar{x}}{z_{25}}\right) - t\bar{z}_{45}\left(\frac{x}{\bar{z}_{25}} - \frac{\bar{x}}{\bar{z}_{35}}\right) + tz_{45}\bar{z}_{45}\left(\frac{x}{z_{35}\bar{z}_{25}} - \frac{\bar{x}}{z_{25}\bar{z}_{35}}\right)\right), \tag{9.8}$$

where we have used the parametrization

$$\omega_4 = t\omega_p, \qquad \omega_5 = (1-t)\omega_p, \tag{9.9}$$

and for $i = \{1, 2, 3\}$ we have

$$\omega_i^* = \epsilon_i\omega_p(\sigma_{i,1} + tz_{45}\sigma_{i,2} + t\bar{z}_{45}\sigma_{i,3} + tz_{45}\bar{z}_{45}\sigma_{i,4}). \tag{9.10}$$

The $\sigma_{i,1}, x, \bar{x}$ are given by

$$\sigma_{1,1} = -\frac{z_{25}\bar{z}_{35}}{z_{12}\bar{z}_{13}}, \tag{9.11}$$

$$\sigma_{2,1} = \frac{z_{15}\bar{z}_{35}}{z_{12}\bar{z}_{23}}, \tag{9.12}$$

$$\sigma_{3,1} = -\frac{z_{25}\bar{z}_{15}}{z_{23}\bar{z}_{13}}, \tag{9.13}$$

$$x = z_{12}z_{35}\bar{z}_{13}\bar{z}_{25}, \qquad \bar{x} = z_{13}z_{25}\bar{z}_{12}\bar{z}_{35}. \tag{9.14}$$

Since, we are only interested in the $\mathcal{O}(1)$ term, we do not need the other $\sigma_{i,j}$'s. However, one can find their expressions in [53].

Now we perform the OPE decomposition of the five-point amplitude (9.7). We apply the strategy of [53]. First, we write $A^{(1)}_{5\,\mathrm{SDYM}}(12345)$, given by (9.6), in terms of $\{\omega_i, z_i, \bar{z}_i\}$ and substitute it into (9.7). Next, using (9.8), we can easily perform the $\omega_i, \{i = 1, 2, 3\}$ integrals. After expanding around $z_{45} = \bar{z}_{45} = 0$, one can then perform the ω_p and t integrals. The leading term is already known in the literature. Here we concentrate on the $\mathcal{O}(1)$ terms. The $\mathcal{O}(1)$ terms are given by

$$\mathcal{M}_{5\,\mathrm{SDYM}}(1^+_{\Delta_1}, 2^+_{\Delta_2}, \ldots, 5^+_{\Delta_5})|_{\mathcal{O}(1)} \\ = \delta\left(\sum_{i=1}^{5}\Delta_i - 5\right)\sum_{k=0}^{2} B(\Delta_4 - 1 + k, \Delta_5 - 1) \\ \times \mathcal{F}_k(\{z_{i\neq 4}, \bar{z}_{i\neq 4}, \Delta_{i\neq 4,5}\}), \tag{9.15}$$

where the explicit forms of the functions $\mathcal{F}_k(\{z_{i\neq 4}, \bar{z}_{i\neq 4}, \Delta_{i\neq 4,5}\})$ are not required for our purpose. For the other 19 terms, one can check that the structure of the B function is the same. Hence, including all of those terms, we write our final color-dressed five-point celestial amplitude as

$$\mathcal{M}_{5\,\mathrm{SDYM}}(1^{a_1,+}_{\Delta_1}, 2^{a_2,+}_{\Delta_2}, \cdots 5^{a_5,+}_{\Delta_5})|_{\mathcal{O}(1)} \\ = \delta\left(\sum_{i=1}^{5}\Delta_i - 5\right)\sum_{k=0}^{2} B(\Delta_4 - 1 + k, \Delta_5 - 1) \\ \times \mathcal{G}_k^{(\{a_i\})}(\{z_{i\neq 4}, \bar{z}_{i\neq 4}, \Delta_{i\neq 4,5}\}). \tag{9.16}$$

Now, to factorize the above five-point SDYM amplitude into a four-point amplitude, we adopt the same strategy as in the case of the self-dual gravity in [53]. We do this by taking different conformally soft limits of the fourth gluon primary and replace the functions $\mathcal{G}_k^{(\{a_i\})}(\{z_{i\neq 4}, \bar{z}_{i\neq 4}, \Delta_{i\neq 4,5}\})$ by correlation functions of the S-algebra descendants of the fifth gluon[7] and the first, second, and third gluons. For example, we can make the fourth gluon leading conformally soft ($\Delta_4 \to 1$) and use the leading soft factorization theorem to replace the function $\mathcal{G}_0^{(\{a_i\})}(\{z_i, \bar{z}_i, \Delta_i\})$, which gives

[7]Note that we are taking the $4 \to 5$ OPE.

$$\mathcal{M}_{5\,\mathrm{SDYM}}(1^{a_1,+}_{\Delta_1},2^{a_2,+}_{\Delta_2},\cdots 5^{a_5,+}_{\Delta_5})|_{\mathcal{O}(1)} = B(\Delta_4-1,\Delta_5-1)R^{1,a_4}_{-1,0}\mathcal{M}_{4\,\mathrm{SDYM}}(1^{a_1,+}_{\Delta_1},2^{a_2,+}_{\Delta_2},3^{a_3,+}_{\Delta_3},5^{a_5,+}_{\Delta_4+\Delta_5-1}) + \delta\left(\sum_{i=1}^{5}\Delta_i-5\right)\sum_{k=1}^{2}B(\Delta_4-1+k,\Delta_5-1)\mathcal{G}^{(\{a_i\})}_k(\{z_i,\bar{z}_i,\Delta_i\}). \tag{9.17}$$

We can repeat this procedure to find other two functions $\mathcal{G}_1$ and $\mathcal{G}_2$ by taking subleading ($\Delta_4\to 0$) and sub-subleading ($\Delta_4\to -1$) conformally soft limits of the fourth gluon primary, respectively, in (9.17). Finally, we can write the five-point SDYM amplitude in the following factorized form:

$$\begin{aligned}\mathcal{M}_{5\,\mathrm{SDYM}}(1^{a_1,+}_{\Delta_1},2^{a_2,+}_{\Delta_2},\cdots 5^{a_5,+}_{\Delta_5})|_{\mathcal{O}(1)} =& \frac{1}{2}\frac{\Gamma(\Delta_4+2)}{\Gamma(\Delta_4)}B(\Delta_4-1,\Delta_5-1)R^{1,a_4}_{-1,0}\mathcal{M}_{4\,\mathrm{SDYM}}(1^{a_1,+}_{\Delta_1},2^{a_2,+}_{\Delta_2},3^{a_3,+}_{\Delta_3},5^{a_5,+}_{\Delta_4+\Delta_5-1})\\ &+\frac{\Gamma(\Delta_4+2)}{\Gamma(\Delta_4+1)}B(\Delta_4,\Delta_5-1)R^{0,a_4}_{-\frac{1}{2},\frac{1}{2}}\mathcal{M}_{4\,\mathrm{SDYM}}(1^{a_1,+}_{\Delta_1},2^{a_2,+}_{\Delta_2},3^{a_3,+}_{\Delta_3},5^{a_5,+}_{\Delta_4+\Delta_5})\\ &+\frac{\Gamma(\Delta_4+2)}{\Gamma(\Delta_4+2)}B(\Delta_4+1,\Delta_5-1)R^{-1,a_4}_{0,1}\mathcal{M}_{4\,\mathrm{SDYM}}(1^{a_1,+}_{\Delta_1},2^{a_2,+}_{\Delta_2},3^{a_3,+}_{\Delta_3},5^{a_5,+}_{\Delta_4+\Delta_5+1}).\end{aligned} \tag{9.18}$$

At the OPE level, it can be written as

$$\begin{aligned}\mathcal{O}^{a_4,+}_{\Delta_4}(z_4,\bar{z}_4)\mathcal{O}^{a_5,+}_{\Delta_5}(z_5,\bar{z}_5)|^{\mathrm{SDYM}}_{\mathcal{O}(1)} &= B(\Delta_4-1,\Delta_5-1)\left[\Delta_4 R^{1,a_4}_{-1,0}\mathcal{O}^{a_5,+}_{\Delta_4+\Delta_5-1}+\frac{\Delta_4-1}{\Delta_4+\Delta_5-2}R^{0,a_4}_{-\frac{1}{2},\frac{1}{2}}\mathcal{O}^{a_5,+}_{\Delta_4+\Delta_5}\right](z_5,\bar{z}_5)\\ &\quad + B(\Delta_4+1,\Delta_5-1)M^{a_4a_5}_1(\Delta_4+\Delta_5)(z_5,\bar{z}_5)\\ &= \mathcal{O}^{a_4,+}_{\Delta_4}(z_4,\bar{z}_4)\mathcal{O}^{a_5,+}_{\Delta_5}(z_5,\bar{z}_5)|^{\mathrm{MHV}}_{\mathcal{O}(1)} + B(\Delta_4+1,\Delta_5-1)M^{a_4a_5}_1(\Delta_4+\Delta_5)(z_5,\bar{z}_5),\end{aligned} \tag{9.19}$$

where M^{ab}_1 is the MHV null state defined in (4.2). Hence, we see that the OPE of two positive-helicity gluon primaries at $\mathcal{O}(1)$ in SDYM theory can be written as the MHV OPE at $\mathcal{O}(1)$ plus an MHV null state.

This precisely matches with our result (6.1) when it is truncated at $n=1$.

X. DISCUSSION

In celestial CFT, the sector with no negative-helicity soft gluon is governed by the infinite-dimensional S algebra. In this paper, we found the most general form of the S-invariant OPE of two positive-helicity gluons at $O(z^0_{12}\bar{z}^0_{12})$. It is given by

$$\begin{aligned}&\mathcal{O}^{a,+}_{\Delta_1}(z,\bar{z})\mathcal{O}^{b,+}_{\Delta_2}(0,0)|_{\mathcal{O}(1)}\\ &= \mathcal{O}^{a,+}_{\Delta_1}(z,\bar{z})\mathcal{O}^{b,+}_{\Delta_2}(0,0)|_{\mathrm{MHV\,OPE\,at\,}\mathcal{O}(1)}\\ &\quad+\sum_{k=1}^{n}B(\Delta_1+k,\Delta_2-1)M^{ab}_k(\Delta_1+\Delta_2).\end{aligned} \tag{10.1}$$

In this equation, $M^{ab}_k(\Delta)$ is an MHV null state, which transforms in a simple manner under the S algebra. We have also shown that for $n=1$, (10.1) gives the correct $O(z^0_{12}\bar{z}^0_{12})$ term in the gluon-gluon OPE in the self-dual Yang-Mills theory. This is an important consistency check for the OPE formula.

Although the OPE coefficients and the descendants $M^{ab}_k(\Delta)$ that can appear at $O(z^0_{12}\bar{z}^0_{12})$ are fixed by the S invariance, the integer n is not fixed by the symmetry. We saw that for $n=1$ we get the OPE of the self -dual Yang-Mills theory, but the underlying theories for $n=2$ and higher are not known. Presumably, they are exactly solvable theories of gluons, which generalize the self-dual Yang-Mills theory. An interesting question for future research is to determine these theories. We also found the KZ-type null states for these (unknown) theories. They may be of some help in the search for these theories.

A celestial dual for the MHV gluon scattering amplitudes was recently found in [64]. The theories that underlie the OPE (10.1) can be thought of as deformations of those in [64], which preserve the S invariance. Our results suggest that there is a *discrete* infinite number of such deformations. It will be interesting to see if this picture is correct.

Another point we would like to emphasize is that for every "theory" there are only a finite number of descendants that contribute to the subleading OPE. This is somewhat unexpected because the spectrum of operator dimensions in celestial CFT is not bounded from below.[8]

[8]See, for example, the recent work [65].

This might point to a reformulation of celestial CFT where operator dimensions are discrete and bounded from below. Proposals along this line have been made in [66,67]. It will be interesting to determine the relation between [66,67] and our observation in this paper.

Let us now discuss some interesting questions whose study we leave to the future.[9]

If there are S-invariant field theories on the celestial sphere that lack bulk Lorentz invariance, can we provide a physical interpretation for these theories? Additionally, can we find reasons, such as mathematical consistency, to rule out S-invariant field theories on the celestial sphere that lack bulk Lorentz invariance? In this paper, we assumed bulk Lorentz invariance from the beginning. However, space-time translational invariance was not assumed.

Could S-invariant non-Lorentz-invariant theories on the celestial sphere emerge from a spontaneous breakdown of Lorentz-invariant theories? Could the Goldstone modes associated with the breakdown of Lorentz invariance be analogous to the soft modes responsible for the breakdown of Bondi-Metzner-Sachs symmetry, as suggested in [68] in the context of the black hole information paradox?

Are there constraints on the Lagrangian formulation of these S-invariant theories? Alternatively, from the perspective of axiomatic CFT, is it conceivable that there is no Lagrangian formulation of a CFT and, instead, the focus should be on verifying whether either $w_{1+\infty}$ or S-invariant celestial CFTs adhere to axioms such as the Osterwalder-Schrader axioms? The null states found in this paper and [38] place tight constraints on the Lagrangian formulation of the celestial dual theories. One way to see this is that in celestial CFT the spectrum of operator dimensions is the *same* for every theory, at least in its current formulation. Therefore, different theories are *not* distinguished by their operator spectrum but by their null states. So any Lagrangian formulation has to produce all of the correct null states, and this may be useful in constraining the form of the Lagrangian. We leave these very interesting questions for future study.

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[9]We are thankful to the anonymous referee for raising these interesting questions.

APPENDIX A: S ALGEBRA PRIMARIES

In this appendix, we write down the conditions on the primary operators that follow from the OPE between two positive-helicity outgoing gluon primaries (2.1). They are obtained by taking different soft limits in (2.1) and comparing both the sides of the OPE:

$$R^{k,a}_{p-\frac{k+1}{2},-q-\frac{k-1}{2}}\mathcal{O}^{b,+}_{\Delta}(0,0)=0,\ p\geq 2$$

$$R^{k,a}_{\frac{1-k}{2},-q-\frac{k-1}{2}}\mathcal{O}^{b,+}_{\Delta}(0,0)=-if^{abc}\frac{(-1)^{k+q+1}}{\Gamma(-k-q+2)}\times\frac{\Gamma(\Delta-1)}{\Gamma(\Delta+q+k-2)}\times\frac{\bar{\partial}^q}{q!}\mathcal{O}^{c,+}_{\Delta+k-1}(0,0), \tag{A1}$$

where $0\leq q\leq 1-k, k=1,0,-1,\ldots$. These conditions are used to write down the transformation properties of the MHV null states. For more details on how to obtain these conditions, one can check Appendix F of [53]. The analyses there were done for $w_{1+\infty}$ primaries, but the methodology is the same for S algebra.

APPENDIX B: TRANSFORMATION PROPERTIES OF THE Ψ^{bc}_j NULL STATES UNDER THE LEADING SOFT GLUON OPERATOR $R^{1,a}_{0,0}$ AND THE SUBLEADING SOFT GLUON OPERATOR $R^{0,a}_{\frac{1}{2},\frac{1}{2}}$

Using (2.5), (4.1), and (A1), one can show that

$$R^{1,a}_{0,0}\Psi^{bc}_j(\Delta)=-if^{abx}\Psi^{xc}_j(\Delta)-if^{acx}\Psi^{bx}_j(\Delta)$$

$$R^{0,a}_{\frac{1}{2},\frac{1}{2}}\Psi^{bc}_j(\Delta)=-(j+2)if^{abx}\Psi^{xc}_{j+1}(\Delta-1)+(\Delta+j-2)if^{acx}\Psi^{bx}_j(\Delta-1)+2\frac{(-1)^j}{\Gamma(j+1)}\frac{\Gamma(\Delta+j-1)}{\Gamma(\Delta-1)}if^{abx}\Psi^{xc}_1(\Delta-1). \tag{B1}$$

These equations are used in Secs. V B and V C.

APPENDIX C: PROOF THAT THE KZ-TYPE NULL STATES ARE CLOSED UNDER THE ACTION OF $R^{0,a}_{\frac{1}{2},\frac{1}{2}}$

We write the second and third terms in (8.8) as

$$\Sigma^{ca}(\Delta)=f^{cby}f^{bax}\left[\sum_{k=1}^{n}(k+1)M^{yx}_k(\Delta)+2E^{yx}(\Delta)\right]+f^{cab}f^{byx}E^{yx}(\Delta). \tag{C1}$$

The above equation can be decomposed into symmetric and antisymmetric parts in the following way:

$$\Sigma^{ca}(\Delta) = \Sigma^{ca}_A(\Delta) + \Sigma^{ca}_S(\Delta), \tag{C2}$$

where

$$\Sigma^{ca}_A(\Delta) = \frac{1}{2}[\Sigma^{ca}(\Delta) - \Sigma^{ac}(\Delta)],$$
$$\Sigma^{ca}_S(\Delta) = \frac{1}{2}[\Sigma^{ca}(\Delta) + \Sigma^{ac}(\Delta)]. \tag{C3}$$

Now, using the Jacobi identity

$$f^{acb} f^{bxy} + f^{xab} f^{bcy} + f^{xcb} f^{aby} = 0, \tag{C4}$$

one can show that

$$\Sigma^{ca}_A(\Delta) = -\frac{1}{2} f^{cab} \chi^{1,b}_n(\Delta) = 0. \tag{C5}$$

We now simplify the symmetric part (C3) and get

$$\Sigma^{ca}_S(\Delta) = \frac{1}{2} f^{cby} f^{bax}\left[\sum_{k=1}^{n}(k+1)(M^{xy}_k(\Delta) + M^{yx}_k(\Delta)) + 2(E^{xy}(\Delta) + E^{yx}(\Delta))\right]. \tag{C6}$$

The leading and subleading soft limits of (8.1) and some straightforward algebra then give

$$\sum_{k=1}^{n}(k+1)(M^{xy}_k(\Delta) + M^{yx}_k(\Delta)) + 2(E^{xy}(\Delta) + E^{yx}(\Delta)) = 0. \tag{C7}$$

Hence, we conclude that

$$\Sigma^{ca}(\Delta) = 0. \tag{C8}$$

[1] A. Strominger, Lectures on the infrared structure of gravity and gauge theory, arXiv:1703.05448.
[2] S. Pasterski, Lectures on celestial amplitudes, Eur. Phys. J. C **81**, 1062 (2021).
[3] L. Donnay, Celestial holography: An asymptotic symmetry perspective, Phys. Rep. **1073**, 1 (2024).
[4] A. Strominger, Asymptotic symmetries of Yang-Mills theory, J. High Energy Phys. 07 (2014) 151.
[5] A. Strominger, On BMS invariance of gravitational scattering, J. High Energy Phys. 07 (2014) 152.
[6] T. He, V. Lysov, P. Mitra, and A. Strominger, BMS supertranslations and Weinberg's soft graviton theorem, J. High Energy Phys. 05 (2015) 151.
[7] A. Strominger and A. Zhiboedov, Gravitational memory, BMS supertranslations and soft theorems, J. High Energy Phys. 01 (2016) 086.
[8] M. Campiglia and A. Laddha, Sub-subleading soft gravitons: New symmetries of quantum gravity?, Phys. Lett. B **764**, 218 (2017).
[9] D. Kapec, P. Mitra, A. M. Raclariu, and A. Strominger, 2D stress tensor for 4D gravity, Phys. Rev. Lett. **119**, 121601 (2017).
[10] D. Kapec, V. Lysov, S. Pasterski, and A. Strominger, Semiclassical Virasoro symmetry of the quantum gravity $\mathcal{S}$-matrix, J. High Energy Phys. 08 (2014) 058.
[11] T. He, D. Kapec, A. M. Raclariu, and A. Strominger, Loop-corrected virasoro symmetry of 4D quantum gravity, J. High Energy Phys. 08 (2017) 050.
[12] S. Banerjee and S. Pasterski, Revisiting the shadow stress tensor in celestial CFT, J. High Energy Phys. 04 (2023) 118.
[13] S. Banerjee, S. Ghosh, and R. Gonzo, BMS symmetry of celestial OPE, J. High Energy Phys. 04 (2020) 130.
[14] S. Banerjee, S. Ghosh, and P. Paul, MHV graviton scattering amplitudes and current algebra on the celestial sphere, J. High Energy Phys. 02 (**2021**) 176.
[15] S. Banerjee and S. Ghosh, MHV gluon scattering amplitudes from celestial current algebras, J. High Energy Phys. 10 (2021) 111.
[16] N. Gupta, P. Paul, and N. V. Suryanarayana, $\widehat{sl_2}$ symmetry of $\mathbb{R}^{1,3}$ gravity, Phys. Rev. D **108**, 086029 (2023).
[17] A. Guevara, E. Himwich, M. Pate, and A. Strominger, Holographic symmetry algebras for gauge theory and gravity, J. High Energy Phys. 11 (2021) 152.
[18] A. Strominger, $w_{1+\infty}$ algebra and the celestial sphere: Infinite towers of soft graviton, photon, and gluon symmetries, Phys. Rev. Lett. **127**, 221601 (2021).
[19] E. Himwich, M. Pate, and K. Singh, Celestial operator product expansions and $\mathrm{w}_{1+\infty}$ symmetry for all spins, J. High Energy Phys. 01 (**2022**) 080.
[20] W. Melton, S. A. Narayanan, and A. Strominger, Deforming soft algebras for gauge theory, J. High Energy Phys. 03 (2023) 233.
[21] T. Adamo, L. Mason, and A. Sharma, Celestial $w_{1+\infty}$ symmetries from twistor space, SIGMA **18**, 016 (2022).
[22] K. Costello and N. M. Paquette, Celestial holography meets twisted holography: 4d amplitudes from chiral correlators, J. High Energy Phys. 10 (**2022**) 193.
[23] K. Costello and N. M. Paquette, Associativity of one-loop corrections to the celestial operator product expansion, Phys. Rev. Lett. **129**, 231604 (2022).

[24] S. Banerjee, S. Ghosh, and S. S. Samal, Subsubleading soft graviton symmetry and MHV graviton scattering amplitudes, J. High Energy Phys. 08 (2021) 067.
[25] L. Donnay and R. Ruzziconi, BMS flux algebra in celestial holography, J. High Energy Phys. 11 (2021) 040.
[26] L. Donnay, S. Pasterski, and A. Puhm, Asymptotic symmetries and celestial CFT, J. High Energy Phys. 09 (2020) 176.
[27] S. Stieberger and T. R. Taylor, Symmetries of celestial amplitudes, Phys. Lett. B **793**, 141 (2019).
[28] K. J. Costello, Bootstrapping two-loop QCD amplitudes, arXiv:2302.00770.
[29] N. Garner and N. M. Paquette, Twistorial monopoles & chiral algebras, J. High Energy Phys. 08 (2023) 088.
[30] S. Stieberger, T. R. Taylor, and B. Zhu, Celestial Liouville theory for Yang-Mills amplitudes, Phys. Lett. B **836**, 137588 (2023).
[31] S. Stieberger, T. R. Taylor, and B. Zhu, Yang-Mills as a Liouville theory, Phys. Lett. B **846**, 138229 (2023).
[32] L. Magnea, Non-Abelian infrared divergences on the celestial sphere, J. High Energy Phys. 05 (2021) 282.
[33] H. A. González and F. Rojas, The structure of IR divergences in celestial gluon amplitudes, J. High Energy Phys. 06 (2021) 171.
[34] S. Pasterski, S. H. Shao, and A. Strominger, Flat space amplitudes and conformal symmetry of the celestial sphere, Phys. Rev. D **96**, 065026 (2017).
[35] S. Pasterski and S. H. Shao, Conformal basis for flat space amplitudes, Phys. Rev. D **96**, 065022 (2017).
[36] S. Banerjee, Null infinity and unitary representation of the Poincare group, J. High Energy Phys. 01 (2019) 205.
[37] S. Banerjee, S. Ghosh, P. Pandey, and A. P. Saha, Modified celestial amplitude in Einstein gravity, J. High Energy Phys. 03 (2020) 125.
[38] S. Banerjee, H. Kulkarni, and P. Paul, An infinite family of $w_{1+\infty}$ invariant theories on the celestial sphere, J. High Energy Phys. 05 (2023) 063.
[39] W. Fan, A. Fotopoulos, and T. R. Taylor, Soft limits of Yang-Mills amplitudes and conformal correlators, J. High Energy Phys. 05 (2019) 121.
[40] M. Pate, A. M. Raclariu, A. Strominger, and E. Y. Yuan, Celestial operator products of gluons and gravitons, Rev. Math. Phys. **33**, 2140003 (2021).
[41] A. Ball, M. Spradlin, A. Yelleshpur Srikant, and A. Volovich, Supersymmetry and the celestial Jacobi identity, J. High Energy Phys. 04 (2024) 099.
[42] S. Ebert, A. Sharma, and D. Wang, Descendants in celestial CFT and emergent multi-collinear factorization, J. High Energy Phys. 03 (2021) 030.
[43] T. Adamo, W. Bu, E. Casali, and A. Sharma, All-order celestial OPE in the MHV sector, J. High Energy Phys. 03 (2023) 252.
[44] L. Ren, A. Schreiber, A. Sharma, and D. Wang, All-order celestial OPE from on-shell recursion, J. High Energy Phys. 10 (2023) 080.
[45] Y. Hu, L. Ren, A. Y. Srikant, and A. Volovich, Celestial dual superconformal symmetry, MHV amplitudes and differential equations, J. High Energy Phys. 12 (2021) 171.
[46] W. Fan, A. Fotopoulos, S. Stieberger, T. R. Taylor, and B. Zhu, Elements of celestial conformal field theory, J. High Energy Phys. 08 (2022) 213.
[47] W. Fan, A. Fotopoulos, S. Stieberger, T. R. Taylor, and B. Zhu, Celestial Yang-Mills amplitudes and $D = 4$ conformal blocks, J. High Energy Phys. 09 (2022) 182.
[48] S. Banerjee, R. Mandal, A. Manu, and P. Paul, MHV gluon scattering in the massive scalar background and celestial OPE, J. High Energy Phys. 10 (2023) 007.
[49] R. Bhardwaj, L. Lippstreu, L. Ren, M. Spradlin, A. Yelleshpur Srikant, and A. Volovich, Loop-level gluon OPEs in celestial holography, J. High Energy Phys. 11 (2022) 171.
[50] H. Krishna, Celestial gluon and graviton OPE at loop level, J. High Energy Phys. 03 (2024) 176.
[51] Y. Hu and S. Pasterski, Celestial recursion, J. High Energy Phys. 01 (2023) 151.
[52] A. Atanasov, W. Melton, A. M. Raclariu, and A. Strominger, Conformal block expansion in celestial CFT, Phys. Rev. D **104**, 126033 (2021).
[53] S. Banerjee, H. Kulkarni, and P. Paul, Celestial OPE in self dual gravity, Phys. Rev. D **109**, 086017 (2024).
[54] E. Casali, W. Melton, and A. Strominger, Celestial amplitudes as AdS-Witten diagrams, J. High Energy Phys. 11 (2022) 140.
[55] L. Donnay, A. Puhm, and A. Strominger, Conformally soft photons and gravitons, J. High Energy Phys. 01 (2019) 184.
[56] M. Pate, A. M. Raclariu, and A. Strominger, Conformally soft theorem in gauge theory, Phys. Rev. D **100**, 085017 (2019).
[57] D. Nandan, A. Schreiber, A. Volovich, and M. Zlotnikov, Celestial amplitudes: Conformal partial waves and soft limits, J. High Energy Phys. 10 (2019) 018.
[58] T. Adamo, L. Mason, and A. Sharma, Celestial amplitudes and conformal soft theorems, Classical Quantum Gravity **36**, 205018 (2019).
[59] A. Puhm, Conformally soft theorem in gravity, J. High Energy Phys. 09 (2020) 130.
[60] A. Guevara, Notes on conformal soft theorems and recursion relations in gravity, arXiv:1906.07810.
[61] Z. Bern, G. Chalmers, L. J. Dixon, and D. A. Kosower, One loop N gluon amplitudes with maximal helicity violation via collinear limits, Phys. Rev. Lett. **72**, 2134 (1994).
[62] R. Monteiro, R. Stark-Muchão, and S. Wikeley, Anomaly and double copy in quantum self-dual Yang-Mills and gravity, J. High Energy Phys. 09 (2023) 030.
[63] P. Chattopadhyay and Y. X. Tao, Celestial self-dual Yang-Mills theory: A new formula and the OPE limit, J. High Energy Phys. 03 (2024) 100.
[64] W. Melton, A. Sharma, A. Strominger, and T. Wang, A celestial dual for MHV amplitudes, arXiv:2403.18896.
[65] L. P. de Gioia and A. M. Raclariu, Celestial sector in CFT: Conformally soft symmetries, arXiv:2303.10037.
[66] J. Cotler, N. Miller, and A. Strominger, An integer basis for celestial amplitudes, J. High Energy Phys. 08 (2023) 192.
[67] L. Freidel, D. Pranzetti, and A. M. Raclariu, A discrete basis for celestial holography, J. High Energy Phys. 02 (2024) 176.
[68] S. W. Hawking, M. J. Perry, and A. Strominger, Soft hair on black holes, Phys. Rev. Lett. **116**, 231301 (2016).