Loop special relativity: Kaluza-Klein area metric as a line element for stringy events

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Let a physical event constitute a simple loop in spacetime. This in turn calls for a generalized loop line element (= distance^2 between two neighboring loops) capable of restoring, at the shrinking loop limit, the special relativistic line element (= distance^2 between the two neighboring centers-of-mass, respectively). Sticking at first stage to a flat Euclidean/Minkowski background, one is led to such a preliminary loop line element, where the role of coordinates is played by the oriented cross sections projected by the loop event. Such cross sections are generically center-of-mass independent, unless (owing to a topological term) the loop events are intrinsically wrapped around a Kaluza-Klein–like compact fifth dimension. Serendipitously, it is the Kaluza-Klein ingredient which, on top of its traditional assignments, is shown to govern the extension of the Pythagoras theorem to loop space. Associated with $M_4 \otimes S_1$ is then a ten-dimensional loop spacetime metric, whose four-dimensional center-of-mass core term is supplemented by a six-dimensional Maxwell-style fine structure. The imperative inclusion of a positive (say Nambu-Goto) string tension within the framework of loop special relativity is fingerprinted by a low periodicity breathing mode. Nash global isometric embedding is conjectured to play a major role in the construction of loop general relativity.

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I. RATIONALE, SETTING, AND PLAN

A mathematical event is by definition a point in spacetime. It marks, for example, the location x^μ of a classical pointlike particle at some given instant. The square of the distance between two such infinitesimally separated pointlike events x^μ and $x^\mu + dx^\mu$, namely

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{1}$$

constitutes the special/general relativistic line element, where $g_{\mu\nu}(x)$ is defined as the metric tensor of the underlying flat/curved spacetime manifold. The line element Eq. (1) has long been recognized as the most fundamental geometrical tool in the service of theoretical physics.

There is no compelling reason, however, why a classical physical event must be inherently pointlike, stripped from any nontrivial (say stringy) microstructure. With this idea in mind, let $x^\mu(\sigma)$ define the so-called simple, that is, devoid of self-intersections or crossings, loop event (a refined definition of simplicity will be given later). As the σ-parameter varies from 0 to 2π, it traces the path of a classical closed string at some given instant. The closed structure of the loop event, formulated by

$$x^\mu(\sigma + 2\pi) = x^\mu(\sigma) \tag{2}$$

gets manifested by means of the Fourier series expansion

$$x^\mu(\sigma) = x^\mu_{cm} + \ell \xi^\mu(\sigma) = x^\mu_{cm} + \ell \sum_{n\neq 0} \xi^\mu_n \, e^{in\sigma} \tag{3}$$

with $\xi^\mu_{-n} = \xi^{\mu\star}_n$. The coefficient ℓ sets the loop length scale, leaving the various ξ^μ_n dimensionless (see Fig. 1).

The immediate question now is the following: Can one consistently construct, using a covariant geometric formalism, a tenable loop line element δS to measure the generalized distance between two such neighboring loop configurations $x^\mu(\sigma)$ and $x^\mu(\sigma) + \delta x^\mu(\sigma)$? The theoretical obstacle is threefold: conceptual, technical, and furthermore dynamical.

(i) On the conceptual level: While obviously dealing with nonlocal (stringy) configurations, the loop line element must nonetheless be local in loop spacetime. To make geometrical sense out of such a requirement, let each individual loop configuration be mapped into a certain point in loop spacetime. And the more so, two neighboring (infinitesimally deformed) loops residing in spacetime must be mapped into two infinitesimally separated points in loop spacetime. This evidently calls for a tenable loop spacetime metric. The precise identifications of

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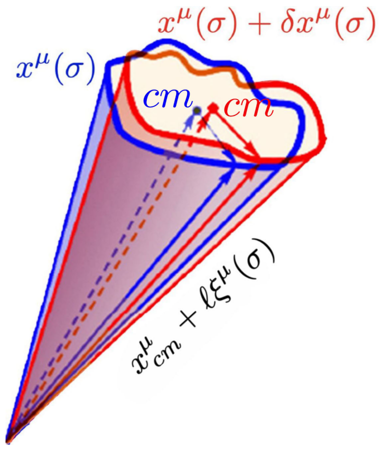

FIG. 1. Two neighboring loops in spacetime are mapped into two infinitely closed points in loop spacetime. A tenable loop line element δS must then reproduce, at the shrinking loop limit $\ell \to 0$, the special/general relativistic line element measuring the distance between the two infinitely separated centers-of-mass, respectively.

the entities, which serve as loop spacetime coordinates, the exact nature of the loop-to-point mapping, and eventually the loop spacetime metric, are to be presented and discussed (see Chapter 2).

(ii) On the technical level: There is a crucial condition which, on self consistency grounds, must be met at the shrinking loop size limit. As $\ell \to 0$, the loop line element δS must reproduce the special/general relativistic line element Eq. (1), which will then measure the distance between the two associated infinitesimally closed centers-of-mass, respectively, that is,

$$\lim_{\ell\to 0}\delta S^2 \to g_{\mu\nu}dx^\mu_{cm}dx^\nu_{cm}. \qquad (4)$$

Fulfilling this limit is a nontrivial technical task. For example, the class of so-called area metrics [1], with which our preliminary loop line element shares some common ingredients, fails to deliver in this respect. The missing ingredient called to the rescue is well known, albeit in a totally different area of physics. We refer to a compact fifth dimension *à la* Kaluza-Klein [2], which plays a novel role in our discussion (see Chapter 3).

(iii) On the dynamical level: The rationale for replacing pointlike events by loop events would not make any sense if the length scale ℓ can grow arbitrarily large. A dynamical physical mechanism, loop shape sensitive, which would account for the natural shrinkage of loop events, seems to be in order. Such a mathematical service can be provided by incorporating positive loop event tension *à la* Nambu-Goto string theory [3]. Once loop dynamics is introduced, one may expect the length scale ℓ to eventually acquire the Planck scale. There is also room, and eventually a necessity, as mentioned earlier, for a nontrivial spacetime topology to enter the game. In this case, relevant for our discussion (see Chapter 3), loop events get wrapped around a Kaluza-Klein–like cylinder.

A local realization of the loop line element idea is expected to pave the way for a corresponding loop special relativity (LSR) theory. With this in mind, we first consider the case of a flat spacetime which admits a Cartesian or pseudo-Cartesian metric η_{ij}, where x^i itself, rather than just the differential dx^i, transforms as a vector. Unfortunately, one immediately notices that the desired special relativity limit Eq. (4) is generically not reachable. Technically, it has to do with the geometrical fact that the loop area is generically center-of-mass independent. The remedy we offer requires a nontrivial topological touch and counterintuitively invokes the introduction of an extra dimension. To be specific, the loop event must be wrapped around a spatial compact Kaluza-Klein–like cylinder in order to activate the explicit entrance of the otherwise hidden center-of-mass coordinate into the loop metric. Put differently, the only loops we are able to handle via our approach are the ones wrapped around the fifth dimension. The physical role, if any, played by unwrapped loop events is not discussed in this paper. While our approach does not seem to have a direct connection with the standard assignments [2] of the Kaluza-Klein idea, it resembles some familiar features. For example, starting from an underlying $M_4 \otimes S_1$ spacetime, the emerging loop line element appears to be ten-dimensional, spanned by four essential center-of-mass coordinates, accompanied by a six-dimensional Maxwell style microstructure.

In Chapter 4 we pose the question whether string dynamics is imperative. In other words, is it necessary to accompany kinematical LSR by (say) dynamical Nambu-Goto? We give a few examples to support our positive answer in a Euclidean background. It turns out, however (see Chapter 5), that the inclusion of Nambu-Goto action within the framework of LSR has a unique fingerprint in the Lorentzian background, namely a low periodicity breathing mode.

The LSR to loop general relativity (LGR) generalization is still at large. One idea in this direction would be to invoke the Nash embedding formalism [4], later adopted by Regge-Teitelboim [5] in its local isometric version within the framework of geodesic brane gravity. For example, given the constraint $x^2+y^2+z^2=1$, curved $ds_2^2 = d\theta^2 + \sin^2\theta d\phi^2$ can be trivially embedded within flat $ds_3^2 = dx^2+dy^2+dz^2$, so that the distance between two loops residing on the S_2 sphere gets translated (subject to the constraint) into the distance between the two loops in the flat E_3 host. A simple example is provided toward the end of Chapter 3. While every arbitrary curved space metric is

Nash embeddable, the embedding procedure itself has several drawbacks: The troubles are that (i) the minimal embeddings are in general local, not global, (ii) the embedding is not necessarily unique, and (iii) the number of embedding dimensions is very much case dependent. Another idea toward constructing LGR calls for geometric gauge invariance of the second type [6]. Regretfully, this line of research lies beyond the scope of the present paper. The same holds for Rosen's bimetric approach [7].

LSR does not seem to show, at least at this stage, any compelling connection with loop quantum gravity (LQG) [8]. Still, encouraged by the four-dimensional center-of-mass resurrection in the loop area formalism, hereby considered as the fingerprint of a compact fifth dimension (see Chapter 3), establishing such a bridge is certainly welcome.

II. PRELIMINARY LOOP LINE ELEMENT

Let our starting point be a loop drawn in a flat plane characterized by the Euclidean metric

$$ds^2 = dx_1^2 + dx_2^2. \tag{5}$$

The area enclosed by a loop is given by

$$A = \frac{1}{2}\oint (x_1 dx_2 - x_2 dx_1) = \frac{1}{2}\int_0^{2\pi} r^2 d\sigma. \tag{6}$$

Being a global quantity, the enclosed area A is not sensitive to the fine local structure of the loop configuration. The mapping from the two-dimensional $\{x_1, x_2\}$ plane onto the one-dimensional A-axis is thus not one-to-one. And most importantly, to be regarded a momentary drawback for our purposes, it has nothing to do with the location of the center-of-mass of the loop.

Equation (6) can be easily generalized for the case of a loop residing within a larger flat space (or spacetime) equipped with a Cartesian (or pseudo-Cartesian) coordinate system. In this case, the projected areas A^{ij} are given by

$$A^{ij} = \frac{1}{2}\oint (x^i dx^j - x^j dx^i) = \frac{1}{2}\int_0^{2\pi} (x^i x'^j - x^j x'^i) d\sigma, \tag{7}$$

where $f' \equiv \frac{\partial f}{\partial \sigma}$. For an n-dimensional spacetime, there are now $\frac{1}{2}n(n-1)$ such projected areas, one per each pair of spacetime indices. It should be emphasized that it is only for the case of a flat rectangular space (or spacetime) that (i) x^i transforms as a vector itself, to be contrasted with the differential dx^i which always does, and (ii) owing to the global nature of the associated Lorentz transformations, the integration over a tensor is mathematically permissible. In other words, A^{ij} as given by Eq. (7) constitutes a rank-2 antisymmetric tensor in flat spacetime.

Now, for any given σ (keeping σ untouched), consider a loop variation

$$x^i(\sigma) \to x^i(\sigma) + \delta x^i(\sigma). \tag{8}$$

Following Euler-Lagrange, and subject to the periodicity condition Eq. (2), we find

$$\delta A^{ij} = \oint (\delta x^i dx^j - \delta x^j dx^i). \tag{9}$$

The increment $\delta x^i(\sigma)$ can be controlled by some parameter τ. In this case, we have $\delta x^i = \dot{x}^i d\tau$, where $\dot{f} \equiv \frac{\partial f}{\partial \tau}$. In turn, the projected areas A^{ij} get shifted by

$$\delta A^{ij} = d\tau \int_0^{2\pi} (\dot{x}^i x'^j - \dot{x}^j x'^i) d\sigma. \tag{10}$$

Note that, owing to the built-in $i \leftrightarrow j$ antisymmetry, the same result would have been obtained had we started from the more general expression $\delta x^i = \dot{x}^i d\tau + x'^i d\sigma$.

While the first derivative $\frac{dA^{ij}}{dt}$, involving the troublesome contour integration, behaves as a tensor solely in flat spacetime, it is the second derivative

$$\frac{d^2 A^{ij}}{d\tau d\sigma} = \dot{x}^i x'^j - \dot{x}^j x'^i \tag{11}$$

that appears to constitute a legitimate tensor even in curved spacetime. This may be the point to start from when attempting to eventually generalize LSR into LGR.

The antisymmetry of the oriented cross sections is a fundamental feature and does not depend on the structure of the spacetime metric. The more so,

$$\epsilon^{\alpha\beta} x^i_{,\alpha} x^j_{,\beta} \quad \{\alpha, \beta\} = \tau, \sigma \tag{12}$$

serves as a set of world sheet scalar densities associated with the world sheet induced metric $\gamma_{\alpha\beta} = \eta_{ij} x^i_{,\alpha} x^j_{,\beta}$. The integrant within Eq. (10) is thus reparametrization invariant. We note in passing that performing a proper reparametrization transformation

$$\begin{cases} \tau \to \tilde{\tau}(\tau, \sigma) = T(\tau) \\ \sigma \to \tilde{\sigma}(\tau, \sigma) = \sigma + \Sigma(\tau) \end{cases}, \tag{13}$$

prior to the integration, a transformation which fully respects the $\Delta\tilde{\sigma} = \Delta\sigma = 2\pi$, loop event periodicity, is equivalent to *a posteriori* gauge fixing $\tau \to T(\tau)$.

Once curvilinear coordinates are being used, while the integration over a tensor is apparently forbidden, there is a simple way out. The trick is to invoke the vierbein formalism, where the curvilinear flat spacetime metric can be written in the form

$$g_{\mu\nu} = \eta_{ij} V^i_{,\mu}(x) V^j_{,\nu}(x), \tag{14}$$

notably involving total derivative vierbeins. Equation (10) can then be easily generalized into

$$\delta A^{ij} = d\tau \int_0^{2\pi} \epsilon^{\alpha\beta} V^i_{,\mu} V^j_{,\nu} x^\mu_{,\alpha} x^\nu_{,\beta} d\sigma = d\tau \int_0^{2\pi} \epsilon^{\alpha\beta} V^i_{,\alpha} V^j_{,\beta} d\sigma, \tag{15}$$

and one is back to the original case, only with the rectangular coordinates $V^i(x)$ replacing the curvilinear coordinates x^μ.

The situation is totally different, however, for a curved spacetime characterized by a nonvanishing Riemann tensor). Having in mind Eq. (15), one may prematurely expect that associated with a curved spacetime metric

$$g_{\mu\nu}(x) = \eta_{ij} V^i_\mu(x) V^j_\nu(x), \tag{16}$$

where the vierbeins have now matured into full gauge fields, are projected loop area increments of the form

$$\delta A^{ij} = d\tau \int_0^{2\pi} \epsilon^{\alpha\beta} V^i_\mu V^j_\nu x^\mu_{,\alpha} x^\nu_{,\beta} d\sigma. \tag{17}$$

Unfortunately, with the exception of the two-dimensional case, this formula does not make sense mathematically. The problem is that it is impossible to erect any single coordinate system that is locally inertial everywhere, unless the spacetime continuum is flat. In other words, integration over a Lorentz tensor does not make then any sense. It is only in two dimensions, owing to the corresponding rank-2 Levi-Civita symbol ϵ_{ij}, that $\delta A^{12} = \frac{1}{2}\epsilon_{ij}\delta A^{ij}$, as defined by Eq. (17), happens to be a Lorentz scalar. From this point forward, on both pedagogical and simplicity grounds, while still lacking a proper formula for $\delta A^{\mu\nu}$ in a curved background, and without invoking either gauge invariance of the second type and/or the Regge-Teitelboim–like embedding technique as potential remedies, we return to the comfortable choice of a Cartesian (or pseudo-Cartesian) spacetime.

Associated with each loop residing in an n-dimensional spacetime there is a corresponding point in the so-called loop spacetime, where the $\frac{1}{2}n(n-1)$ independent projected areas A^{ij} play the role of coordinates (see Fig. 2). Two such loops are then mapped into two points in loop spacetime, and in principle, provided the corresponding loop metric is specified, a geodesic trajectory can be drawn. If the two points are infinitesimally closed, we pay tribute to the antisymmetric structure of A^{ij}, imitate the general relativistic structure of the Maxwell kinetic term, and accordingly define the scalar loop line element

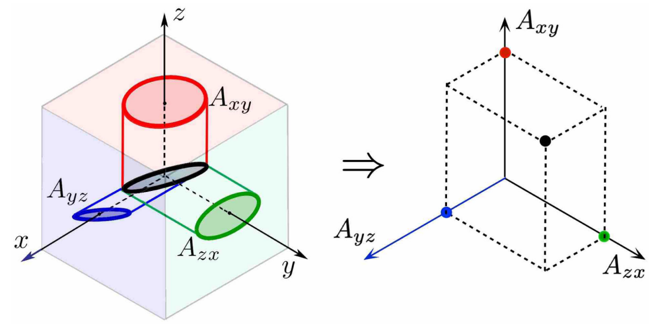

FIG. 2. Associated with every simple loop event (black) residing in an n-dimensional flat space or spacetime, there are $\frac{1}{2}n(n-1)$ simple projected areas (red, blue, green, and so on), thereby mapping a loop event into a point $\{A_{xy}, A_{yz}, A_{yz}, \ldots\}$ in loop space.

$$\boxed{\delta S^2 = \frac{\eta_{ik}\eta_{jl}}{16\pi^2\ell^2}\delta A^{ij}\delta A^{kl},} \tag{18}$$

properly normalized for future assignments. Note that for the special case $n = 4$, there exists the option of supplementing Eq. (18), or even replacing it, by the dual term proportional to $\epsilon_{ijkl}\delta A^{ij}\delta A^{kl}$. Equation (18) falls into the category of area metrices [1]. Note in passing that areas also play an important role in Regge [9] calculus, and oriented areas enclosed by string constitute a vital part of the Clifford space metric [10].

Unfortunately, the special relativistic limit Eq. (4) is still unattainable at this stage. Substituting the Fourier expansion Eq. (3) into Eq. (10), we obtain

$$\delta A^{ij} = \ell^2 \int_0^{2\pi} (\dot{\xi}^i \xi'^j - \dot{\xi}^j \xi'^i) d\tau d\sigma, \tag{19}$$

and immediately we notice that the center-of-mass $x^i_{cm}(\tau)$ and its τ-derivative $\dot{x}^i_{cm}(\tau)$ do not enter the game. This poses a major drawback, as all pointlike (= shrinking loop) events become practically indistinguishable, and pile at the origin. Until the missing ingredient is found, and the center-of-mass comes out of hiding, Eq. (18) has to be regarded incomplete.

III. KALUZA-KLEIN TO THE RESCUE

A. Center-of-mass resurrection

Counterintuitively, the remedy comes from topology. The idea is to supplement spacetime by an extra Kaluza-Klein–like closed dimension. This has nothing to do though with the original Kaluza-Klein idea. For some reason soon to be clarified, this extra dimension must be spacelike. On historical grounds we generically refer to such an extra dimension as x^5, and we consider loop events wrapped around this cylindrical fifth dimension. The previously introduced spacetime coordinates x^μ are now accompanied by a new x^5, subject to the periodicity condition

$$\Delta x^5 = 2\pi R. \tag{20}$$

The game changer element is then the presence of the topological term $R\sigma$ in the modified Fourier expansion. To be contrasted with Eq. (3), we now have

$$x^5(\sigma) = x^5_{cm} + R\sigma + \ell\xi^5(\sigma), \tag{21}$$

where as usual $\xi^5(\sigma) = \sum_{n\neq 0} \xi^5_n e^{in\sigma}$.

Owing to the slight yet significant modification in the corresponding partial derivative expansions, that is,

$$\begin{cases} \dot{x}^5 = \dot{x}^5_{cm} + \ell\dot{\xi}^5 \\ x'^5 = R + \ell\xi'^5 \end{cases} \tag{22}$$

in comparison with the former

$$\begin{cases} \dot{x}^i = \dot{x}^i_{cm} + \ell\dot{\xi}^i \\ x'^i = \ell\xi'^i \end{cases}, \tag{23}$$

the projected areas split into two distinguishable categories. While, as before, the center-of-mass derivative $\dot{x}^i_{cm}$ is still absent from

$$\frac{\delta A^{ij}}{2\pi} = \left(\frac{\ell^2}{2\pi}\int_0^{2\pi}(\dot{\xi}^i\xi'^j - \dot{\xi}^j\xi'^i)d\sigma\right)d\tau, \tag{24}$$

it makes its resurrection via

$$\frac{\delta A^{i5}}{2\pi} = \left(R\dot{x}^i_{cm} + \frac{\ell^2}{2\pi}\int_0^{2\pi}(\dot{\xi}^i\xi'^5 - \dot{\xi}^5\xi'^i)d\sigma\right)d\tau. \tag{25}$$

Notably, as expected, and in contrast with the presence of $\dot{x}^i_{cm}$, irrelevant $\dot{x}^5_{cm}$ stays completely out of the game.

Equipped with Eqs. (24) and (25), one can now reconstruct the loop line element Eq. (18). In its modified version, reflecting the underlying $M_n \otimes S_1$ spacetime, it takes the final form

$$\boxed{\delta S^2 = \frac{1}{8\pi^2R^2}\left(\eta_{ij}\eta_{55}\delta A^{i5}\delta A^{j5} + \frac{1}{2}\eta_{ik}\eta_{jl}\delta A^{ij}\delta A^{kl}\right)}. \tag{26}$$

Here, by requiring $\eta_{ij}\eta_{55} = \eta_{ij}$ on consistency grounds, in order to account for the mandatory $\eta_{ij}x^i_{cm}x^j_{cm}$ term, we are finally led to the tenable signature choice

$$\eta_{55} = +1, \tag{27}$$

a posteriori justifying the spacelike nature of x^5 (nothing to do with the original Kaluza-Klein theory). On the other hand, given the underlying η_{ij} metric, the coefficients $\eta_{ik}\eta_{jl}$ are uniquely fixed.

B. Pythagoras theorem in loop space

The special case we now discuss in detail is the simplest, yet the most fundamental case in hand. And as such, it has been moved from the appendix level to the main body of our paper. To be specific, let us calculate the distance between two arbitrary loops residing in a flat two-dimensional plane

$$ds^2 = dx^2 + dy^2. \tag{28}$$

Following our prescription, our first step is to add the Kaluza-Klein ingredient to the game, and deal with a larger yet flat three-dimensional space governed by

$$d\bar{s}^2 = ds^2 + dx_5^2, \qquad \Delta x_5 = 2\pi R. \tag{29}$$

On simplicity and pedagogical grounds, we consider the evolution (parametrized by λ) of a circular loop. The loop is of radius $r(\lambda)$, is centered at $\{x_{cm}(\lambda), y_{cm}(\lambda)\}$

$$\begin{aligned} x(\lambda,\sigma) &= x_{cm}(\lambda) + r(\lambda)\cos\sigma \\ y(\lambda,\sigma) &= y_{cm}(\lambda) + r(\lambda)\sin\sigma \\ x_5(\lambda,\sigma) &= x_{5\mathrm{cm}}(\lambda) + R\sigma, \end{aligned} \tag{30}$$

and is furthermore wrapped, precisely once at this stage, around the Kaluza-Klein cylinder. The three projected areas are given explicitly by

$$\frac{dA^{xy}}{d\lambda} = 2\pi r\frac{dr}{d\lambda}, \tag{31}$$

$$\frac{dA^{x5}}{d\lambda} = 2\pi R\frac{dx_{cm}}{d\lambda}, \tag{32}$$

$$\frac{dA^{y5}}{d\lambda} = 2\pi R\frac{dy_{cm}}{d\lambda}. \tag{33}$$

Substituting the latter into Eq. (26), we immediately find out that

$$dS^2 = dx^2_{cm} + dy^2_{cm} + \frac{r^2}{R^2}dr^2, \tag{34}$$

and recalling the loop area $A^{xy} = \pi r^2$, brings us to the final Pythagoras formula

$$\boxed{dS^2 = ds^2_{cm} + \left(\frac{dA}{2\pi R}\right)^2}. \tag{35}$$

We have thus reached a flat three-dimensional loop space, with $\frac{A}{2\pi R}$ serving as a third dimension. It is an open

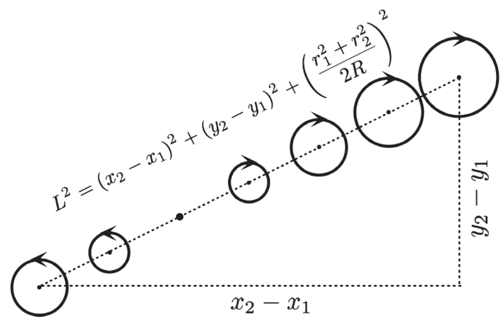

FIG. 3. Generalized Pythagoras theorem: Let L denote the geodesic distance between two loops, demonstrated here for circles of opposite sides, so that $A_2 - A_1 = \pi(r_2^2 - (-r_1^2))$. While the shape of the loop stays unconstrained in the absence of loop dynamics, the center-of-mass location, as well as the loop area, evolves linearly.

dimension, to be contrasted with the compact nature of x^5, thus mimicking the role of z_{cm}. Altogether (see Fig. 3), the geodesic distance L between two loops, of areas A_1 and A_2, centered at $\{x_1, y_1\}$ and $\{x_2, y_2\}$, respectively, is given by

$$L^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + \left(\frac{A_2 - A_1}{2\pi R}\right)^2. \qquad (36)$$

One may now wonder under what circumstances is the (area difference^2) term negligible? The trivial answer would be when $A_1 = A_2$, the case of a fixed loop area (the rigid loop case obviously included). The generic case calls for $A^{xy} \sim \ell^2$ to pick up the only length scale floating around. In general, however, as far as the geometry is concerned, the area difference $A_1 - A_2$ stays at this stage unbounded.

C. Light cone in loop spacetime

Had one started from a Lorentzian spacetime M_2 enriched by a compact ($\Delta x_5 = 2\pi R$) fifth dimension

$$d\bar{s}^2 = -dt^2 + dz^2 + dx_5^2, \qquad (37)$$

and analogously considered the simple τ-evolution

$$\begin{aligned} t(\tau,\sigma) &= t_{cm}(\tau) + a(\tau)\cos\sigma \\ z(\tau,\sigma) &= z_{cm}(\tau) + b(\tau)\sin\sigma \\ x_5(\tau,\sigma) &= x_{5\mathrm{cm}}(\tau) + R\sigma, \end{aligned} \qquad (38)$$

one would have once again ended up with the fundamental Eq. (35), only with $A^{tz} = \pi ab$. Note that A^{ij} is not sensitive to the time/spacelike nature of the ij-indices.

Altogether, as long as loop dynamics is not switched on, the kinematical evolution of a loop event, residing in M_2 and being wrapped around S_1, is described by means of the geodesic evolution of a pointlike event in flat M_3. While the Poincare-like symmetry associated with the loop spacetime metric

$$dS^2 = -dt_{cm}^2 + dz_{cm}^2 - \left(\frac{dA}{2\pi R}\right)^2 \qquad (39)$$

is well established, it contains some novel elements such as a $\{z_{cm}, A\}$ boost and a $\{t_{cm}, A\}$ rotation. The timelike nature of the A^{tz}-dimension stems from the opposite time/spacelike assignments of t and z, as expressed via $\eta_{tt}\eta_{zz} = -1$, in accordance with $\eta_{tt}\eta_{55} = -1$. The accompanying energy/momentum relation takes then the form

$$m^2 = E_{cm}^2 + E_A^2 - p_{cm}^2, \qquad (40)$$

where the total energy now has two independent sources $E^2 = E_{cm}^2 + E_A^2$. The second energy operator

$$E_A = -2\pi i R \frac{\partial}{\partial A} \qquad (41)$$

is identified as the generator of loop area expansions in the $\{t, z\}$-plane. As a trivial consistency check one may verify that a loop of a constant area, not necessarily of a constant shape, would return the familiar formulas describing a point particle moving in $1 + 1$ dimensions.

D. From special relativity to loop special relativity

The loop spacetime line element Eq. (26) and the subsequent special cases discussed bring us one step closer to our second goal, which is formulating a LSR theory capable of supporting the special relativity (SR) limit

$$\delta S^2 = \eta_{ij} dx_{cm}^i dx_{cm}^j + O[\ell^2]. \qquad (42)$$

The idea of trapping the loop around the compact fifth dimension can be conveniently realized by executing the partial gauge choice (σ-redefinition)

$$x^5(\tau,\sigma) = x_{cm}^5(\tau) + R\sigma. \qquad (43)$$

This way, starting from the most general Fourier expansions for

$$x^i(\tau,\sigma) = x_{cm}^i(\tau) + \ell \sum_{n\neq 0} \xi_n^i(\tau) e^{in\sigma}, \qquad (44)$$

we can significantly simplify the explicit expressions for the associated infinitesimal projected areas. Absorbing for simplicity the $\frac{\ell}{R}$ ratio within ξ^i, we find

$$\frac{1}{2\pi R}\frac{dA^{i5}}{d\tau} = \frac{d}{d\tau} x_{cm}^i, \qquad (45)$$

$$\frac{1}{2\pi R}\frac{dA^{ij}}{d\tau} = 2R \frac{d}{d\tau} \sum_{n>0} \Im(n \xi_n^i \xi_n^{j\star}), \qquad (46)$$

which can finally be substituted into Eq. (26). Altogether, the LSR line element takes the elegant form

$$\boxed{ds^2_{\mathrm{LSR}} = ds^2_{\mathrm{SR}} - R^2(ds^2_E - ds^2_B)}. \tag{47}$$

Our anchor, the $(3+1)$-dimensional center-of-mass SR line element, is hereby supplemented by a $(3+3)$-dimensional Maxwell-like structure. It should be noted that (i) all $O[\ell]$ mixed pieces have been dropped away by the specific gauge choice Eq. (43), and consequently that (ii) $x^5_{cm}(\tau)$ appears irrelevant as it has no realization in our formalism.

Naturally, attention is devoted to the light cone structure. The special relativistic $ds^2_{\mathrm{SR}} = 0$ is replaced by its loop special relativistic $ds^2_{\mathrm{LSR}} = 0$. Thus, from the point of view of an observer, unfamiliar with (or just insensitive to) LSR, as is evident from

$$ds^2_{\mathrm{LSR}} = 0 \Rightarrow ds^2_{\mathrm{SR}} = R^2(ds^2_E - ds^2_B), \tag{48}$$

the center-of-mass light cone acquires a six-dimensional (hopefully Planck scale) Maxwell-like fine structure. Three of the extra dimensions are “electric” (timelike) and the other three are “magnetic” (spacelike). They are supposed to fade away as $R \to 0$. This seems to constitute the main physics fingerprint of LSR.

E. Toward loop general relativity: Nash global isometric embedding

Attempting to go beyond flatness, one would now like to calculate the geodesic distance L between two loops which reside in a curved background, say two parallel loops living on a two-dimensional sphere of constant radius ℓ. The crucial point is that the two-sphere can be globally embedded within a flat three-dimensional space, that is,

$$\begin{aligned} x(\theta,\sigma) &= \ell \sin\theta\cos\sigma \\ y(\theta,\sigma) &= \ell \sin\theta\sin\sigma \\ z(\theta,\sigma) &= \ell\cos\theta, \end{aligned} \tag{49}$$

subject to the global constraint $x^2 + y^2 + z^2 = \ell^2$. The only nonvanishing projected area increment is then

$$\delta A^{xy} = 2\pi\ell^2 \sin\theta\cos\theta d\theta. \tag{50}$$

Following our prescription, one invokes the Kaluza-Klein topological term

$$x_5(\theta,\sigma) = R\sigma, \tag{51}$$

and consequently finds

$$L = \frac{\ell}{R}\int_{\theta_1}^{\theta_2} \sqrt{\ell^2\cos^2\theta + R^2}\sin\theta d\theta. \tag{52}$$

However, from a variety of reasons outlined toward the end of the Introduction, this is not necessarily a recipe for LGR. Still, one cannot rule out the possibility, make it a conjecture, that Nash local/global embedding may eventually play some role in constructing LGR.

IV. IS LOOP DYNAMICS IMPERATIVE?

A. Introducing the Planck scale

So far, we have demonstrated how to map a loop event, residing in $M_n \otimes S_1$ Kaluza-Klein spacetime, into a pointlike event residing in a larger $\frac{1}{2}n(n+1)$-dimensional flat loop spacetime whose center-of-mass submetric is supplemented by a Maxwell style higher dimensional companion. We are also aware of the topological advantage that, being wrapped around the fifth dimension, the loop event cannot really shrink to a pointlike event. However, as it stands, while the mandatory SR limit has been nontrivially recovered, the overall picture is still not fully satisfactory. The reasons are fourfold:

(i) Arbitrary loop shape: The loop-to-point mapping is unfortunately not one-to-one, and only captures the projected areas involved. Equation (26) is incapable of telling one loop configuration from the other as long as their projected areas are the same. The challenge would be to convert such a residual degree of freedom into a physically tamed shape uncertainty.

(ii) Unbounded loop size: Whereas the fifth dimension invoked is compact by definition, characterized by its tiny Kaluza-Klein (KK) radius R, the loop projected areas A^{ij} can in principle take arbitrarily large values. In turn, unless the projected areas are themselves of order $\mathcal{O}[R^2]$, the loop line element δS^2 will be dominated by the Maxwell-like term rather than by the center-of-mass term.

(iii) Multiple timelike dimensions: Once η_{ij} is specified, and x^5 is assigned spacelike, the signatures of Maxwell-like terms are not a matter of choice. To be specific, a single timelike spacetime coordinate t gives rise to $(n-1)$ timelike loop spacetime coordinates A^{it} [and of course to $\frac{1}{2}(n-1)(n-2)$ spacelike loop spacetime coordinates A^{ij}]. This opens the door for problematic mathematical as well as philosophical cause-and-effect issues, arguing that the behavior of physical systems could not be predicted reliably. Saying this, note that several theories, F-theory and 2T-theory [11] among them, do host multiple timelike (and necessarily accompanied by spacelike) dimensions.

(iv) Multiple KK wrappings: Recalling that $\pi_1(S_1) = \mathcal{Z}$, the closed loop event can carry an arbitrary integer winding number $w = n \neq 0$. We can of course always generalize our simple loop assumption, that is, allow no self-intersections of the loop's oriented projections (which surround the projected areas A^{ij}), but it is not natural to do so in the presence of a nontrivial topology. After all, a winding number $w = n \neq \pm 1$ loop gives rise to $(|n| - 1)$ self-intersections on the KK cylinder. A presumably quantum field theoretical self-interaction mechanism, capable of decomposing a $w = n$ loop into n separated $w = 1$ simple loops, is certainly in order, but unfortunately stays beyond the scope of the present paper.

Appreciating the above potential drawbacks, one is left with two apparently contradictable options:

Option I: The exact configuration of the loop event does not show up at the classical level at all. Had R vanished as $\hbar \to 0$, such an option would have been encouraged by quantum mechanics, with every loop configuration carrying its own amplitude. In fact, this option favors the entrance of the Planck scale into the game via

$$R \sim \ell_P = \sqrt{\frac{\hbar G}{c^3}}, \tag{53}$$

implying that R should vanish at the $G \to 0$ limit as well and of course as $c \to \infty$.

Option II: The exact configuration of the loop event does acquire a physical meaning already at the classical level. The introduction of dynamics via some string theoretical action seems then unavoidable, with the main goal being to encourage loop events to shrink (positive string tension). Such a dynamical approach would furthermore account for the assumption that $A^{ij} \sim R^2$ and that the perimeter $2\pi R$ of the fifth dimension share the one and the same length scale.

While the second option may seem easier to utilize on technical grounds, it is the first option which actually catches our imagination. Thus, we choose to adopt the Planck length scale Eq. (53), but without giving up the idea of loop event self-dynamics. In other words, we attempt to make a compromise, and choose the following.

Option I + II: Translated into the Lagrangian formalism, we propose

$$\boxed{\mathcal{I} = \mathcal{I}_{\rm LSR} + \Lambda \mathcal{I}_{\rm NG}}, \tag{54}$$

where $\mathcal{I}_{\rm LSR} = \int dS$ is the loop spacetime geodesic action, and $\mathcal{I}_{\rm NG} = \int \sqrt{-g_2}\, d\tau d\sigma$ stands for the familiar string theoretical Nambu-Goto action (or alternatively for its Polyakov variant). The first ingredient contains the SR limit, but is insensitive to the detailed structure of the loop. The second ingredient does not have an SR limit, but forcefully governs the inner loop dynamics. Λ is a dimensionless coefficient, which may be eventually elevated in some stage to the level of a Lagrange multiplier. To show our point we discuss now in some detail the simplest pedagogical case of sufficient complexity, namely a two-dimensional soap world sheet embedded within a three-dimensional Euclidean space.

B. No-go Nambu-Goto

We now prove, as was claimed before, that the Nambu-Goto action (as well as its Polyakov variant) cannot consistently serve as a measure of the "distance" between two loops. To stand on familiar geometrical grounds, we choose to make our point using the simplest pedagogical case of a two-dimensional soap world sheet embedded within a three-dimensional Euclidean space

$$x^1(z,\sigma) = r(z)\cos\sigma \quad x^2(z,\sigma) = r(z)\sin\sigma \quad x^3(z,\sigma) = z, \tag{55}$$

connecting two circles of equal radii

$$r(-h) = r(h) = \ell. \tag{56}$$

The corresponding Nambu-Goto action reads

$$I_{\rm NG} = 2\pi \int_{-h}^{h} r(z)\sqrt{1 + r'(z)^2}\, dz. \tag{57}$$

The Euler-Lagrange equation and the corresponding analytic solution are given by

$$rr'' - r'^2 - 1 = 0 \Rightarrow r(z) = \frac{\cosh kz}{\cosh kh}\ell, \tag{58}$$

subject to the symmetric boundary condition

$$\cosh kh = k\ell. \tag{59}$$

It can be numerically verified that there is no solution for $\frac{\ell}{h} < 1.508$, with the critical value marking a phase transition, a phenomenon which can be experimentally demonstrated with soap films. The critical point is associated with the extra mathematical condition

$$\sinh kh = \frac{\ell}{h}. \tag{60}$$

In other words, the Nambu-Goto action cannot be interpreted as distance between loops separated too far (beyond criticality) apart.

C. Kaluza-Klein modified Nambu-Goto

We now show that the trick of adding a compact fifth dimension and wrapping the loop event once around it, that is,

$$\begin{aligned} x^1(z,\sigma) &= r(z)\cos\sigma \\ x^2(z,\sigma) &= r(z)\sin\sigma \\ x^3(z,\sigma) &= z \\ x^5(z,\sigma) &= R\sigma, \end{aligned} \tag{61}$$

while having some advantage over the plain Nambu-Goto (NG) case, is still quite problematic. The former Nambu-Goto action Eq. (57) gets now generalized into

$$I_{\mathrm{NG+KK}} = 2\pi\int_{-h}^{h}\sqrt{(r(z)^2+R^2)(1+r'(z)^2)}dz, \tag{62}$$

and subsequently, the modified Euler-Lagrange equation takes the form

$$(r^2+R^2)r'' - r(1+r'^2) = 0. \tag{63}$$

The analytic solution looks very much like the one given by Eq. (58), save for the modified boundary condition

$$\cosh\, kh = \frac{k\ell}{\sqrt{1-k^2R^2}}, \tag{64}$$

replacing the former Eq. (59).

A critical $h_c = 2.790R$ plays now a major role. For $h < h_c$, there is exactly one real solution for every a. For $h > h_c$, there can still be a single real solution provided $\ell < \ell_{\mathrm{min}}$ or $\ell > \ell_{\mathrm{max}}$. However, for $h > h_c$ and $\ell_{\mathrm{min}} < \ell < \ell_{\mathrm{max}}$, where $\ell_{\mathrm{min,max}}$ are the local extrema values of Eq. (64), there appear to be three real solutions. Not only is uniqueness lost, but furthermore, one unexpectedly faces a hysteresis phenomenon.

There is, however, some encouraging news to report on. This has to do with the large loop separation region $h \gg R$, a region which plain $\mathcal{I}_{\mathrm{NG}}$ Eq. (57) simply could not reach. In this case, we derive the long distance behavior

$$\frac{\mathcal{I}}{2\pi R} \simeq 2h + \frac{\ell^2}{R}\tanh\frac{h}{R} + \cdots, \tag{65}$$

showing a small $\mathcal{O}(\frac{\ell^2}{R})$ correction to the classical large value of $2h$. For some classical wrapped string solutions see Ref. [12].

D. Combining LSR with Nambu-Goto

Sticking to Eqs. (61), we now substitute the various $x^i(z,\sigma)$ into the action Eq. (54) to arrive at

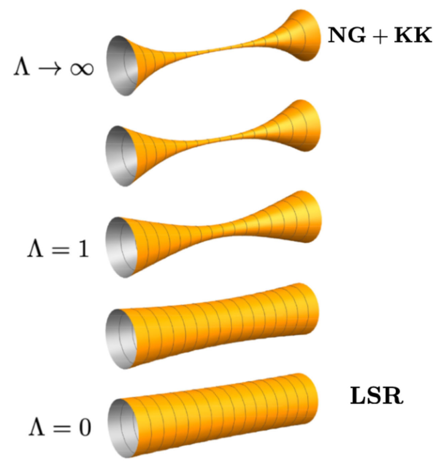

FIG. 4. From LSR to NG + KK: Soap branes connecting two identical circular rings, residing in a three-dimensional space and separated $2h$ apart. As Λ grows, while holding the KK radius R fixed, the LSR cylinder transforms into an NG + KK narrow waist candlestick (owing its stability to $R \neq 0$).

$$\mathcal{I}_\Lambda = 2\pi\int_{-h}^{h}\left(\sqrt{R^2+r^2r'^2} + \Lambda\sqrt{(R^2+r^2)(1+r'^2)}\right)dz. \tag{66}$$

Here, Λ is just a dimensionless coefficient, not a Lagrange multiplier. $\Lambda \to \infty$ marks the NG-limit, whereas $\Lambda \to 0$ takes us back to the LSR territory (see Fig. 4). At the first glance, the associated Euler-Lagrange equation looks quite cumbersome, but once recasted into the form $r'' = f(r, r')$, with R and h serving as parameters, it can be numerically handled straightforwardly.

The numerical lesson is twofold:

(i) $\Lambda \geq 0$ on z-evolutionary grounds, as otherwise we are necessarily driven into an undesirable $r = 0$ collapse.

(ii) $\Lambda \ll 1$ on self-consistency grounds, as otherwise we loose track of LSR, which becomes merely a perturbation on the classical string action. For such a small mixing parameter Λ, we obtain

$$\frac{\mathcal{I}_\Lambda}{2\pi R} = 2h\left(1+\sqrt{1+\frac{\ell^2}{R^2}\Lambda}\right) + \cdots \quad \text{for}\ \Lambda \ll 1, \tag{67}$$

to be fully contrasted with

$$\frac{\mathcal{I}_\Lambda}{2\pi R} = 2h\Lambda\xi + \cdots \quad \text{for}\ \Lambda \gg 1, \tag{68}$$

where ξ is some geometrical $\mathcal{O}[1]$ factor.

V. LOW FREQUENCY BREATHING MODE

In a Lorentzian $M_4 \otimes S_1$ background, with the following loop assignments:

$$
\begin{aligned}
x^0(\tau,\sigma) &= \tau \\
x^1(\tau,\sigma) &= r(\tau)\cos\sigma \\
x^2(\tau,\sigma) &= r(\tau)\sin\sigma \\
x^3(\tau,\sigma) &= 0 \\
x^5(\tau,\sigma) &= R\sigma,
\end{aligned}
\tag{69}
$$

corresponding to a circular loop evolving in the xy-plane, the action Eq. (66) is traded for

$$
\mathcal{I}_\Lambda = 2\pi \int \left(\sqrt{R^2 - r^2\dot{r}^2} + \Lambda\sqrt{(R^2+r^2)(1-\dot{r}^2)}\right) d\tau. \tag{70}
$$

Associated with the latter action is the Euler-Lagrange equation

$$
\frac{\ddot{r}}{r} = -\frac{\dot{r}^2 + \Lambda(1-\dot{r}^2)m(r,\dot{r})}{r^2 + \Lambda(R^2+r^2)m(r,\dot{r})}, \tag{71}
$$

where $m(r,\dot{r})$ is given explicitly by the ratio

$$
m(r,\dot{r}) = \frac{(R^2 - r^2\dot{r}^2)^{\frac{3}{2}}}{R^2(R^2+r^2)^{\frac{1}{2}}(1-\dot{r}^2)^{\frac{3}{2}}}. \tag{72}
$$

Without loosing generality, the accompanying initial conditions are $r(0) = \ell$, $\dot{r}(0) = 0$.

The solutions $r(\tau)$ of Eq. (71) are in general oscillatory in τ, with frequencies of the general form

$$
\omega_\Lambda = \frac{1}{R} f_\Lambda\left(\frac{\ell}{R}\right), \tag{73}
$$

which we now proceed to extract. There are no oscillatory solutions, however, for

$$
-\frac{\ell^2}{R\sqrt{R^2+\ell^2}} \le \Lambda \le 0, \tag{74}
$$

an important observation when the small Λ regime is one's preference.

A remark is in order. The fact that the oscillatory solutions pass through $r = 0$, corresponding to an orientation flip of the loop in the xy-plane, is of no special concern. Owing to the underlying topology, that is, being wrapped around the Kaluza-Klein cylinder, the loop cannot really shrink to a singular point.

Three trivial cases can be immediately verified:

(i) $\Lambda = 0$ marks the LSR limit. As explained earlier, the area $\sim r^2$ evolves linearly with τ, and given our initial conditions, we face

$$
r(\tau) = a. \tag{75}
$$

In light of the forthcoming analysis, we assign

$$
\omega_{\mathrm{LSR}} = 0. \tag{76}
$$

(ii) $\Lambda \to \infty$ is recognized as the NG + KK limit. In this case, we recover radial oscillations via $r(\tau) = \ell\cos\omega\tau$, with frequency

$$
\omega_{\mathrm{NG}} = \frac{1}{\sqrt{R^2+\ell^2}}. \tag{77}
$$

(iii) The limit $\frac{\ell}{R} \to 0$ is trustfully translated into $\frac{r(\tau)}{R} \to 0$. As the scale of $r(\tau)$ shrinks away, one immediately notices that $m(r,\dot{r}) \to 1$. Altogether, as could have been expected, and for any finite Λ, we are back to NG + KK only with

$$
\omega_\Lambda \to \frac{1}{R}. \tag{78}
$$

For a positive string tension $\Lambda > 0$, we now prove the existence of a periodic breathing mode, and attempt to study the entire range $\omega_{\mathrm{LSR}} < \omega_\Lambda < \omega_{\mathrm{NG}}$. Our interest lies, however, with the low frequency mode associated with the LSR governed case $\Lambda \ll 1$. The transition from the low to high frequency regimes is depicted (for the special case $\ell = R$) on a log-log graph; see Fig. 5.

(iv) $\Lambda \ll 1$ is our case of interest. On simplicity and pedagogical grounds, we choose to make our points using the prototype case $\ell = R$. The periodic numerical solution then suggests $\omega_\Lambda \sim \frac{\sqrt{\Lambda}}{R}$, a result which we now extract semianalytically. We start by noticing that $m(\tau)$ is a shallow function of τ. It starts at $m(0) = 1/\sqrt{2}$ and stays below 1 up to an extremely narrow yet finite peak when $r(\tau)$ passes through zero. We then replace $m(\tau)$ by its τ-average

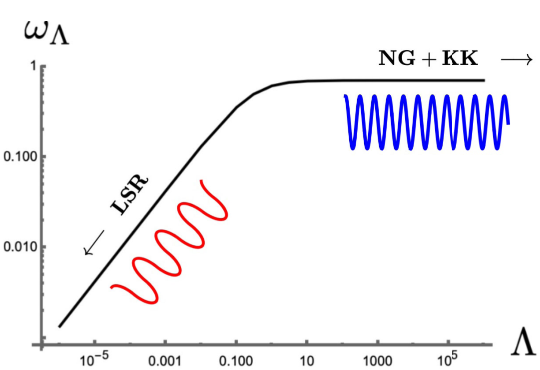

FIG. 5. High to law frequency transition as a function of the string tension Λ (plotted for the special case $\ell = R = 1$). While ω_Λ is Λ-independent at the high-ω regime, it is $\sim\sqrt{\Lambda}$ at the low-ω regime.

constant value $\frac{1}{\sqrt{2}} < m < 1$ (a tenable value is $m \simeq 0.8$). And finally, throwing away two terms of tiny numerical contributions, we arrive at the approximated equation of motion

$$\frac{\ddot{r}}{r} \simeq -\frac{\dot{r}^2 + \Lambda m}{r^2 + R^2 \Lambda m}. \tag{79}$$

Not only does the latter equation admit an analytic solution, but it forcefully captures the essence (shape, extrema, zeros) of the original numerical solution of Eq. (71). The semianalytic solution is given explicitly by

$$\frac{\tau}{R} = E\left[\frac{\pi}{2}, \frac{-1}{\Lambda m}\right] - E\left[\arcsin\frac{r(\tau)}{R}, \frac{-1}{\Lambda m}\right], \tag{80}$$

where $E[\phi, \chi] \equiv \int_0^\phi \sqrt{1 - \chi \sin^2\theta}\, d\theta$ stands for the elliptic integral of the second kind.

The role of the constant term in Eq. (80) is to reassure that $r(0) = R$. Furthermore, the radial velocity

$$\dot{r}(\tau) \simeq -\frac{\sqrt{1 - \frac{r^2}{R^2}}}{\sqrt{1 + \frac{r^2}{R^2 \Lambda m}}} \tag{81}$$

not only confirms that $\dot{r}(0) = 0$ but also correctly takes care of the physical upper bound $|\dot{r}(\tau)| \leq 1$. Our main bonus is now the frequency formula

$$\omega_\Lambda = \frac{2\pi}{\Delta\tau} = \frac{\pi}{2\mathrm{RE}[\frac{\pi}{2}, \frac{-1}{\Lambda m}]}. \tag{82}$$

Appreciating the fact that $E[\frac{\pi}{2}, \frac{-1}{x}]$ behaves as $\frac{1}{\sqrt{x}}$ for $x \ll 1$, we end up with the low frequency limit

$$\boxed{\omega_\Lambda \simeq \frac{\pi\sqrt{\Lambda m}}{2R} \quad \text{for } \Lambda \ll 1}. \tag{83}$$

Note that for $\Lambda \gg 1$ we recapture Eq. (78).

VI. EPILOGUE

This paper focuses on an unexpected role played by the compact fifth dimension, serving as the missing topological ingredient which allows for the conversion of an area metric into a legitimate line element in loop spacetime. Ironically, it is the Kaluza-Klein ansatz which paves the way for LSR to exhibit the indispensable SR limit. Associated with $M_4 \otimes S_1$ is then a ten-dimensional loop spacetime metric, whose four-dimensional center-of-mass core term is supplemented by a six-dimensional Maxwell-style Planck-scale fine structure which drops away at the shrinking loop limit.

There are, however, as listed in Chapter 4, four flies in the LSR ointment, so to speak. Being kinematical in nature, LSR suffers from arbitrary loop shapes, unbounded loop sizes, multiple timelike dimensions, and multiple KK wrappings. Some of these problems require a dynamical solution, suggesting the presence of a positive string tension. But dynamical Nambu-Goto (or Polyakov) by itself will not do either, as it clearly lacks the SR limit in the technical sense of Eq. (4). The action marriage Eq. (54) is perhaps an elegant, and by far the simplest, way out. It comes with a clear signature, that is, a low-frequency breathing mode. And finally, just a reminder that the generalization of LSR into LGR is still in order. Nash global isometric embedding may play a key role in constructing the latter theory.

ACKNOWLEDGMENTS

The idea presented here was shaped up back in 1996, during a visit to N. K. Nielsen (University of Southern Denmark) and has later served to seed the M.Sc. thesis of Nadav Barkai (Ben Gurion University, 2007, unpublished) which dealt with “Gauge invariance of the second type.”

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