Higher-point gauge-theory couplings of massive spin-2 states in four-dimensional string theories

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We explicitly compute the Neveu-Schwarz sector conventional type-I superstring tree-level amplitudes at five points after compactifying to 4D, express the QFT building block in the helicity basis, and give several attempts toward arbitrary n points. More specifically, we consider the interaction of one first excited level and otherwise massless states of conventional type-I superstrings, where the four-dimensional states can, for instance, be realized via D3 branes. We construct the amplitude by using the Berends-Giele currents. From the recursion of Berends-Giele currents, we can generate the higher point amplitude. We also apply the BCFW recursion with massive external legs shifted and get the amplitude for arbitrary n points.

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I. INTRODUCTION

The study of scattering amplitudes is one of the historical origins of string theory [1]. For example, Veneziano amplitude, a candidate amplitude for hadron scattering, is often referred to as the first equation of string theory. Also, some important features of string theory and quantum field theory are hidden in the amplitudes. The computation of string amplitudes is closely related to the correlation functions of vertex operators on the so-called world sheet. Technically, this is a two-dimensional conformal field theory (CFT) on the world sheet. The CFT approach leads to many interesting properties, and one of them is the famous KLT relation [2]: For closed strings, the genus-zero correlators and in fact, even their integrals over the vertex points, can be factorized into left movers and right movers, also called holomorphic and antiholomorphic building blocks. KLT relation indicates the tree-level double-copy relation, especially between perturbative gravity and gauge theories.

In recent decades, numerous studies have been conducted on this topic (see, for instance, [2–4]), and the KLT and double-copy relation at tree level have already become one of the essential features of string amplitudes. There are also attempts toward loop-level generalizations [5–8]. In the past 10 years, the discoveries of additional double-copy structures indicated that, if all external states are massless, the tree-level coupling of the type-I superstring [9,10] and the open bosonic string [11,12] can be factorized into the scalar integrals on disk (the disk integral is also known as Z-theory amplitudes) and quantum field theory (QFT) building blocks. Some recent papers even generalized this relation to the coupling with 1 external mass-level-1 state and found the QFT building block of this coupling [13–17]. This is closely related to the heterotic version of the chiral or twisted string, also called the twisted heterotic string [18,19]. The twisted heterotic string amplitudes have been studied using the field theory methods in the context of conformal supergravity amplitudes and their double copy constructions [20–23]

When analyzing the QFT amplitude, the complexity and computational difficulty of the traditional Feynman diagrammatic approach increase rapidly as the number of external particles grows [24]. In contrast, the development of modern amplitude methods in the past few decades has directed an alternative route to arrive at otherwise intractable results, both classical and quantum [24,25], and various recursion relations. Apart from the famous on-shell BCFW recursion [26], there is also a semi-on-shell recursion relation based on the Berend-Giele (B-G) currents [27–29]. There are also applications to amplitudes of various theories at tree and loop level [30–33].

Since the KLT and the additional double-copy relation are key features of string amplitudes and KLT can work for arbitrary excited states of strings, it is really important to understand how the additional double-copy structure behaves when excited states are involved. Since all the information on polarization and momentum is inside the QFT building block, it is important to learn the dynamical structure of the QFT building block, especially in 4D after compactification or due to the D branes. One of the most

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powerful tools in researching 4D QFT amplitudes is the so-called spinor helicity formalism. Thus, our aim in this paper is to produce the spinor helicity form of the QFT building block for conventional type-I string amplitude. We first reviewed the basic idea of spinor helicity, BCFW recursion, dimension-agnostic Berends-Giele currents, and B-G recursion. Then generalized the result involving one of the universal Regge excitation states and up to three massless gluons [34] to one excitation state coupling with arbitrary $n-1$ massless gluons, which result in a very nice n-point amplitude formula Eq. (2.54) for a specific configuration of external gluon helicities. This helicity configuration resembles the so-called maximal helicity violation (MHV) in pure Yang-Mills. We analytically proved the five-point case and gave a numerical check for the six-point case. We also applied BCFW to the QFT building blocks of the conventional type-I superstring amplitude. The final result is in agreement with Eq. (2.54).

II. NOTATION AND CONVENTION

A. Spinor helicity in four dimensions

Spinor helicity formalism [25] is another way[1] of expressing the QFT amplitude. It has extremely simplified the calculation of the scattering amplitudes in four dimensions [24]. There is also spinor helicity formalism in other dimensions, for instance, three, six, and 10 dimensions [35], but our discussion is specified to four dimensions; thus, we only introduce the four-dimensional spinor helicity formalism in this paper.

In this subsection, we will introduce the conventions for the spinor helicity formalism:

We will mostly keep those in Elvang and Huang's Scattering amplitudes textbook [25]. The metric is chosen to be the "mostly plus" metric, $\eta_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$. We define the σ matrices[2] as

$$(\sigma^\mu)_{\alpha\dot\beta} = (1,\sigma^i)_{\alpha\dot\beta}, \qquad (\bar\sigma^\mu)^{\dot\alpha\beta} = (1,-\sigma^i)^{\dot\alpha\beta}, \qquad (2.1)$$

where σ^i, $i = 1, 2, 3$ are Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (2.2)$$

The two sets of spinor indices are raised and lowered individually using the $SU(2)$ invariant tensor, also known as the Levi-Civita tensor:

$$\varepsilon^{\alpha\beta} = \varepsilon^{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\varepsilon_{\alpha\beta} = -\varepsilon_{\dot\alpha\dot\beta}, \qquad (2.3)$$

and obey $\varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma} = \delta_\alpha{}^\gamma$

We contract $(\sigma^\mu)_{\alpha\dot\alpha}$ with k^μ to get the 2×2 matrix $k_{\alpha\dot\alpha} = k_\mu \sigma^\mu_{\alpha\dot\alpha}$ and also notice $\det(k_{\alpha\dot\alpha}) = m^2$, which leads to the obvious difference between the massive external states and the massless external states; we will come to this later.

1. Spinor helicity for massless particles

For massless particles, we have $\det(k_{\alpha\dot\alpha}) = 0$, and thus, the matrix $k_{\alpha\dot\alpha}$ is of rank 1. We can then write it as the direct product form:

$$k_{\alpha\dot\alpha} = -\lambda_\alpha \tilde\lambda_{\dot\alpha}; \qquad (2.4)$$

we also write λ_α and $\tilde\lambda_{\dot\alpha}$ as $|k]_\alpha$ and $\langle k|_{\dot\alpha}$. So, we can also write the momentum in a matrix form:

$$k_{\alpha\dot\alpha} = -|k]_\alpha \langle k|_{\dot\alpha}. \qquad (2.5)$$

Consider the Dirac equation in the massless case. The Dirac equation would decouple into the Weyl equation without mass. Thus, when $m = 0$, we have

$$\not{k} v_\pm(k) = 0, \qquad \bar u_\pm(k)\not{k} = 0, \qquad (2.6)$$

where $v_\pm(p)$ and $\bar u_\pm(p)$ are wave functions associated with an outgoing antifermion and fermions. The wave functions are related as $u_\pm = v_\mp$ and $\bar v_\pm = \bar u_\mp$

We can now write the two independent solutions of the Dirac equation as

$$v_+(k) = \begin{pmatrix} |k]_\alpha \\ 0 \end{pmatrix} \qquad v_-(k) = \begin{pmatrix} 0 \\ |k\rangle^{\dot\alpha} \end{pmatrix}, \qquad (2.7)$$

and

$$\bar u_-(k) = (\,0, \quad \langle k|_{\dot\alpha}\,) \qquad \bar u_+(k) = (\,[k|^\alpha, \quad 0\,). \qquad (2.8)$$

The angle and square spinors are two-component commuting spinors. After defining the spinor, we can write the massless Weyl equation:

$$k^{\dot\alpha\beta}|k]_\beta = 0, \qquad k_{\alpha\dot\beta}|k\rangle^{\dot\beta} = 0, \qquad [k|^\beta k_{\beta\dot\alpha} = 0, \qquad \langle k|_{\dot\beta} k^{\dot\beta\alpha} = 0. \qquad (2.9)$$

We have two sets of important identities. First, the Fierz identities:

[1] With respect to the covariant way, which preserves the locality and Lorentz invariance in each Feynman diagram.

[2] We will omit the spinor indices $(\alpha,\beta,\ldots)$ later on when it leads to no misunderstanding. When we come to the massive spinor part, we use $(a,b,\ldots)$ as the little group indices. We will come to the details in that section.

$$(\sigma^\mu)_{\alpha\dot\alpha}(\sigma_\mu)_{\beta\dot\beta} = -2\varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta},$$
$$(\sigma^\mu)_{\alpha\dot\alpha}(\bar\sigma_\mu)^{\dot\beta\beta} = -2\delta_\alpha{}^\beta\delta_{\dot\alpha}{}^{\dot\beta},$$
$$[a|\sigma^\mu|b\rangle[c|\sigma_\mu|d\rangle = 2[ac]\langle bd\rangle,$$
$$[a|\sigma^\mu|b\rangle\langle c|\bar\sigma_\mu|d] = 2\langle bc\rangle[ad], \tag{2.10}$$

and second, the Schuten identity:

$$|i\rangle\langle jk\rangle + |j\rangle\langle ki\rangle + |k\rangle\langle ij\rangle = 0. \tag{2.11}$$

2. Spinor helicity for massive particles

We now consider the massive spinor helicity formalism [36,37]. The spinor helicity formalism of massive particles has no essential difference from the one of massless particles. We can regard the massive momentum as the linear combination of two massless momenta:

$$k_i^\mu = a_i^\mu + b_i^\mu, \tag{2.12}$$

where a_i^μ, b_i^μ have the same spatial direction as k_i^μ, which are along z axis. We can write down the components of k, a, b now:

$$k_i^\mu = (k_i^0, 0, 0, k_i^3) \qquad a_i^\mu = \left(\frac{k_i^0 + k_i^3}{2}, 0, 0, \frac{k_i^0 + k_i^3}{2}\right)$$
$$b_i^\mu = \left(\frac{k_i^0 - k_i^3}{2}, 0, 0, -\frac{k_i^0 - k_i^3}{2}\right). \tag{2.13}$$

Or, more generally, (both direction and the length), we can parametrize the massive momentum k_i^μ:

$$k_i^\mu = a_i^\mu - \frac{m_i^2}{2a_i\cdot b_i} b_i^\mu, \tag{2.14}$$

and we can contract both side of this equation with σ^μ and decompose a_i^μ and b_i^μ on the right-hand side into spinor helicity form using Eq. (2.5) so that we have

$$k_{\alpha\dot\alpha} = |a]_\alpha\langle a|_{\dot\alpha} + |b]_\alpha\langle b|_{\dot\alpha}. \tag{2.15}$$

This is simply a rank 2 matrix $k_{\alpha\dot\alpha}$, which can now be written as

$$k_{\alpha\dot\alpha} = \lambda_\alpha{}^a\tilde\lambda_{\dot\alpha a}, \tag{2.16}$$

where $a = 1,\ 2$ corresponds to $|a]$, $|a\rangle$ and $|b]$, $|b\rangle$ separately. We now rewrite λ and $\tilde\lambda$ as a matrix and regard α and a as the matrix index. We then have:

$$k^2 = -m^2 \to \det\lambda\times\det\tilde\lambda = -m^2 \tag{2.17}$$

where we set[3] $\det\lambda = \det\tilde\lambda = m$. We can also raise or lower the indices a, b by using ϵ^{ab} and ϵ_{ab} so that we can write

$$k_{\alpha\dot\alpha} = \lambda_\alpha{}^a\tilde\lambda_{\dot\alpha}{}^b\epsilon_{ab}. \tag{2.18}$$

Also, notice that we have the Dirac equation:

$$k_{\alpha\dot\alpha}\tilde\lambda^{\dot\alpha a} = \lambda_\alpha{}^b\tilde\lambda_{\dot\alpha b}\tilde\lambda^{\dot\alpha a} = \lambda_\alpha{}^b\tilde\lambda_{\dot\alpha b}\epsilon^{\dot\alpha\dot\beta}\tilde\lambda_{\dot\beta c}\epsilon^{ca}$$
$$= \lambda_\alpha{}^b\det(\tilde\lambda)\delta_b{}^a = m\lambda_\alpha{}^a. \tag{2.19}$$

Similarly, we also have

$$k_{\alpha\dot\alpha}\lambda^{\alpha a} = -m\tilde\lambda_{\dot\alpha}{}^a. \tag{2.20}$$

These two equations are equivalent to the Dirac equation [38].

By using the [39] decomposition of massive momentum, we can also write down the component of massive spinor[4]:

$$|k^a\rangle = \begin{pmatrix} |b\rangle\frac{m}{\langle ab\rangle} \\ |a\rangle \end{pmatrix} \qquad |k^a] = \begin{pmatrix} |a] \\ |b]\frac{m}{[ab]} \end{pmatrix}. \tag{2.21}$$

In other references, people also define $\langle\boldsymbol{k}|$ as the massive spinor. In our notation, it is simply z_a that is used to absorb the $SU(2)$ index.

$$\langle\boldsymbol{k}| = \langle k^a|z_a. \tag{2.22}$$

We have a constrain on z_a, $\bar z_b$ and the antisymmetric tensor ϵ^{ab}:

$$z_a\epsilon^{ab}\bar z_b = -1. \tag{2.23}$$

We can construct the polarization tensor[5] for the spin-2 massive particles as follows[6]:

$$\Phi_i^{\mu\nu}(k_i,-2) = \frac{1}{2m^2}[a|\bar\sigma^\mu|b\rangle[a|\bar\sigma^\nu|b\rangle, \tag{2.24}$$

[3]This is a trivial convention; $\det\lambda$ and $\det\tilde\lambda$ are not necessarily equal to each other. There could be a phase factor that makes $\det\lambda$ and $\det\tilde\lambda$ both different from m but preserve the constraint $\det\lambda\times\det\tilde\lambda = m^2$. Let us take the trivial one as an example; we will come to this very soon.

[4]This convention is more non-trivial than the one mentioned before. We now still have $\langle ab\rangle[ab] = m^2$, but the $\langle ab\rangle = [ab] = m$ relationship no longer exists.

[5]We will shortly see that this is simply the single particle B-G current reduced to a specific four-dimensional choice of polarization.

[6]The $(k_i,-2)$ in $\Phi^{\mu\nu}(k_i,-2)$ means the polarization is a function of the momentum; for spin choice -2, we will omit this bracket when it leads to no confusion.

where $\Phi^{\mu\nu}$ is a traceless symmetric tensor. The momentum k is decomposed as $k^\mu = a^\mu + b^\mu$. For a massive spin j particle, there exist $2j+1$ spin degrees of freedom, the spin quantization axis is chosen as the direction of a in the rest frame. Each spin choice corresponds to a state, we express all of them by $|m, j\rangle$, where $m = -j, -j+1, \ldots, j-1, j$, and the $2j+1$ choices of m exactly correspond to the $2j+1$ degrees of freedom we found for spin j particle.

We can relate the $2j+1$ states by acting the raising and lowering operator on one state and raising or lowering m:

$$J^{+1}|m-1, j\rangle = \sqrt{\frac{(j+m)(j-m+1)}{2}}|m, j\rangle,$$
$$J^{-1}|m, j\rangle = \sqrt{\frac{(j+m)(j-m+1)}{2}}|m-1, j\rangle, \tag{2.25}$$

where the boundary requirements are $J^{+1}|j, j\rangle = 0$, $J^{-1}|-j, j\rangle = 0$.

Similar to the operator acting on states, we can define a set of raising and lowering operators acting on the polarization tensor:

$$\begin{aligned} O^{+1}\Phi^{\mu_1\mu_2\cdots\mu_j}(k, m-1) &= \frac{1}{\sqrt{2}}\left(+|a\rangle\frac{\partial}{\partial|b\rangle} - [b|\frac{\partial}{\partial[a|}\right)\Phi^{\mu_1\mu_2\cdots\mu_j}(k, m-1) \\ &= N(m, j)\Phi^{\mu_1\mu_2\cdots\mu_j}(k, m), \\ O^{-1}\Phi^{\mu_1\mu_2\cdots\mu_j}(k, m) &= \frac{1}{\sqrt{2}}\left(-|a\rangle\frac{\partial}{\partial|b\rangle} + [b|\frac{\partial}{\partial[a|}\right)\Phi^{\mu_1\mu_2\cdots\mu_j}(k, m-1) \\ &= N(m, j)\Phi^{\mu_1\mu_2\cdots\mu_j}(k, m-1), \end{aligned} \tag{2.26}$$

where we used $N(m, j)$ as a shorthand of $\sqrt{\frac{(j+m)(j-m+1)}{2}}$, and get the polarization tensor corresponding to $|-1, 2\rangle, |0, 2\rangle|+1, 2\rangle|+2, 2\rangle$ by acting the raising and lowering operator on the $|-2, 2\rangle$ state[7]:

$$\begin{aligned} \Phi^{\mu\nu}(k, -1) &= \frac{1}{\sqrt{2}}O^{+1}\Phi^{\mu\nu}(k, -2) \\ &= \frac{1}{4m^2}[([a|\bar{\sigma}^\mu|a\rangle - [b|\bar{\sigma}^\mu|b\rangle)[a|\bar{\sigma}^\nu|b\rangle + [a|\bar{\sigma}^\mu|b\rangle([a|\bar{\sigma}^\nu|a\rangle - [b|\bar{\sigma}^\nu|b\rangle)], \\ \Phi^{\mu\nu}(k, 0) &= \frac{1}{\sqrt{3}}O^{+1}\Phi^{\mu\nu}(k, -1) \\ &= \frac{1}{2m^2\sqrt{6}}[([a|\bar{\sigma}^\mu|a\rangle - [b|\bar{\sigma}^\mu|b\rangle)([a|\bar{\sigma}^\nu|a\rangle - [b|\bar{\sigma}^\nu|b\rangle) - [a|\bar{\sigma}^\mu|b\rangle[a|\bar{\sigma}^\nu|b\rangle - [b|\bar{\sigma}^\mu|a\rangle[b|\bar{\sigma}^\nu|a\rangle]. \end{aligned} \tag{2.27}$$

B. Dimension-agnostic Berends-Giele recursions

From now on, we need to deal with the multiparticle Berends-Giele currents and field strength. We use Latin letters $P, Q, X, Y, \ldots$ to denote different sets of particles.

The so-called Berends-Giele (B-G for short) recursions [27,40] is an effective approach to determining the tensor structure of arbitrary D-dimensional tree amplitudes in pure Yang-Mills theory, introduced by Berends and Giele in 1987 [27]. The idea of B-G recursions is to recursively combine all color-ordered Feynman diagrams with multiple external on-shell legs and one single off-shell leg using the B-G currents $\epsilon^\mu_{12,\ldots,p}$. They can be regarded as functions of dynamical variables such as polarization vectors ϵ^μ_i and null momentum vectors k^μ_i of the external particles $i = 1, 2, \ldots, p$ constrained by the following on-shell conditions:

$$\epsilon_i \cdot k_i = k_i \cdot k_i = 0, \tag{2.28}$$

where $i = 1, 2, \ldots, p$ refer to external-state labels, and the Lorentz-indices are denoted by Greek letters $\mu, \nu, \ldots = 0, 1, \ldots, D-1$.

The B-G recursion of the Yang-Mills amplitude is done via the recursion[8] of the B-G current [27]:

$$s_P\epsilon^\mu_P = \sum_{XY=P}[\epsilon_X, \epsilon_Y]^\mu + \sum_{XYZ=P}\{\epsilon_X, \epsilon_Y, \epsilon_Z\}^\mu, \tag{2.29}$$

[7]Here, we only write two examples.

[8]The boundary constraint is that the single particle current equals the single particle polarization.

where capital Latin letters $P, Q, X, Y, \ldots$ are multiple particle labels, also known as the rank of the B-G current. The length of, for example, $P = 12\ldots p$, is denoted by $|P| = p$. $[\epsilon_X, \epsilon_Y]^\mu$ and $\{\epsilon_X, \epsilon_Y, \epsilon_Z\}^\mu$ are defined as

$$[\epsilon_X, \epsilon_Y]^\mu = (k_Y \cdot \epsilon_X)\epsilon_Y^\mu - (k_X \cdot \epsilon_Y)\epsilon_X^\mu + \frac{1}{2}(k_X^\mu - k_Y^\mu)(\epsilon_X \cdot \epsilon_Y), \tag{2.30}$$

$$\{\epsilon_X, \epsilon_Y, \epsilon_Z\}^\mu = (\epsilon_X \cdot \epsilon_Z)\epsilon_Y^\mu - \frac{1}{2}(\epsilon_X \cdot \epsilon_Y)\epsilon_Z^\mu - \frac{1}{2}(\epsilon_Y \cdot \epsilon_Z)\epsilon_X^\mu, \tag{2.31}$$

and the Mandelstam variable with multiple particle indices s_P is defined as

$$s_P = \frac{1}{2}k_P^2, \tag{2.32}$$

where the multiple particle momentum $k^\mu_{P=12\ldots p} = k_1^\mu + k_2^\mu + \cdots + k_p^\mu$.

To review the recursion of B-G current, we need to define the division of multiparticle labels $P = 12\ldots p$. The summation over $XY = P$ means dividing P into nonempty sets $X = 12\ldots j$, $Y = j+1\ldots p$, where X, Y nonempty sets indicate that $1 \leq j \leq p-1$; thus, this summation has $|P| - 1 = p - 1$ terms. The same discussion can be applied to the summation over $XYZ = P$. We can also define the field strength F:

$$F_P^{\mu\nu} = k_P^\mu \epsilon_P^\nu - k_P^\nu \epsilon_P^\mu - \sum_{XY=P}(\epsilon_X^\mu \epsilon_Y^\nu - \epsilon_X^\nu \epsilon_Y^\mu), \tag{2.33}$$

and get the simpler form of the B-G current:

$$\epsilon_P^\mu = \frac{1}{2s_P}\sum_{XY=P}[(k_Y \cdot \epsilon_X)\epsilon_Y^\mu + \epsilon_X^\nu F_Y^{\nu\mu} - (X \leftrightarrow Y)]. \tag{2.34}$$

The color-ordered on-shell amplitudes at $n = p+1$ points are recovered by taking the off-shell leg in the rank-p B-G current ϵ_P^μ on shell. This is done by:

(i) Contracting with the polarization vector of particle n: ϵ_n^μ, which is also a B-G current.
(ii) Removing the propagator $s^{-1}_{12\ldots p}$ in the p-particle channel of ϵ_P^μ, which would diverge when taken particle n on shell.

Thus, we have

$$\mathcal{A}(1, 2, \ldots, n-1, n) = s_{12\ldots n-1}\epsilon^\mu_{12\ldots n-1}\epsilon_n^\nu \eta_{\mu\nu}. \tag{2.35}$$

C. Twisted heterotic string and conventional type-I superstrings

The twisted heterotic string [18,19,41,42] is a special kind of string that satisfies the twisted level-matching condition. One of the most important features of a twisted heterotic string is that the spectrum is finite as opposed to the infinite excited states for type II-A, II-B, or type I superstring theory. The physical vertex operators represent the following three multiplets of 10D $\mathcal{N} = 1$ supersymmetry:

(i) A gauge multiplet involving gluon (A) and gluino ($\mathcal{X}$),

$$\mathcal{V}_A^a = \bar{V}_{\bar{J}}^a \otimes V_\epsilon e^{ik\cdot X} \qquad \mathcal{V}_{\mathcal{X}}^a = \bar{V}_{\bar{J}}^a \otimes V_\chi e^{ik\cdot X}, \tag{2.36}$$

(ii) A supergravity multiplet involving graviton, B field and dilaton ($\bar{V}_{\bar{\epsilon}} \otimes V_\epsilon$) as well as gravitino and dilatino ($\bar{V}_{\bar{\epsilon}} \otimes V_\chi$),
(iii) A massive multiplet with $k^2 = -\frac{4}{\alpha'}$ comprising a spin-2 field $\Phi_{\mu\nu}$, a 3-form $E_{\mu\nu\rho}$ and a spin-$\frac{3}{2}$ field Ψ_μ^α,

$$\mathcal{V}_{\{\Phi,E,\Psi\}} = \bar{V}_T \otimes V_{\{\Phi,e,\psi\}} e^{ik\cdot X}, \tag{2.37}$$

where the massive states can be viewed as a double copy of a tachyon, $\bar{V}_T = 1$, with the first mass level of the open superstring [42]. The Lagrangian description of the amplitude with one external massive states and otherwise gauge multiplets is given in [13]. We only discuss the tree-level couplings of the conventional type-I superstring with only one massive multiplet $\underline{n}$:

$$\mathcal{M}_{\text{type-I}}(1, 2, \ldots, n-1, \underline{n}) = \sum_{\rho\in S_{n-3}} F^\rho(\mathbf{s}_n)\mathcal{A}(1, \rho(2, \ldots, n-2), n-1|\underline{n})|_{\alpha'\to 4\alpha'}, \tag{2.38}$$

where $\mathcal{A}(\ldots)$ is a rational function of the external momenta as usual for QFT amplitudes. Hence, $\mathcal{A}(\ldots)$ in Eq. (2.38) will be later on referred to as "QFT building blocks." Moreover, the disk integral $F^\rho(\mathbf{s}_n)$ is given by

$$F^\rho(\mathbf{s}_n) = (2\alpha')^{n-3}\int_\Omega \mathrm{d}z_2\mathrm{d}z_3\ldots\mathrm{d}z_{n-2}\prod_{1\leq i<j}^{n-1}|z_{ij}|^{2\alpha' s_{ij}}\rho\left\{\frac{s_{21}}{z_{21}}\left(\frac{s_{31}}{z_{31}}+\frac{s_{32}}{z_{32}}\right)\cdots\left(\frac{s_{n-2,1}}{z_{n-2,1}}+\cdots+\frac{s_{n-2,n-3}}{z_{n-2,n-3}}\right)\right\}, \tag{2.39}$$

where $z_{ij} = z_i - z_j$, Ω stands for the integration area. Here, it is $0 < z_2 < z_3 < \ldots < z_{n-2} < 1$ because we have fixed $(z_1, z_{n-1}, z_n) \to (0, 1, \infty)$.

We identify multiparticle polarizations with B-G currents [27] mentioned in Sec. II B and write down the recursion rules of B-G currents. Similar with Eqs. (2.33) and (2.34), we have

$$\Phi_P^{\mu\nu} = \sqrt{\alpha'} \sum_{P=QR} F^{\mu}_{Q\rho} F^{\rho\nu}_{R} + \mathrm{cyc}_P, \tag{2.40}$$

where the $F_P^{\mu\nu}$ and the ϵ_P^{μ} has the same form as Yang-Mills B-G current, following the discussion in [13]. cyc_P instructs one to add cyclic permutations in $P = 1, 2, \ldots, p$. For example, $A_P = B_P + \mathrm{cyc}_P$ where $P = 1$, 2, 3, 4 means

$$A_P = B_P + \mathrm{cyc}_P = B_{1,2,3,4} + B_{2,3,4,1} + B_{3,4,1,2} + B_{4,1,2,3}. \tag{2.41}$$

The amplitude can be constructed from B-G currents:

$$\mathcal{A}(1, 2, \ldots, n-1|\underline{n}) = (\Phi_{12\ldots n-1})^{\mu\rho} (\Phi_n)_{\mu\rho}. \tag{2.42}$$

For the lowest or highest spin state of the only massive particle, the single particle B-G current[9] $\Phi^{\mu\nu}$ is simply a direct product of the polarization vector of the lowest or highest spin state of spin-1 massive particle: $\epsilon^{\mu}\epsilon^{\nu}$. Our aim is to prove this n-point amplitude is equal to Eq. (2.54).

D. Amplitudes compactified to 4D

We impose the external polarization and momentum to lie in four-dimensional Minkowski space and convert the amplitude into spinor helicity form.

Some useful expressions are stated below:

$$k_{i\mu}\sigma^{\mu}_{\alpha\dot\beta} = \not{k}_{i\alpha\dot\beta} = |i]\langle i| \qquad \epsilon_i^{\mu-1} = \frac{[r|\sigma^{\mu}|i\rangle}{[ir]} \qquad \epsilon_i^{\mu+1} = \frac{\langle r|\bar\sigma^{\mu}|i]}{\langle ri\rangle}, \tag{2.43}$$

where i denotes the particle label, and μ denotes the Lorentz index. The ± 1 on ϵ_i means the polarization of particle i is ± 1. r denotes the reference spinor, which can be arbitrarily chosen to be any spinor not proportion to that of i.

Suppose we choose different reference spinor: $[r| = [2|$ or $[r| = [3|$ for $\epsilon_i^{\mu-1}$:

$$\epsilon_i^{\mu-1} = \frac{[2|\sigma^{\mu}|i\rangle}{[i2]} \qquad \tilde\epsilon_i^{\mu-1} = \frac{[3|\sigma^{\mu}|i\rangle}{[i3]}, \tag{2.44}$$

and compute the difference:

$$\epsilon_i^{\mu-1} - \tilde\epsilon_i^{\mu-1} = \frac{[2|\sigma^{\mu}|i\rangle}{[i2]} - \frac{[3|\sigma^{\mu}|i\rangle}{[i3]} = \frac{[2|\sigma^{\mu}|i\rangle[i3] - [3|\sigma^{\mu}|i\rangle[i2]}{[i2][i3]}. \tag{2.45}$$

Using simple Clifford algebra, we can conclude $\delta\epsilon_i^{\mu-1} = \epsilon_i^{\mu-1} - \tilde\epsilon_i^{\mu-1} \propto k_i^{\mu}$. This transformation on polarization is simply a linearized gauge transformation and does not have physical meaning. For convenience, we always choose the reference spinor to be the spinor of another particle and simplify our calculation. Take the four-point Yang-Mills amplitude $\mathcal{A}_{\mathrm{YM}}(1^+, 2^-, 3^-, 4^-)$ as an example. Since it breaks the MHV requirement, it is supposed to be 0. We can choose the reference spinor of particle 2, 3, 4 to be λ_1, and set λ_2 as the reference spinor of particle 1. Take $\epsilon_2 \cdot \epsilon_3$ as an example:

$$\epsilon_2 \cdot \epsilon_3 = \frac{[1|\sigma^{\mu}|2\rangle}{[21]} \frac{[1|\sigma^{\nu}|3\rangle}{[31]} \eta_{\mu\nu} \propto [11] = 0; \tag{2.46}$$

this can be easily generalized to any dot products among polarization vectors.[10] By momentum power counting of the numerators of n-point Yang-Mills amplitude, there is at most $n-2$ momentum. Together with n polarization vectors of external legs contracting with metric, there must be at least one dot product between polarization vectors, which leads to the fact that amplitudes with all-minus helicity and single-plus helicity vanish.

Now that we can consider the interaction we are interested in, we will still start from a four-point example.

1. Pure gluon example

The pure gluon example is also known as the Yang-Mills interaction, one of the cases in which the amplitude can be expressed using the closed form of spinor helicity. We can generate the current from the polarization. To calculate the amplitude more easily, we take the spinor helicity form. After applying the recursion rule of B-G currents and the spinor helicity form of the polarization vector, the B-G current that has all on-shell gluons with the same helicity can be expressed as [27]

$$\epsilon^{\mu}(i^+, i+1^+, \ldots, n^+) = \frac{\langle r|\bar\sigma^{\mu}\not{k}_{i,i+1,\ldots,n}|r\rangle}{\sqrt{2}\langle ri\rangle\langle i, i+1\rangle \cdots \langle n-1, n\rangle\langle nr\rangle}, \tag{2.47}$$

where the $\langle r|$ and $|r\rangle$ stand for the reference spinor. We have chosen the reference spinor for each on-shell leg to be the same. Using the recursion relation, we can get the B-G current where the first on-shell gluon has negative helicity. Here, we choose the reference spinor of particle 1 to be λ_2 and the reference spinor for particle $2, 3, \ldots, n$ as λ_1. We get [27]

[9]Now it can be regarded as polarization tensor.

[10]We will prove this relation for any two polarization vectors later on using Fierz identity Eq. (2.10).

$$\epsilon^\mu(1^-,2^+,\ldots,n^+) = \frac{\langle 1|\bar{\sigma}^\mu \not{k}_{2,3,\ldots,n}|1\rangle}{\sqrt{2}\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle} \times \sum_{m=3}^{n} \frac{\langle 1|\not{k}_m \not{k}_{1,2,\ldots,m}|1\rangle}{k^2_{1,2,\ldots,m-1} k^2_{1,2,\ldots,m}}. \quad (2.48)$$

After contracting with the negative helicity gluon n and some simple simplification, we have the well-known Parker-Taylor formula [24]:

$$\mathcal{A}(1^-,2^+,3^+,\ldots,n-1^+,n^-) = \frac{\langle 1n\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}. \quad (2.49)$$

2. Four-point QFT building block example

Consider the QFT building block of the four-point type-I super string $\mathcal{A}(1^-,2^+,3^+|\underline{4}^{-2})$; the underline denotes the massive leg. The momentum of the particle is k_4, which can be decomposed into the summation of two null vectors, denoted by a and b:

$$k_4^\mu = a^\mu + b^\mu \qquad a^2 = b^2 = 0. \quad (2.50)$$

We can always choose the reference spinor of particle 1 to be λ_2; different reference spinor choices can always bring us different cancellations, but the final result of the amplitude is always the same.

Let us set the reference spinor of particle 2,3 to be λ_1. The cancellation table of the dot product between polarization vectors is.

$$\epsilon_i \dot{\epsilon}_j = 0; \qquad i.j = 1,2,3. \quad (2.51)$$

The four-point string amplitude can be written as [Eq. (V.74)] in [34], as described in Eq. (2.38), and we focus on the QFT building block. In our reference spinor choice, we can easily find $\epsilon_i\cdot\epsilon_j = \epsilon_2\cdot k_1 = \epsilon_3\cdot k_1 = \epsilon_1\cdot k_2 = 0$, where i, $j = 1, 2, 3$. We can simplify the four-point QFT building block to[11]

$$\begin{aligned}\mathcal{A}(1^-,2^+,3^+|\underline{4}^{-2}) = \Phi_{\mu\nu}\Big\{&(\epsilon_3\cdot k_2)(\epsilon_1^\mu\epsilon_2^\nu) - (\epsilon_2\cdot k_3)(\epsilon_1^\mu\epsilon_3^\nu) + (\epsilon_1\cdot k_3)(\epsilon_2^\mu\epsilon_3^\nu) + \frac{1}{s_{1,3}}(\epsilon_1\cdot k_3)(\epsilon_3\cdot k_2)(\epsilon_2^\mu k_1^\nu)\\ &+\frac{1}{s_{1,3}}(\epsilon_1\cdot k_3)(\epsilon_3\cdot k_2)(\epsilon_2^\mu k_3^\nu) + \frac{1}{s_{1,3}}(\epsilon_1\cdot k_3)(\epsilon_2\cdot k_3)(\epsilon_3^\mu k^\nu)\\ &+\frac{1}{s_{2,3}}(\epsilon_1\cdot k_3)(\epsilon_3\cdot k_2)(\epsilon_2^\mu k_1^\nu) - \frac{1}{s_{2,3}}(\epsilon_1\cdot k_3)(\epsilon_2\cdot k_3)(\epsilon_3^\mu k_1^\nu)\Big\},\end{aligned} \quad (2.52)$$

and we can now rewrite it into the spinor helicity form:

$$\mathcal{A}(1^-,2^+,3^+|\underline{4}^{-2}) = \frac{\langle 1b\rangle^4[ab]^2}{m^2\langle 12\rangle\langle 23\rangle\langle 31\rangle}. \quad (2.53)$$

The central result of this work is a conjectural generalization to arbitrary n-point QFT building blocks:

$$\mathcal{A}(1^-,2^+,\ldots,n-2^+,n-1^+|\underline{n}^{-2}) = \frac{[ab]^2}{2m^2}\frac{\langle 1b\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n-1,1\rangle}. \quad (2.54)$$

After applying the spin raising operator, we can easily get

$$\begin{aligned}\mathcal{A}(1^-,2^+,\ldots,n-2^+,n-1^+|\underline{n}^{-1}) &= \frac{[ab]^2}{m^2}\frac{\langle 1b\rangle^3\langle 1a\rangle}{\langle 12\rangle\langle 23\rangle\cdots\langle n-1,1\rangle},\\ \mathcal{A}(1^-,2^+,\ldots,n-2^+,n-1^+|\underline{n}^{0}) &= \frac{\sqrt{6}[ab]^2}{2m^2}\frac{\langle 1b\rangle^2\langle 1a\rangle^2}{\langle 12\rangle\langle 23\rangle\cdots\langle n-1,1\rangle},\\ \mathcal{A}(1^-,2^+,\ldots,n-2^+,n-1^+|\underline{n}^{+1}) &= \frac{[ab]^2}{m^2}\frac{\langle 1b\rangle\langle 1a\rangle^3}{\langle 12\rangle\langle 23\rangle\cdots\langle n-1,1\rangle},\\ \mathcal{A}(1^-,2^+,\ldots,n-2^+,n-1^+|\underline{n}^{+2}) &= \frac{[ab]^2}{2m^2}\frac{\langle 1a\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n-1,1\rangle}.\end{aligned} \quad (2.55)$$

[11] We assume all constant factors are hidden in the disk integration building block and thus will not show up in our QFT amplitude.

The aim of this paper is to prove these formulas. For simplification, we will focus on the coupling with n^{-2}. Others can be generated by acting spin raising operator on the polarization.

3. Comparing helicity configurations

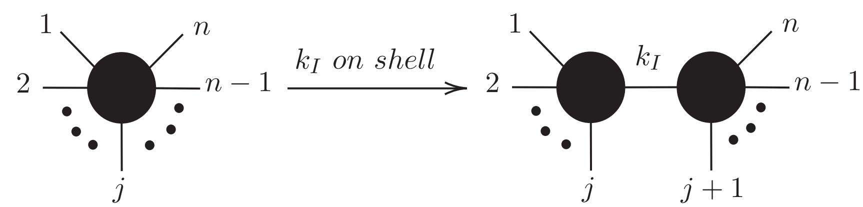
In the previous section, we gave the spinor helicity form of one specific helicity configuration Eq. (2.54), which is similar to the MHV helicity configuration in Yang-Mills theory Eq. (2.49). The helicity configuration in Eq. (2.54) will be referred to as MHV-like helicity configuration. There are also other helicity configurations, and changing helicities in Eq. (2.54) to all-plus would arrive at another QFT building block, which turns out to vanish:

$$\mathcal{A}(1^+, 2^+, \ldots, n-2^+, n-1^+|\underline{n}^{-2}) = 0. \tag{2.56}$$

The vanishing of all-plus QFT building blocks to all multiplicities n will be proven in Appendix A and is analogous to the vanishing of single-minus and all-plus helicity amplitudes in Yang-Mills theory. There are also helicity configurations similar with the NMHV, N^2MHV, … in Yang-Mills theory, $\mathcal{A}(1^-, 2^-, 3^+\ldots, n-1^+|\underline{n})$, for example. The spinor helicity form of them are expected to be much more complicated in the same way as NMHV, N^2MHV, etc. Helicity configurations in pure Yang-Mills theory give rise to more lengthy amplitude formulas than the MHV sector.

E. Basic idea of BCFW recursion

BCFW method [26,43] aims to construct higher point QFT amplitudes using lower point QFT amplitudes. It is based on the factorization property of amplitudes,[12] also known as the unitary requirement of amplitudes. The factorization property can be expressed as



(2.57)

$$\mathcal{A}_n \xrightarrow{k_I \;\; on \;\; shell} \mathcal{A}_L \frac{1}{k_I} \mathcal{A}_R,$$

where $\mathcal{A}_n$ denotes an amplitude with n external legs. While $\mathcal{A}_L$ and $\mathcal{A}_R$ denote subamplitudes on the left-hand side and right-hand side separately, momentum k_I is the internal momentum that flows from left to right or right to left, up to our choice. Suppose the external momentum of the left-hand side subamplitude is $k_1, k_2, \cdots k_i$ and define set I:

$$I = \{1, 2, 3, \ldots i\}. \tag{2.58}$$

We set the direction of all n external momentum outward, and k_I flows from left to right. After applying momentum conservation, we have $k_I = k_1 + k_2 + \cdots + k_i$. On the other hand, the external legs of $\mathcal{A}_R$ are denoted by set J:

$$J = \{i+1, i+2, \cdots n\}, \tag{2.59}$$

for the same reason, $k_I = -k_{i+1} - k_{i+2} - \cdots - k_n$.

For most of the amplitudes we might need, k_I is off shell.[13] After complex shifting external momentum,

$$\hat{k}_i^\mu = k_i^\mu + z r_i^\mu \qquad z \in \mathbb{C} \qquad i \in I, \tag{2.60}$$

and with some constraints on r, we can solve z_I which makes k_I on shell. The on-shell condition of external momentum and momentum conservation is preserved. k_I is thus shifted as

$$\hat{k}_I = k_I + z \sum_{i \in I} r_i. \tag{2.61}$$

The complex shift $z r_i$ should not influence the fundamental property of the amplitude and the shifted momentum. Thus, we want the external legs still satisfy their on-shell condition with mass unchanged. This will lead to some constraints on r_i. All r_i satisfying the constraints can be the

[12]When the propagator becomes on shell, the whole amplitude will decompose into products of two lower-multiplicity amplitudes.
[13]Which is easily seen using momentum conservation and the on-shell requirement for external momentum.

shift we use. To give a clean and convenient form of the expressions, we always choose the r_i as simple as possible and choose the suitable z, making $\hat{k}_I^2 = 0$ for some I, and factorize the amplitude we need into the multiplication of two subamplitudes and a pole on the z plane. Each propagator corresponds to a pole on the complex plane. According to the Cauchy theorem, we have

$$0 = \sum_{z_I} \mathrm{Res}_{z=z_I} \frac{\hat{\mathcal{A}}_n(z)}{z} - B_n, \qquad (2.62)$$

where the index I in z_I denotes the pole corresponding to k_I on shell, and B_n is the residual of the pole at the infinite point.

Especially, $z_I = 0$ stands for the amplitude without shifting external momentum, which is the original amplitude. We have

$$\mathcal{A}_n = -\sum_{z_I \neq 0} \mathrm{Res}_{z=z_I} \frac{\hat{\mathcal{A}}_n(z)}{z} + B_n. \qquad (2.63)$$

III. RESULT AT FIVE POINTS

In this section, we will take the five-point case as an example. Using the B-G recursion, we can analytically prove that the five-point QFT building blocks are

$$\begin{aligned}
\mathcal{A}(1^{-1},2^{+1},3^{+1},4^{+1}|\underline{5}^{-2}) &= \frac{[ab]^2}{2m^2}\frac{\langle 1b\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle},\\
\mathcal{A}(1^{-1},2^{+1},3^{+1},4^{+1}|\underline{5}^{-1}) &= \frac{[ab]^2}{m^2}\frac{\langle 1b\rangle^3\langle 1a\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle},\\
\mathcal{A}(1^{-1},2^{+1},3^{+1},4^{+1}|\underline{5}^{0}) &= \frac{\sqrt{6}[ab]^2}{2m^2}\frac{\langle 1b\rangle^2\langle 1a\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle},\\
\mathcal{A}(1^{-1},2^{+1},3^{+1},4^{+1}|\underline{5}^{+1}) &= \frac{[ab]^2}{m^2}\frac{\langle 1b\rangle\langle 1a\rangle^3}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle},\\
\mathcal{A}(1^{-1},2^{+1},3^{+1},4^{+1}|\underline{5}^{+2}) &= \frac{[ab]^2}{2m^2}\frac{\langle 1a\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}. \qquad (3.1)
\end{aligned}$$

By using B-G currents, we construct the QFT building block for $|-2,2\rangle$ state coupling with four gluons and generate the coupling of other spins by applying the raising and lowering operator Eq. (2.26) on polarization tensor $\Phi^{\mu\nu}$.

The underlined external leg stands for the massive leg. Here is particle 5; we decompose the momentum p_5^μ into $p_5^\mu = a^\mu + b^\mu$, where a and b are both massless. After we work out the result for $\mathcal{A}(1^{-1}, 2^{+1}, 3^{+1}, 4^{+1}|\underline{5}^{-2})$, we would use the raising and lowering operator on the amplitude and get the result for different spin state of particle 5. Thus, in this section, we only work out the coupling with $|-2,2\rangle$ state of particle 5. For the $|-2,2\rangle$ state, the polarization $\Phi^{\mu\nu}$ can be decomposed as

$$\Phi_{\mu\nu}^{(-2)} = \underline{\epsilon}_\mu^{(-1)}\underline{\epsilon}_\mu^{(-1)}, \qquad (3.2)$$

where the superscript (-2) and (-1) is just an index identifying the spin choice of the polarization. Since we only analyze the coupling with spin -2 state of particle 5, we will omit this index.

According to the definition Eq. (2.42), we have

$$\begin{aligned}
\mathcal{A}(1^{-1},2^{+1},3^{+1},4^{+1}|\underline{5}^{-2}) &= (\Phi_{1234})^{\mu\nu}(\Phi_5)_{\mu\nu}\\
&= \sqrt{\alpha'}\sum_{P=QR} F^{\mu}_{Q\rho}F^{\rho\nu}_{R}(\Phi_5)_{\mu\nu} + \mathrm{cyc}_P,
\end{aligned} \qquad (3.3)$$

where $P = 1234$.

A. Some basic facts

Some equations can help us simplify our calculation. Before we move on, let us derive them first.[14]

As before, we take the reference spinor of particle 1 to be λ_2, and the reference spinor of all other massless particles[15] $r_2, r_3 \ldots, r_j = \lambda_1$. Thus, all polarization vectors of massless particles have $\langle 1|$ or $|1\rangle$ in their numerator; by using the Fierz identities Eq. (2.10), we can easily find that the contraction between any two polarization vectors of a massless particle gives zero. This special choice of reference spinor can bring us more information.

First, we can express the Berends-Giele currents in terms of spinor helicities. Two examples are Eqs. (2.47) and (2.48). With our reference spinor choice, we rewrite the closed form of the B-G current for massless states in the QFT building block of type-I superstring as follows:

$$\begin{aligned}
\epsilon^\mu(i^+, i+1^+, \ldots, n^+)|_{r^\mu_{i,i+1,\ldots,n}=k_1^\mu} &= \frac{\langle 1|\bar{\sigma}^\mu \not{k}_{i,i+1,\ldots,n}|1\rangle}{\sqrt{2}\langle 1i\rangle\langle i, i+1\rangle\cdots\langle n-1, n\rangle\langle n1\rangle}\\
\epsilon^\mu(1^-, 2^+, \ldots, n^+)|_{r^\mu_{2,3,\ldots,n}=k_1^\mu, r_1=k_2^\mu} &= \frac{\langle 1|\bar{\sigma}^\mu \not{k}_{2,3,\ldots,n}|1\rangle}{\sqrt{2}\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}\sum_{m=3}^{n}\frac{\langle 1|\not{k}_m\not{k}_{1,2,\ldots,m}|1\rangle}{k^2_{1,2,\ldots,m-1}k^2_{1,2,\ldots,m}}.
\end{aligned} \qquad (3.4)$$

[14] These equations work for arbitrary n.

[15] There is no need to define a reference spinor for a massive particle.

Under this reference spinor choice, using B-G recursion, we can easily show that the spinor helicity closed form of a B-G current always has a similar numerator structure $\langle 1|\bar{\sigma}^{\mu}\not{k}_{i,j}|1\rangle$. Therefore, if we contract any two B-G currents, using Fierz identity Eq. (2.10), we immediately get a result proportional to $\langle 11\rangle = 0$.

Second, let us contract any B-G current with momentum k_1^{μ}; we can always get a result proportional to $\langle 1|\not{k}_1$. We can expand $\not{k}_1 = -|1\rangle[1|$. The final result is proportional to $\langle 11\rangle = 0$. For the same reason, $\epsilon_1 \cdot k_2 = 0$.

Third, the B-G current we use is the same as the one for Yang-Mills amplitude, which is $\epsilon_{12}^{\mu}|_{r_2^{\mu}=k_1^{\mu},r_1=k_2^{\mu}} = \epsilon_{21}^{\mu}|_{r_2^{\mu}=k_1^{\mu},r_1=k_2^{\mu}} = 0$:

$$\begin{aligned}\epsilon_{12}^{\mu}|_{r_2^{\mu}=k_1^{\mu},r_1=k_2^{\mu}} &= \frac{1}{2s_{12}}[(k_2\cdot\epsilon_1)\epsilon_2^{\mu}-(k_1\cdot\epsilon_2)\epsilon_1^{\mu}]\Big|_{r_2^{\mu}=k_1^{\mu},r_1=k_2^{\mu}} \\ &= \frac{1}{2s_{12}}[(k_2\cdot\epsilon_1)\epsilon_2^{\mu}-(k_1\cdot\epsilon_2)\epsilon_1^{\mu}]\Big|_{r_2^{\mu}=k_1^{\mu},r_1=k_2^{\mu}} = 0.\end{aligned} \tag{3.5}$$

B. Contributions of each configuration

In Eq. (3.3), $P = 1, 2, 3, 4$ are divided into Q and R. Thus, we can write down all the configurations of Q and R before we calculate each of them[16]:

Configurations	1	2	3	4	5	6
Q	1	1, 2	1, 2, 3	2	2, 3	3
R	2, 3, 4	3, 4	4	3, 4, 1	4, 1	4, 1, 2

. (3.6)

1. The only nonzero configuration

Among all six configurations in Eq. (3.6), there is only one configuration that nontrivially contributes to the final result of the five-point QFT building block, which is configuration 1. Consider the coupling with $|-2,2\rangle$ state for computation simplicity:

$$\begin{aligned}F_{1\rho}^{\mu}F_{234}^{\rho\nu}(\Phi_5)_{\mu\nu} &= (k_1^{\mu}\epsilon_{1\rho}-k_{1\rho}\epsilon_1^{\mu})[k_{234}^{\rho}\epsilon_{234}^{\nu}-k_{234}^{\nu}\epsilon_{234}^{\rho}-(\epsilon_2^{\rho}\epsilon_{34}^{\nu}-\epsilon_2^{\nu}\epsilon_{34}^{\rho}+\epsilon_{23}^{\rho}\epsilon_4^{\nu}-\epsilon_{23}^{\nu}\epsilon_4^{\rho})](\Phi_5)_{\mu\nu} \\ &= (k_1^{\mu}\epsilon_{1\rho}-k_{1\rho}\epsilon_1^{\mu})[k_{234}^{\rho}\epsilon_{234}^{\nu}-k_{234}^{\nu}\epsilon_{234}^{\rho}](\Phi_5)_{\mu\nu},\end{aligned} \tag{3.7}$$

where we applied the basic rules and concluded in Sec. III A in the second step. After converting into the spinor helicity form and applying Eq. (2.47), we get

$$\begin{aligned}F_{1\rho}^{\mu}F_{234}^{\rho\nu}(\Phi_5)_{\mu\nu} &= \frac{\langle b|\not{k}_1|a]}{\sqrt{2}m}\frac{[2|\not{k}_{234}|1\rangle}{\sqrt{2}[12]}\frac{\langle 1b\rangle[a|\not{k}_{234}|1\rangle}{m\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}+2\frac{[2a]\langle b1\rangle}{2[12]m}(k_1\cdot k_{234})\frac{\langle 1b\rangle[a|\not{k}_{234}|1\rangle}{m\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \\ &= \frac{\langle 1b\rangle^4[ab]^2}{2m^2\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle},\end{aligned} \tag{3.8}$$

where this is exactly the five-point amplitude we claimed.

2. Other configurations vanish individually

We can analytically prove that except for the configuration mentioned in Sec. III B 1, all other five configurations vanish individually. We will show these one by one:

[16]Notice $(\Phi_5)_{\mu\rho}$ is symmetric in m and p indices. Thus, we can exchange Q and R without changing the final result. This decreases the number of configurations by a factor of 2.

(i) Configuration 2
Configuration 2, $F^{\mu}_{12\rho}F^{\rho\nu}_{34}(\Phi_5)_{\mu\nu}$ equals to

$$\begin{aligned} F^{\mu}_{12\rho}F^{\rho\nu}_{34}(\Phi_5)_{\mu\nu} &= [k^{\mu}_{12}\epsilon_{12\rho} - k_{12\rho}\epsilon^{\mu}_{12} - (\epsilon^{\mu}_{1}\epsilon_{2\rho} - \epsilon_{1\rho}\epsilon^{\mu}_{2})][k^{\rho}_{34}\epsilon^{\nu}_{34} - k^{\nu}_{34}\epsilon^{\rho}_{34} - (\epsilon^{\rho}_{3}\epsilon^{\nu}_{4} - \epsilon^{\nu}_{3}\epsilon^{\rho}_{4})](\Phi_5)_{\mu\nu} \\ &= -(\epsilon^{\mu}_{1}\epsilon_{2\rho} - \epsilon_{1\rho}\epsilon^{\mu}_{2})k^{\rho}_{34}\epsilon^{\nu}_{34}(\Phi_5)_{\mu\nu} = 0. \end{aligned} \tag{3.9}$$

(ii) Configuration 3
Configuration 3, $F^{\mu}_{123\rho}F^{\rho\nu}_{4}(\Phi_5)_{\mu\nu}$ equals to

$$\begin{aligned} F^{\mu}_{123\rho}F^{\rho\nu}_{4}(\Phi_5)_{\mu\nu} &= [k^{\mu}_{123}\epsilon_{123\rho} - k_{123\rho}\epsilon^{\mu}_{123} - (\epsilon^{\mu}_{1}\epsilon_{23\rho} - \epsilon_{1\rho}\epsilon^{\mu}_{23} + \epsilon^{\mu}_{12}\epsilon_{3\rho} - \epsilon_{12\rho}\epsilon^{\mu}_{3})][k^{\rho}_{4}\epsilon^{\nu}_{4} - k^{\nu}_{4}\epsilon^{\rho}_{4}](\Phi_5)_{\mu\nu} \\ &= -\frac{4\langle 1b\rangle^2[24](k_{14}\cdot k_{23}\langle 1|\not{k}_{23}|a][4a] + \langle 1|\not{k}_{23}|4]\langle b|\not{k}_{14}|a][ba])}{m^2\langle 12\rangle\langle 13\rangle\langle 14\rangle\langle 23\rangle[14][21]} \\ &\quad + \frac{4\langle b1\rangle^2[4a]^2[24][23]}{m^2\langle 12\rangle\langle 13\rangle[14][21]} - \frac{4\langle b1\rangle^2[24]\langle 1|\not{k}_{23}|a]^2}{m^2\langle 12\rangle\langle 13\rangle\langle 14\rangle\langle 23\rangle[21]} = 0. \end{aligned} \tag{3.10}$$

(iii) Configuration 4
Configuration 4, $F^{\mu}_{2\rho}F^{\rho\nu}_{341}(\Phi_5)_{\mu\nu}$ equals to

$$\begin{aligned} F^{\mu}_{2\rho}F^{\rho\nu}_{341}(\Phi_5)_{\mu\nu} &= [k^{\mu}_{2}\epsilon_{2\rho} - k_{2\rho}\epsilon^{\mu}_{2}][k^{\rho}_{341}\epsilon^{\nu}_{341} - k^{\nu}_{341}\epsilon^{\rho}_{341} - (\epsilon^{\rho}_{34}\epsilon^{\nu}_{1} - \epsilon^{\nu}_{34}\epsilon^{\rho}_{1} + \epsilon^{\rho}_{3}\epsilon^{\nu}_{41} - \epsilon^{\nu}_{3}\epsilon^{\rho}_{41})](\Phi_5)_{\mu\nu} \\ &= -\frac{4\langle 1b\rangle^2[2a]^2(\langle 13\rangle[14][32] + 2k_4\cdot k_{13}[42])}{m^2\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle[14][21]} + \frac{4\langle 1b\rangle^2[2a]^2([41]\langle 1|\not{k}_{34}|2] + 2k_3\cdot k_4[42])}{m^2\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle[14][21]} = 0. \end{aligned} \tag{3.11}$$

(iv) Configuration 5
Configuration 5, $F^{\mu}_{23\rho}F^{\rho\nu}_{41}(\Phi_5)_{\mu\nu}$ equals to

$$\begin{aligned} F^{\mu}_{23\rho}F^{\rho\nu}_{41}(\Phi_5)_{\mu\nu} &= [k^{\mu}_{23}\epsilon_{23\rho} - k_{23\rho}\epsilon^{\mu}_{23} - (\epsilon^{\mu}_{2}\epsilon_{3\rho} - \epsilon_{2\rho}\epsilon^{\mu}_{3})][k^{\rho}_{41}\epsilon^{\nu}_{41} - k^{\nu}_{41}\epsilon^{\rho}_{41} - (\epsilon^{\rho}_{4}\epsilon^{\nu}_{1} - \epsilon^{\nu}_{4}\epsilon^{\rho}_{1})](\Phi_5)_{\mu\nu} \\ &= -\frac{4\langle 1b\rangle^2[24](\langle 1|\not{k}_{23}|4]\langle b|\not{k}_{23}|a][ab] + k_{14}\cdot k_{23}\langle 1|\not{k}_{23}|a][4a])}{m^2\langle 12\rangle\langle 13\rangle\langle 14\rangle\langle 23\rangle[14][21]} \\ &\quad + \frac{4\langle b1\rangle^2[4a]^2[24][23]}{m^2\langle 12\rangle\langle 13\rangle[14][21]} - \frac{4\langle b1\rangle^2[24]\langle 1|\not{k}_{23}|a]^2}{m^2\langle 12\rangle\langle 13\rangle\langle 14\rangle\langle 23\rangle[21]} = 0. \end{aligned} \tag{3.12}$$

(v) Configuration 6
Configuration 6, $F^{\mu}_{3\rho}F^{\rho\nu}_{412}(\Phi_5)_{\mu\nu}$ equals to

$$\begin{aligned} F^{\mu}_{3\rho}F^{\rho\nu}_{412}(\Phi_5)_{\mu\nu} &= (k^{\mu}_{3}\epsilon_{3\rho} - k_{3\rho}\epsilon^{\mu}_{3})[k^{\rho}_{412}\epsilon^{\nu}_{412} - k^{\nu}_{412}\epsilon^{\rho}_{412} - (\epsilon^{\rho}_{4}\epsilon^{\nu}_{12} - \epsilon^{\nu}_{4}\epsilon^{\rho}_{12} + \epsilon^{\rho}_{41}\epsilon^{\nu}_{2} - \epsilon^{\nu}_{41}\epsilon^{\rho}_{2})](\Phi_5)_{\mu\nu} \\ &= -\frac{4\langle 1b\rangle^2[24]^2[3a](\langle 1|\not{k}_{24}|a]k_3\cdot k_5 + \langle 1|\not{k}_{24}|3]\langle 3|\not{p}|a])}{m^2k^2_{12}k^2_{14}\langle 13\rangle k^2_{124}} + \frac{4\langle 1b\rangle^2[24]^2[3a]^2}{m^2k^2_{12}k^2_{14}} = 0. \end{aligned} \tag{3.13}$$

IV. THE N-POINT AMPLITUDE

One can easily apply Eqs. (2.33), (2.34) and the definition of Φ_P Eq. (2.40) for multiple particles to the n-point amplitude Eq. (2.42) and count the number of terms in the B-G recursions. More specifically, we shall count the number of configurations $F^{\mu}_{Q\rho}F^{\rho\nu}_{R}$ compatible with the cyclic ordering of $\mathcal{A}(1,2,\ldots,n-1|\underline{n})$. Similar to Eq. (3.6), the number of configurations for n-point amplitude is $\frac{n^2-3n+2}{2}$, which increases at order of $O(n^2)$. From the computation of five points, one can also tell that the complexity of each configuration increases when considering higher point amplitudes. Thus, we cannot analytically work out the closed form one by one. However, there is some special configuration that we can easily work out for arbitrary

n point, and one of them equals to the n point QFT building block we claimed:

$$\mathcal{A}(\hat{1}^-, 2^+, \ldots, n-2^+, n-1^+|\underline{\hat{n}}^{-2}) = \frac{[ab]^2}{2m^2}\frac{\langle 1b\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n-1,1\rangle}. \tag{4.1}$$

A. The nonzero configuration

Similar to the five-point example, one configuration contributes to the closed form we expected for the n-point amplitude:

$$\begin{aligned} F^{\mu}_{1\rho}F^{\rho\nu}_{23\ldots n-1}(\Phi_n)_{\mu\nu} &= (k_1^\mu\epsilon_{1p} - k_{1p}\epsilon_1^\mu)[k^\rho_{23\ldots n-1}\epsilon^\nu_{23\ldots n-1} - k^\nu_{23\ldots n-1}\epsilon^\rho_{23\ldots n-1} - (\epsilon_2^\rho\epsilon^\nu_{3\ldots n} - \epsilon_2^\nu\epsilon^\rho_{3\ldots n} + \cdots)](\Phi_n)_{\mu\nu} \\ &= (k_1^\mu\epsilon_{1p} - k_{1p}\epsilon_1^\mu)[k^\rho_{23\ldots n-1}\epsilon^\nu_{23\ldots n-1} - k^\nu_{23\ldots n-1}\epsilon^\rho_{23\ldots n-1}](\Phi_n)_{\mu\nu}, \end{aligned} \tag{4.2}$$

where in the last step, we used the important facts in Sec. III A that any two B-G currents contract to zero, and k_1 contracting with any B-G currents gives zero. After applying the spinor helicity form, the QFT building block of $|-2,2\rangle$ state coupling with $n-1$ gluons equals to

$$\begin{aligned} F^{\mu}_{1\rho}F^{\rho\nu}_{23\ldots n-1}(\Phi_n)_{\mu\nu} &= \frac{\langle b|\not{k}_1|a]}{\sqrt{2}m}\frac{[2|\not{k}_{23\ldots n-1}|1\rangle}{\sqrt{2}[12]}\frac{\langle 1b\rangle[a|\not{k}_{23\ldots n-1}|1\rangle}{m\langle 12\rangle\langle 23\rangle\cdots\langle n-2,n-1\rangle\langle n-1,1\rangle} \\ &\quad + 2\frac{[2a]\langle b1\rangle}{2[12]m}(k_1\cdot k_{23\ldots n-1})\frac{\langle 1b\rangle[a|\not{k}_{23\ldots n-1}|1\rangle}{m\langle 12\rangle\langle 23\rangle\cdots\langle n-2,n-1\rangle\langle n-1,1\rangle} \\ &= \frac{[ab]^2}{2m^2}\frac{\langle 1b\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n-2,n-1\rangle\langle n-1,1\rangle}, \end{aligned} \tag{4.3}$$

where we used Schouten identity, momentum conservation, and Fierz identity. It is exactly what we claimed in Eq. (2.54)

B. Other configurations

For other configurations, we believe they are individually equal to zero as it goes for a five-point amplitude. We can prove one of them equals zero analytically, but for others, what we can do so far is numerically check up to six points. This numerical check provided the confirmation of Eq. (2.54) at $n = 6$. Similarly, one can give a numerical check for a $n = 7$ or higher point QFT building block.

We can prove that one configuration goes to zero individually for n-point amplitude. This is a generalization of configuration 2 in Eq. (3.6). More specifically, we divide $1, 2, 3, \ldots, n$ into 1,2 and $3, \ldots, n$. Still, we focus on the QFT building block of spin -2 massive particle coupling with gluons:

$$\begin{aligned} F^{\mu}_{12\rho}F^{\rho\nu}_{3\ldots n-1}(\Phi_n)_{\mu\nu} &= [k^\mu_{12}\epsilon_{12\rho} - k_{12\rho}\epsilon^\mu_{12} - (\epsilon_1^\mu\epsilon_{2\rho} - \epsilon_{1\rho}\epsilon_2^\mu)] \\ &\quad\times[k^\rho_{3\ldots,n-1}\epsilon^\nu_{3\ldots,n-1} - k^\nu_{3\ldots,n-1}\epsilon^\rho_{3\ldots,n-1} - (\epsilon_3^\rho\epsilon^\nu_{4\ldots,n} - \epsilon_3^\nu\epsilon^\rho_{4\ldots,n} + \cdots)](\Phi_n)_{\mu\nu} \\ &= -(\epsilon_1^\mu\epsilon_{2\rho} - \epsilon_{1\rho}\epsilon_2^\mu)k^\rho_{3\ldots,n-1}\epsilon^\nu_{3\ldots,n-1}(\Phi_n)_{\mu\nu} \\ &\propto\left(\frac{\langle 1b\rangle[a2]\langle 1|\not{k}_{3\ldots,n-1}|2]}{k_{12}^2} - \frac{\langle 1|\not{k}_{3\ldots,n-1}|2]\langle 1b\rangle[a2]}{k_{12}^2}\right) \\ &= 0, \end{aligned} \tag{4.4}$$

and as for other configurations, we can numerically check they vanish separately for a six-point amplitude. Thus, we believe Eq. (4.3) is the only nonzero configuration that contributes to the expected result for an arbitrary n point, and all other configurations sum to zero. We will later on prove this using the BCFW recursion.

V. APPLICATION OF BCFW RECURION ON THE QFT BUILDING BLOCK

Considering the helicity configuration, the naive BCFW shifted leg should be 1 and n, and the external leg n is massive; thus, the BCFW shift for the massive particle is needed. We will show in the appendix that this kind of shift can indeed provide us with the expected result, but there exists a tentative boundary term. In this section, we will show if we shift leg 2 and 3 instead, we can still find the expected result without any BCFW boundary.

The momentum is shifted as

$$\hat{k}_2^\mu = k_2^\mu + z r^\mu \qquad \hat{k}_3^\mu = k_3^\mu - z r^\mu, \tag{5.1}$$

where the $\not{r}$ needs to satisfy was chosen to be

$$\not{r} = |3]\langle 2| + |2\rangle[3|. \tag{5.2}$$

Shifting the momentum leads to the following shifted spinor helicities:

$$|\hat{2}] = |2] + z|3] \\ |\hat{3}\rangle = |3\rangle - z|2\rangle, \tag{5.3}$$

where other spinor helicity is unshifted, $|\hat{2}\rangle = |2\rangle$ and $|\hat{3}] = |3]$. We have the following relations:

$$\langle \hat{2}\,\hat{3}\rangle = \langle 23\rangle \qquad [\hat{2}\,\hat{3}] = \langle 23\rangle. \tag{5.4}$$

We can now rewrite the four-point amplitude into spinor helicity form:

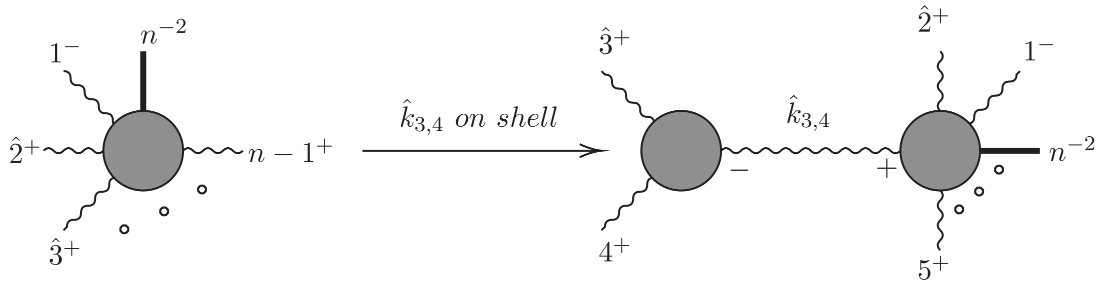
$$\mathcal{A}(1^-, 2^+, 3^+|\underline{4}^{-2}) = \frac{[ab]^2}{2m^2}\frac{\langle 1b\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}. \tag{5.5}$$

To prove the expected form, we need to assume the $n-1$ point amplitudes satisfy

$$\mathcal{A}(1^-, 2^+, k_I^+, 5^+, \ldots, n-2^+, n-1^+|\underline{n}^{-2}) \\ = \frac{[ab]^2}{2m^2}\frac{\langle 1b\rangle^4}{\langle 12\rangle\langle 2k_I\rangle\langle k_I 5\rangle\cdots\langle n-1,1\rangle} \tag{5.6}$$

and show that the existence of the expected form for any $n-1$ point always leads to the existence of n point since we already proved the five-point case using B-G recursion; if such $n-1 \to n$ always exists, such form would be true for an arbitrary n-point amplitude. The shifted n-point amplitude equals the sum of all residues of poles on the z plane, which corresponds to the sum of different configurations of subamplitudes.

However, one can easily see that there exists only one pole that gives a nonvanishing contribution when we shift 2 and 3, which is:



The 12-channel here does not contribute because of the special three-point kinematics, which is:

$$0 = k_{1,2}^2 = \langle 1\hat{2}\rangle[1\hat{2}] = \langle 12\rangle[1\hat{2}]. \tag{5.7}$$

The only way to make the RHS vanish is to set $[1\hat{2}] = 0$, but it leads to the subamplitude $\mathcal{A}(1^-, 2^+, -k_I^+) = 0$. Other poles either have all-plus helicity for three-point Yang-Mills amplitude or have single-minus helicity for four or higher-point Yang-Mills amplitude, which vanishes due to the mostly helicity violation (MHV) requirement.

The three-point pure Yang-Mills amplitude can be written as

$$\hat{\mathcal{A}}(\hat{3}^+, 4^+, -k_I^-) = \frac{[\hat{3}4]^4}{[\hat{3}4][4\hat{k}_I][\hat{k}_I\hat{3}]}, \tag{5.8}$$

where $I = 3, 4$ and $k_I = k_{3,4}$, and we construct the n-point amplitude from $n-1$ point amplitude using the three-point pure Yang-Mills amplitude as a building block:

$$\mathcal{A}(1^-, 2^+, 3^+\ldots, n-2^+, n-1^+|\underline{n}^{-2}) = \hat{\mathcal{A}}_L\frac{1}{k_I^2}\hat{\mathcal{A}}_R \\ = \frac{[ab]^2}{2m^2}\frac{\langle 1b\rangle^4}{\langle 1\hat{2}\rangle\langle \hat{2}\hat{k}_I\rangle\langle \hat{k}_I 5\rangle\cdots\langle n-1,1\rangle}\times\frac{1}{\langle \hat{3}4\rangle[\hat{3}4]} \\ \times\frac{[\hat{3}4]^4}{[\hat{3}4][4\hat{k}_I][\hat{k}_I\hat{3}]}. \tag{5.9}$$

We can simplify the above equation using

$$[\hat{3}\hat{k}_I]\langle\hat{k}_I 5\rangle = [34]\langle 45\rangle$$
$$\langle\hat{2}\hat{k}_I\rangle[\hat{k}_I\hat{4}] = \langle 23\rangle[34]. \tag{5.10}$$

Finally, we get the n-point amplitude:

$$\mathcal{A}(1^-,2^+,\ldots,n-2^+,n-1^+|\underline{n}^{-2}) = \frac{[ab]^2}{2m^2}\frac{\langle 1b\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n-1,1\rangle}, \tag{5.11}$$

where the boundary comes from the z dependence of the amplitude. More specifically, the large z limit of the amplitude corresponds to the boundary. Since we shift momentum 2, 3, the naive z dependence we can read from the amplitude exists in $\langle 34\rangle$. Thus, the naive z dependence is $\frac{1}{z}$, which leads to the vanishing boundary. One can produce the n-point generalization of Eq. (3.1):

$$\mathcal{A}(1^-,2^+,\ldots,n-2^+,n-1^+|\underline{n}^{-2}) = \frac{[ab]^2}{2m^2}\frac{\langle 1b\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n-1,1\rangle},$$
$$\mathcal{A}(1^-,2^+,\ldots,n-2^+,n-1^+|\underline{n}^{-1}) = \frac{[ab]^2}{m^2}\frac{\langle 1b\rangle^3\langle 1a\rangle}{\langle 12\rangle\langle 23\rangle\cdots\langle n-1,1\rangle},$$
$$\mathcal{A}(1^-,2^+,\ldots,n-2^+,n-1^+|\underline{n}^{0}) = \frac{\sqrt{6}[ab]^2}{2m^2}\frac{\langle 1b\rangle^2\langle 1a\rangle^2}{\langle 12\rangle\langle 23\rangle\cdots\langle n-1,1\rangle},$$
$$\mathcal{A}(1^-,2^+,\ldots,n-2^+,n-1^+|\underline{n}^{+1}) = \frac{[ab]^2}{m^2}\frac{\langle 1b\rangle\langle 1a\rangle^3}{\langle 12\rangle\langle 23\rangle\cdots\langle n-1,1\rangle},$$
$$\mathcal{A}(1^-,2^+,\ldots,n-2^+,n-1^+|\underline{n}^{+2}) = \frac{[ab]^2}{2m^2}\frac{\langle 1a\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n-1,1\rangle}, \tag{5.12}$$

using the raising and lowering operator as mentioned before.

VI. CONCLUSION

In this paper, we reviewed connections between tree-level amplitudes of twisted heterotic strings and conventional type-I superstrings [13]. By using B-G currents and their recursion, we explicitly worked out the formula for the QFT building block of four-point and five-point cases. After that, we conjectured a compact formula Eq. (2.54) for the QFT building block of n-point conventional type-I superstring amplitude of the spin-2 state at the first mass level coupled to $n-1$ gluons in spinor helicity basis. In Sec. V, we finally used the recursive construction to approach the n-point formula and arrived at Eq. (5.11), which is exactly Eq. (2.54); this provided the proof of the desired formula using BCFW recursion.

This formula can be regarded as a n-point generalization from the four-point coupling of one single spin-2 massive state and massless states in [34]. More importantly, by the choice of gluon helicities $(1^-,2^+,3^+,\ldots,n-1^+)$, the n-point conjecture Eq. (2.54) can be viewed as a massive extension of the Parker-Taylor formula [24] for pure-gluon tree amplitudes with MHV helicities. We also offered various pieces of evidence: We provided an analytic proof at five points and a numerical check at six points.

One possible generalization of this topic is toward amplitudes with more massive states or higher excited states, and one can find detailed studies for mass level 2 states in [44,45]. What is more, if we generalize to two higher excited massive states, such higher-spin amplitudes were recently used to study classical Kerr black-hole scattering [39,46].

Another direct generalization is to consider the NMHV-like QFT building block of conventional type-I string amplitude, which is having two or more massless particles having minus helicity. We have proven that the QFT building block with all massless particles having plus helicity vanishes; this is similar to the behavior of MHV violated Yang-Mills amplitude. One can expect that the NMHV-like amplitude would be more complicated, as how NMHV amplitude behaves in Yang-Mills theory.

The massive double copy mentioned in this work emerges from the consistency constraints at low multiplicity [47]. The all-multiplicity result could serve as an important consistency check for future studies.

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APPENDIX A: PROOF OF VANISHING ALL-PLUS HELICITY CONFIGURATION

In this appendix, we will discuss the QFT building block with all-plus helicity configuration[17]: $\mathcal{A}(1^+,2^+,\ldots,n-2^+,n-1^+|\underline{n}^{-2})$. We claim that the QFT building block with this helicity choice equals zero.

We cannot use the old reference spinor helicity choice as before since all the helicities of gluons are the same. In this appendix, we choose all the reference spinors of gluons to be $\langle b|$, and the polarization vectors can be written as

$$\epsilon_i^\mu|_{r^\mu=b^\mu}=\frac{\langle b|\bar{\sigma}^\mu|i]}{\langle \mathrm{bi}\rangle}. \tag{A1}$$

By using B-G currents Eq. (2.42) and its definition Eq. (2.40), we can express the QFT building block as we did in Eq. (3.3), but with $P=1,2,3,\ldots,n-1$ this time:

$$\mathcal{A}(1^+,2^+,\ldots,n-2^+,n-1^+|\underline{n})=(\Phi_P)^{\mu\nu}(\Phi_n)_{\mu\nu}$$
$$=\sqrt{\alpha'}\sum_{P=QR}F^\mu_{Q\rho}F^{\rho\nu}_R(\Phi_n)_{\mu\nu}+\mathrm{cyc}_P, \tag{A2}$$

where we can apply Eq. (2.33) and expand $F^\mu_{Q\rho}$ and $F^{\rho\nu}_R$. Since we are considering the all-plus helicity configuration, we can apply Eq. (3.4) to ϵ^μ_X, where X can be any multiple-particle label:

$$\epsilon^\mu(i^+,i+1^+,\ldots,n^+)|_{r^\mu_{i,i+1,\ldots,n}=b^\mu}$$
$$=\frac{\langle b|\bar{\sigma}^\mu \not{k}_{i,i+1,\ldots,n}|b\rangle}{\sqrt{2}\langle \mathrm{bi}\rangle\langle i,i+1\rangle\cdots\langle n-1,n\rangle\langle nb\rangle}. \tag{A3}$$

By using Fierz identity Eq. (2.10), we can easily find any contraction among polarization vectors or B-G currents equal to zero. For the same reasons, reference spinors $|r_i\rangle=|b\rangle$ lead to vanishing contractions $\epsilon^\mu_P(\Phi_n)_{\mu\nu}$ for the arbitrary multiparticle P. As a consequence of our choice of reference spinors,

$$\epsilon_P\cdot\epsilon_Q|_{r_i^\mu=b^\mu}=0, \qquad \epsilon^\mu_P(\Phi_n)_{\mu\nu}|_{r_i^\mu=b^\mu}=0. \tag{A4}$$

Each contribution $F^\mu_{Q\rho}F^{\rho\nu}_R(\Phi_n)_{\mu\nu}$ to the B-G formula Eq. (2.42) vanishes in the all-plus helicity configuration:

$$F^\mu_{Q\rho}F^{\rho\nu}_R(\Phi_n)_{\mu\nu}|_{r_i^\mu=b^\mu}$$
$$=\left[k^\mu_Q\epsilon_{Q\rho}-k_{Q\rho}\epsilon^\mu_Q-\sum_{XY=Q}(\epsilon^\mu_X\epsilon_{Y\rho}-\epsilon_{X\rho}\epsilon^\mu_Y)\right]$$
$$\times\left[k^\rho_R\epsilon^\nu_R-k^\nu_R\epsilon^\rho_R-\sum_{ZW=R}(\epsilon^\rho_Z\epsilon^\nu_W-\epsilon^\nu_Z\epsilon^\rho_W)\right](\Phi_n)_{\mu\nu}|_{r_i^\mu=b^\mu}$$
$$=-k^\mu_Qk^\nu_R(\epsilon_Q\cdot\epsilon_R)(\Phi_n)_{\mu\nu}|_{r_i^\mu=b^\mu}=0, \tag{A5}$$

where the second step follows from discarding any $\epsilon^\mu_P(\Phi_n)_{\mu\nu}$, and we finally use the vanishing of $\epsilon_Q\cdot\epsilon_R$ in the last step. Thus, the QFT building block with all plus helicity choice vanishes.

Or we can use a similar power counting as in Sec. II D; there are $n-1$ momentum vectors in each term, but there exist $n-1$ polarization vectors and one polarization tensor, which has two indices. There are $n+1$ indices from polarizations in total. Thus, there exists at least one contraction between two Lorentz indices of the polarizations in each term. We can conclude that the QFT building block with all plus helicity choice vanishes.

APPENDIX B: BCFW RECURSION WITH 1 AND N SHIFTED

We can generalize the discussion to two or even more massive states in the future. Thus, knowing how to apply the shift with massive particles to the QFT building block is useful, although we have tentative boundaries in this shift choice.

1. BCFW shift for massive momentum

The BCFW shift for all massless external momentum is already discussed in the review section. For the amplitude with massive legs, we need to shift the massive momentum k_j and the massless momentum k_i as follows [48]:

$$\hat{k}^\mu_j=k^\mu_j+zr^\mu$$
$$\hat{k}^\mu_i=k^\mu_i-zr^\mu. \tag{B1}$$

The momentum conservation is naturally satisfied. We expand k^μ_j into $k^\mu_j=a^\mu+b^\mu$.

The on-shell condition becomes

$$\hat{k}^2_j+m^2=0$$
$$\hat{k}^2_i=0. \tag{B2}$$

Next, we choose a suitable z, which makes the complex shifted propagator on shell. We can find the z we want by solving the mass shell equation:

[17]Still, the other spin choice of massive particle n are generated by using the raising and lowering operator as we did in Eq. (2.27).

$$\hat{k}_I^2 + m_I^2 = \left(\sum_{i\in I}\hat{k}_i\right)^2 + m_I^2 = 0$$
$$\hat{k}_J^2 + m_J^2 = \left(\sum_{j\in J}\hat{k}_j\right)^2 + m_J^2 = 0, \tag{B3}$$

where m_I^2 and m_J^2 denote the mass of k_I^μ and k_J^μ. Since the only two shifted momentums are k_j^μ and k_i^μ, the complex shifts of k_I^μ and k_J^μ are also zr^μ. Only single first-order singularity contributes residual. We need to set the second order and higher order to zero, which means

$$r^2 = 0. \tag{B4}$$

The first constraint on r^μ is that r^μ has to be a null vector. Plugging back to the on-shell condition of momentum 1 and n, we have:

$$k_i \cdot r = k_j \cdot r = 0. \tag{B5}$$

The second constrain on r^μ is that r^μ has to be orthogonal to both k_i^μ and k_j^μ.

Momentum conservation is automatically satisfied. The simplest nontrivial r^μ found is

$$r^\mu = -\frac{1}{2m}\langle i|_{\dot\beta} k_j^{\dot\beta\alpha}\sigma^\mu_{\alpha\dot\alpha}|i\rangle^{\dot\alpha}, \tag{B6}$$

and we can also define $\not{r}$ by contracting r^μ with σ_μ:

$$\not{r}_{\alpha\dot\beta} = r^\mu(\sigma^\nu)_{\alpha\dot\beta}\eta_{\mu\nu} = -\frac{1}{2m}\langle i|_{\dot\gamma} k_j^\nu \bar\sigma_\nu^{\dot\gamma\delta}\sigma^\mu_{\delta\dot\delta}|i\rangle^{\dot\delta}\sigma^\rho_{\alpha\dot\beta}\eta_{\mu\rho}$$
$$= -\frac{1}{m}(\not{k}_j|i\rangle)_\alpha\langle i|_{\dot\beta}$$
$$= \frac{1}{m}(|a]_\alpha\langle ai\rangle + |b]_\alpha\langle \mathrm{bi}\rangle)\langle i|_{\dot\beta}. \tag{B7}$$

2. Application to the QFT building block

We can now apply our new BCFW recursion with massive external legs shifted into the QFT building block.

The momentum is shifted as

$$\hat{k}_1^\mu = k_1^\mu + zr^\mu \qquad \hat{k}_n^\mu = k_n^\mu - zr^\mu, \tag{B8}$$

where we still expand the massive leg into a and b:

$$k_n^\mu = a^\mu + b^\mu, \tag{B9}$$

where n is the only massive leg. Plugging back to Eq. (B6), we get

$$r^\mu = -\frac{1}{2m}\langle 1|_{\dot\beta} k_n^{\dot\beta\alpha}\sigma^\mu_{\alpha\dot\alpha}|i\rangle^{\dot\alpha}. \tag{B10}$$

The $\not{r}$ is

$$\not{r}_{\alpha\dot\beta} = r^\mu(\sigma^\nu)_{\alpha\dot\beta}\eta_{\mu\nu} = -\frac{1}{2m}\langle 1|_{\dot\gamma} k_n^\nu \bar\sigma_\nu^{\dot\gamma\delta}\sigma^\mu_{\delta\dot\delta}|1\rangle^{\dot\delta}\sigma^\rho_{\alpha\dot\beta}\eta_{\mu\rho}$$
$$= -\frac{1}{m}(\not{k}_n|1\rangle)_\alpha\langle 1|_{\dot\beta} = \frac{1}{m}(|a]_\alpha\langle a1\rangle + |b]_\alpha\langle b1\rangle)\langle 1|_{\dot\beta}. \tag{B11}$$

As for the shifted spinor helicity, we have

$$\hat{\not{k}}_1 = \not{k}_1 + z\not{r} = |1]\langle 1| + \frac{z}{m}(|a]\langle a1\rangle + |b]\langle b1\rangle)\langle 1|$$
$$\hat{\not{k}}_n = \not{a} + \not{b} - z\not{r} = |a]\langle a| + |b]\langle b| - \frac{z}{m}(|a]\langle a1\rangle + |b]\langle b1\rangle)\langle 1|, \tag{B12}$$

where momentum p_n is decomposed into $p_n = a + b$. Thus, we have

$$|\hat{1}] = |1] + \frac{z}{m}(|a]\langle a1\rangle + |b]\langle b1\rangle)$$
$$\langle\hat{a}| = \langle a| - \frac{z}{m}\langle a1\rangle\langle 1|$$
$$\langle\hat{b}| = \langle b| - \frac{z}{m}\langle b1\rangle\langle 1|, \tag{B13}$$

where other shifted spinor helicity is unchanged. We have several special relations:

$$\langle\hat{a}\,\hat{1}\rangle = \langle a1\rangle \qquad \langle\hat{b}\,\hat{1}\rangle = \langle b1\rangle$$
$$[\hat{a}\,\hat{1}] = [a1] \qquad [\hat{b}\,\hat{1}] = [b1]. \tag{B14}$$

After rewriting four-point amplitude into spinor helicity form, we can get

$$\mathcal{A}(1^-, 2^+, 3^+|\underline{4}^{-2}) = \frac{[ab]^2}{2m^2}\frac{\langle 1b\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}, \tag{B15}$$

where $k_4 = a + b$, the same as the discussion before for general n.

Suppose the $n-1$ point amplitudes take the form:

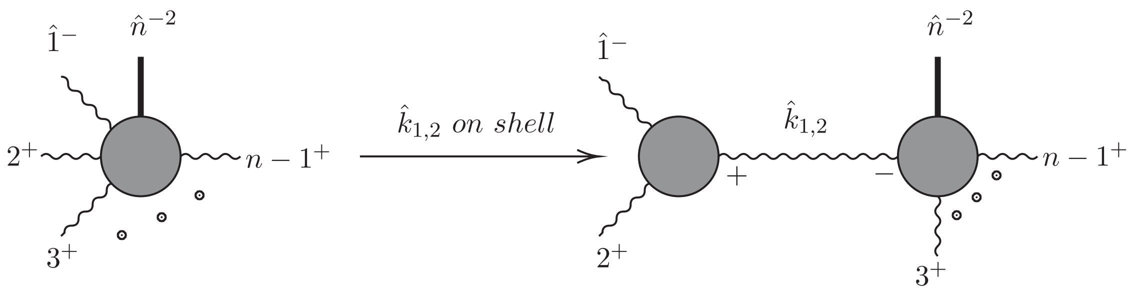
$$\mathcal{A}(k_I^-, 3^+, \ldots, n-2^+, n-1^+|\underline{n}^{-2})$$
$$= \frac{[ab]^2}{2m^2}\frac{\langle k_I b\rangle^4}{\langle k_I 3\rangle\langle 34\rangle\cdots\langle n-1, k_I\rangle}. \tag{B16}$$

The three-point pure Yang-Mills amplitude can be written as

$$\mathcal{A}(1^-, 2^+, -k_I^+) = \frac{[2k_I]^4}{[12][2k_I][k_I 1]}, \tag{B17}$$

where $I = 1, 2$ and $k_I = k_{1,2}$, and we can always construct the n-point amplitude from lower point amplitudes.

Interestingly, there is still only one pole giving nonzero contribution:



The helicity choice on the propagator has to be plus on the left and minus to the right because we have shown that the all-plus configuration vanishes. We can only have a three-point Yang-Mills amplitude on the left, due to the MHV.

$$\mathcal{A}(\hat{1}^-, 2^+, \ldots, n-2^+, n-1^+|\underline{\hat{n}}^{-2}) = \frac{[ab]^2}{2m^2}\frac{\langle \hat{k}_I \hat{b}\rangle^4}{\langle \hat{k}_I 3\rangle\langle 34\rangle\cdots\langle n-1, \hat{k}_I\rangle} \times \frac{1}{\langle \hat{1}2\rangle[\hat{1}2]} \times \frac{[2\hat{k}_I]^4}{[\hat{1}2][2\hat{k}_I][\hat{k}_I\hat{1}]}, \tag{B18}$$

where we ignore the boundary term for now. We can simplify the above equation using the following:

$$[2\hat{k}_I]\langle \hat{k}_I \hat{b}\rangle = -[2|\hat{1}|\hat{a}\rangle = [2\hat{1}]\langle 1b\rangle$$
$$[\hat{1}\hat{k}_I]\langle \hat{k}_I 3\rangle = [\hat{1}2]\langle 23\rangle$$
$$\langle n-1, \hat{k}_I\rangle[\hat{k}_I 2] = \langle n-1, \hat{1}\rangle[\hat{1}2]. \tag{B19}$$

Finally, we get the n-point amplitude:

$$\mathcal{A}(\hat{1}^-, 2^+, \ldots, n-2^+, n-1^+|\underline{\hat{n}}^{-2}) = \frac{[ab]^2}{2m^2}\frac{\langle 1b\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n-1, 1\rangle}. \tag{B20}$$

This is what we expected for the n-point amplitude. It agrees with the nonzero configuration we analytically worked out in Sec. IV A.

However, there is one more thing noticeable: We also need to consider the boundary contribution. We can find the naive boundary in Eq. (B20).

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