Functional renormalization group approach to dipolar fixed point which is scale invariant but nonconformal

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A dipolar fixed point introduced by Aharony and Fisher is a physical example of interacting scale-invariant but nonconformal field theories. We find that the perturbative critical exponents computed in ϵ expansions violate the conformal bootstrap bound. We formulate the functional renormalization group equations *à la* Wetterich and Polchinski to study the fixed point. We present some results in three dimensions within (uncontrolled) local potential approximations (with or without perturbative anomalous dimensions).

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I. INTRODUCTION

Conformal invariance has played a central role in understanding critical phenomena not only in two dimensions but also in higher dimensions. For instance, conformal invariance is powerful enough to determine the critical exponents of the three-dimensional Ising model in six digits by using the recently developed numerical conformal bootstrap method [1–5]. There are many other critical phenomena studied by using the conformal bootstrap (see, e.g., [6] for a review).

While powerful enough, it seems mysterious that the critical phenomena show enhanced conformal symmetry rather than mere scale invariance. It is indeed quite challenging to prove that the Ising model at criticality shows conformal invariance. On the other hand, it is surprisingly hard to find examples of scale-invariant but not conformal field theories in theory [7–9], let alone in physical examples (see, e.g., [10] for a review).

In [11], it was discussed that an (isotropic) dipolar magnet [12] is one of the rare examples of an interacting scale but not conformal field theory.[1] Because it is not conformal invariant, we cannot use the numerical conformal bootstrap method to investigate its critical exponents. Indeed, in this paper, we will show that the perturbative critical exponents computed in ϵ expansions violate the conformal bootstrap bound.

[1]Subsequently, the details of this theory, including the nonrenormalization property of the virial current from the (hidden) shift symmetry, was developed in [13].

With this situation in mind, we investigate the functional renormalization group approaches to the dipolar fixed point. The functional renormalization group is regarded as a nonperturbative method to study the renormalization group flow and its fixed point (see, e.g., [14–16] for reviews). Since it does not rely on the conformal symmetry, unlike the conformal bootstrap method, it can be applied to the dipolar magnet. In this paper, we use the Wetterich equation [17] as well as the Polchinski equation [18] to investigate the dipolar fixed point. We first show that both approaches reproduce the lowest order ϵ expansions in the local potential approximation with the perturbative truncation. We then present some (nonperturbative) results in three dimensions within (uncontrolled) local potential approximations.

II. FUNCTIONAL RENORMALIZATTION GROUP APPOACHES TO DIPOLAR FIXED POINT

A. Dipolar fixed point and violation of bootstrap bound

In the Landau-Ginzburg description, the Heisenberg magnet in d dimensions is described by the effective action

$$S = \int d^d x \left(\frac{1}{2} \partial_\mu \phi_i \partial_\mu \phi_i + t\phi_i^2 + \lambda(\phi_i^2)^2 \right), \qquad (1)$$

where $i = 1, \ldots, d$. It has the global $O(d)$ symmetry [as well as the $O(d)$ spatial rotational symmetry] since the exchange interaction relevant to the Heisenberg magnet acts only on internal spin, not on the orbital spin.[2] The renormalization group fixed point of this effective

[2]Strictly speaking, the magnetization is not a "vector" in $d \neq 3$ dimensions (rather it is a two-form), but we will continue the dimensionality here in order to set up a simple ϵ expansion.

action describes the critical behavior of the Heisenberg magnet.

A dipolar interaction breaks the separation of the spin rotation and the orbital rotation, resulting in the explicit symmetry breaking of $O(d) \times O(d)$ down to $O(d)$. In the Landau-Ginzburg description, it is described by the effective action

$$S = \int d^d x \left(\frac{1}{2} \partial_\mu \phi_\nu \partial_\mu \phi_\nu + \xi (\partial_\mu \phi_\mu)^2 + t\phi_\mu^2 + \lambda (\phi_\mu^2)^2 \right). \quad (2)$$

We will assume $\xi = \infty$ so that the vector ϕ_μ is purely transverse.[3] Alternatively, one may use the Lagrange multiplier formulation

$$S = \int d^d x \left(\frac{1}{2} \partial_\mu \phi_\nu \partial_\mu \phi_\nu + U \partial_\mu \phi_\mu + t\phi_\mu^2 + \lambda (\phi_\mu^2)^2 \right), \quad (3)$$

where U is the Lagrange multiplier. In this picture, it is easier to see that the transverse condition is not renormalized because of the shift symmetry of U. The critical behavior of the dipolar magnet is described by the renormalization group fixed point of this action.

Aharony and Fisher did the perturbative studies of the renormalization group flow in $d = 4 - \epsilon$ dimensions. We quote their results [12,19,20] (see also [21] for three-loop results directly in three dimensions). The scaling dimension of the lowest nontrivial singlet operator Δ_t is given by

$$\Delta_t = 2 - \frac{8}{17}\epsilon. \quad (4)$$

The scaling dimension of the lowest vector operator Δ_ϕ is given by

$$\Delta_\phi = \frac{2-\epsilon}{2} + \frac{10}{867}\epsilon^2. \quad (5)$$

In comparison, let us also quote the scaling dimensions of the corresponding operator in the critical $O(N)$ model

$$\Delta_t = 2 - \frac{6}{N+8}\epsilon,$$
$$\Delta_\phi = \frac{2-\epsilon}{2} + \frac{(N+2)}{4(N+8)^2}\epsilon^2. \quad (6)$$

We can also systematically investigate the scaling dimensions as well as the unitarity bound of the critical $O(N)$ models by using the numerical conformal bootstrap. We show the bound of the scaling dimensions of Δ_t as a function of Δ_ϕ in an $O(d)$ model in $d = 3.98$ dimensions in

[3] Within perturbative ϵ expansions, it turns out that $\xi = \infty$ is an unstable IR fixed point, but there is a (hidden) symmetry that makes it possible to set $\xi = \infty$ under the renormalization group flow. See [13] for a complete analysis of the story.

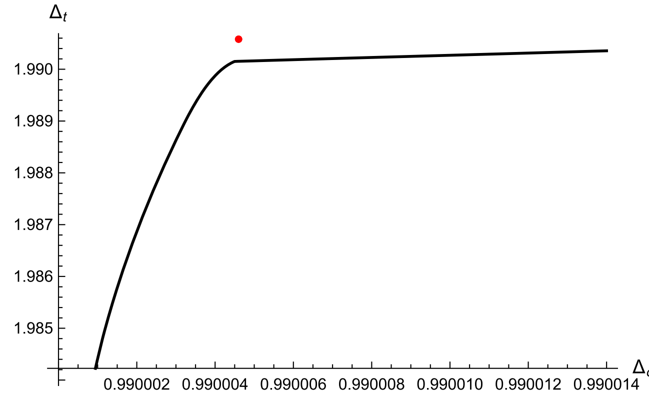

FIG. 1. Unitarity bound for Δ_t in $O(d)$ symmetric conformal field theory in $d = 3.98$ dimensions. The red dot represents the dipolar fixed point (in ϵ expansion).

Fig. 1 by dimensionally continuing the parameters d and N.[4] It is interesting to observe that within ϵ expansions, the scaling dimensions of the dipolar fixed point computed by Aharony and Fisher violate the bootstrap bound. Of course, this is not a contradiction, because the dipolar fixed point does not possess conformal invariance or reflection positivity, but it is indicative that, in a real experiment, we might obtain the number that violates the conformal bootstrap bound, which could result from scale but nonconformal interactions.

A couple of comments are in order. In Fig. 1, we find an interesting kink whose location is numerically very close to the $O(3.98)$ fixed point located at $(\Delta_\phi, \Delta_t) = (0.99000416, 1.98998)$, which is computed at the leading order in the ϵ expansions. Strictly speaking, the $O(N)$ fixed point in noninteger dimensions and noninteger N is not reflection positive, and the conformal bootstrap bound based on the unitarity may not apply. In practice, however, the violation of the unitarity is extremely weak (see, e.g., [24] for detailed discussions), and the numerical conformal bootstrap has not yet seen any indication of the violation of the unitarity. It should be contrasted with the dipolar fixed point, where the large violation is seen.

In Fig. 1, we observe that, above the kink, the bound on Δ_t changes much less than below the kink. This persists whenever ϵ is sufficiently small. With this observation, we can argue that for sufficiently small ϵ, where the higher loop correction to the ϵ expansion is negligible, the dipolar fixed point from the ϵ expansion violates the numerical conformal bootstrap bound. To argue this, we first note $\Delta_\phi^{\text{dipolar}} > \Delta_\phi^{O(d)}$ when ϵ is small [see (5) and (6)]. Then, assuming that the numerical conformal bootstrap bound is saturated by the $O(d)$ fixed point at the kink and that the bound on Δ_t does not change above the kink, we see that

[4] We used CBOOT [22] with SDPB [23] to generate the plot.

the numerical bootstrap bound is indeed violated because $\Delta_t^{\text{dipolar}} > \Delta_t^{O(d)}$ at one loop [see (4) and (6)].

After investigating the functional renormalization group approach to the dipolar fixed point, in Sec. 2.3 we will come back to the comparison with a bootstrap bond for the Heisenberg model in three dimensions.

B. Wetterich version

In the following, we would like to study the functional renormalization group approaches to study the dipolar fixed point. We begin our studies with the local potential approximation of the Wetterich equation. The schematic form of the Wetterich equation is

$$k\partial_k\Gamma = \frac{1}{2}\mathrm{Tr}(\partial_k R_k(\partial_\phi^2\Gamma + R_k)^{-1}), \tag{7}$$

where R_k is the regularization functional and we will often use the Litim (or optimal) regulator $R_k = (k^2 - p^2)\theta(k^2 - p^2)$ [25].

Within the local potential approximation, the effective action for the dipolar magnet is truncated as

$$\Gamma = \int d^dx\left(\frac{1}{2}\partial_\mu\phi_\nu\partial_\mu\phi_\nu + \xi(\partial_\mu\phi_\mu)^2 + V(\phi_\mu^2)\right). \tag{8}$$

We assume that $\xi = \infty$ is a fixed point under the renormalization group flow and we do not consider its renormalization, as can be justified in the Lagrange multiplier formulation.

Noting that the inverse of the kinetic term $(p^2\delta_{\mu\nu} + 2\xi p_\mu p_\nu)^{-1}$ at $\xi = \infty$ is formally given by the Landau gauge propagator $\frac{\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}}{p^2} = \frac{1}{p^2}P_{\mu\nu}$ with the projector $P_{\mu\nu}$, the Wetterich equation with the local potential approximation becomes

$$k\partial_k V = \int\frac{d^dp}{(2\pi)^d}\partial_k R_k P_{\mu\nu}(p^2\delta_{\nu\mu} + 2(V'\delta_{\nu\rho} + 2V''\phi_\nu\phi_\rho)P_{\rho\mu}) + R_k P_{\nu\mu})^{-1}. \tag{9}$$

With the Litim type regulator, the integration over p can be formerly performed,

$$k\partial_k V = k^{d+1}\mu_d\langle P_{\mu\nu}(k^2\delta_{\nu\mu} + 2(V'\delta_{\nu\rho} + 2V''\phi_\nu\phi_\rho)P_{\rho\mu}))^{-1}\rangle_n, \tag{10}$$

where we still have to evaluate the angular average of the projectors $P_\mu = \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}$. For example,

$$\left\langle\frac{p_\mu p_\nu}{p^2}\right\rangle_n = \frac{1}{d}\delta_{\mu\nu},$$
$$\left\langle\frac{p_\mu p_\nu p_\rho p_\sigma}{p^4}\right\rangle_n = \frac{1}{d(d+2)}(\delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho}),$$
$$\left\langle\frac{p_\mu p_\nu p_\rho p_\sigma p_\alpha p_\beta}{p^6}\right\rangle = \frac{1}{d(d+2)(d+4)} \times (\delta_{\mu\nu}\delta_{\rho\sigma}\delta_{\alpha\beta} + 14\,\text{terms}). \tag{11}$$

Since it is in the denominator with a noncommuting matrix $\phi_\mu\phi_\nu$, the explicit evaluation is nontrivial. We can, however, always expand the denominator in perturbation theory, as we will see.

As our first study, we show how to reproduce the earlier results in ϵ expansions in $d = 4 - \epsilon$ dimensions. For this purpose, we truncate the effective action

$$V = t\phi_\mu^2 + \lambda(\phi_\mu^2)^2 \tag{12}$$

and work in perturbation theory with respect to λ (and t).

Within the perturbation theory, one can expand the matrix in the denominator and evaluate the angular average up to ϕ^4. The beta function is obtained as

$$\dot{t} = -2t - \left(2(d-1) + 4 - \frac{4}{d}\right)\mu_d 2\lambda + 2\left(2(d-1) + 4 - \frac{4}{d}\right)\mu_d 4\lambda t + \cdots,$$
$$\dot{\lambda} = -\epsilon\lambda + 4\cdot 4\lambda^2\mu_d\left(d + 7 - \frac{12}{d} + \frac{12}{d(d+2)}\right) + \cdots, \tag{13}$$

with the fixed point $\lambda_* = \frac{\epsilon}{4\cdot 34\mu_d} + O(\epsilon)^2$. The critical exponent $y_t = d - \Delta_t$ can be computed as

$$y_t = 2 - \frac{9}{17}\epsilon + O(\epsilon^2) \tag{14}$$

by linearizing the beta functions at the fixed point and diagonalizing the Hessian matrix $\partial_a\beta^b$. This reproduces the result by Fisher and Aharony [12].

In principle, we may study nonperturbative fixed points in $d = 3$ dimensions within the local potential approximation. Here, we present just one example of (uncontrolled) truncation at the next order in the space of coupling constants. We truncate the effective action

$$V = t\phi_\mu^2 + \lambda(\phi_\mu^2)^2 + g(\phi_\mu^2)^3 \tag{15}$$

and demand vanishing of beta functions of t, λ, and g. We also neglect the anomalous dimensions of ϕ.[5] Explicitly we have

[5] In ϵ expansion, it is fixed by the momentum-dependent wave function renormalization of $O(\epsilon^2)$.

$$\dot{t} = -2t - \frac{(2(d-1)+4-\frac{4}{d})\mu_d 2\lambda}{(1+2t)^2},$$
$$\dot{\lambda} = -(4-d)\lambda + \frac{4\cdot 4\lambda^2\mu_d(d+7-\frac{12}{d}+\frac{12}{d(d+2)})}{(1+2t)^3} - \mu_d\frac{(d-1)+(4-\frac{4}{d})}{(1+2t)^2}6g,$$
$$\dot{g} = -(6-2d)g + 48\mu_d g\lambda\frac{d-1+6(1-\frac{1}{d})+8(1-\frac{2}{d}+\frac{3}{d(d+2)})}{(1+2t)^3},$$
$$-64\mu_d\lambda^3\frac{d-1+6(1-\frac{1}{d})+12(1-\frac{2}{d}+\frac{3}{d(d+2)})+8(1-\frac{3}{d}+\frac{9}{d(d+2)}-\frac{15}{d(d+2)(d+4)}))}{(1+2t)^4}. \quad (16)$$

(Here we have omitted some terms that are at higher orders in ϵ expansions.) Substituting $d=3$ and linearizing the renormalization group equation around the fixed point, we obtain the lowest renormalization group eigenvalue as

$$y_t = 1.529. \quad (17)$$

In comparison, let us quote the lowest renormalization group eigenvalue in the $O(3)$ model in $d=3$ dimensions with the same local potential approximation. It is given by

$$y_t = 1.553. \quad (18)$$

Note that the scaling dimension Δ_t obtained here is larger in the dipolar fixed point than in the Heisenberg fixed point, which seems consistent with the perturbation theory.[6]

We could actually write down the full functional form of the renormalization group equation in $d=3$ dimensions.[7] We first evaluate the effective propagator in the Wetterich equation:

$$G_{\mu\nu} = \tilde{A}P_{\mu\nu} + \tilde{C}P_{\mu\alpha}\phi_\alpha P_{\nu\beta}\phi_\beta, \quad (19)$$

where

$$\tilde{A} = \frac{1}{p^2+2V'} = \frac{1}{\bar{p}^2},$$
$$\tilde{C} = -\frac{p^2}{\bar{p}^2}\frac{4V''}{p^2(\bar{p}^2+4V''\phi_\mu^2)-4V''(p_\mu\phi_\mu)^2}. \quad (20)$$

Let us now perform the angular average of p integration on the right-hand side of the Wetterich equation in $d=3$. It is effectively given by

$$\frac{2}{\bar{p}^2}+\frac{1}{2}\int_{-1}^{1}d(\cos\theta)\frac{-p^2}{\bar{p}^2}\frac{4V''\phi_\mu^2(1-\cos\theta^2)}{p^2(\bar{p}^2+4V''\phi_\mu^2)-4V''p^2\phi_\mu^2\cos^2\theta}$$
$$=\frac{2}{\bar{p}^2}-\frac{p^2}{\bar{p}^2}\frac{2V''\phi_\mu^2}{p^2\bar{p}^2+4V''p^2\phi_\mu^2}\int_{-1}^{1}dx\frac{1-x^2}{1-\frac{4V''p^2\phi_\mu^2}{p^2\bar{p}^2+4V''p^2\phi_\mu^2}x^2}$$
$$=\frac{2}{\bar{p}^2}-\frac{p^2}{\bar{p}^2}\frac{2V''\phi_\mu^2}{p^2\bar{p}^2+4V''p^2\phi_\mu^2}\frac{2a+\frac{-1+a^2}{2}\log(\frac{1+a}{1-a})^2}{a^3}, \quad (21)$$

where $a^2 = \frac{4V''p^2\phi_\mu^2}{p^2\bar{p}^2+4V''p^2\phi_\mu^2}$. By performing the polar integration with the optimal regulator, we get

$$k\partial_k V = \frac{2}{\bar{k}^2}-\frac{k^2}{\bar{k}^2}\frac{2V''\phi_\mu^2}{k^2\bar{k}^2+4V''k^2\phi_\mu^2}\frac{2\bar{a}+\frac{-1+\bar{a}^2}{2}\log(\frac{1+\bar{a}}{1-\bar{a}})^2}{\bar{a}^3}$$
$$=\frac{2}{\bar{k}^2}-\frac{k^2}{\bar{k}^2}\frac{2V''\phi_\mu^2}{k^2\bar{k}^2+4V''k^2\phi_\mu^2}\sum_{n=1}\frac{4\bar{a}^{2n-2}}{4n^2-1}, \quad (22)$$

with $\bar{k}^2 = k^2+2V'$ and $\bar{a}^2 = \frac{4V''k^2\phi_\mu^2}{k^2\bar{k}^2+4V''k^2\phi_\mu^2}$. One can verify that it reproduces the beta functions we obtained perturbatively above.

C. LPA′ and more results

One may incorporate the effect of the anomalous dimensions within the functional renormalization group approach. We do not attempt the evaluation of the wave function renormalization in a self-consistent manner, which is technically more involved. Here we take an approach called local potential approximation′ (LPA′) and get the effect of the wave function renormalization "by hand." In this approach, the net effect of the wave function renormalization is given by replacing (9) with

$$k\partial_k V = \partial_k(k^{d+2}Z_k)\mu_d\langle P_{\mu\nu}(Z_k k^2\delta_{\nu\mu} + (2V'\delta_{\nu\rho}+2V''\phi_\nu\phi_\rho)P_{\rho\mu}))^{-1}\rangle_n, \quad (23)$$

where we assume that $Z_k \sim k^{-2\gamma_\phi}$, with γ_ϕ being the anomalous dimension of ϕ_i that can be computed separately. Within the LPA′ approach, where we determine the

[6] We cannot trust the actual number very much. For example, the conformal bootstrap suggests that $y_t = 1.406$ for the $O(3)$ model in $d=3$ dimensions.

[7] The following observation was first suggested by K. Fukushima.

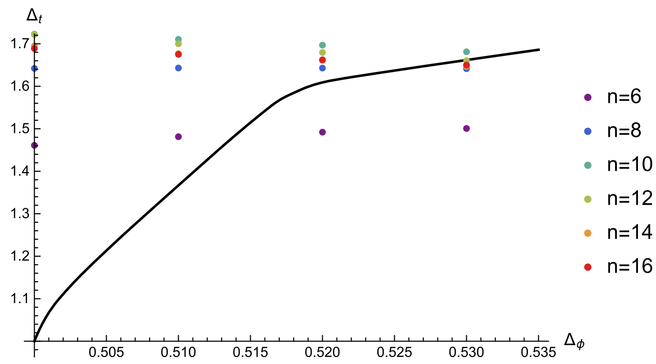

FIG. 2. Critical exponents obtained from LPA′ truncation of the functional renormalization group in three dimensions. For comparison, we showed the conformal bootstrap bound on the Heisenberg model as a black curve.

value of γ_ϕ by hand, the resulting renormalization group equations are almost the same as (16), except that the coefficient of the first term is modified: for $g_n\phi^n$ coupling, we replace $-(n - \frac{dn}{2} + d)g_n$ with $-(n - \frac{dn}{2} + d - n\gamma_\phi)g_n$.

The values of γ_ϕ can be taken from the perturbative computations based on the epsilon expansions (or any other methods). At $d = 3$, we have $\gamma_\phi \sim 0.01(1)$, which gives only a tiny modification of the (lowest) renormalization group eigenvalues y_t (of order γ_ϕ; see Fig. 2).

We report the evaluation of $\Delta_t = 3 - y_t$ as a function of $\Delta_\phi = \frac{1}{2} + \gamma_\phi$ in the Aharony-Fisher model (in $d = 3$) within the LPA′ approximation by changing the truncation of the potential in Fig. 2. To quote some numbers here, if we truncate the potential up to ϕ^6, we obtain $y_t = 1.508$ or, if we truncate the potential up to ϕ^{16}, we obtain $y_t = 1.33$ (at $\gamma_\phi = 0.02$). The small dependence on γ_ϕ can be extrapolated from Fig. 2.

The prediction from y_t by increasing the truncation order of the potential seems to converge rapidly, but this does not mean that we can trust the actual number that we have obtained. While we cannot estimate the systematic error in the Ahanorny-Fisher fixed point, with the same truncation, we obtain $y_t = 1.31$ in the $O(3)$ model, whose accurate value should be $y_t = 1.406$. It is therefore expected that the systematic error of our prediction of y_t could be as large as 0.1 irrespective of the convergence of the polynomial truncations within the LPA′. See also [16,28] for similar comparisons in $O(N)$ models. Note that the effect of the truncation (of the other terms that we neglect in LPA′) seems much more severe than the effect of the anomalous dimensions γ_ϕ.

Let us finally quote the predictions of y_t (or $\Delta_t = 3 - y_t$) from various other approaches. The three-loop computations of the renormalization group directly in three dimensions [21] gave $\Delta_\phi = 0.5165(40)$ and $\Delta_0 = 1.576(10)$. The experimental values (more than 40 years ago) in EuO and EuS gave $\Delta_0 = 1.58(5)$ and 1.59(5), respectively [29].

D. Polchinski version

Next, let us study the local potential approximation of the Polchinski equation as another functional renormalization group approach to the dipolar fixed point. The schematic form of the Polchinski equation for the Aharony-Fisher model is given by

$$\dot{S} = -\frac{\delta S}{\delta\phi_\mu(p)} P_{\mu\nu} \frac{\delta S}{\delta\phi_\nu(-p)} + \mathrm{Tr} P_{\mu\nu} \frac{\delta^2 S}{\delta\phi_\mu(p)\phi_\nu(-p)}. \tag{24}$$

One apparent advantage of the Polchinski equation (compared with the Wetterich equation) is the absence of the denominator.

The important difference compared with the standard scalar ϕ^4 theory is to keep the projector $P_{\mu\nu} = \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}$ in the interaction vertex even in the local potential approximation. We also perform the angular average when we take the trace in the second term of (24), but we do not perform the average in the first term. This makes the solution of the Polchinski equation much more complicated, but it is necessary even in the perturbation theory.

As our first application, let us study a perturbative fixed point in $d = 4 - \epsilon$ dimensions. In order to make the renormalization group equation closed within the perturbation theory, we make the ansatz[8]

$$V(\phi) = t\phi_\mu P_{\mu\nu}\phi_\nu + \lambda\phi_\mu\phi_\mu\phi_\nu\phi_\nu + g\phi_\mu\phi_\mu\phi_\nu P_{\nu\sigma}\phi_\sigma\phi_\rho\phi_\rho. \tag{25}$$

Note that the six-point vertex has a specific projector.[9] The fixed point equation for g at the lowest order becomes

$$0 = -16\lambda^2\phi_\mu\phi_\mu\phi_\nu P_{\nu\sigma}\phi_\sigma\phi_\rho\phi_\rho - 2g\phi_\mu\phi_\mu\phi_\nu P_{\nu\sigma}\phi_\sigma\phi_\rho\phi_\rho, \tag{26}$$

which indeed shows the necessity of the projector.

Similarly, for t, we have

$$0 = 2t\phi_\mu\phi_\mu + \left(2(d-1) + 4\left(1 - \frac{1}{d}\right)\right)\lambda\phi_\mu\phi_\mu - 4t^2\phi_\mu P_{\mu\nu}\phi_\nu, \tag{27}$$

We should note that for the two-point vertex, there is no distinction between $\phi_\mu\phi_\mu$ and $\phi_\mu P_{\mu\nu}\phi_\nu$, so we can combine all these terms and demand the vanishing of the coefficient.

The fixed point equation for λ has two contributions. One is the one-particle reducible one

[8]We need the six-point vertex to reproduce the standard ϵ expansions in the standard Wilson-Fisher fixed point from the Polchinski equation.

[9]Note that if the projector is connected to only one ϕ (i.e., in the t term), it does nothing because the external line is always transverse. On the other hand, if the projector connects more fields (i.e., in the g term), then it makes a difference.

$$- 16t\lambda(\phi_\mu\phi_\mu\phi_\nu)P_{\nu\sigma}\phi_\sigma + \left(2g(d-1) + 4g\left(1-\frac{1}{d}\right)\right) \times (\phi_\mu\phi_\mu\phi_\nu)P_{\nu\sigma}\phi_\sigma \qquad (28)$$

and the other is the one-particle irreducible one

$$g\left(d+7-\frac{12}{d}+\frac{12}{d(d+1)}\right)\phi_\mu\phi_\mu\phi_\rho\phi_\rho. \qquad (29)$$

At the fixed point, we see that the one-particle reducible contributions sum up to zero and only the one-particle irreducible one remains. The fixed point equation for λ becomes

$$\dot{\lambda} = \epsilon\lambda - 8\lambda^2\left(d+7-\frac{12}{d}+\frac{12}{d(d+2)}\right), \qquad (30)$$

with the fixed point value of $\lambda_* = \frac{\epsilon}{4\cdot 17}$ (and $g_* = -8\lambda_*^2$ and $t_* = -\frac{9}{2}\lambda_*$). We can compute the renormalization group eigenvalues, and we correctly obtain $y_t = 2 - \frac{9}{17}\epsilon$.

Our original hope was that the Polchinski equation may work better to study the nonperturbative renormalization group fixed point in the Aharony-Fisher model (at least within the local potential approximation) because of the absence of the denominator. Unfortunately, it may not be that simple. Owing to the existence of the projector, we may have to introduce more and more terms,

$$V = t\phi^2 + \lambda_0\phi^2\phi^2 + \lambda_1\phi\phi P\phi\phi + g_0\phi^2\phi^2\phi^2 + g_1\phi^2\phi P\phi\phi^2 + g_2\phi\phi P\phi\phi P\phi\phi + \cdots, \qquad (31)$$

to write the effective action. It is not obvious how to truncate such potentials or make any nonperturbative ansatz that is closed under the renormalization group flow.

III. $d = 2$ AND MULTICRITICAL POINTS

The physical motivation of the dipolar fixed point resides mainly in $d = 3$ dimensions, but we may also be able to find a nontrivial fixed point in $d = 2$ dimensions. Note that the ordinary $O(2)$ model does not show spontaneous symmetry breaking in $d = 2$ dimensions, due to the Coleman-Mermin-Wagner theorem, but it does not apply to the Aharony-Fisher model, because the global symmetry is mixed with the rotational symmetry.

In two dimensions, the transverse vector can be replaced by a scalar with a (gauged) shift symmetry,

$$\phi_\mu = \epsilon_{\mu\nu}\partial_\nu\varphi, \qquad (32)$$

where φ, the Landau-Ginzburg effective action for the Aharony-Fisher model, can be represented as

$$S = \int d^2x(\partial^2\varphi\partial^2\varphi + V(\partial_\mu\varphi\partial_\mu\varphi) + \cdots). \qquad (33)$$

When $V = 0$, the theory is globally conformal invariant but not Virasoro invariant [26,27]. It is not obvious if nontrivial multicritical fixed points with $V \neq 0$ admit (global) conformal invariance. Presumably, they do not,[10] but in either case, we may find these nontrivial renormalization group fixed points.

While we may study nontrivial fixed points from the functional renormalization group directly in the original variable ϕ_μ which is transverse, we may also study them from the new variable φ without any constraint. In the local potential approximation with the optimal regulator, the Wetterich equation of this model is given by

$$k\partial_k V = k^{d+1}\left\langle \frac{1}{k^2 + 2V'(\partial_\mu\phi\partial_\mu\phi) + 4k^{-2}V''(\partial_\mu\phi\partial_\mu\phi)\partial_\rho\phi\partial_\sigma\phi k_\rho k_\sigma}\right\rangle_n. \qquad (34)$$

This is similar, but slightly different from, the equations discussed before in terms of ϕ_μ. Since the truncation we are using here is equally uncontrolled, we cannot say which would give a more reasonable result. Note that here again we have to expand the denominator to evaluate the angular average, and the computational difficulty has not been alleviated.

Actually, we can perform the angular average in $d = 2$. It is given by

$$k\partial_k V = k^3\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \frac{1}{k^2 + 2V' + 4(\partial_\mu\varphi)^2 V''\cos^2\theta} = k^3\frac{1}{\sqrt{k^2+2V'}\sqrt{k^2+2V'+4(\partial_\mu\varphi)^2V''}}. \qquad (35)$$

This may give a starting point to study the functional analysis of the fixed point potential V.

As in conformal minimal models in $d = 2$ dimensions, we expect that the model admits (infinitely many) multicritical fixed points by fine-tuning V. They can be regarded as scale but nonconformal analogs of minimal models. It would be very interesting to study their properties and the renormalization group flow among them.

IV. DISCUSSION

In this paper, we have presented our first attempt to use the functional renormalization group method to study the

[10] In [13], it is conjectured that an interacting fixed point with shift symmetry (like the one here) is scale invariant only without conformal invariance, based on the genericity argument.

critical exponents of the dipolar fixed point. There are a couple of directions to be explored. One is to conduct a systematic search for the nonperturbative fixed point without doing a brute-force truncation of the potential even within the local potential approximation.

Another important direction is to introduce the effect of the wave function renormalization to compute the critical exponent η. Even in perturbation theory, it is nontrivial to compute η in the functional renormalization group approach [30–32], and it requires the field-dependent wave function renormalization to be computed at the dipolar fixed point. In the perturbative functional renormalization group, η can be related to terms such as $Z_{\phi^2}(\phi^2)\partial_\mu\phi_\nu\partial_\mu\phi_\nu$ in the one-loop effective action. Now Z_{ϕ^2} itself is of order λ^2 in the one-loop integral of the bare Lagrangian, so η is of order λ^2 corresponding to the effective two-loop integral. It is crucial to obtain η nonperturbatively in $d = 3$ dimensions in order to see if the dipolar fixed point really violates the conformal bootstrap bound for the $O(3)$ models in $d = 3$ dimensions.

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