

First-order formalism for β functions in bosonic sigma models from supersymmetry breaking

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We consider the renormalization group flow equation for the two-dimensional sigma models with the Kähler target space. The first-order formulation allows us to treat perturbations in these models as current-current deformations. We demonstrate, however, that the conventional first-order formalism misses certain anomalies in the measure, and should be amended. We reconcile beta functions obtained within the conformal perturbation theory for the current-current deformations with traditional “geometric” results obtained in the background field methods, in this way resolving the peculiarities pointed out in O. Gamayun *et al.* [Peculiarities of beta functions in sigma models, *J. High Energy Phys.* **10** (2023) 097]. The result is achieved by the supersymmetric completion of the first-order sigma model.

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Beta functions in quantum field theories define the dependence of the coupling constant on the renormalization scale (the RG flow, see e.g., [1]). For two-dimensional sigma models, this flow has a rich geometric meaning, which is the main reason for their successful application in string theory and statistical mechanics [2]. In [3] it was shown how to cast a traditional bosonic sigma model into the so-called first-order form, which allows one, in particular, to treat metric perturbations as conformal perturbation theory.

In Ref. [4] we have considered a special type of metric deformations dubbed Lie-algebraic sigma models [5], which correspond to the current-current deformations. This allowed us to compare our results with the rich research history of beta functions for such deformations [7–13]. We have established that the first-order sigma models proposed in [3] when applied to β function calculations works perfectly at the leading order. However, it leads to results incompatible with the standard geometric (background field method) calculations starting from the second order. In particular, in [4] we considered a

Lie-algebraic generalization of the $\mathbb{C}P^1$ model on the Kähler space of one complex parameter φ . The metric (with the upper indices) was a finite polynomial of $\varphi, \bar{\varphi}$ parametrized by a number of generally speaking complex parameters n_i . A straightforward calculation of higher loops in this formalism predicts that the higher loops must be polynomial too, which contradicts the geometric result already at two loops.

We have formulated a hypothesis to explain the discrepancy as follows: the loss of polynomiality in the second and higher loops is due to an infrared effect which in turn reflects the loss of symmetry in the measure not explicitly seen in the path integral. In this work we will demonstrate that this is indeed the case. We consider a more general case of Kählerian target space of arbitrary dimension. Our starting observation is as follows: if we endow the bosonic model at hand by $\mathcal{N} = (2, 2)$ supersymmetry, which is always possible, then all contributions to the β function beyond the first loop vanish, and simultaneously the measure is regularized. Next, we make superpartners’ mass large and integrate them out. Remarkably, we observe a leftover—a finite effect which can be viewed as an anomaly. This effect violates polynomiality.

Our observation is somewhat similar to the situation in super-Yang-Mills (without matter). If we start from $\mathcal{N} = 2$ theory, its perturbative β function contains only one loop which does not violate holomorphy in the complexified gauge coupling $1/g^2$. Now, if we add the mass term to the

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scalar superfield (including the “second” gluino) we reduce $\mathcal{N} = 2$ down to $\mathcal{N} = 1$ breaking holomorphy starting from the second loop [14,15] as a result of an anomaly in the measure [16].

Technically, the failure of the first-order formalisms in higher loops in [4] and the successful resolution which will be reported below is due to the following circumstance. In [4] the original $\beta\gamma$ system was defined classically, i.e., with the flat metric, while our perturbation used a curved metric. Now, through additional supersymmetry and heavy fermion masses we obtain the bosonic $\beta\gamma$ system at the quantum level, taking into account a nonflat metric in the measure.

More specifically, by the bosonic action of the unperturbed $\beta\gamma$ system, we understand the following sigma-model

$$S_0 = \int_{\Sigma} \frac{d^2z}{\pi} (p_a \bar{\partial} \varphi^a + \bar{p}_{\bar{a}} \partial \bar{\varphi}^{\bar{a}}). \quad (1)$$

Here the scalar fields φ^a and $\bar{\varphi}^{\bar{a}}$ represent coordinates of the D-dimensional target space. [17]. The fields $p_a, p_{\bar{a}}$ are (1, 0) and (0, 1) forms on Σ correspondingly. For our purposes, it will be enough to consider $\Sigma = \mathbb{C}P^1$. This theory is classically invariant under the diffeomorphism transformations $\varphi^a \rightarrow \varphi^a - \epsilon V_A^a(\varphi)$, generated by the current $J_A = p_a V_A^a$. Henceforth the capital Latin letters enumerate all possible vector fields, while the lowercase is reserved for the target space indices (cf., with [4]). On the quantum level this symmetry becomes *anomalous* which is reflected in the following operator product expansion (OPE) of the currents

$$J_A(z) J_B(0) = -\frac{\eta_{AB}}{z^2} + \frac{f_{AB}^C J_C(0) - \Omega_{AB}}{z} + \text{reg}. \quad (2)$$

Here we assume that the vector fields form an algebra with the structure constants f_{AB}^C . The structures η_{AB} and Ω_{AB} in a chosen coordinate frame are

$$\eta_{AB} = \partial_k V_A^c \partial_c V_B^k, \quad \Omega_{AB} = \partial_c V_B^k \partial_m \partial_k V_A^c \partial \varphi^m. \quad (3)$$

In the general case, both η and Ω are functions on the target space (depending on both φ and $\bar{\varphi}$). The operator product expansion (2) does not represent a chiral *current* algebra. Moreover, it is not even a vertex operator algebroid studied in [18–20], because of $\bar{\varphi}$ dependence. This way, we have a *new structure* not encountered before. However, in the general case η_{AB} depends on the choice of the coordinate frame and does not transform as a scalar under the diffeomorphisms of the target (similar to [18–20]). Therefore, it is even more surprising that under the current-current deformation of the theory S_0 , specified by the action S_G

$$S_G = \int \frac{d^2z}{\pi} G^{A\bar{A}} J_A(\varphi) J_{\bar{A}}(\bar{\varphi}) \quad (4)$$

the corresponding beta function (understood as a flow of the “couplings” $G^{A\bar{A}}$) in the first two loops reads as

$$[\beta_2^{\text{algebra}}]^{A\bar{A}} = \frac{1}{2} G^{B\bar{B}} G^{C\bar{C}} f_{BC}^A f_{\bar{B}\bar{C}}^{\bar{A}}, \quad (5)$$

$$[\beta_3^{\text{algebra}}]^{A\bar{A}} = \frac{\alpha'}{2} G^{C\bar{C}} G^{B\bar{B}} G^{F\bar{F}} (f_{CF}^D f_{BD}^A \eta_{\bar{C}\bar{B}} + \text{c.c.}). \quad (6)$$

The subscripts of the beta functions correspond to the power of the perturbation operators (in conformal perturbation methods [4]) and have to be identified with a number of loops +1. For instance, β_2^{algebra} corresponds to the first loop, β_3^{algebra} to the second, and so on. The superscript indicates that these expressions can be obtained solely using the current *algebra* (2). Additionally, we have restored parameter α' . In Ref. [4] expressions (A1) and (6) were obtained in a more general version of the deformation (4) [21]

$$S_g = \int \frac{d^2z}{\pi} g^{a\bar{a}}(\varphi, \bar{\varphi}) p_a \bar{p}_{\bar{a}}. \quad (7)$$

In this case the integration over p_a and $p_{\bar{a}}$ transforms the first-order sigma model into the traditional second-order geometric representation [3]. This allows us to compare two beta functions. Specifically, in the first two loops for a generic Kähler metric $g^{a\bar{a}}$, the corresponding beta function is well defined via the geometric objects of the target space [22,23]

$$\beta_2^{\text{geometry}} = R^{a\bar{a}}, \quad \beta_3^{\text{geometry}} = \frac{\alpha'}{2} R^a_{m\bar{p}b} R^{\bar{a}m\bar{p}b}. \quad (8)$$

One can show (see Appendix A) that expressions for the one-loop beta functions do coincide,

$$\beta_2^{\text{algebra}} = \beta_2^{\text{geometry}}, \quad (9)$$

after the proper identification of $g^{a\bar{a}}$ and $G^{A\bar{A}}$ via equating (4) and (7). This can be attributed to the fact that the structure constants f_{AB}^C do not require any extra structures to be defined. In the two-loop case (i.e., for β_3) this is no longer true and extra care is needed to address the special nature of the structure η_{AB} mentioned above. Alternatively, this can be considered as subtleties of integration over momenta p_a and $p_{\bar{a}}$ (see [24–27]).

To avoid these subtleties and reconcile algebraic and geometric answers we introduce a supersymmetric generalization of the original $\beta\gamma$ system. Namely, we introduce fermions π_a ($\bar{\pi}_{\bar{a}}$) and ψ^a ($\bar{\psi}^{\bar{a}}$) and modify S_0 as

$$S_0 \rightarrow S_0 - \int \frac{d^2z}{\pi} (\pi_a \bar{\partial} \psi^a + \bar{\pi}_{\bar{a}} \partial \bar{\psi}^{\bar{a}}). \quad (10)$$

The currents J_A are promoted to the supersymmetric ones

$$J_A \rightarrow \mathcal{J}_A = p_a V_A^a(\varphi) - \pi_a \partial_b V_A^a \psi^b. \quad (11)$$

Their current algebra is no longer anomalous

$$\mathcal{J}_A(z) \mathcal{J}_B(0) = \frac{f_{AB}^C \mathcal{J}_C(0)}{z} + \text{reg.} \quad (12)$$

To be able to mode out the fermions we additionally introduce fermionic mass terms

$$\delta S_m = m \int \frac{d^2 z}{2\pi} M^{a\bar{a}} \pi_a \bar{\pi}_{\bar{a}} + m \int \frac{d^2 z}{2\pi} M_{\bar{a}a} \bar{\psi}^{\bar{a}} \psi^a. \quad (13)$$

So far we do not require any symmetric properties of the matrices $M^{a\bar{a}}$ and $M_{\bar{a}a}$ and their relation to the deformations (7), although we assume for simplicity that they are inverse of each other

$$M_{\bar{a}a} M^{a\bar{b}} = \delta_{\bar{a}}^{\bar{b}}, \quad M^{\bar{b}a} M_{\bar{a}b} = \delta_{\bar{a}}^{\bar{b}}. \quad (14)$$

The mass-deformed action $S_m = S_0 + \delta S_m$ is invariant with respect to the diffeomorphisms generated by the vector field $V_A^a(\varphi)$ provided that the deformations transform *covariantly*,

$$\delta_A M_{\bar{a}a} = V_A^b \partial_b M_{\bar{a}a} + M_{\bar{a}b} \partial_a V_A^b, \quad (15)$$

$$\delta_A M^{a\bar{a}} = V_A^b \partial_b M^{a\bar{a}} - M^{b\bar{a}} \partial_b V_A^a. \quad (16)$$

The quantity η_{AB} previously defined via the OPE (2) can be alternatively defined via the two-point function of the currents. Indeed, in the purely bosonic theory, we have

$$\langle J_A(z) J_B(0) \rangle_{S_0} = -\frac{\partial_a V_A^b \partial_b V_B^a}{z^2} = -\frac{\eta_{AB}}{z^2}. \quad (17)$$

One might expect that in the supersymmetric version with the mass-deformation switched on, all fermionic degrees of freedom decouple at the distances $m|z| \gg 1$, and we immediately recover the first expression in Eq. (C4),

$$\langle \mathcal{J}_A(z) \mathcal{J}_B(0) \rangle_{S_m} \stackrel{m|z| \gg 1}{\approx} -\frac{\partial_a V_A^b \partial_b V_B^a}{z^2}. \quad (18)$$

However, this would be a hasty conclusion. It turns out to be true only for the constant matrices $M_{\bar{a}a}$.

We claim that the derivatives of $M_{\bar{a}a}$ modify this expression already in the leading order in $m|z|$ by introducing the covariant derivatives associated with $M_{\bar{a}a}$ [instead of the partial derivatives as in (17)],

$$\langle \mathcal{J}_A(z) \mathcal{J}_B(0) \rangle_{S_m} \stackrel{m|z| \gg 1}{\approx} -\frac{\nabla_a V_A^b \nabla_b V_B^a}{z^2}. \quad (19)$$

Here

$$\nabla_a V_A^b = \partial_a V_A^b + M^{b\bar{a}} \partial_c M_{\bar{a}a} V_A^c. \quad (20)$$

Notice that this object transforms as a tensor upon the diffeomorphisms shown in (15). So, η_{AB} is now a proper scalar

$$\begin{aligned} \eta_{AB} &= \nabla_a V_A^b \nabla_b V_B^a \\ &= \partial_a V_A^b \partial_b V_B^a + M^{b\bar{a}} \partial_c M_{\bar{a}a} V_A^c \partial_b V_B^a \\ &\quad + M^{b\bar{a}} \partial_c M_{\bar{a}a} V_B^c \partial_b V_A^a - V_B^b V_A^a \partial_a G_{\bar{c}c} \partial_b M^{c\bar{c}}. \end{aligned} \quad (21)$$

The outline of the derivation of this statement is presented in Appendix C, where we rigorously derive terms linear in the derivatives, while quadratic terms are recovered in the special perturbative regime.

This connection reminds the Hermitian connection

$$\nabla_a^H V_A^b = \partial_a V_A^b + M^{b\bar{a}} \partial_c M_{\bar{a}c} V_A^c. \quad (22)$$

However, the Hermitian connection is compatible with the metric

$$\nabla_a^H M^{b\bar{k}} = 0, \quad (23)$$

while for our connection we have

$$\nabla_a M^{b\bar{k}} = \partial_a M^{b\bar{k}} + M^{b\bar{a}} \partial_c M_{\bar{a}a} M^{c\bar{k}}. \quad (24)$$

This is nonzero unless the metric is Kählerian. For the Kählerian metrics both connections reduce to the Levi-Civita connection. Moreover, as we demonstrate in Appendix B, for the case when the fermions mass metric coincides with the deformed Kähler metric $M^{a\bar{a}} = g^{a\bar{a}}$ the algebraic beta function (6) coincides with the geometric one (8)

$$\beta_3^{\text{algebra}} = \beta_3^{\text{geometry}}. \quad (25)$$

Notice that the condition $M^{a\bar{a}} = g^{a\bar{a}}$ is rather natural; in the traditional second-order sigma model, it is manifested in the coincidence of the target space metric in the action with the metric that defines the integration measure [2].

The example of (25) considered in [4], Eq. (2.52)—one-dimensional Kähler space (a Lie-algebraic generalization of $\mathbb{C}P^1$)—reads

$$\beta_{G_{i\bar{i}}} = G_{i\bar{i}} \left[\frac{1}{4\pi} \mathcal{R} + \left(\frac{1}{4\pi} \mathcal{R} \right)^2 + \dots \right], \quad (26)$$

where \mathcal{R} is the scalar curvature and $\alpha' = 1/(2\pi)$. Our conjecture amounts to summing the geometric progression in Eq. (26). The third and higher-order loops are scheme-dependent, however. We plan to address this issue in the subsequent publication.

Conclusion. We conclude with the statement that the first-order formalism in 2D sigma models must be amended to take into account an anomaly in the measure, as was conjectured in Ref. [4]. In this work we present the pros this fact in a particular regularization, namely, supersymmetry-based regularization. We calculated the second loop of the beta function using this regularized first-order formalism. We demonstrated that sending the fermion masses to infinity leaves a finite trace in the bosonic model. The above residual nonvanishing contribution amends the first-order formalism result and makes it identical to the “geometric” calculation. A new understanding gained in this study is uncovering the anomalous nature of the second and higher loops. This indicates that the exact all-order beta function most probably can be recovered on Kählerian target spaces.

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APPENDIX A: ONE-LOOP COMPARISON

Let us demonstrate the equivalence of the geometric and algebraic beta function for the Kähler metric in one loop level. First we transform the algebraic beta function

$$[\beta_2^{\text{algebra}}]^{C\bar{C}} = \frac{1}{2} G^{A\bar{A}} G^{B\bar{B}} f_{AB}^C f_{\bar{A}\bar{B}}^{\bar{C}} \quad (\text{A1})$$

in the form of the metric $g^{a\bar{a}} = G^{A\bar{A}} V_A^a \bar{V}_{\bar{A}}^{\bar{a}}$. Contracting the beta function in Eq. (A1) with the $V_C^a \bar{V}_{\bar{C}}^{\bar{a}}$ we obtain

$$\begin{aligned} [\beta_2^{\text{algebra}}]^{a\bar{a}} &= [\beta_2^{\text{algebra}}]^{C\bar{C}} V_C^a \bar{V}_{\bar{C}}^{\bar{a}} \\ &= \frac{1}{2} G^{A\bar{A}} G^{B\bar{B}} V_{[A}^k \partial_k V_{B]}^a \bar{V}_{[\bar{A}}^{\bar{k}} \partial_{\bar{k}} \bar{V}_{\bar{B}}^{\bar{a}}] \\ &= g^{k\bar{k}} \partial_k \partial_{\bar{k}} g^{a\bar{a}} - \partial_{\bar{k}} g^{k\bar{a}} \partial_k g^{a\bar{k}}. \end{aligned} \quad (\text{A2})$$

$$\begin{aligned} G^{A\bar{A}} G^{B\bar{B}} G^{C\bar{C}} f_{\bar{A}\bar{C}}^{\bar{D}} f_{\bar{B}\bar{D}}^{\bar{E}} \eta_{AB} V_C^a \bar{V}_{\bar{E}}^{\bar{a}} &= G^{A\bar{A}} G^{B\bar{B}} G^{C\bar{C}} \left(V_B^{\bar{k}} V_A^{\bar{s}} \partial_{\bar{k}} \partial_{\bar{s}} V_C^{\bar{a}} + V_B^{\bar{k}} \partial_{\bar{k}} V_A^{\bar{s}} \partial_{\bar{s}} V_C^{\bar{a}} - V_B^{\bar{k}} \partial_{\bar{k}} V_C^{\bar{s}} \partial_{\bar{s}} V_A^{\bar{a}} - V_B^{\bar{k}} V_C^{\bar{s}} \partial_{\bar{k}} \partial_{\bar{s}} V_A^{\bar{a}} \right. \\ &\quad \left. + V_C^{\bar{k}} \partial_{\bar{k}} V_A^{\bar{s}} \partial_{\bar{s}} V_B^{\bar{a}} - V_A^{\bar{k}} \partial_{\bar{k}} V_C^{\bar{s}} \partial_{\bar{s}} V_B^{\bar{a}} \right) (\partial_k V_A^b \partial_b V_B^k + g^{b\bar{c}} \partial_c g_{\bar{c}k} V_A^c \partial_b V_B^k \\ &\quad \left. + g^{b\bar{c}} \partial_c g_{\bar{c}k} V_B^c \partial_b V_A^k - V_B^b V_A^k \partial_k g_{\bar{c}c} \partial_b g^{c\bar{c}} \right) V_C^a \end{aligned} \quad (\text{B2})$$

Here we have used the definition of the commutator of the vector fields and the modified definition of η_{AB} (See (21) in the main text). Using $A \leftrightarrow B$ symmetry we can further rewrite

The geometric beta function is given by the Ricci tensor

$$[\beta_2^{\text{geometry}}]^{a\bar{a}} = R^{a\bar{a}} \quad (\text{A3})$$

For the Kähler metric g we can use the following definition of the Ricci tensor

$$\begin{aligned} -R_{i\bar{j}} &= \partial_i \partial_{\bar{j}} \log(g) \\ &= \partial_i (g^{k\bar{l}} \partial_{\bar{j}} g_{k\bar{l}}) \\ &= \partial_i g^{k\bar{l}} \partial_{\bar{j}} g_{k\bar{l}} + g^{k\bar{l}} \partial_i \partial_{\bar{j}} g_{k\bar{l}} \\ &= \partial_i g^{k\bar{l}} \partial_{\bar{l}} g_{k\bar{j}} + g^{k\bar{l}} \partial_i \partial_{\bar{j}} g_{k\bar{l}} \end{aligned} \quad (\text{A4})$$

where in the last line we have used the Kähler property. One can easily prove the following lemma

$$\partial_k \partial_{\bar{l}} g^{a\bar{a}} + g^{i\bar{a}} G^{a\bar{j}} \partial_k \partial_{\bar{l}} g_{i\bar{j}} + g^{i\bar{a}} \partial_{\bar{l}} g_{i\bar{j}} \partial_k g^{a\bar{j}} + g^{i\bar{a}} \partial_k g_{i\bar{j}} \partial_{\bar{l}} g^{a\bar{j}} = 0, \quad (\text{A5})$$

which allows us to show that

$$R^{a\bar{a}} = g^{k\bar{l}} \partial_k \partial_{\bar{l}} g^{a\bar{a}} - \partial_{\bar{j}} g^{k\bar{a}} \partial_k g^{a\bar{j}}. \quad (\text{A6})$$

This way, we conclude that

$$\boxed{\beta_2^{\text{algebra}} = \beta_2^{\text{geometry}}}. \quad (\text{A7})$$

APPENDIX B: TWO-LOOP COMPARISON

In this section, we compare an algebraic and geometric beta function at two-loop level. The algebraic beta function reads

$$[\beta_3^{\text{algebra}}]^{E\bar{C}} = \frac{1}{2} G^{A\bar{A}} G^{B\bar{B}} G^{C\bar{C}} f_{AC}^D f_{BD}^E \eta_{\bar{A}\bar{B}} + \text{c.c.} \quad (\text{B1})$$

Let us present this expression in terms of the metric of the target space, which for simplicity, is assumed to be (i) Kähler; (ii) equivalent to the metric of the mass deformation.

We start by rewriting identically

$$\begin{aligned}
G^{A\bar{A}}G^{B\bar{B}}G^{C\bar{C}}f_{\bar{A}\bar{C}}^{\bar{D}}f_{\bar{B}\bar{D}}^{\bar{E}}\eta_{AB}V_C\bar{V}_{\bar{E}}^{\bar{a}} &= G^{A\bar{A}}G^{B\bar{B}}G^{C\bar{C}}\left(V_{\bar{B}}^{\bar{k}}V_{\bar{A}}^{\bar{s}}\partial_{\bar{k}}\partial_{\bar{s}}V_{\bar{C}}^{\bar{a}} - V_{\bar{B}}^{\bar{k}}V_{\bar{C}}^{\bar{s}}\partial_{\bar{k}}\partial_{\bar{s}}V_{\bar{A}}^{\bar{a}} + V_{\bar{B}}^{\bar{k}}\partial_{\bar{k}}V_{\bar{A}}^{\bar{s}}\partial_{\bar{s}}V_{\bar{C}}^{\bar{a}} - 2V_{\bar{B}}^{\bar{k}}\partial_{\bar{k}}V_{\bar{C}}^{\bar{s}}\partial_{\bar{s}}V_{\bar{A}}^{\bar{a}}\right. \\
&+ V_{\bar{C}}^{\bar{k}}\partial_{\bar{k}}V_{\bar{B}}^{\bar{s}}\partial_{\bar{s}}V_{\bar{A}}^{\bar{a}}\left.\right)\left(\partial_kV_A^b\partial_bV_B^k + g^{b\bar{c}}\partial_cg_{\bar{c}k}V_A^c\partial_bV_B^k + g^{b\bar{c}}\partial_cg_{\bar{c}k}V_B^c\partial_bV_A^k\right. \\
&\left. - V_B^bV_A^k\partial_kg_{\bar{c}c}\partial_bg^{\bar{c}\bar{c}}\right)V_C^a
\end{aligned} \tag{B3}$$

or equivalently

$$\begin{aligned}
G^{A\bar{A}}G^{B\bar{B}}G^{C\bar{C}}f_{\bar{A}\bar{C}}^{\bar{D}}f_{\bar{B}\bar{D}}^{\bar{E}}\eta_{AB}V_C\bar{V}_{\bar{E}}^{\bar{a}} &= G^{A\bar{A}}G^{B\bar{B}}\left(V_{\bar{B}}^{\bar{k}}V_{\bar{A}}^{\bar{s}}\partial_{\bar{k}}\partial_{\bar{s}}g^{a\bar{a}} - V_{\bar{B}}^{\bar{k}}g^{a\bar{s}}\partial_{\bar{k}}\partial_{\bar{s}}V_{\bar{A}}^{\bar{a}} + V_{\bar{B}}^{\bar{k}}\partial_{\bar{k}}V_{\bar{A}}^{\bar{s}}\partial_{\bar{s}}g^{a\bar{a}} - 2V_{\bar{B}}^{\bar{k}}\partial_{\bar{k}}g^{a\bar{s}}\partial_{\bar{s}}V_{\bar{A}}^{\bar{a}}\right. \\
&\left. + g^{a\bar{k}}\partial_{\bar{k}}V_{\bar{B}}^{\bar{s}}\partial_{\bar{s}}V_{\bar{A}}^{\bar{a}}\right)\left(\partial_kV_A^b\partial_bV_B^k + g^{b\bar{c}}\partial_cg_{\bar{c}k}V_A^c\partial_bV_B^k + g^{b\bar{c}}\partial_cg_{\bar{c}k}V_B^c\partial_bV_A^k - V_B^bV_A^k\partial_kg_{\bar{c}c}\partial_bg^{\bar{c}\bar{c}}\right).
\end{aligned} \tag{B4}$$

This way we obtain

$$\begin{aligned}
G^{A\bar{A}}G^{B\bar{B}}G^{C\bar{C}}f_{\bar{A}\bar{C}}^{\bar{D}}f_{\bar{B}\bar{D}}^{\bar{E}}\eta_{AB}V_C\bar{V}_{\bar{E}}^{\bar{a}} &= g^{a\bar{s}}\nabla_k\partial_{\bar{k}}g^{b\bar{a}}\nabla_b\partial_{\bar{s}}g^{k\bar{k}} + \nabla_bg^{k\bar{s}}\left(\partial_{\bar{k}}g^{a\bar{a}}\nabla_k\partial_{\bar{s}}g^{b\bar{k}} - 2\partial_{\bar{s}}g^{a\bar{k}}\nabla_k\partial_{\bar{k}}g^{b\bar{a}}\right. \\
&\left. + \partial_{\bar{k}}\partial_{\bar{s}}g^{a\bar{a}}\nabla_kg^{b\bar{k}} - g^{a\bar{k}}\nabla_k\partial_{\bar{k}}\partial_{\bar{s}}g^{b\bar{a}}\right).
\end{aligned} \tag{B5}$$

Where

$$\nabla_bg^{k\bar{s}} = \partial_bg^{k\bar{s}} + g^{k\bar{c}}\partial_cg_{\bar{c}b}g^{c\bar{s}} \tag{B6}$$

$$\nabla_b\bar{\partial}\dots\bar{\partial}g^{k\bar{s}} = \partial_b\bar{\partial}\dots\bar{\partial}g^{k\bar{s}} + g^{k\bar{c}}\partial_cg_{\bar{c}b}\bar{\partial}\dots\bar{\partial}g^{c\bar{s}}. \tag{B7}$$

These computations are valid for any metric $g^{a\bar{a}}$. For Kähler metric they can be simplified even further. In particular, in this case, the covariant derivative of a metric vanishes

$$\nabla_bg^{k\bar{s}} = 0, \tag{B8}$$

while for the ‘‘double’’ derivative we have the following presentation

$$\begin{aligned}
\nabla_k\partial_{\bar{k}}g^{b\bar{a}} &= \partial_k\partial_{\bar{k}}g^{b\bar{a}} + \Gamma_{k\bar{c}}^b\partial_{\bar{k}}g^{c\bar{a}} \\
&= \partial_{\bar{k}}(\nabla_kg^{b\bar{a}}) - g^{c\bar{a}}\partial_{\bar{k}}\Gamma_{k\bar{c}}^b \\
&= -R^b{}_{k\bar{k}c}g^{c\bar{a}} = R_{\bar{b}k\bar{k}c}g^{b\bar{b}}g^{c\bar{a}} \\
&= -R_{c\bar{k}\bar{k}b}g^{b\bar{b}}G^{c\bar{a}} = -R^{\bar{a}}{}_{\bar{k}\bar{k}b}g^{b\bar{b}}.
\end{aligned} \tag{B9}$$

And in a similar way

$$\begin{aligned}
\nabla_b\partial_{\bar{s}}g^{k\bar{k}} &= -R^k{}_{m\bar{s}b}g^{m\bar{k}} = -R^k{}_{b\bar{s}m}g^{m\bar{k}} \\
&= -R_{\bar{p}b\bar{s}m}g^{m\bar{k}}g^{k\bar{p}} = -R_{\bar{s}m\bar{p}b}g^{m\bar{k}}g^{k\bar{p}}.
\end{aligned} \tag{B10}$$

Altogether we get

$$\begin{aligned}
G^{A\bar{A}}G^{B\bar{B}}G^{C\bar{C}}f_{\bar{A}\bar{C}}^{\bar{D}}f_{\bar{B}\bar{D}}^{\bar{E}}\eta_{AB} &= R^{\bar{a}}{}_{\bar{k}\bar{k}b}R^a{}_{m\bar{p}b}g^{m\bar{k}}g^{k\bar{p}}g^{b\bar{b}} \\
&= R^a{}_{m\bar{p}b}R^{\bar{a}m\bar{p}b},
\end{aligned} \tag{B11}$$

which demonstrates that even on the two-loop level

$$\beta_3^{\text{algebra}} = \beta_3^{\text{geometry}}. \tag{B12}$$

APPENDIX C: CURRENT-CURRENT CORRELATION FUNCTION

In this section, we outline proof of Eqs. (19)–(21) in the main text.

For convenience let us reformulate the problem here. We are interested in the computation of the following current-current correlation function

$$\langle \mathcal{J}_A(z)\mathcal{J}_B(0) \rangle_{S_m} \equiv -\frac{\eta_{AB}}{z^2} \tag{C1}$$

in the limit $m|z| \gg 1$.

The currents are given by the normal ordered expressions $\mathcal{J}_A(z) = p_aV_A^a(\varphi) - \pi_a\partial_bV_A^a\psi^b$, while the mass deformed action reads

$$\begin{aligned}
S_m &= \underbrace{\int \frac{d^2z}{\pi} (p_a\bar{\partial}\varphi^a - \pi_a\bar{\partial}\psi^a + \text{c.c.})}_{S_0} \\
&+ \underbrace{m \int \frac{d^2z}{2\pi} M^{a\bar{a}}(\varphi)\pi_a\bar{\pi}_{\bar{a}} + m \int \frac{d^2z}{2\pi} M_{\bar{a}a}(\varphi)\bar{\psi}^{\bar{a}}\psi^a}_{\delta S_m}.
\end{aligned} \tag{C2}$$

The coefficients are chosen in such a way that at $m = 0$ the OPE would read as

$$\begin{aligned}
p_a(z)\varphi^b(w)|_{m=0} &= \frac{\delta_a^b}{z-w} + \text{reg}, \\
\pi_a(z)\psi^b(w)|_{m=0} &= \frac{\delta_a^b}{z-w} + \text{reg}.
\end{aligned} \tag{C3}$$

We are going to prove that

$$\begin{aligned}
\eta_{AB} &= \nabla_a V_A^b \nabla_b V_B^a \\
&= \partial_a V_A^b \partial_b V_B^a + M^{b\bar{a}} \partial_c M_{\bar{a}a} V_A^c \partial_b V_B^a \\
&\quad + M^{b\bar{a}} \partial_c M_{\bar{a}a} V_B^c \partial_b V_A^a - V_B^b V_A^a \partial_a M_{\bar{c}c} \partial_b M^{c\bar{c}}.
\end{aligned} \tag{C4}$$

First let us notice that in the correlators that do not involve p , one can easily ignore φ dependence in $M_{a\bar{a}}$ and $M^{a\bar{a}}$ in S_m . In particular, the fermions correlators could be computed explicitly (we drop S_m subscript in the correlators),

$$\begin{aligned}
\langle \pi_a(z)\psi^b(0) \rangle &= \delta_a^b \frac{mz}{|z|} K_1(m|z|), \\
\langle \bar{\pi}_{\bar{a}}(z)\bar{\psi}^{\bar{b}}(0) \rangle &= \delta_{\bar{a}}^{\bar{b}} \frac{m\bar{z}}{|z|} K_1(m|z|) \\
\langle \pi_a(z)\bar{\pi}_{\bar{a}}(0) \rangle &= mM_{\bar{a}a} K_0(m|z|), \\
\langle \bar{\psi}^{\bar{a}}(z)\psi^a(0) \rangle &= mM^{a\bar{a}} K_0(m|z|).
\end{aligned} \tag{C5}$$

Here K_0 and K_1 are the modified Bessel function of the second kind. The rest of fermionic two-point functions are zero. All these correlators are exponentially small for $m|z| \gg 1$. This way, the fermion-fermion current terms do not contribute in the correlator (C1)

$$\langle \pi_a \partial_b V_A^a \psi^b(z) \pi_c \partial_n V_B^c \psi^n(0) \rangle = O(e^{-m|z|}). \tag{C7}$$

Now let us have a look at mixed terms. In this case, the field p from the bosonic part of the current can be contracted with the φ in the deformation terms in the mass part of S_m (C2). Namely,

$$\begin{aligned}
&\langle p_a V_A^a(z) \pi_c \partial_n V_B^c \psi^n(0) \delta S_m \rangle \\
&= 2V_A^a \partial_n V_B^c M^{n\bar{a}} \partial_a M_{\bar{a}c} \int \frac{d^2w}{2\pi} \frac{m^3}{z-w} \frac{\bar{w}}{|w|} \\
&\quad \times K_1(m|w|) K_0(m|w|).
\end{aligned} \tag{C8}$$

Notice that the above integral does not have UV divergences, to compute it we use the radial coordinates and rescale the absolute value $|w|$

$$\begin{aligned}
&\int_0^\infty d|w| \int_{-\pi}^\pi \frac{d\varphi}{\pi} \frac{m|w| e^{-i\varphi}}{z - e^{i\varphi}|w|/m} K_0(|w|) K_1(|w|) \\
&= \frac{2}{z^2} \int_0^{m|z|} |w|^2 K_0(|w|) K_1(|w|) d|w|.
\end{aligned} \tag{C9}$$

For $m|z| \gg 1$ we can replace the upper limit to infinity, which will yield us some constant

$$\int_0^\infty |w|^2 K_0(|w|) K_1(|w|) d|w| = \frac{1}{2}. \tag{C10}$$

This gives

$$\langle p_a V_A^a(z) \pi_c \partial_n V_B^c \psi^n(0) \delta S_m \rangle_{S_0} \stackrel{m|z| \gg 1}{=} \frac{V_A^a \partial_n V_B^c M^{n\bar{a}} \partial_a M_{\bar{a}c}}{z^2}. \tag{C11}$$

This way, we recover VV' terms in (C4).

Now let us turn to the boson-boson contributions. To simplify computations we consider a small deformation on top of the constant metric $M_{\bar{a}a} = [M_0]_{\bar{a}a} + \delta M_{\bar{a}a}$, and correspondingly $M^{a\bar{a}} = [M_0]^{a\bar{a}} - \delta M^{a\bar{a}}$, where

$$\delta M^{\bar{a}a} \equiv M_0^{\bar{a}c} \delta M_{\bar{c}c} M_0^{ca} + O(\delta M^2). \tag{C12}$$

Quadratic in $\delta M_{\bar{a}a}$ terms are

$$\left\langle p_a V_A^a(z) p_b V_B^b(0) \frac{(\delta S_m)^2}{2} \right\rangle = I_{\pi\pi} + I_{\pi\psi} + I_{\psi\psi} \tag{C13}$$

where

$$I_{\pi\pi} = \frac{m^2}{2} \int \frac{d^2w_1}{2\pi} \int \frac{d^2w_2}{2\pi} \langle p_a V_A^a(z) p_b V_B^b(0) \delta M^{c\bar{c}} \pi_c \bar{\pi}_{\bar{c}}(w_1) \delta M^{k\bar{k}} \pi_k \bar{\pi}_{\bar{k}}(w_2) \rangle_{M_0} \tag{C14}$$

$$I_{\psi\psi} = \frac{m^2}{2} \int \frac{d^2w_1}{2\pi} \int \frac{d^2w_2}{2\pi} \langle p_a V_A^a(z) p_b V_B^b(0) \delta M_{\bar{c}c} \bar{\psi}^{\bar{c}} \psi^c(w_1) \delta M_{\bar{k}k} \bar{\psi}^{\bar{k}} \psi^k(w_2) \rangle_{M_0} \tag{C15}$$

$$I_{\pi\psi} = -m^2 \int \frac{d^2w_1}{2\pi} \int \frac{d^2w_2}{2\pi} \langle p_a V_A^a(z) p_b V_B^b(0) \delta M^{c\bar{c}} \pi_c \bar{\pi}_{\bar{c}}(w_1) \delta M_{\bar{k}k} \bar{\psi}^{\bar{k}} \psi^k(w_2) \rangle_{M_0}. \tag{C16}$$

Here the correlators are computed with the action S_m with metric M_0 . Now let us focus on terms that are proportional to $V_A^a V_B^b$ i.e., where vectors fields are not differentiated, meaning that the fields p are contracted with φ in the perturbation. Let us additionally introduce

$$h_a^c = M_0^{\bar{c}c} \delta M_{\bar{c}a} \quad (\text{C17})$$

then we obtain

$$\begin{aligned} \left\langle p_a V_A^a(z) p_b V_B^b(0) \frac{(\delta S)^2}{2} \right\rangle_{VV} &= -V_A^a V_B^b \int \frac{d^2 w_1}{2\pi} \int \frac{d^2 w_2}{2\pi} m^4 [K_1(m|w_1 - w_2|)^2 + K_0(m|w_1 - w_2|)^2] \\ &\times \left(\frac{\partial_a h_c^d \partial_b h_d^c}{(z - w_1) w_2} + \frac{h_d^c \partial_a \partial_b h_c^d}{(z - w_1) w_1} + (w_1, a \leftrightarrow w_2, b) \right). \end{aligned} \quad (\text{C18})$$

Identically we can present this expression as

$$\left\langle p_a V_A^a(z) p_b V_B^b(0) \frac{(\delta S)^2}{2} \right\rangle_{VV} = V_A^a V_B^b \partial_a h_c^d \partial_b h_d^c C_1 - V_A^a V_B^b \partial_a \partial_b (h_c^d h_d^c) C_2 \quad (\text{C19})$$

where

$$\begin{aligned} C_1 &= \int \frac{d^2 w_1}{2\pi} \int \frac{d^2 w_2}{2\pi} \frac{2m^4 w_{21} (K_1(m|w_{21}|)^2 + K_0(m|w_{21}|)^2)}{(z - w_1) w_1 w_2} \\ &= \int \frac{d^2 w_1}{2\pi} \int \frac{d^2 w_2}{2\pi} \frac{2m^4 w_2 (K_1(m|w_2|)^2 + K_0(m|w_2|)^2)}{(z - w_1) w_1 (w_2 - w_1)} \end{aligned} \quad (\text{C20})$$

$$C_2 = \int \frac{d^2 w_1}{2\pi} \int \frac{d^2 w_2}{2\pi} \frac{2m^4 (K_1(m|w_{21}|)^2 + K_0(m|w_{21}|)^2)}{(z - w_1) w_1} = \int \frac{d^2 w_1}{2\pi} \frac{C_3}{(z - w_1) w_1} \quad (\text{C21})$$

with $w_{12} = w_1 - w_2$, and C_3 equals to

$$C_3 = \int \frac{d^2 w_2}{2\pi} 2m^4 (K_1(m|w_2|)^2 + K_0(m|w_2|)^2). \quad (\text{C22})$$

This integral is divergent logarithmically, but luckily it can be regularized by adding counterterms to the action

$$S_m \rightarrow S_m - C_3 \int h_c^d h_d^c(z) d^2 z. \quad (\text{C23})$$

Notice this regularization is also needed for VV' terms. The constant C_1 is finite and can be computed as follows. First, we perform integration over the angle of w_2

$$\int \frac{d^2 w_2}{2\pi} \frac{w_2 f(|w_2|)}{w_2 - w_1} = \int_{|w_1|}^{\infty} |w_2| f(|w_2|) d|w_2| \quad (\text{C24})$$

and then similarly over angle of w_1

$$\int \frac{d^2 w_1}{2\pi} \frac{f(|w_1|)}{(z - w_1) w_1} = \frac{1}{z^2} \int_0^{|z|} |w_1| f(|w_1|) d|w_1|, \quad (\text{C25})$$

which leaves us with the two-dimensional integral over absolute values (we also rescale $|w_i| \rightarrow |w_i|/m$)

$$\begin{aligned} C_1 &= \frac{2}{z^2} \int_0^{m|z|} d|w_1| |w_1| \\ &\times \int_{|w_1|}^{\infty} |w_2| (K_1(|w_2|)^2 + K_0(|w_2|)^2) d|w_2|. \end{aligned} \quad (\text{C26})$$

For $m|z| \gg 1$ we replace the upper limit with ∞ and then exchange integration over $|w_1|$ and $|w_2|$, in the end we arrive at

$$C_1 = \frac{1}{z^2} \int_0^{\infty} |w_2|^3 (K_1(|w_2|)^2 + K_0(|w_2|)^2) d|w_2| = \frac{1}{z^2}. \quad (\text{C27})$$

This way, without C_2 and with $C_1 = 1/z^2$ from (C19) we recover quadratic terms in the expansion of (C4) around the constant metric M_0 on terms without derivative of the vector fields.

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