Gravitational chiral anomaly at finite temperature and density

Claudio Corianò,[1,2] Mario Cretì,[1,3] Stefano Lionetti,[1] and Riccardo Tommasi[1]

[1]*Dipartimento di Matematica e Fisica, Università del Salento and INFN Sezione di Lecce, Via Arnesano 73100 Lecce, Italy and National Center for HPC, Big Data and Quantum Computing, Via Magnanelli 2, 40033 Casalecchio di Reno, Italy*

[2]*Institute of Nanotechnology, National Research Council (CNR-NANOTEC), Lecce 73100, Italy*

[3]*Center for Biomolecular Nanotechnologies, Istituto Italiano di Tecnologia, Via Barsanti 14, 73010 Arnesano, Lecce, Italy*

(Received 2 May 2024; accepted 3 June 2024; published 12 July 2024)

We investigate the gravitational anomaly vertex $\langle TTJ_5\rangle$ (graviton—graviton—axial current) under conditions of finite density and temperature. Through a direct analysis of perturbative contributions, we demonstrate that neither finite temperature nor finite fermion density affects the gravitational chiral anomaly. These results find application in several contexts, from topological materials to the early Universe plasma. They affect the decay of any axion or axionlike particle into gravitational waves, in very dense and hot environments.

DOI: 10.1103/PhysRevD.110.025008

I. INTRODUCTION

The interest on gravitational anomalies has been continuous due to their connections with various theories, including ordinary gauge theories [1,2], supergravity [3], and self-dual antisymmetric fields in string theory [4]. The gravitational anomaly $R\tilde{R}$ can manifest in different scenarios, with its implications varying from benign to critical. For example, consider a scenario involving a Dirac fermion interacting with gravity and a vector gauge field. The anomaly, that in this case appears in the divergence of J_5, was first computed by Kimura, Delbourgo, and Salam [1,2]. This specific anomaly poses no threat and holds relevance in phenomenology.

Another instance involves a chiral model incorporating a Weyl fermion $\psi_{L/R}$ interacting with gravity and a gauge field. In this case, the anomaly emerges in the divergence of $J_{L/R}$, potentially endangering unitarity and renormalization, unless it is canceled [4,5]. In the Standard Model, when J_5 is the non-Abelian $SU(2)$ gauge current or the hypercharge gauge current, the gravitational anomaly cancel out by summing over the chiral spectrum of each fermion generation. This feature is usually interpreted as an indication of the compatibility of the Standard Model when coupled to a gravitational background, providing an essential constraint on its possible extensions. The correlator $\langle TTJ_5\rangle$, where T denotes the stress energy tensor, under examination in this work, has been recently re-investigated [6] using conformal field theory (CFT) in momentum space methods [7–9], previously studied in [10] using coordinate space methods.

Correlators influenced by chiral and conformal anomalies, as well as discrete anomalies, play a vital role in condensed matter theory, particularly in the context of topological materials [11–20]. The gravitational anomaly has been investigated, in the same context, in other interesting works [21,22]. Crucial, in this analysis, is the correspondence between thermal stresses and gravity, as summarized by Luttinger's relation [23] connecting a gravitational potential to a thermal gradient [24,25]. Understanding these phenomena is essential for unraveling the intricate properties of such materials. Since their dynamical contribution in the evolution of topological matter, in the realistic case, is characterized by both thermal effecs and by Fermi surfaces, which break the charge conjugation C invariance of the vacuum, the quantification of such corrections becomes crucial for phenomenology.

A. Thermal and density effects

Considerable attention has been dedicated to examining the effects of finite temperature and density on the axial gauge anomaly in several other contexts [26–33]. Despite the diverse approaches taken by the various authors to address this issue, a unanimous consensus emerges: such anomaly remains insensitive to corrections from finite temperature and density.

In our previous work [31], we have investigated the general structure of the chiral anomaly vertex $\langle AVV\rangle$, in the presence of chemical potentials in perturbation theory. We have classified the minimal number of tensorial structures for the $\langle AVV\rangle$ parametrization, and we have provided a direct perturbative identification of their corresponding form factors. Moreover, when the photons are on-shell, we have shown that the entire correlator reduces to the longitudinal anomalous sector.

The gravitational axial anomaly has yet to be explored in the context of finite temperature and density. In this paper, we aim to fill this gap by investigating this interaction. Our approach involves utilizing the real-time Green's function method [34]. Through this technique, we will compute the amplitude of thermal Feynman diagrams. Our objective is to demonstrate that the gravitational axial anomaly remains unchanged, with no contributions from either density or temperature effects.

II. REVIEW: DECOMPOSITION OF THE CORRELATOR AND THE PERTURBATIVE EXPANSION AT ZERO TEMPERATURE AND DENSITY

In this section, we provide an overview of the general structure for the $\langle TTJ_5\rangle$ correlator in the vacuum, using the longitudinal/transverse-traceless decomposition, commonly referred to as the "L/T" decomposition [8]. This allows us to gain insight into its global tensorial structure under zero temperature and density conditions. The decomposition has been used in the case of parity-even anomaly correlators in several previous works involving both 3-point and 4-point functions [9,35–37] and extended to the simplest parity-odd correlator, the AVV in [38]. Nonconserved vector currents, have been discussed by the same method in [39]. The analysis presented here, at zero T and μ, is nonperturbative since it relies on the solution of the conformal constraints by the inclusion of an anomaly pole in order to fix the longitudinal sector of the correlator, and does not require a Lagrangian realization. The structure becomes considerably more intricate in the case of finite T and μ, primarily due to the inclusion of a velocity four-vector η representing the heat bath. The exhaustive analysis of the vertex entails a tensorial expansion, which is particularly demanding, even in the simpler AVV diagram case [31].

We start by decomposing the energy-momentum tensor $T^{\mu\nu}$ and the current J_5^μ in terms of their transverse-traceless part and longitudinal ones (also called "local"),

$$T^{\mu_i\nu_i}(p_i) = t^{\mu_i\nu_i}(p_i) + t_{\rm loc}^{\mu_i\nu_i}(p_i), \tag{2.1}$$

$$J_5^{\mu_i}(p_i) = j_5^{\mu_i}(p_i) + j_{5loc}^{\mu_i}(p_i), \tag{2.2}$$

where

$$\begin{aligned} t^{\mu_i\nu_i}(p_i) &= \Pi^{\mu_i\nu_i}_{\alpha_i\beta_i}(p_i)T^{\alpha_i\beta_i}(p_i), \qquad & t_{\rm loc}^{\mu_i\nu_i}(p_i) &= \Sigma^{\mu_i\nu_i}_{\alpha_i\beta_i}(p)T^{\alpha_i\beta_i}(p_i), \\ j_5^{\mu_i}(p_i) &= \pi^{\mu_i}_{\alpha_i}(p_i)J_5^{\alpha_i}(p_i), \qquad & j_{5{\rm loc}}^{\mu_i}(p_i) &= \frac{p_i^{\mu_i}p_{i\alpha_i}}{p_i^2}J_5^{\alpha_i}(p_i), \end{aligned} \tag{2.3}$$

having introduced the transverse-traceless (Π), transverse (π) and longitudinal (Σ) projectors, given respectively by

$$\pi^\mu_\alpha = \delta^\mu_\alpha - \frac{p^\mu p_\alpha}{p^2}, \tag{2.4}$$

$$\Pi^{\mu\nu}_{\alpha\beta} = \frac{1}{2}(\pi^\mu_\alpha\pi^\nu_\beta + \pi^\mu_\beta\pi^\nu_\alpha) - \frac{1}{d-1}\pi^{\mu\nu}\pi_{\alpha\beta}, \tag{2.5}$$

$$\Sigma^{\mu_i\nu_i}_{\alpha_i\beta_i} = \frac{p_{i\beta_i}}{p_i^2}\left[2\delta^{(\nu_i}_{\alpha_i}p_i^{\mu_i)} - \frac{p_{i\alpha_i}}{(d-1)}\left(\delta^{\mu_i\nu_i} + (d-2)\frac{p_i^{\mu_i}p_i^{\nu_i}}{p_i^2}\right)\right] + \frac{\pi^{\mu_i\nu_i}(p_i)}{(d-1)}\delta_{\alpha_i\beta_i}. \tag{2.6}$$

Such decomposition allows to split the vacuum correlation function into the following terms:

$$\begin{aligned} \langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}J_5^{\mu_3}\rangle &= \langle t^{\mu_1\nu_1}t^{\mu_2\nu_2}j_5^{\mu_3}\rangle + \langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}j_{5\rm loc}^{\mu_3}\rangle + \langle T^{\mu_1\nu_1}t_{\rm loc}^{\mu_2\nu_2}J_5^{\mu_3}\rangle + \langle t_{\rm loc}^{\mu_1\nu_1}T^{\mu_2\nu_2}J_5^{\mu_3}\rangle \\ &\quad - \langle T^{\mu_1\nu_1}t_{\rm loc}^{\mu_2\nu_2}j_{5\rm loc}^{\mu_3}\rangle - \langle t_{\rm loc}^{\mu_1\nu_1}t_{\rm loc}^{\mu_2\nu_2}J_5^{\mu_3}\rangle - \langle t_{\rm loc}^{\mu_1\nu_1}T^{\mu_2\nu_2}j_{5\rm loc}^{\mu_3}\rangle + \langle t_{\rm loc}^{\mu_1\nu_1}t_{\rm loc}^{\mu_2\nu_2}j_{5\rm loc}^{\mu_3}\rangle, \end{aligned} \tag{2.7}$$

where the first contribution on the rhs of the expression above identifies the transverse-traceless part. Using the (anomalous) conservation and trace WIs, it is then possible to completely fix all the longitudinal parts, i.e. the terms containing at least one j_{5loc}^μ or $t_{\rm loc}^{\mu\nu}$. The nonanomalous equations are

$$\begin{aligned} \delta_{\mu_i\nu_i}\langle T^{\mu_1\nu_1}(p_1)T^{\mu_2\nu_2}(p_2)J_5^{\mu_3}(p_3)\rangle &= 0, \qquad i=\{1,2\} \\ p_{i\mu_i}\langle T^{\mu_1\nu_1}(p_1)T^{\mu_2\nu_2}(p_2)J_5^{\mu_3}(p_3)\rangle &= 0, \qquad i=\{1,2\}. \end{aligned} \tag{2.8}$$

Thanks to these WIs, we can eliminate most of terms on the right-hand side of Eq. (2.7), ending up with only two terms,

$$\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}J_5^{\mu_3}\rangle = \langle t^{\mu_1\nu_1}t^{\mu_2\nu_2}j_5^{\mu_3}\rangle + \langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}j_{5\mathrm{loc}}^{\mu_3}\rangle = \langle t^{\mu_1\nu_1}t^{\mu_2\nu_2}j_5^{\mu_3}\rangle + \langle t^{\mu_1\nu_1}t^{\mu_2\nu_2}j_{5\mathrm{loc}}^{\mu_3}\rangle. \tag{2.9}$$

The remaining local term in the expression above is then fixed by the anomalous WI of J_5,

$$p_{3\mu_3}\langle T^{\mu_1\nu_1}(p_1)T^{\mu_2\nu_2}(p_2)J_5^{\mu_3}(p_3)\rangle = 4ia_2(p_1\cdot p_2)\left\{\left[\varepsilon^{\nu_1\nu_2p_1p_2}\left(g^{\mu_1\mu_2} - \frac{p_1^{\mu_2}p_2^{\mu_1}}{p_1\cdot p_2}\right) + (\mu_1\leftrightarrow\nu_1)\right] + (\mu_2\leftrightarrow\nu_2)\right\}, \tag{2.10}$$

where a_2 is the anomaly constant. Thanks to the equation above, we can write

$$\langle t^{\mu_1\nu_1}t^{\mu_2\nu_2}j_{5loc}^{\mu_3}\rangle = 4ia_2\frac{p_3^{\mu_3}}{p_3^2}(p_1\cdot p_2)\left\{\left[\varepsilon^{\nu_1\nu_2p_1p_2}\left(g^{\mu_1\mu_2} - \frac{p_1^{\mu_2}p_2^{\mu_1}}{p_1\cdot p_2}\right) + (\mu_1\leftrightarrow\nu_1)\right] + (\mu_2\leftrightarrow\nu_2)\right\}. \tag{2.11}$$

Notice that in order to identify in field space the effective action associated with this contribution, we need to multiply the stress energy tensors by the gravitational fluctuations of the metric $h_{\mu\nu}$ and the axial-vector source A_μ on the current J_5, in the form,

$$\mathcal{S}_{\mathrm{anom}} \sim a_2\int d^4x d^4y \partial\cdot A(x)\Box^{-1}(x,y)R\tilde{R}(y). \tag{2.12}$$

In other words, Eq. (2.11) can be generated from $\mathcal{S}_{\mathrm{anom}}$ by the functional differentiation of the latter twice with respect to the metric and once with respect to A_μ, followed by an ordinary Fourier transform. The terms $1/p_3^2$ is the anomaly pole, introduced in order to solve (2.10).

Notice that this four-dimensional procedure is quite straightforward in the case of chiral anomaly diagrams, since the conformal Ward identities are exact; i.e. they are not affected by the anomaly. The reason for such behavior is quite simple, and is due to the fact that there is no renormalzation needed in order to define the correlator at spacetime dimension four. The total anomaly effective action can indeed be decomposed in the form,

$$\mathcal{S}_{\mathrm{eff}} = \mathcal{S}_{\mathrm{anom}} + \mathcal{S}_{\perp}. \tag{2.13}$$

The only remaining term in the reconstruction of the entire correlator is the transverse-traceless part $\langle t^{\mu_1\nu_1}t^{\mu_2\nu_2}j_5^{\mu_3}\rangle$ related to $\mathcal{S}_\perp$. Its explicit form is given by the inclusion of transverse and transverse-traceless projectors π and Π acting on a tensor structure that is parametrized in terms of a minimal number of independent form factors,

$$\langle t^{\mu_1\nu_1}(p_1)t^{\mu_2\nu_2}(p_2)j_5^{\mu_3}(p_3)\rangle = \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(p_1)\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(p_2)\pi^{\mu_3}_{\alpha_3}(p_3)X^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3}, \tag{2.14}$$

where $X^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3}$ is a general rank five tensor built by products of metric tensors, momenta and ε tensors with the appropriate choice of indices. Indeed, as a consequence of the projectors in (2.14), $X^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3}$ can not be constructed by using $g_{\alpha_i\beta_i}$, nor by $p_{i\alpha_i}$ with $i=\{1,2,3\}$. Using the Schouten identities and imposing the symmetries of the correlators, the general structure of the transverse-traceless part is given by the simplified minimal expression [6],

$$\begin{aligned}&\langle t^{\mu_1\nu_1}(p_1)t^{\mu_2\nu_2}(p_2)j_5^{\mu_3}(p_3)\rangle\\ &= \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(p_1)\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(p_2)\pi^{\mu_3}_{\alpha_3}(p_3)\Big[\\ &\quad A_1\varepsilon^{p_1\alpha_1\alpha_2\alpha_3}p_2^{\beta_1}p_3^{\beta_2} - A_1(p_1\leftrightarrow p_2)\varepsilon^{p_2\alpha_1\alpha_2\alpha_3}p_2^{\beta_1}p_3^{\beta_2}\\ &\quad + A_2\varepsilon^{p_1\alpha_1\alpha_2\alpha_3}\delta^{\beta_1\beta_2} - A_2(p_1\leftrightarrow p_2)\varepsilon^{p_2\alpha_1\alpha_2\alpha_3}\delta^{\beta_1\beta_2}\\ &\quad + A_3\varepsilon^{p_1p_2\alpha_1\alpha_2}p_2^{\beta_1}p_3^{\beta_2}p_1^{\alpha_3} + A_4\varepsilon^{p_1p_2\alpha_1\alpha_2}\delta^{\beta_1\beta_2}p_1^{\alpha_3}\Big],\end{aligned} \tag{2.15}$$

where A_3 and A_4 are antisymmetric under the exchange $(p_1\leftrightarrow p_2)$. The expressions of the form factors A_i can be derived in two ways. Either from the solutions of the conformal Ward identities (CWIs), which are expressed in terms of 3K integrals, i.e. parametric integrals of three Bessel functions, or they can be extracted by resorting to the perturbative expansion. The system of equations derived from the CWIs is rather involved, but one can show that the transverse-traceless sector is completely determined in terms of the same coefficient a_2 characterising the anomaly constraint. Details can be found in [6].

Both the perturbative and nonperturbative procedures are in complete agreement. In the next section, we will review the former approach.

A. The action and the perturbative realization

The perturbative evaluation of the correlator at one-loop, can be performed by working in a specific regularization scheme. We define the vacuum partition function,

$$e^{i\mathcal{S}[g]} \equiv \int [d\Phi]e^{iS_0[\Phi,g]}, \tag{2.16}$$

with the action given by

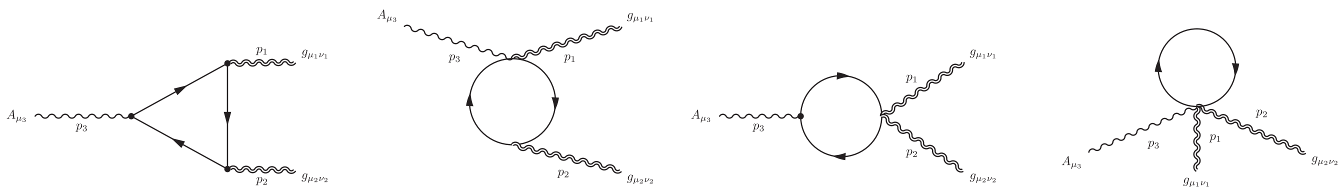


FIG. 1. Topologies of Feynman diagrams appearing in the perturbative computation.

$$S_0 = \int d^d x \frac{e}{2} e^\mu_a [i\bar{\psi}\gamma^a (D_\mu \psi) - i(D_\mu \bar{\psi})\gamma^a \psi], \tag{2.17}$$

where e^μ_a is the vielbein, e is its determinant and D_μ is the covariant derivative defined as

$$D_\mu \psi = (\nabla_\mu + ig\gamma_5 A_\mu)\psi = \left(\partial_\mu + ig\gamma_5 A_\mu + \frac{1}{2}\omega_{\mu ab}\Sigma^{ab}\right)\psi,$$
$$D_\mu \bar{\psi} = (\nabla_\mu - ig\gamma_5 A_\mu)\bar{\psi} = \left(\partial_\mu - ig\gamma_5 A_\mu - \frac{1}{2}\omega_{\mu ab}\Sigma^{ab}\right)\bar{\psi}. \tag{2.18}$$

Σ^{ab} are the generators of the Lorentz group in the case of a spin 1/2-field, while the spin connection is given by

$$\omega_{\mu ab} \equiv e^\nu_a (\partial_\mu e_{\nu b} - \Gamma^\lambda_{\mu\nu} e_{\lambda b}). \tag{2.19}$$

The Latin and Greek indices are related to the (locally) flat basis and the curved background respectively. Using the explicit expression of the generators of the Lorentz group one can reexpress the action S_0 as follows:

$$S_0 = \int d^d x e \left[\frac{i}{2}\bar{\psi} e^\mu_a \gamma^a (\partial_\mu \psi) - \frac{i}{2}(\partial_\mu \bar{\psi}) e^\mu_a \gamma^a \psi - g A_\mu \bar{\psi} e^\mu_a \gamma^a \gamma_5 \psi + \frac{i}{4}\omega_{\mu ab} e^\mu_c \bar{\psi}\gamma^{abc}\psi\right], \tag{2.20}$$

with

$$\gamma^{abc} = \{\Sigma^{ab}, \gamma^c\}. \tag{2.21}$$

The explicit expressions of the vertices for such action are reported in Appendix A. Taking a first variation of the action with respect to the metric one can construct the energy momentum tensor as

$$T^{\mu\nu} = -\frac{i}{2}[\bar{\psi}\gamma^{(\mu}\nabla^{\nu)}\psi - \nabla^{(\mu}\bar{\psi}\gamma^{\nu)}\psi - g^{\mu\nu}(\bar{\psi}\gamma^\lambda \nabla_\lambda \psi - \nabla_\lambda \bar{\psi}\gamma^\lambda \psi)] - g\bar{\psi}(g^{\mu\nu}\gamma^\lambda A_\lambda - \gamma^{(\mu}A^{\nu)})\gamma_5 \psi. \tag{2.22}$$

We now proceed with the perturbative evaluation. For this, we use the Breitenlohner-Maison regularization scheme. The topology of the diagrams is shown in Fig. 1.

The contribution of the triangle diagrams is given by

$$V_1^{\mu_1\nu_1\mu_2\nu_2\mu_3} = -i^3 \int \frac{d^d l}{(2\pi)^d} \frac{\mathrm{tr}\left[V^{\mu_1\nu_1}_{g\bar{\psi}\psi}(l-p_1, l)(\not{l} - \not{p}_1) V^{\mu_3}_{A\bar{\psi}\psi}(\not{l} + \not{p}_2) V^{\mu_2\nu_2}_{g\bar{\psi}\psi}(l, l+p_2)\not{l}\right]}{(l-p_1)^2 (l+p_2)^2 l^2} + \text{exchange}, \tag{2.23}$$

while the bubble diagrams are

$$V_2^{\mu_1\nu_1\mu_2\nu_2\mu_3} = -i^2 \int \frac{d^d l}{(2\pi)^d} \frac{\mathrm{tr}\left[V^{\mu_1\nu_1\mu_3}_{gA\bar{\psi}\psi}(\not{l} + \not{p}_2) V^{\mu_2\nu_2}_{g\bar{\psi}\psi}(l, l+p_2)\not{l}\right]}{(l+p_2)^2 l^2} + \text{exchange}, \tag{2.24}$$

and

$$V_3^{\mu_1\nu_1\mu_2\nu_2\mu_3} = -i^2 \int \frac{d^d l}{(2\pi)^d} \frac{\mathrm{tr}\left[V^{\mu_1\nu_1\mu_2\nu_2}_{gg\bar{\psi}\psi}(p_1, p_2, l-p_1-p_2, l)(\not{l} - \not{p}_1 - \not{p}_2) V^{\mu_3}_{A\bar{\psi}\psi}\not{l}\right]}{(l-p_1-p_2)^2 l^2}. \tag{2.25}$$

After performing the integration, one can verify that $V_2^{\mu_1\nu_1\mu_2\nu_2\mu_3}$ vanishes. Lastly, the tadpole diagram is given by

$$V_4^{\mu_1\nu_1\mu_2\nu_2\mu_3} = -i\int \frac{d^d l}{(2\pi)^d}\frac{\mathrm{tr}\left[V_{ggA\bar{\psi}\psi}^{\mu_1\nu_1\mu_2\nu_2\mu_3}\, \not{l}\right]}{l^2}. \qquad (2.26)$$

This last diagram vanishes since it contains the trace of two γ's and a γ_5. The perturbative realization of the correlator will be written down as the sum of all these amplitudes, formally given by the expression,

$$\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}J_5^{\mu_3}\rangle = 4\sum_{i=1}^{4} V_i^{\mu_1\nu_1\mu_2\nu_2\mu_3}. \qquad (2.27)$$

The perturbative realization of the correlator satisfies the (anomalous) conservation and trace WIs. Thus, we can decompose the perturbative correlator as described in previous section in order to match the coefficient of the anomaly a_2 with the outcome obtained from the perturbative evaluation. Therefore we reintroduce the two terms of the L/T decomposition, now implemented at the perturbative level,

$$\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}J_5^{\mu_3}\rangle_{\mathrm{pert}} = \langle t^{\mu_1\nu_1}t^{\mu_2\nu_2}j_5^{\mu_3}\rangle_{\mathrm{pert}} + \langle t^{\mu_1\nu_1}t^{\mu_2\nu_2}j_{5\mathrm{loc}}^{\mu_3}\rangle_{\mathrm{pert}}. \qquad (2.28)$$

In particular, the anomalous pole is found to be

$$\langle t^{\mu_1\nu_1}t^{\mu_2\nu_2}j_{5\mathrm{loc}}^{\mu_3}\rangle_{\mathrm{pert}} = \frac{g}{96\pi^2}\frac{p_3^{\mu_3}}{p_3^2}(p_1\cdot p_2) \times\left\{\left[\varepsilon^{\nu_1\nu_2 p_1 p_2}\left(g^{\mu_1\mu_2} - \frac{p_1^{\mu_2}p_2^{\mu_1}}{p_1\cdot p_2}\right) + (\mu_1\leftrightarrow\nu_1)\right] + (\mu_2\leftrightarrow\nu_2)\right\}, \qquad (2.29)$$

which corresponds to Eq. (2.11) when we set

$$a_2 = -\frac{ig}{384\pi^2}. \qquad (2.30)$$

The transverse-traceless part $\langle t^{\mu_1\nu_1}t^{\mu_2\nu_2}j_5^{\mu_3}\rangle$ can be expressed in terms of four form factors as described in Eq. (2.15). The perturbative calculation in four dimensions gives

$$\begin{aligned}
A_1 &= \frac{gp_2^2}{24\pi^2\lambda^4}\left\{A_{11} + A_{12}\log\left(\frac{p_1^2}{p_2^2}\right) + A_{13}\log\left(\frac{p_1^2}{p_3^2}\right) + A_{14}C_0(p_1^2,p_2^2,p_3^2)\right\},\\
A_2 &= \frac{gp_2^2}{48\pi^2\lambda^3}\left\{A_{21} + A_{22}\log\left(\frac{p_1^2}{p_2^2}\right) + A_{23}\log\left(\frac{p_1^2}{p_3^2}\right) + A_{24}C_0(p_1^2,p_2^2,p_3^2)\right\},\\
A_3 &= 0,\\
A_4 &= 0,
\end{aligned} \qquad (2.31)$$

where C_0 in Minkowski space is the master integral,

$$C_0(p_1^2,p_2^2,p_3^2) \equiv \frac{1}{i\pi^2}\int d^d l\frac{1}{l^2(l-p_1)^2(l+p_2)^2}, \qquad (2.32)$$

and we have introduced the functions A_{ij}, reported in the Appendix B. At this stage, after performing the match of the pole contribution with the perturbative expansion, we proceed by decorating the diagrams with the new propagators at finite density and temperature.

III. THE PERTURBATIVE REALIZATION AT FINITE TEMPERATURE AND DENSITY

In order to compute the finite density and temperature corrections to the gravitational anomaly we need to consider the same Feynman diagrams defined in the vacuum case. However, now we have to replace the usual fermionic propagator with its generalization in a hot medium in the real time formulation. We use the expression,

$$S_F \equiv (\not{k}+m)G_F = (\not{k}+m)\left\{\frac{1}{k^2-m^2} + 2\pi i\delta(k^2-m^2)\left[\frac{\theta(k_0)}{e^{\beta(E-\mu)}+1} + \frac{\theta(-k_0)}{e^{\beta(E+\mu)}+1}\right]\right\}, \qquad (3.1)$$

with the fermion mass m set to zero. Such expression can also be formulated covariantly by introducing the four-vector η, defining the velocity of the heath-bath. The contribution of the triangle diagrams is now given by

$$V_1^{\mu_1\nu_1\mu_2\nu_2\mu_3} = -i^3\int\frac{d^d l}{(2\pi)^d}\mathrm{tr}[V_{g\bar{\psi}\psi}^{\mu_1\nu_1}(l-p_1,l)(\not{l}-\not{p}_1)V_{A\bar{\psi}\psi}^{\mu_3}(\not{l}+\not{p}_2)V_{g\bar{\psi}\psi}^{\mu_2\nu_2}(l,l+p_2)\not{l}]G_F(l-p_1)G_F(l+p_2)G_F(l) + \text{exchange}, \qquad (3.2)$$

while the bubble diagrams are

$$V_2^{\mu_1\nu_1\mu_2\nu_2\mu_3} = -i^2 \int \frac{d^d l}{(2\pi)^d} \mathrm{tr}\left[V_{gA\bar{\psi}\psi}^{\mu_1\nu_1\mu_3}(\slashed{l} + \slashed{p}_2) V_{g\bar{\psi}\psi}^{\mu_2\nu_2}(l, l+p_2)\slashed{l}\right] G_F(l+p_2) G_F(l) + \text{exchange}, \tag{3.3}$$

and

$$V_3^{\mu_1\nu_1\mu_2\nu_2\mu_3} = -i^2 \int \frac{d^d l}{(2\pi)^d} \mathrm{tr}\left[V_{gg\bar{\psi}\psi}^{\mu_1\nu_1\mu_2\nu_2}(p_1, p_2, l-p_1-p_2, l)(\slashed{l} - \slashed{p}_1 - \slashed{p}_2) V_{A\bar{\psi}\psi}^{\mu_3}\slashed{l}\right] G_F(l-p_1-p_2) G_F(l). \tag{3.4}$$

As previously mentioned, in the case of zero temperature and density, V_2 vanishes. The proof relies on Lorentz symmetry, which is violated by temperature and density effects. Consequently, V_2 now exhibits nonvanishing thermal contributions that cannot be discarded. Furthermore, the presence of V_2 is essential to demonstrate the cancellation of all diagram contributions to the gravitational anomaly. Lastly, similar to the scenario of zero temperature and density, the tadpole diagram vanishes as it contains the trace of two γ's and a γ_5.

We can now decompose G_F into a standard contribution to the Fermi propagator G_0 and the finite density and temperature corrections G_1,

$$G_F = G_0 + G_1, \qquad G_0 = \frac{1}{k^2 - m^2},$$
$$G_1 = 2\pi i \delta(k^2 - m^2)\left[\frac{\theta(k_0)}{e^{\beta(E-\mu)} + 1} + \frac{\theta(-k_0)}{e^{\beta(E+\mu)} + 1}\right]. \tag{3.5}$$

Then, we can split the $\langle TTJ_5\rangle$ correlator into four different parts depending on the number of G_1 contained in the loop integrals,

$$\langle TTJ_5\rangle = \langle TTJ_5\rangle^{(0)} + \langle TTJ_5\rangle^{(1)} + \langle TTJ_5\rangle^{(2)} + \langle TTJ_5\rangle^{(3)}. \tag{3.6}$$

$\langle TTJ_5\rangle^{(0)}$ represents the zero density and temperature part that was computed in the previous section, $\langle TTJ_5\rangle^{(1)}$ contains only one G_1 and so on. Note that the triangles diagrams contribute to all the four terms on the right-hand side of Eq. (3.6), while the bubble diagrams to not contribute to $\langle TTJ_5\rangle^{(3)}$ since they contain two fermionic propagators G_F.

IV. THE GRAVITATIONAL ANOMALY AT FINITE TEMPERATURE AND DENSITY

In this section, we examine the longitudinal anomalous sector of J_5, showing that it is protected from finite density and temperature effects. To achieve this, we dissect the terms associated with finite density and temperature corrections in Eq. (3.6) separately. We will illustrate that each of these terms vanishes upon contraction with the momentum of the axial current, $p_3^{\mu_3}$. The sole surviving term is the zero density and temperature one which showcases the effect of the gravitational anomaly,

$$\begin{aligned} p_{3\mu_3}\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}J_5^{\mu_3}\rangle &= p_{3\mu_3}\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}J_5^{\mu_3}\rangle^{(0)} \\ &= 4ia_2(p_1\cdot p_2)\left\{\left[\varepsilon^{\nu_1\nu_2 p_1 p_2}\left(g^{\mu_1\mu_2} - \frac{p_1^{\mu_2}p_2^{\mu_1}}{p_1\cdot p_2}\right) + (\mu_1 \leftrightarrow \nu_1)\right] + (\mu_2 \leftrightarrow \nu_2)\right\}. \end{aligned} \tag{4.1}$$

The proof of such statement may appear miraculous, as exceedingly long expressions cancel out. Furthermore, in order to achieve this, we do not need to specify the explicit form of G_1, except for the fact that it contains a Dirac delta. There is also no requirement to perform the loop integral since the cancellations occur within the integrand itself.

This procedure has been previously applied to the more simple case of the gauge chiral anomaly [26,30]. As we will see, in the case of the gravitational chiral anomaly, the computations are significantly lengthier and require the use of Schouten identities, which relate different tensor structures. We now proceed with our proof.

A. $\langle \mathbf{TTJ_5}\rangle^{(1)}$

We start by considering $\langle TTJ_5\rangle^{(1)}$ (see Fig. 2). The terms contributing to $\langle TTJ_5\rangle^{(1)}$ contain only one thermal correction G_1 but with different momenta as arguments,

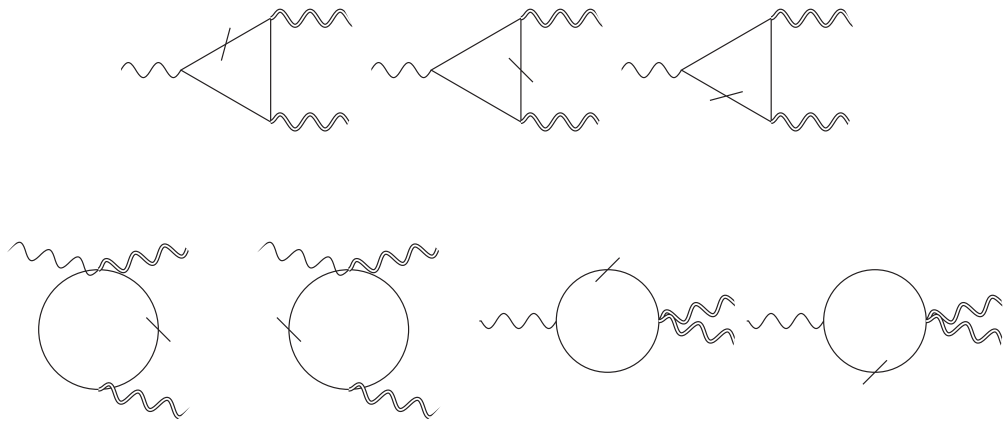


FIG. 2. Topologies of diagrams contributing to $\langle TTJ_5\rangle^{(1)}$. The bar on the fermions' lines denotes the insertion of a hot propagator G_1.

$$\begin{aligned} &G_1(l), \quad G_1(l+p_1), \quad G_1(l-p_1), \quad G_1(l+p_2), \quad G_1(l-p_2), \\ &G_1(l+p_3), \quad G_1(l-p_3), \quad G_1(l-p_1+p_2), \quad G_1(l+p_1-p_2). \end{aligned} \tag{4.2}$$

However, since we are dealing with finite integrals, we can perform multiple shifts in the loop momentum to ensure that the terms inside $\langle TTJ_5\rangle^{(1)}$ only have one single dependence, for example $G_1(l)$. We can then write the contracted correlator in the following form:

$$p_{3\mu_3}\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}J_5^{\mu_3}\rangle^{(1)} = \int \frac{d^d l}{(2\pi)^d} W_1^{\mu_1\nu_1\mu_2\nu_2}(p_1,p_2,l)G_1(l). \tag{4.3}$$

The dependence on the temperature and chemical potential in the integral above is contained only in $G_1(l)$. $W_1^{\mu_1\nu_1\mu_2\nu_2}$ is a parity-odd tensor constructed with the momenta p_1 p_2 and l. It is symmetric under the exchange $\{\mu_1 \leftrightarrow \nu_1\}$ and $\{\mu_2 \leftrightarrow \nu_2\}$ and $\{(\mu_1,\nu_1,p_1) \leftrightarrow (\mu_2,\nu_2,p_2)\}$. The explicit initial expression for $W_1^{\mu_1\nu_1\mu_2\nu_2}$ is extremely long but one can significantly simplify it by utilizing a set of tensorial relations, known as Schouten identities, which are detailed in Appendix C. These identities arise from the dimensional degeneracies of tensor structures, given that we are working in $d=4$.

Surprisingly, by applying all the Schouten identities reported in the Appendix C and then setting $l^2=0$ due to the $\delta(l^2)$ contained in $G_1(l)$, one is able to prove that

$$0 = W_1^{\mu_1\nu_1\mu_2\nu_2}|_{l^2=0}. \tag{4.4}$$

Therefore,

$$p_{3\mu_3}\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}J_5^{\mu_3}\rangle^{(1)} = 0, \tag{4.5}$$

which means that the $\langle TTJ_5\rangle^{(1)}$ contributions do not modify the axial anomalous Ward identity.

B. $\langle \mathbf{TTJ_5}\rangle^{(2)}$

The procedure for $\langle TTJ_5\rangle^{(2)}$ is very similar to the one followed in the previous subsection (see Fig. 3). Considering our parametrization of the loop integral, the combinations in which the thermal corrections G_1 appear in $\langle TTJ_5\rangle^{(2)}$ are

$$\begin{aligned} &G_1(l)G_1(l+p_1), \quad G_1(l)G_1(l+p_2), \quad G_1(l)G_1(l+p_3), \\ &G_1(l)G_1(l-p_2), \quad G_1(l)G_1(l-p_1), \quad G_1(l+p_1)G_1(l-p_2), \quad G_1(l-p_1)G_1(l+p_2). \end{aligned} \tag{4.6}$$

Since the integrals are finite, we can perform multiple shifts to the loop momentum, reducing all the combinations above to only three terms,

$$\begin{aligned} p_{3\mu_3}\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}J_5^{\mu_3}\rangle^{(2)} = \int \frac{d^d l}{(2\pi)^d} & W_{2,1}^{\mu_1\nu_1\mu_2\nu_2}(p_1,p_2,l)G_1(l)G_1(l+p_1) \\ & + W_{2,2}^{\mu_1\nu_1\mu_2\nu_2}(p_1,p_2,l)G_1(l)G_1(l+p_2) + W_{2,3}^{\mu_1\nu_1\mu_2\nu_2}(p_1,p_2,l)G_1(l)G_1(l+p_3). \end{aligned} \tag{4.7}$$

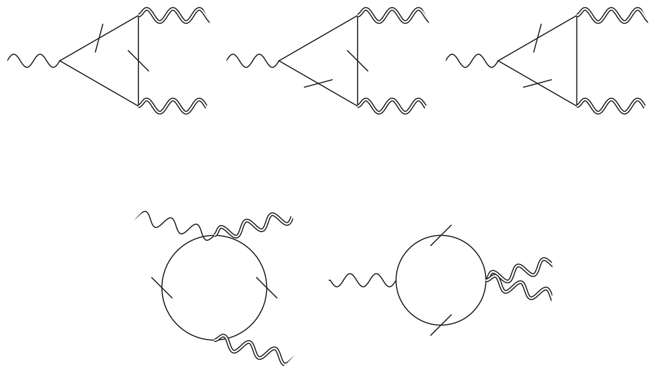


FIG. 3. Topologies of diagrams contributing to $\langle TTJ_5\rangle^{(2)}$. They are characterized by the insertion of two hot propagators G_1.

The dependence on the temperature and chemical potential in the integrals above is contained in G_1. The tensors $W_{2,i}^{\mu_1\nu_1\mu_2\nu_2}$ are constructed with the momenta p_1 p_2 and l. They are parity-odd and symmetric under the exchange $\{\mu_1 \leftrightarrow \nu_1\}$ and $\{\mu_2 \leftrightarrow \nu_2\}$. Moreover, due to the Bose symmetry, we can write

$$W_{2,1}^{\mu_1\nu_1\mu_2\nu_2}(p_1,p_2,l) = W_{2,2}^{\mu_2\nu_2\mu_1\nu_1}(p_2,p_1,l),$$
$$W_{2,3}^{\mu_1\nu_1\mu_2\nu_2}(p_1,p_2,l) = W_{2,3}^{\mu_2\nu_2\mu_1\nu_1}(p_2,p_1,l). \tag{4.8}$$

The explicit initial expression for $W_{2,i}^{\mu_1\nu_1\mu_2\nu_2}$ is extremely long but one can significantly simplify it by utilizing the Schouten identities. Indeed, by applying all the identities reported in the Appendix C and using the fact that G_1 contains delta functions, we can prove that all three terms in Eq. (4.12) vanish individually,

$$0 = W_{2,1}|_{l^2=(l+p_1)^2=0}, \qquad 0 = W_{2,2}|_{l^2=(l+p_2)^2=0},$$
$$0 = W_{2,3}|_{l^2=(l+p_3)^2=0}. \tag{4.9}$$

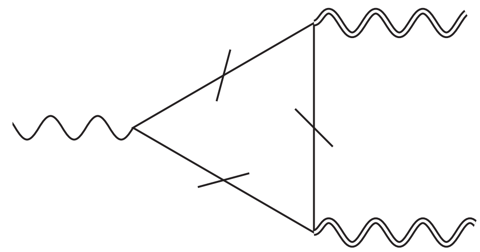


FIG. 4. Topology of diagram contributing to $\langle TTJ_5\rangle^{(3)}$. Such diagram is characterized by the insertion of three hot propagators G_1.

Therefore, we have

$$p_{3\mu_3}\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}J_5^{\mu_3}\rangle^{(2)} = 0, \tag{4.10}$$

which means that the $\langle TTJ_5\rangle^{(2)}$ contributions do not modify the axial anomalous Ward identity.

C. $\langle \mathbf{TTJ_5}\rangle^{(3)}$

Only triangle diagrams contribute to $\langle TTJ_5\rangle^{(3)}$ since the bubble diagrams have only two fermionic propagators (see Fig. 4).
The combination of momenta that appear as arguments of G_1 are

$$G_1(l)G_1(l-p_1)G_1(l+p_2),$$
$$G_1(l)G_1(l+p_1)G_1(l-p_2). \tag{4.11}$$

Such combination of G_1 can not be further reduced by shift in the loop momentum as in the previous case. Therefore, we can write

$$p_{3\mu_3}\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}J_5^{\mu_3}\rangle^{(3)} = \int \frac{d^dl}{(2\pi)^d} W_{3,1}^{\mu_1\nu_1\mu_2\nu_2}(p_1,p_2,l)G_1(l)G_1(l-p_1)G_1(l+p_2)$$
$$+ W_{3,2}^{\mu_1\nu_1\mu_2\nu_2}(p_1,p_2,l)G_1(l)G_1(l+p_1)G_1(l-p_2). \tag{4.12}$$

$W_{3,i}^{\mu_1\nu_1\mu_2\nu_2}$ are parity-odd tensors that depend on the momenta p_1 p_2 and l. They are symmetric under the exchange $\{\mu_1 \leftrightarrow \nu_1\}$ and $\{\mu_2 \leftrightarrow \nu_2\}$. Moreover, they are related to each other due to the Bose symmetry,

$$W_{3,1}^{\mu_1\nu_1\mu_2\nu_2}(p_1,p_2,l) = W_{3,2}^{\mu_2\nu_2\mu_1\nu_1}(p_2,p_1,l). \tag{4.13}$$

Once again, we can use the Schouten identities and the fact that G_1 contains delta functions in order to prove that

$$0 = W_{3,1}|_{l^2=(l-p_1)^2=(l+p_2)^2=0}, \quad 0 = W_{3,2}|_{l^2=(l+p_1)^2=(l-p_2)^2=0}. \tag{4.14}$$

Therefore, we have

$$p_{3\mu_3}\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}J_5^{\mu_3}\rangle^{(3)} = 0. \tag{4.15}$$

This completes the proof that the longitudinal axial WI is not modified with respect to the vacuum part and the solution of the longitudinal equation is still given by the exchange of an anomaly pole, as shown in (4.1).

V. CONCLUSIONS

In this paper, we have examined the gravitational anomaly vertex $\langle TTJ_5\rangle$ in a hot and dense medium. We have shown that the anomaly is protected from finite density and temperature corrections.

It would be interesting to investigate how dilatations and special conformal transformations are broken in this context [40].

Our result has application to the decay of an axion or any axionlike particle into gravitational waves, as well as in the production of chiral currents from gravitational waves, in very dense and hot environments. Furthermore, the protection of the gravitational axial anomaly against finite density and temperature corrections presents intriguing experimental opportunities within condensed matter theory, particularly in the context of topological materials. We hope to return to the investigation of these topics in another work.

ACKNOWLEDGMENTS

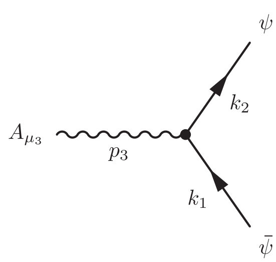
This work is partially funded by the European Union, Next Generation EU, PNRR Project "National Centre for HPC, Big Data and Quantum Computing," Project Code No. CN00000013; by INFN, inziativa specifica *QG-sky* and by the Grant No. PRIN 2022BP52A MUR "The Holographic Universe for all Lambdas" Lecce-Naples.

APPENDIX A: VERTICES

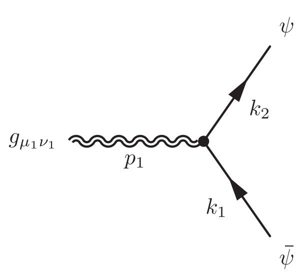
In this section we list the explicit expression of all the vertices needed for the perturbative analysis of the $\langle TTJ_5\rangle$ correlator. The momenta of the gravitons and the axial boson are all incoming as well as the momentum indicated with k_1. The momentum k_2 instead is outgoing. In order to simplify the notation, we introduce the tensor components,

$$
\begin{aligned}
A^{\mu\nu\rho\sigma} &\equiv g^{\mu\nu}g^{\rho\sigma} - \frac{1}{2}(g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})\\
B^{\mu\nu\rho\sigma\alpha\beta} &\equiv g^{\alpha\beta}g^{\mu\nu}g^{\rho\sigma} - g^{\alpha\beta}(g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})\\
C^{\mu\nu\rho\sigma\alpha\beta} &\equiv \frac{1}{2}g^{\mu\nu}(g^{\alpha\rho}g^{\beta\sigma} + g^{\alpha\sigma}g^{\beta\rho}) + \frac{1}{2}g^{\rho\sigma}(g^{\alpha\mu}g^{\beta\nu} + g^{\alpha\nu}g^{\beta\mu})\\
D^{\mu\nu\rho\sigma\alpha\beta} &\equiv \frac{1}{2}(g^{\alpha\sigma}g^{\beta\mu}g^{\nu\rho} + g^{\alpha\rho}g^{\beta\mu}g^{\nu\sigma} + g^{\alpha\sigma}g^{\beta\nu}g^{\mu\rho} + g^{\alpha\rho}g^{\beta\nu}g^{\mu\sigma})\\
&\quad + \frac{1}{4}(g^{\alpha\mu}g^{\beta\sigma}g^{\nu\rho} + g^{\alpha\mu}g^{\beta\rho}g^{\nu\sigma} + g^{\alpha\nu}g^{\beta\sigma}g^{\mu\rho} + g^{\alpha\nu}g^{\beta\rho}g^{\mu\sigma})\\
G^{\alpha\beta\gamma} &\equiv \gamma^\alpha\gamma^\beta\gamma^\gamma - \gamma^\beta\gamma^\alpha\gamma^\gamma + \gamma^\gamma\gamma^\alpha\gamma^\beta - \gamma^\gamma\gamma^\beta\gamma^\alpha.
\end{aligned}
\tag{A1}
$$

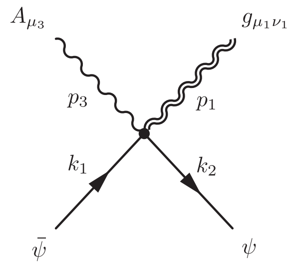
The vertices can then be written as



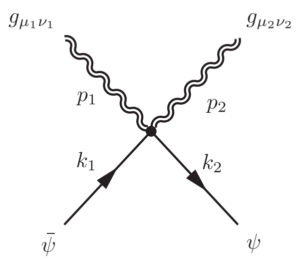
$$V^{\mu_3}_{A\bar{\psi}\psi} = -ig\gamma^{\mu_3}\gamma_5$$



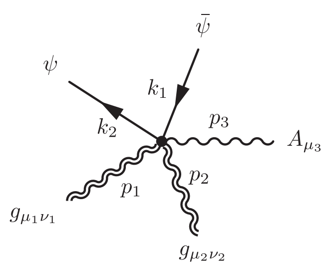
$$V^{\mu_1\nu_1}_{g\bar{\psi}\psi} = \frac{i}{4}A^{\mu_1\nu_1\rho\sigma}(k_1+k_2)_\rho\gamma_\sigma$$



$$V^{\mu_1\nu_1\mu_3}_{gA\bar\psi\psi} = -\frac{ig}{2}A^{\mu_1\nu_1\mu_3\rho}\gamma_\rho\gamma_5$$



$$\begin{aligned}V^{\mu_1\nu_1\mu_2\nu_2}_{gg\bar\psi\psi} &= \\ &-\frac{i}{8}\left(B^{\mu_1\nu_1\mu_2\nu_2\alpha\beta} - C^{\mu_1\nu_1\mu_2\nu_2\alpha\beta} + D^{\mu_1\nu_1\mu_2\nu_2\alpha\beta}\right)\gamma_\alpha(k_1+k_2)_\beta \\ &+\frac{i}{128}G^{\alpha\beta\gamma}A^{\mu_1\nu_1\gamma\rho}p_2^\sigma \times \\ &\left(g^{\alpha\mu_2}g^{\beta\sigma}g^{\nu_2\rho} + g^{\alpha\nu_2}g^{\beta\sigma}g^{\mu_2\rho} - g^{\alpha\sigma}g^{\beta\nu_2}g^{\mu_2\rho} - g^{\alpha\sigma}g^{\beta\mu_2}g^{\nu_2\rho}\right)\end{aligned}$$



$$\begin{aligned}V^{\mu_1\nu_1\mu_2\nu_2\mu_3}_{ggA\bar\psi\psi} &= \\ &-\frac{ig}{4}\left(B^{\mu_1\nu_1\mu_2\nu_2\mu_3\lambda} - C^{\mu_1\nu_1\mu_2\nu_2\mu_3\lambda} + D^{\mu_1\nu_1\mu_2\nu_2\mu_3\lambda}\right)\gamma_\lambda\gamma_5\end{aligned}$$

APPENDIX B: DEFINITIONS FOR THE SOLUTION OF THE VACUUM CORRELATOR

In this section, we provide a list of definitions for the functions introduced in Eq. (2.31). They are described as follows:

$$\begin{aligned}A_{11} = -\lambda\Big[&2p_1^{10} - p_1^8(p_2^2+p_3^2) - 2p_1^6(5p_2^4 - 48p_2^2p_3^2 + 5p_3^4) + 4p_1^4(p_2^2+p_3^2)(4p_2^4 - 23p_2^2p_3^2 + 4p_3^4) \\ &- 8p_1^2(p_2^2-p_3^2)^2(p_2^4 + 4p_2^2p_3^2 + p_3^4) + (p_2^2-p_3^2)^4(p_2^2+p_3^2)\Big]\end{aligned}$$

$$\begin{aligned}A_{12} = +2p_2^2\Big[&p_3^2(p_3^2-p_2^2)^5 + p_1^{10}(38p_3^2 - 12p_2^2) + p_1^8(18p_2^4 + 41p_2^2p_3^2 - 121p_3^4) - 4p_1^6(3p_2^6 + 46p_2^4p_3^2 - 38p_2^2p_3^4 - 26p_3^6) \\ &+ p_1^4(p_2-p_3)(p_2+p_3)(3p_2^6 + 95p_2^4p_3^2 + 215p_2^2p_3^4 + 11p_3^6) + 14p_1^2p_3^2(p_2^2-p_3^2)^3(p_2^2+p_3^2) + 3p_1^{12}\Big]\end{aligned}$$

$$\begin{aligned}A_{13} = +2p_3^2\Big[&3p_1^{12} + 2p_1^{10}(19p_2^2 - 6p_3^2) + p_1^8(-121p_2^4 + 41p_2^2p_3^2 + 18p_3^4) + 4p_1^6(26p_2^6 + 38p_2^4p_3^2 - 46p_2^2p_3^4 - 3p_3^6) \\ &- 14p_1^2p_2^2(p_2^2-p_3^2)^3(p_2^2+p_3^2) - p_1^4(p_2-p_3)(p_2+p_3)(11p_2^6 + 215p_2^4p_3^2 + 95p_2^2p_3^4 + 3p_3^6) + p_2^2(p_2^2-p_3^2)^5\Big]\end{aligned}$$

$$\begin{aligned}A_{14} = -24p_1^4p_2^2p_3^2\Big[&(p_1^2-p_2^2)^3(2p_1^2 + 3p_2^2) - 3p_3^4(p_1^4 + 4p_1^2p_2^2 - 4p_2^4) - 3p_3^2(p_1^6 - 6p_1^4p_2^2 + 4p_1^2p_2^4 + p_2^6) \\ &- 3p_3^8p_3^6(7p_1^2 - 3p_2^2)\Big]\end{aligned}$$

$$A_{21} = -\lambda\Big[2p_3^6(3p_1^2+p_2^2) + 4p_1^2p_3^4(3p_2^2 - 2p_1^2) + (p_1^2-p_2^2)^4 + 2p_3^2(p_1-p_2)(p_1+p_2)(p_1^4 + 8p_1^2p_2^2 + p_2^4) - p_3^8\Big]$$

$$A_{22} = -2p_2^2p_3^2\Big[-17p_1^8 + p_1^6(28p_2^2 + 26p_3^2) - 4p_1^4(p_2^4 + 15p_2^2p_3^2) + (p_2^2-p_3^2)^4 - 2p_1^2(p_2^2-p_3^2)^2(4p_2^2 + 5p_3^2)\Big]$$

$$A_{23} = +2p_3^2\Big[2p_1^{10} - p_1^8(p_2^2+6p_3^2) + p_1^6(-10p_2^4+46p_2^2p_3^2+6p_3^4) - 2p_1^2(4p_2^2+5p_3^2)(p_2^3-p_2p_3^2)^2 + p_2^2(p_2^2-p_3^2)^4 + 2p_1^4(8p_2^6-21p_2^4p_3^2-18p_2^2p_3^4-p_3^6)\Big]$$

$$A_{24} = -12p_1^4p_2^2p_3^2\Big[p_3^4(3p_2^2-5p_1^2) + (p_1^2-p_2^2)^3 + p_3^2(p_1^4+4p_1^2p_2^2-5p_2^4) + 3p_3^6\Big], \tag{B1}$$

with the Källen λ-function given by

$$\lambda \equiv \lambda(p_1,p_2,p_3) = (p_1-p_2-p_3)(p_1+p_2-p_3)(p_1-p_2+p_3).(p_1+p_2+p_3). \tag{B2}$$

APPENDIX C: SCHOUTEN IDENTITIES

When examining the gravitational anomaly corrections at finite density and temperature, we have introduced the rank-4 tensors $W_{i,j}^{\mu_1\nu_1\mu_2\nu_2}$. Such tensors are parity-odd and depend on the graviton momenta p_1 and p_2, as well as the momentum of the fermionic loop l. There is a set of tensorial relations, known as Schouten identities, which can be used to reduce the expression of $W_{i,j}^{\mu_1\nu_1\mu_2\nu_2}$. These identities arise from the dimensional degeneracies of tensor structures, given that we are working in $d=4$. In particular, in this case we can construct such identities starting from the following format:

$$0 = \varepsilon^{[l p_1 p_2 \mu_1}\delta^{\mu_2]\alpha}. \tag{C1}$$

Since we are antisymmetrizing over five indices and we are working in four dimensions, the result must vanish. The index α can be contracted with a momentum, obtaining

$$0 = \varepsilon^{[l p_1 p_2 \mu_1}p_1^{\mu_2]}$$
$$0 = \varepsilon^{[l p_1 p_2 \mu_1}p_2^{\mu_2]}$$
$$0 = \varepsilon^{[l p_1 p_2 \mu_1}l^{\mu_2]}, \tag{C2}$$

or we can pick $\alpha = \{\nu_1, \nu_2\}$,

$$0 = \varepsilon^{[l p_1 p_2 \mu_1}\delta^{\mu_2]\nu_1}$$
$$0 = \varepsilon^{[l p_1 p_2 \mu_1}\delta^{\mu_2]\nu_2}. \tag{C3}$$

Note that we do not need to consider Schouten identities where both μ_1 and ν_1 are antisymmetrized since the energy-momentum tensor (and therefore $W_{i,j}^{\mu_1\nu_1\mu_2\nu_2}$) is symmetric under the exchange $\mu_1 \leftrightarrow \nu_1$. The same is true for the indices μ_2 and ν_2.

The identity (C2) and (C3) relates tensors with rank less than four. Therefore, we need to complete them with the remaining indices. As an example we can pick the first identity in Eq. (C3), which is a rank-3 equation, and we multiply it with $p_1^{\nu_2}$, $p_2^{\nu_2}$ or l^{ν_2} ending up with three relations between rank-4 tensors. Proceeding in such way, we end up with 36 identities. Moreover, we can also get new identities from this 36 ones by exchanging $\mu_1 \leftrightarrow \nu_1$ and/or $\mu_2 \leftrightarrow \nu_2$. Therefore, the total number of Schouten identities is $36 \times 4 = 144$. All these identities can be used to simplify the structures appearing in the W tensors.

[1] T. Kimura, Divergence of axial-vector current in the gravitational field, Prog. Theor. Phys. **42**, 1191 (1969).
[2] R. Delbourgo and A. Salam, The gravitational correction to PCAC, Phys. Lett. **40B**, 381 (1972).
[3] N. K. Nielsen, M. T. Grisaru, H. Romer, and P. van Nieuwenhuizen, Approaches to the gravitational spin 3/2 axial anomaly, Nucl. Phys. **B140**, 477 (1978).
[4] L. Alvarez-Gaume and E. Witten, Gravitational anomalies, Nucl. Phys. **B234**, 269 (1984).
[5] L. Bonora, *Fermions and Anomalies in Quantum Field Theories*, Theoretical and Mathematical Physics (Springer, Cham, 2023).
[6] C. Corianò, S. Lionetti, and M. M. Maglio, Parity-violating CFT and the gravitational chiral anomaly, Phys. Rev. D **109**, 045004 (2024).
[7] C. Corianò, L. Delle Rose, E. Mottola, and M. Serino, Solving the conformal constraints for scalar operators in momentum space and the evaluation of Feynman's master integrals, J. High Energy Phys. 07 (2013) 011.
[8] A. Bzowski, P. McFadden, and K. Skenderis, Implications of conformal invariance in momentum space, J. High Energy Phys. 03 (2014) 111.
[9] A. Bzowski, P. McFadden, and K. Skenderis, Renormalised CFT 3-point functions of scalars, currents and stress tensors, J. High Energy Phys. 11 (2018) 159.

[10] J. Erdmenger, Gravitational axial anomaly for four-dimensional conformal field theories, Nucl. Phys. **B562**, 315 (1999).
[11] M. Z. Hasan and C. L. Kane, Topological insulators, Rev. Mod. Phys. **82**, 3045 (2010).
[12] M. N. Chernodub, A. Cortijo, A. G. Grushin, K. Landsteiner, and M. A. H. Vozmediano, Condensed matter realization of the axial magnetic effect, Phys. Rev. B **89**, 081407 (2014).
[13] V. E. Ambrus and M. N. Chernodub, Helical vortical effects, helical waves, and anomalies of Dirac fermions, Eur. Phys. J. C **83**, 111 (2023).
[14] M. N. Chernodub, Y. Ferreiros, A. G. Grushin, K. Landsteiner, and M. A. H. Vozmediano, Thermal transport, geometry, and anomalies, Phys. Rep. **977**, 1 (2022).
[15] R. Arouca, A. Cappelli, and T. H. Hansson, Quantum field theory anomalies in condensed matter physics, SciPost Phys. Lect. Notes **62**, 1 (2022).
[16] C. Tutschku, F. S. Nogueira, C. Northe, J. van den Brink, and E. M. Hankiewicz, Temperature and chemical potential dependence of the parity anomaly in quantum anomalous Hall insulators, Phys. Rev. B **102**, 205407 (2020).
[17] K. Landsteiner, Anomalous transport of Weyl fermions in Weyl semimetals, Phys. Rev. B **89**, 075124 (2014).
[18] J. Fröhlich, Gauge invariance and anomalies in condensed matter physics, arXiv:2303.14741.
[19] E. Mottola, A. V. Sadofyev, and A. Stergiou, Axions and superfluidity in Weyl semimetals, Phys. Rev. B **109**, 134512 (2024).
[20] E. Mottola and A. V. Sadofyev, Chiral waves on the Fermi-Dirac sea: Quantum superfluidity and the axial anomaly, Nucl. Phys. **B966**, 115385 (2021).
[21] J. Nissinen and G. E. Volovik, Anomalous chiral transport with vorticity and torsion: Cancellation of two mixed gravitational anomaly currents in rotating chiral $\mathrm{p}+\mathrm{ip}$ Weyl condensates, Phys. Rev. D **106**, 045022 (2022).
[22] J. Nissinen and G. E. Volovik, Elasticity tetrads, mixed axial-gravitational anomalies, and $3+1$d quantum Hall effect, Phys. Rev. Res. **1**, 023007 (2019).
[23] J. M. Luttinger, Theory of thermal transport coefficients, Phys. Rev. **135**, A1505 (1964).
[24] B. Bermond, M. Chernodub, A. G. Grushin, and D. Carpentier, Anomalous Luttinger equivalence between temperature and curved spacetime: From black hole's atmosphere to thermal quenches, arXiv:2206.08784.
[25] M. N. Chernodub, C. Corianò, and M. M. Maglio, Anomalous gravitational TTT vertex, temperature inhomogeneity, and pressure anisotropy, Phys. Lett. B **802**, 135236 (2020).
[26] C. Contreras and M. Loewe, The triangle diagram at finite temperature, Z. Phys. C **40**, 253 (1988).
[27] M. Reuter and W. Dittrich, Chiral Jacobians and QED in two-dimensions at finite temperature, Phys. Rev. D **32**, 513 (1985).
[28] S. Chaturvedi, N. Gupte, and V. Srinivasan, Axial anomaly at finite temperature, J. Phys. A **18**, L963 (1985).
[29] A. K. Das and A. Karev, The derivative expansion and t he anomaly at finite temperature, Phys. Rev. D **36**, 623 (1987).
[30] Z.-X. Qian, R.-K. Su, and P. K. N. Yu, Axial anomaly at finite temperature and finite density, Z. Phys. C **63**, 651 (1994).
[31] C. Corianò, M. Cretì, S. Lionetti, and R. Tommasi, Axion-like quasiparticles and topological states of matter: Finite density corrections of the chiral anomaly vertex, arXiv:2402.03151.
[32] K. Kamada, N. Yamamoto, and D.-L. Yang, Chiral effects in astrophysics and cosmology, Prog. Part. Nucl. Phys. **129**, 104016 (2023).
[33] T. Kanazawa and N. Yamamoto, U(1) axial symmetry and Dirac spectra in QCD at high temperature, J. High Energy Phys. 01 (2016) 141.
[34] A. K. Das, *Finite Temperature Field Theory* (World Scientific, New York, 1997).
[35] C. Corianò and M. M. Maglio, Exact correlators from conformal Ward identities in momentum space and the perturbative TJJ vertex, Nucl. Phys. **B938**, 440 (2019).
[36] C. Corianò and M. M. Maglio, The general 3-graviton vertex (TTT) of conformal field theories in momentum space in $d=4$, Nucl. Phys. **B937**, 56 (2018).
[37] C. Corianò, M. M. Maglio, and R. Tommasi, Four-point functions of gravitons and conserved currents of CFT in momentum space: Testing the nonlocal action with the TTJJ, Eur. Phys. J. C **83**, 427 (2023).
[38] C. Corianò, S. Lionetti, and M. M. Maglio, Parity-odd 3-point functions from CFT in momentum space and the chiral anomaly, Eur. Phys. J. C **83**, 502 (2023).
[39] R. Marotta, K. Skenderis, and M. Verma, Momentum space CFT correlators of non-conserved spinning operators, J. High Energy Phys. 03 (2023) 196.
[40] E. Marchetto, A. Miscioscia, and E. Pomoni, Broken (super) conformal Ward identities at finite temperature, J. High Energy Phys. 12 (2023) 186.