Flowing between string vacua for the critical non-Abelian vortex with a deformation of $\mathcal{N}=2$ Liouville theory

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It has been shown that non-Abelian solitonic vortex string supported in four-dimensional (4D) $\mathcal{N}=2$ supersymmetric QCD (SQCD) with the U(2) gauge group and $N_f=4$ quark flavors becomes a critical superstring. This string propagates in the ten-dimensional space formed by a product of the flat 4D space and an internal space given by a Calabi-Yau noncompact threefold, namely, the conifold. The spectrum of low-lying closed string states was found and interpreted as a spectrum of hadrons in 4D $\mathcal{N}=2$ SQCD. In particular, the lowest string state appears to be a massless Bogomol'nyi-Prasad-Sommerfield (BPS) baryon associated with the deformation of the complex structure modulus b of the conifold. It was recently shown that the Coulomb branch of the associated string sigma model which opens up at strong coupling can be described by $\mathcal{N}=2$ Liouville theory. Building on these results we switch on quark masses in 4D $\mathcal{N}=2$ SQCD and study the interpolation of the initial U(2) SQCD with $N_f=4$ quarks to the final SQCD with the U(4) gauge group and $N_f=8$ quarks. To find the true string vacuum which arises due to the mass deformation we solve the effective supergravity equations of motion associated with the deformed world sheet Liouville theory. We show that the massless BPS baryon b survives the deformation and that finding of the spectrum of low-lying massive hadrons in the final SQCD is linked to the Calogero problem.

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I. INTRODUCTION

It was shown in [1] that the non-Abelian solitonic vortex string in four-dimensional (4D) $\mathcal{N}=2$ supersymmetric QCD (SQCD) with the U($N=2$) gauge group and $N_f=2N=4$ flavors of quark hypermultiplets behaves as a critical superstring. Non-Abelian vortices were first found in $\mathcal{N}=2$ SQCD with the gauge group U(N) and $N_f \geq N$ flavors of quarks [2–5]. The non-Abelian vortex string is 1/2 Bogomol'nyi-Prasad-Sommerfield (BPS) saturated and, therefore, has $\mathcal{N}=(2,2)$ supersymmetry on its world sheet. In addition to four translational moduli the non-Abelian string carries orientational moduli, as well as the size moduli if $N_f > N$ [2–5] (see Refs. [6–9] for reviews). Their dynamics are described by the effective two-dimensional (2D) sigma model on the string world sheet, the so-called $\mathcal{N}=(2,2)$ supersymmetric weighted CP model [$\mathbb{WCP}(N, N_f - N)$].

For $N_f=2N$ the world sheet sigma model becomes conformal. Moreover, for $N=2$ the number of the orientational/size moduli is six and they can be combined with four translational moduli to form a ten-dimensional space required for a superstring to become critical [1,10]. In this case the target space of the world sheet theory on the non-Abelian vortex string is $\mathbb{R}^4 \times Y_6$, where Y_6 is a noncompact six-dimensional Calabi-Yau (CY) manifold, the conifold [11], see Ref. [12] for a review. Moreover, the theory of the critical vortex string at hand was identified as the superstring theory of type IIA [10]. The spectrum of low-lying closed string excitation was found in [10,13].

A version of the string-gauge duality for 4D SQCD was proposed [1]; at weak coupling this theory is in the Higgs phase and can be described in terms of quarks and Higgsed gauge bosons, while at strong coupling hadrons of this theory can be understood as closed string states formed by the non-Abelian vortex string. We call this approach a "solitonic string-gauge duality".

Most of massless and massive string modes have non-normalizable wave functions over the conifold Y_6, i.e., they are not localized in 4D and cannot be interpreted as dynamical states in 4D theory, in particular there are no massless 4D gravitons in the physical spectrum [10]. However, an excitation associated with the deformation of the complex structure modulus b of Y_6 has (logarithmically) normalizable wave function and was interpreted as a massless baryon in the spectrum of hadrons of 4D $\mathcal{N} = 2$ SQCD [10].

To analyze the massive states, a different approach was chosen similar to the one, used for little string theories (see Ref. [14] for a review). It is based on the equivalence [15] between the critical string on the conifold and non-critical $c = 1$ string containing the Liouville field and a compact scalar at the self-dual radius (united into a complex scalar of $\mathcal{N} = 2$ Liouville theory [16,17]).[1] Later similar correspondence was proposed (and treated as a holographic AdS/CFT-type duality) for the critical string on certain non-compact CY spaces with isolated singularity in so-called double scaling limit, and noncritical $c = 1$ string with an additional Ginzburg-Landau $\mathcal{N} = 2$ superconformal system [18,19]. In the conifold case, this extra Ginzburg-Landau conformal field theory (CFT) is absent.

Recently this equivalence was demonstrated in a more direct way. Namely, it was shown in [20] that Coulomb branches of world sheet $\mathbb{WCP}(N,N)$ models on non-compact toric CY manifolds with an isolated singularity which open up at strong coupling can be described by $\mathcal{N} = 2$ Liouville theory with background charge depending on N. This was shown first in the large-N approximation and then extrapolated to an exact equivalence.

The above equivalence was used in [13,21] to find the low-lying spectrum of hadrons in 4D $\mathcal{N} = 2$ SQCD with gauge group U(2) and $N_f = 4$ quark flavors.

Although the solitonic string-gauge duality program looks rather generic, one of its main limitations is that so far only one example of 4D gauge theory supporting non-Abelian strings which can be quantized using well-developed string theory methods was found, namely $\mathcal{N} = 2$ SQCD with gauge group U(2) and $N_f = 4$ quark flavors. The reason is that the string sigma model [$\mathbb{WCP}(N,N)$] living on the world sheet of non-Abelian string is conformal for $N_f = 2N$ and its target space has critical dimension equal to ten for $N = 2$ [1].

In this paper we make a step towards broadening of the class of 4D $\mathcal{N} = 2$ SQCDs where the solitonic string-gauge duality can be applied, see also [22,23]. To this end we introduce quark masses in $\mathcal{N} = 2$ SQCD and changing values of mass parameters interpolate between SQCDs with different gauge groups and numbers of quark flavors.

Quark masses in 4D SQCD induce so-called twisted masses of fields in the world sheet $\mathbb{WCP}(N,N)$ model breaking its conformal invariance. We repeat the derivation in [20] and reduce the Coulomb branch of the mass-deformed $\mathbb{WCP}(N,N)$ model to the deformed $\mathcal{N} = 2$ Liouville theory, where the mass deformation boils down to a nontrivial metric of the target space. However, this "classical" metric still cannot be used for the string quantization since the conformal invariance of the model is broken by the mass deformation.

To find a true string vacuum we solve effective supergravity equations using the classical metric of the mass-deformed $\mathcal{N} = 2$ Liouville theory only as initial conditions for the true metric at large values of the Liouville field ϕ, where the mass deformation is small.

Solving the gravity equations of motion and finding the true quantum metric and the dilaton for the mass-deformed $\mathcal{N} = 2$ Liouville theory allows us to interpolate between two 4D SQCDs. Namely, starting from $\mathcal{N} = 2$ SQCD with gauge group U(2) and $N_f = 4$ which supports critical non-Abelian vortex string and reducing the mass parameter "integrating extra quarks in" we interpolate to $\mathcal{N} = 2$ SQCD with gauge group U(4) and $N_f = 8$ quark flavors. We show that the b-baryon survives the deformation and remains massless in the final SQCD. Instead massive states "feel" the naked singularity which is present in the metric and finding of their spectrum is linked to the Calogero problem with the "falling to the center" $1/\phi^2$-type potential associated with the singularity.

The paper is organized as follows. In Sec. II we review 4D $\mathcal{N} = 2$ SQCD which supports non-Abelian strings and $\mathbb{WCP}(N,N)$ models arising as world sheet theories on these strings focusing on conformal cases $N_f = 2N$. Next, we describe our mass deformation which interpolates from SQCD with U(2) gauge group and $N_f = 4$ to SQCD with U(4) gauge group and $N_f = 8$. We also review the massless b-baryon associated with the complex structure modulus of the conifold in the former theory. In Sec. III we review the derivation of $\mathcal{N} = 2$ Liouville theory from the $\mathbb{WCP}(N,N)$ world sheet model at strong coupling and then describe its mass deformation. In Sec. IV we study effective supergravity equations of motion associated with the mass-deformed Liouville world sheet model and find their solution which describes the true string vacuum. In Sec. V we continue using the gravity approach and consider the tachyon equation of motion for the vertex operators of the mass-deformed theory. We show that the b-baryon remains massless in the mass-deformed theory and discuss qualitatively the structure of the spectrum of massive string states pointing out that finding of this spectrum is related to the Calogero problem. Section VI contains our conclusions.

II. NON-ABELIAN VORTEX STRING

A. Four-dimensional $\mathcal{N} = 2$ SQCD

As was already mentioned, non-Abelian vortex strings were first found in 4D $\mathcal{N} = 2$ SQCD with the gauge group U(N) and $N_f \geq N$ quark flavors supplemented by the Fayet-Iliopoulos (FI) term [24] with parameter ξ [2–5], see for example, [8] for a detailed review of this theory. The field content is as follows. The $\mathcal{N} = 2$ vector multiplet consists of the U(1) gauge field and the SU(N) gauge field which can be combined in the matrix $(A_\mu)^k_l$ plus complex

[1] In [15] this equivalence was shown for topological versions of the string theories.

scalar fields a^k_l, and their Weyl fermion superpartners. The N_f quark multiplets of the $\mathrm{U}(N)$ theory consist of the complex scalar fields q^{kA} and $\tilde{q}_{Ak}$ (squarks) and their fermion superpartners, all in the fundamental representation of the $\mathrm{SU}(N)$ gauge group. Here $\mu = 1, \ldots, 4$ is the 4D Minkowski index, $k, l = 1, \ldots, N$ are color indices, while A is the flavor index, $A = 1, \ldots, N_f$. Below we briefly describe how we can interpolate between SQCDs with different gauge groups and number of quark flavors considering different limits of quark masses.

At weak coupling $g^2 \ll 1$, this theory is in the Higgs phase in which adjoint scalars develop vacuum expectation values (VEVs)

$$\langle a \rangle = -\frac{1}{\sqrt{2}} \begin{pmatrix} m_1 & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & m_N \end{pmatrix}, \tag{2.1}$$

where we select a vacuum where the first N quark flavors are massless at zero FI parameter ξ, while m_A are quark masses. Adjoint VEVs break $\mathrm{U}(N)$ gauge group down to $\mathrm{U}(1)^N$, masses of off-diagonal gauge bosons are given by $M^k_l \sim |m_k - m_l|$. If certain quark masses coincide adjoint VEVs leave certain non-Abelian subgroups of $\mathrm{U}(N)$ unbroken, see below. Masses of q^{kA} and $\tilde{q}_{Ak}$ quarks are equal to $|m_k - m_A|$.

At nonzero ξ following diagonal components of squarks also develop VEVs

$$\langle q^{kk} \rangle = \sqrt{\xi} \quad \text{no summation}, \quad k = 1, \ldots, N, \tag{2.2}$$

with all other components equal to zero.

These VEVs break the $\mathrm{U}(N)$ gauge group Higgsing all gauge bosons. The Higgsed gauge bosons combine with the screened quarks to form long $\mathcal{N} = 2$ multiplets with mass $m_G \sim g\sqrt{\xi}$ in the limit of zero quark masses.

In this limit the global flavor $\mathrm{SU}(N_f)$ is also broken down by quark VEVs to the so-called color-flavor locked group. The resulting global symmetry is

$$\mathrm{SU}(N)_{C+F} \times \mathrm{SU}(N_f - N) \times \mathrm{U}(1)_B, \tag{2.3}$$

see Ref. [8] for more details.

The unbroken global $\mathrm{U}(1)_B$ factor above is identified with a baryonic symmetry. Note that what is usually identified as the baryonic U(1) charge is a part of our 4D theory gauge group. Our $\mathrm{U}(1)_B$ is an unbroken by squark VEVs combination of two U(1) symmetries; the first is a subgroup of the flavor $\mathrm{SU}(N_f)$, and the second is the global U(1) subgroup of $\mathrm{U}(N)$ gauge symmetry.

In this paper we consider SQCDs with $N_f = 2N$. In these cases coupling constants in both the 4D SQCD and the world sheet $\mathbb{WCP}(N, N)$ model on the non-Abelian string does not run. However, the conformal invariance of the 4D theory is explicitly broken by the FI parameter ξ, which defines the VEVs of quarks. The FI parameter is not renormalized. We also assume that N is even, $N = 2K$, where K is integer.

Below we consider a special choice of quark masses,

$$m_{A+N} = m_A, \quad A = 1, \ldots, N, \tag{2.4}$$

which ensures that "extra" quarks with $A = (N+1), \ldots, 2N$ have the same masses as the first N ones. We also assume that

$$m_A = 0, \quad \text{for } A = 1, \ldots, K; \qquad m_A = M, \quad \text{for } A = (K+1), \ldots, N, \tag{2.5}$$

so we have only one quark mass parameter M to interpolate between different SQCDs.

Consider first the limit $M \to \infty$. In this limit gauge fields $(A_\mu)^k_{k'}$ and $(A_\mu)^l_{l'}$ are massless (at $\xi = 0$), while $(A_\mu)^k_l$ and $(A_\mu)^l_k$ together with their superpartners become infinitely heavy and decouple, $k, k' = 1, \ldots, K$, $l, l' = (K+1), \ldots, N$. Therefore, the gauge group $\mathrm{U}(N)$ is broken down to $\mathrm{U}(K) \times \mathrm{U}(K)$. Similarly quarks q^{kB} and q^{lA} together with their superpartners becomes infinitely heavy and decouple, $k = 1, \ldots, K$, $l = (K+1), \ldots, N$ and $A = 1, \ldots, K$, $B = (K+1), \ldots, N$. As a result in this limit our SQCD with gauge group $\mathrm{U}(N)$ and $N_f = 2N$ flavors splits in two noninteracting SQCDs with gauge groups $\mathrm{U}(K)$ and $N_f = 2K$ flavors. Eventually we will put $K = 2$ so the starting point of our interpolation process will be $\mathcal{N} = 2$ SQCD with the gauge group U(2) and $N_f = 4$ quarks, which as we mentioned in the Introduction supports the critical non-Abelian vortex string.

The final point of our interpolation process is the limit $M = 0$. In this limit our theory for $K = 2$ becomes massless $\mathcal{N} = 2$ SQCD with gauge group U(4) and $N_f = 8$ quark flavors.

B. World sheet sigma model

The presence of the color-flavor locked group $\mathrm{SU}(N)_{C+F}$ in 4D $\mathcal{N} = 2$ SQCD with gauge group $\mathrm{U}(N)$ is the reason for the formation of non-Abelian vortex strings [2–5]. The most important feature of these vortices is the presence of the orientational zero modes. As was already mentioned, in $\mathcal{N} = 2$ SQCD these strings are 1/2 BPS saturated and preserve $\mathcal{N} = (2, 2)$ supersymmetry on the world sheet. Their tension is determined exactly by FI parameter,

$$T = 2\pi\xi. \tag{2.6}$$

Let us briefly review the model emerging on the world sheet of the non-Abelian string [8].

The translational moduli fields are described by the Nambu-Goto action and decouple from all other moduli. Below we focus on internal moduli.

If $N_f = N$ the dynamics of the orientational zero modes of the non-Abelian vortex, which become orientational moduli fields on the world sheet, are described by 2D $\mathcal{N} = (2,2)$ supersymmetric $\mathbb{CP}(N-1)$ model.

If one adds additional quark flavors, non-Abelian vortices become semilocal and they acquire size moduli [25]. In particular, for the non-Abelian semilocal vortex in U(N) $\mathcal{N} = 2$ SQCD with $2N$ flavors, in addition to the complex orientational moduli n^i (here $i = 1, \ldots, N$), we must add N complex size moduli ρ^j [where $j = (N+1), \ldots, 2N$], see Refs. [2,5,25–28].

The effective theory on the string world sheet is a two-dimensional $\mathcal{N} = (2,2)$ supersymmetric $\mathbb{WCP}(N,N)$ model, see review [8] for details. This model can be defined as a low-energy limit of the U(1) gauge theory [29]. The fields n^i and ρ^j have charges $+1$ and -1 respectively with respect to the U(1) gauge field. The bosonic part of this gauge linear sigma model (GLSM) action reads,

$$S = \int d^2x \left\{ |\nabla_\alpha n^i|^2 + |\tilde{\nabla}_\alpha \rho^j|^2 - \frac{1}{4e_0^2} F_{\alpha\beta}^2 + \frac{1}{e_0^2} |\partial_\alpha \sigma|^2 + \frac{1}{2e_0^2} D^2 - \left|\sqrt{2}\sigma + m_i\right|^2 |n^i|^2 - \left|\sqrt{2}\sigma + m_j\right|^2 |\rho^j|^2 + D(|n^i|^2 - |\rho^j|^2 - \mathrm{Re}\beta) - \frac{\vartheta}{2\pi} F_{01} \right\},$$
$$\alpha, \beta = 1, \ldots, 2, \quad i = 1, \ldots, N, \quad j = (N+1), \ldots, 2N, \tag{2.7}$$

where

$$\nabla_\alpha = \partial_\alpha - iA_\alpha, \quad \tilde{\nabla}_\alpha = \partial_\alpha + iA_\alpha. \tag{2.8}$$

The complex scalar σ is a superpartner of the U(1) gauge field A_α and D is the auxiliary field in the vector supermultiplet. These fields can be written in terms of the twisted chiral superfield Σ [29][2]

$$\Sigma = \sigma + \sqrt{2}\theta_R \bar{\lambda}_L - \sqrt{2}\bar{\theta}_L \lambda_R + \sqrt{2}\theta_R \bar{\theta}_L (D - iF_{01}). \tag{2.9}$$

The low-energy limit in this model corresponds to $e_0^2 \to \infty$ when components of the vector supermultiplet classically decouple due to the Higgs mechanism.

The complexified inverse coupling in (2.7),

$$\beta = \mathrm{Re}\beta + i\frac{\vartheta}{2\pi}, \tag{2.10}$$

is defined via 2D FI term (twisted superpotential),

$$-\frac{\beta}{2} \int d^2\tilde{\theta}\, \sqrt{2}\, \Sigma = -\frac{\beta}{2}(D - iF_{01}). \tag{2.11}$$

Twisted masses m_i and m_j of fields n^i and ρ^j in (2.7) coincide with quark masses of 4D SQCD, namely with masses m_A of the first N flavors, $A = 1, \ldots, N$ and "extra" flavors, $A = (N+1), \ldots, 2N$, respectively.

In the massless limit the number of real bosonic degrees of freedom in the model (2.7) defines the dimension of its target space (Higgs branch), given by

$$\dim_{\mathbb{R}} \mathcal{H} = 4N - 1 - 1 = 2(2N - 1), \tag{2.12}$$

where $4N$ is the number of real degrees of freedom of (n^i, ρ^j) fields and we subtract one real D-term constraint,

$$|n^i|^2 - |\rho^j|^2 = \mathrm{Re}\beta, \tag{2.13}$$

in the limit $e_0^2 \to \infty$, and one gauge phase is eaten by the Higgs mechanism.

On the quantum level, the coupling β does not run in this theory because the sum of charges of n and ρ fields vanishes. Hence, it is superconformal in the limit of zero masses. Therefore, its target space is Ricci-flat and [being Kähler due to $\mathcal{N} = (2,2)$ supersymmetry] represents a (noncompact) Calabi-Yau manifold, see Refs. [12,30] for reviews on toric geometry.

The dimension of the Higgs branch (2.12) determines the central charge of the 2D conformal field theory (CFT) of the CY manifold

$$\hat{c}_{\mathrm{CY}} \equiv \frac{c_{\mathrm{CY}}}{3} = \dim_{\mathbb{C}} \mathcal{H} = 2N - 1, \tag{2.14}$$

just equal to its complex dimension. In $N = 2$ case these $\dim_{\mathbb{R}} \mathcal{H} = 2(2N-1) = 6$ internal degrees of freedom can be combined with four translational moduli of the non-Abelian vortex to form a 10D target space of a critical superstring [1,10].

Consider now the classical vacuum structure of the $\mathbb{WCP}(N,N)$ model (2.7). At $\mathrm{Re}\beta > 0$ we have N vacua

$$\sqrt{2}\sigma = -m_{i_0}, \quad |n^{i_0}|^2 = \mathrm{Re}\beta, \quad i_0 = 1, \ldots, N. \tag{2.15}$$

Fields n^i, $i \neq i_0$ and fields ρ^j have masses $|m_i - m_{i_0}|$ and $|m_j - m_{i_0}|$, respectively. The number of vacua stay intact in the quantum theory because it is protected by Witten index which is equal to N.

[2]Here spinor indices are written as subscripts, say $\theta^L = \theta_R$, $\theta^R = -\theta_L$. We also defined the twisted measure $d^2\tilde{\theta} = \frac{1}{2} d\bar{\theta}_L d\theta_R$ to ensure that $\int d^2\tilde{\theta}\tilde{\theta}^2 = \int d\bar{\theta}_L d\theta_R \theta_R \bar{\theta}_L = 1$.

Let us discuss how our interpolation process is seen in the world sheet theory. Consider the choice of masses given by (2.4) and (2.5). In the limit $M \to \infty$ we have K vacua with $\sigma = 0$ where one of n^{i_0}, $i_0 = 1, \ldots, K$ develop VEV. In these vacua fields n^i with $i = (K+1), \ldots, N$ and ρ^j, $j = (N+K+1), \ldots, 2N$ become infinitely heavy (with mass $|M|$) and decouple. The model (2.7) reduces to the $\mathbb{WCP}(K, K)$ model with massless fields n^i, $i = 1, \ldots, K$ and ρ^j, $j = (N+1), \ldots, (N+K)$. For $K = 2$ the central charge $\hat{c}_{\rm CY} = 3$ and the $\mathbb{WCP}(2, 2)$ model becomes a sigma model on the conifold Y_6. In this case the non-Abelian vortex becomes a critical superstring. This will be the starting point of our interpolating process.

The final point corresponds to the limit $M = 0$. The $\mathbb{WCP}(4, 4)$ model is still conformal, but does not have the right central charge. We will show that the quantum world sheet model for $N = 4$ case is not given by $\mathbb{WCP}(4, 4)$ model. As we already mentioned we will find the true string vacuum solving effective gravity equations.

Note, that in the limit $M \to \infty$ we could also consider another K vacua with $\sqrt{2}\sigma = -M$. This would give another $\mathbb{WCP}(K, K)$ model as a starting point. In 4D this is associated with U(N) SQCD reducing to two noninteracting SQCDs with U(K) gauge groups, see the previous subsection. Below for definiteness we will consider the first option above.

To conclude this subsection we note that at $\beta = 0$ the world sheet $\mathbb{WCP}(N, N)$ model develop the Coulomb branch with arbitrary value of σ in the massless limit. This can be shown using the exact twisted superpotential for the $\mathbb{WCP}(N, N)$ models as a function of twisted superfield Σ. This exact twisted superpotential is a generalization [31,32] of the $CP(N-1)$ model superpotential [29,33–35] of the Veneziano-Yankielowicz type [36]. The vacuum equation for σ obtained by differentiating of this superpotential with respect to σ reads [20],

$$\prod_{i=1}^{N} (\sqrt{2}\sigma + m_i) = e^{-2\pi\beta} \prod_{j=N+1}^{2N} (\sqrt{2}\sigma + m_j). \tag{2.16}$$

At generic values of masses it gives just N distinct vacua with certain fixed values of σ. In the limit $m_i = m_j = 0$ one gets,

$$\sigma^N = e^{-2\pi\beta}\sigma^N, \tag{2.17}$$

with the N-degenerate vacuum solution $\sigma = 0$ for any nonvanishing β. This means that fields n and ρ remain massless (see two first terms in the second line in (2.7)) and live on the Higgs branch of the theory. However, for both $\beta = 0$ and vanishing twisted masses the complex scalar σ can have arbitrary value making n^i and ρ^j massive. This solution describes the Coulomb branch, which opens up at $\beta = 0$. As was shown in [20] this Coulomb branch can be effectively described in terms of $\mathcal{N} = 2$ Liouville theory.

Note also that the value $\beta = 0$ at strong coupling is exactly what we are interested in for $N = 2$ case. The so-called "thin string conjecture" put forward in [1,10] implies that only at $\beta = 0$ we expect that the solitonic string-gauge duality works and the world sheet $\mathbb{WCP}(2, 2)$ model defines the right string theory for the critical non-Abelian vortex in $\mathcal{N} = 2$ 4D SQCD.

C. Massless 4D baryon

In this section we consider the conifold case taking $N = 2$ in the massless $\mathbb{WCP}(N, N)$ model (2.7) and briefly review the only 4D massless state found in the string theory of the critical non-Abelian vortex [10]. It is associated with the deformation of the conifold complex structure. As was already mentioned, all other massless string modes have non-normalizable wave functions over the conifold. In particular, the 4D graviton associated with a constant wave function over the conifold Y_6 is absent as expected [10].

We can construct the U(1) gauge-invariant "mesonic" variables,

$$w^{ij} = n^i\rho^j, \quad i = 1, 2, \quad j = 3, 4. \tag{2.18}$$

These variables are subject to the constraint

$$\det w^{ij} = 0. \tag{2.19}$$

Equation (2.19) defines the conifold Y_6. It has the Ricci-flat Kähler metric and represents a noncompact Calabi-Yau manifold [11,12,29]. It is a cone which can be parametrized by the noncompact radial coordinate,

$$\tilde{r}^2 = \mathrm{Tr}\,\bar{w}w, \tag{2.20}$$

and five angles, see Ref. [11]. Its section at fixed $\tilde{r}$ is $S_2 \times S_3$.

At $\beta = 0$ the conifold develops a conical singularity, so both spheres S_2 and S_3 can shrink to zero. The conifold singularity can be smoothed out in two distinct ways; by deforming the Kähler form or by deforming the complex structure. The first option is called the resolved conifold and amounts to keeping a nonzero value of β in (2.13). This resolution preserves the Kähler structure and Ricci-flatness of the metric. If we put $\rho^K = 0$ in (2.13) we get the $\mathbb{CP}(1)$ model with the sphere S_2 as a target space (with the radius $\sqrt{\beta}$). The resolved conifold has no normalizable zero modes. In particular, the Kähler modulus β which becomes a scalar field in four dimensions has a non-normalizable (quadratically divergent) wave function over the Y_6 and therefore is not dynamical [10].

If $\beta = 0$ (i.e., exactly when the Coulomb branch opens up) another option exists, namely a deformation of the complex structure [12]. It preserves the Kähler structure and Ricci flatness of the conifold and is usually referred to as the deformed conifold. It is defined by the deformation of Eq. (2.19), namely,

$$\det w^{ij} = b, \tag{2.21}$$

where b is a complex parameter. Now the sphere S_3 can not shrink to zero, its minimal size is determined by b.

We see that the resolved conifold corresponds to the Higgs branch of the GLSM (2.7) at $N = 2$, while the deformed conifold is associated with the Coulomb branch of this theory, which opens up at $\beta = 0$ [20].

The modulus b becomes a 4D complex scalar field. The effective action for this field was calculated in [10] using the explicit metric on the deformed conifold [11,37,38],

$$S_{\rm kin}(b) = T\int d^4x|\partial_\mu b|^2 \log\frac{\tilde{R}^2_{\rm IR}}{|b|}, \tag{2.22}$$

where $\tilde{R}_{\rm IR}$ is the maximal value of the radial coordinate $\tilde{r}$ introduced as an infrared (IR)regularization of the logarithmically divergent b-field norm. Here the logarithmic integral at small $\tilde{r}$ is cut off by the minimal size of S_3, which is equal to $|b|$.

We see that the norm of the modulus b turns out to be logarithmically divergent in the infrared [10,39]. Such states at the borderline between normalizable and non-normalizable modes are considered as physical states "localized" in the 4D.

The field b being massless can develop a VEV. Thus, we have a new Higgs branch in 4D $\mathcal{N} = 2$ SQCD which is developed only for the critical value of the 4D coupling constant $\tau_{\rm SW} = 1$ associated with $\beta = 0$ [40].

In [10] the massless state b was interpreted as a baryon of 4D $\mathcal{N} = 2$ QCD. Its charge with respect to the baryonic $U(1)_B$ symmetry in (2.3) is $Q_B(b) = 2$ [10].

To conclude this section, we make a comment on non-normalizable strings modes. As was mentioned above most of string modes have non-normalizable wave functions over the conifold, i.e., they are not localized in 4D and cannot be interpreted as dynamical states in 4D theory. Technically this happens because infinite normalization factor over the internal space appears in 4D kinetic terms for these states making them nondynamical. These modes play a role of coupling constants in 4D theory, see Ref. [41] where the nature of non-normalizable string modes which appear upon Calabi-Yau compactifications was discussed. The example of such a coupling constant in the theory at hand is the inverse coupling β of the world sheet sigma model (2.7) (it is related to 4D gauge coupling [40]). As we discussed above it corresponds to the Kähler form modulus of the conifold and has quadratically non-normalizable wave function.

III. $\mathcal{N} = 2$ LIOUVILLE THEORY FROM $\mathbb{WCP}(N,N)$ MODEL

In this section we first briefly review the derivation of the $\mathcal{N} = 2$ Liouville theory from the world sheet $\mathbb{WCP}(N,N)$ model at $\beta = 0$ [20] and then consider its deformation upon switching on masses in the $\mathbb{WCP}(N,N)$ model.

A. Massless theory

Consider, first, the massless $\mathbb{WCP}(N,N)$ model (2.7) in the large-N limit, $N \to \infty$. As we discussed in Sec. II B at $\beta = 0$ the complex scalar σ can take arbitrary values on the Coulomb branch of the theory. For $\sigma \neq 0$ this makes the fields n and ρ massive, and one can integrate them out. For both nonsupersymmetric and $\mathcal{N} = (2,2)$ supersymmetric $CP(N-1)$ models this was done by Witten [42] (see also [43]). He showed that the bare gauge coupling e_0^2 taken to be infinite in the classical limit is renormalized at one loop and becomes finite. This means that the U(1) gauge field introduced as an auxiliary field in the GLSM formulation acquires a finite kinetic term and becomes physical.

Almost the same calculation for $\mathbb{WCP}(N,N)$ model gives the effective action for the vector multiplet (see Ref. [20]). Focusing here on the most important kinetic term for σ we get

$$S^{\sigma}_{\rm eff} = \int d^2x \frac{1}{e^2}|\partial_\alpha\sigma|^2, \tag{3.1}$$

where the classical gauge coupling e_0^2 is corrected by the one-loop contribution

$$\frac{1}{e^2} = \left(\frac{1}{e_0^2} + \frac{2N}{4\pi}\frac{1}{2|\sigma|^2}\right)\Bigg|_{e_0^2\to\infty} = \frac{2N}{4\pi}\frac{1}{2|\sigma|^2}. \tag{3.2}$$

The wave function renormalization comes from n^i and ρ^j fields (with their fermionic superpartners) propagating in the loop. The loop integral is finite in the ultraviolet (UV) region and is saturated in the IR region at momenta of order of n and ρ "mass" $\sqrt{2}|\sigma|$ [see Eq. (2.7)].[3] The loop graph contains two vertices, each proportional to the electric charge of a given n or ρ field (equal to ± 1). Therefore, this graph is proportional to the sum of squires of these electric charges, which is equal to $2N$. The result (3.2) gives the leading term in the $1/N$ expansion. We have

$$S^{\sigma}_{\rm eff} = \frac{2N}{4\pi}\int d^2x \frac{1}{2}\frac{|\partial_\alpha\sigma|^2}{|\sigma|^2} \tag{3.3}$$

[3] We put "mass" in quotation marks, since in 2D theory σ does not have a definite VEV, instead the ground-state wave function is spread over the whole Coulomb branch, cf. [44].

with the tube metric.[4] Making a change of variables

$$\sigma = e^{-\frac{\phi+iY}{Q}}, \tag{3.4}$$

where we parametrized the modulus of σ by the real scalar field ϕ (which will be the Liouville field) and its phase by the real compact scalar Y with the periodicity condition,

$$Y + 2\pi Q \sim Y, \tag{3.5}$$

we arrive to the bosonic part of the effective action

$$S_{\mathrm{eff}} = \frac{1}{4\pi}\int d^2x\left(\frac{1}{2}(\partial_\alpha\phi)^2 + \frac{1}{2}(\partial_\alpha Y)^2\right), \tag{3.6}$$

where the radius of the compact dimension

$$Q \underset{N\to\infty}{\approx} \sqrt{2N}\,. \tag{3.7}$$

Other components of the vector supermultiplet can be considered similarly, see Ref. [20]. For example, the U(1) gauge field has no physical degrees of freedom in two dimensions and can be integrated out together with the D-field, see Ref. [20] for details.

Repeating the above calculation on the curved world sheet we can restore the background charge of the Liouville field [20],

$$S_{\mathrm{eff}} = \frac{1}{4\pi}\int d^2x\sqrt{h}\left(\frac{1}{2}h^{\alpha\beta}(\partial_\alpha\phi\partial_\beta\phi + \partial_\alpha Y\partial_\beta Y) - \frac{Q}{2}\phi R^{(2)}\right), \tag{3.8}$$

where $h_{\alpha\beta}$ is the world sheet metric, $R^{(2)}$ is the world sheet Ricci scalar and $h = \det(h_{\alpha\beta})$.

This is exactly the bosonic part of the $\mathcal{N} = 2$ Liouville action (see Ref. [45] for a review). Note the linear dilaton in (3.8),

$$\Phi = -\frac{Q}{2}\phi, \tag{3.9}$$

with the background charge Q for the Liouville field ϕ which coincides with the radius of the compact dimension (as it should in the $\mathcal{N} = 2$ Liouville theory). In the large N approximation Q is given by (3.7).

The action in (3.8) leads to the following holomorphic stress tensor of the bosonic part of the theory,

$$T = -\frac{1}{2}[(\partial_z\phi)^2 + Q\partial_z^2\phi + (\partial_z Y)^2]. \tag{3.10}$$

[4]The metric looks singular, but actually there is no singularity at the origin [44].

The $\mathcal{N} = 2$ Liouville interaction superpotential (see Ref. [45]) comes from the 2D FI term (2.11) in the $\mathbb{WCP}(N,N)$ model,

$$L_{\mathrm{int}} = \mu\int d^2\tilde{\theta}\Sigma = \mu\int d^2\tilde{\theta}e^{-\frac{\phi+iY}{Q}}, \tag{3.11}$$

where we use parametrization (3.4) and promote scalars ϕ and Y to (twisted) chiral superfields, see Ref. [20] for details.[5]

This superpotential is a marginal deformation of the $\mathcal{N} = 2$ Liouville theory (3.8). The conformal dimension of σ is

$$\Delta\left(\sigma = e^{-\frac{\phi+iY}{Q}}\right) = \left(\frac{1}{2},\frac{1}{2}\right), \tag{3.12}$$

which can be easily checked using the stress tensor (3.10).

The above outlined equivalence of the Coulomb branch of $\mathbb{WCP}(N,N)$ model and $\mathcal{N} = 2$ Liouville theory obtained in the large N approximation can be promoted to the exact equivalence. It was argued in [20] that the σ-dependence of the effective action (3.3) is fixed on dimensional grounds and integrating fields n^i and ρ^j exactly rather then in the large N approximation we would arrive to the same action (3.3) with the coefficient $2N$ replaced by the exact coefficient $Q^2(N)$. To find the exact dependence of $Q^2(N)$ on N we can demand that central charges of both CFTs ($\mathbb{WCP}(N,N)$ model and $\mathcal{N} = 2$ Liouville theory) should coincide. The central charge of the $\mathcal{N} = 2$ Liouville theory is

$$\hat{c}_L = 1 + Q^2. \tag{3.13}$$

Requiring that it should be equal to the central charge $\hat{c}_{\mathrm{CY}}$ (2.14) gives the exact relation,

$$Q(N) = \sqrt{2(N-1)}, \tag{3.14}$$

which reduces to (3.7) in the large-N approximation.

Note also that as we already mentioned for the case $N = 2$ the Coulomb branch of $\mathbb{WCP}(2,2)$ model is associated with the deformed conifold. Therefore, the coefficient μ in front of the marginal deformation (3.11) should be identified with the conifold complex structure parameter b [18–20],[6]

$$\mu \sim b. \tag{3.15}$$

[5]The fact that the Liouville interaction is given by a twisted superpotential is just a matter of conventions since there are no untwisted chiral fields in the effective theory.

[6]The unit power of b in the rhs of (3.15) was fixed in [13] using the baryonic U(1) symmetry.

On the CY side, parameter b smooths the conifold singularity at small $\tilde{r}$, i.e., provides a UV regularization. In the Liouville theory the Liouville superpotential (the Liouville wall) at nonzero μ also provides a UV regularization preventing field ϕ from penetrating to the region of large negative values. With the identification (3.15) the conifold complex structure modulus which was not seen in the GLSM description (2.7) becomes manifest in the Liouville description.

To conclude this subsection, we note that the dilaton has a linear dependence on the Liouville coordinate ϕ, see Eq. (3.9). Therefore, the string coupling constant $g_s = e^{\Phi}$ would become large at large negative ϕ. On the other hand at nonzero b the Liouville wall prevents field ϕ from penetrating to the region of large negative values. In fact, the maximum value of the string coupling is $g_s \sim 1/|b|$ for $Q = \sqrt{2}$. In this paper we keep b large to ensure that the string coupling is small and the string perturbation theory is reliable, see Refs. [18,21]. In particular, we can use the tree-level approximation to obtain the string spectrum.

In terms of 4D SQCD taking b large means moving along the Higgs branch far away from the origin.

B. Primary operators

In this subsection we review primary operators in the $\mathcal{N}=2$ Liouville theory. For $N=2$ case they describe physical string states interpreted as hadrons in 4D SQCD, see Ref. [13] for details.

Primary operators for the $\mathcal{N}=2$ Liouville theory are constructed in [18], see also [19,46]. For large positive ϕ (where the Liouville interaction is small) primaries take the form,

$$T_{j;m_L,m_R} = e^{Q[j\phi + i(m_L Y_L - m_R Y_R)]}, \tag{3.16}$$

where we split Y into left and right-moving parts. Parameters m_L and m_R for left-moving and right-moving sectors are given by

$$m_L = \frac{1}{2}(n_1 + kn_2), \quad m_R = \frac{1}{2}(n_1 - kn_2), \tag{3.17}$$

where n_2 and n_1 are integers corresponding to momentum and winding numbers along the compact dimension Y.

The primary operator (3.16) is related to the wave function on the target space as follows:

$$T_{j;m_L,m_R} = g_s \Psi_{j;m_L,m_R}(\phi, Y), \tag{3.18}$$

where the string coupling $g_s = e^{\Phi}$ depends on ϕ, see Eq. (3.9). Thus,

$$\Psi_{j;m_L,m_R}(\phi, Y) \sim e^{Q(j+\frac{1}{2})\phi + iQ(m_L Y_L - m_R Y_R)}. \tag{3.19}$$

We will look for string states with normalizable along the noncompact Liouville dimension wave functions. These states are localized in 4D and can be interpreted as hadrons in 4D SQCD. The condition for the states to have normalizable wave functions reduces to

$$j \le -\frac{1}{2}. \tag{3.20}$$

We include the case $j = -\frac{1}{2}$ which is at the borderline between normalizable and non-normalizable states.

The conformal dimension of the primary operator (3.16) is

$$\Delta_{j,m} = \frac{Q^2}{2}\{m^2 - j(j+1)\}. \tag{3.21}$$

Unitarity implies that the conformal dimension (3.21) should be positive,

$$\Delta_{j,m} > 0. \tag{3.22}$$

Moreover, to ensure that conformal dimensions of left- and right-moving parts of the vertex operator (3.16) are the same we impose that $m_R = \pm m_L$.

The $\mathcal{N}=2$ Liouville theory has a mirror description [47] in terms of a supersymmetric version of the two-dimensional black hole with the cigar geometry [48], which is the $\mathcal{N}=2$ SL(2, ℝ)/U(1) coset Wess-Zumino-Novikov-Witten (WZNW) theory [15,18,46,49] at the level

$$k = \frac{2}{Q^2}. \tag{3.23}$$

of the Kač-Moody algebra.

The spectrum of the allowed values of j and m in (3.16) was exactly determined using the Kač-Moody algebra for the mirror description of the theory in [46,50–53], see Ref. [54] for a review. Both discrete and continuous representations were found. Parameters j and m determine the global quadratic Casimir operator and the projection of the spin on the third axis,

$$J^2|j,m\rangle = -j(j+1)|j,m\rangle, \quad J^3|j,m\rangle = m|j,m\rangle. \tag{3.24}$$

We have

(i) Discrete representations with

$$j = -\frac{1}{2}, -1, -\frac{3}{2}, \ldots, \quad m = \pm\{j, j-1, j-2, \ldots\}. \tag{3.25}$$

(ii) Principal continuous representations with

$$j=-\frac{1}{2}+is, \quad m=\text{integer} \ \text{ or } \ m=\text{half-integer}, \tag{3.26}$$

where s is a real parameter.

We see that discrete representations include normalizable and borderline-normalizable states localized near the tip of the cigar. This nicely matches our qualitative expectations.

Consider now the $\mathcal{N}=2$ Liouville theory with $N=2$, $Q=\sqrt{2(N-1)}=\sqrt{2}$. This corresponds to $k=1$ in the mirror description on the cigar. Take the primary operator (3.16) with $j=-1/2$ and $m_L=\pm 1/2$. Its conformal dimension is

$$\Delta_{j=-\frac{1}{2},m=\pm\frac{1}{2}}=\frac{1}{2}, \tag{3.27}$$

[see Eq. (3.21)] so it is marginal and describes a massless string state in 4D. As was noticed in [13] this massless state corresponds to the complex structure modulus b for the string compactification on the conifold. Two possible values of $m=\pm 1/2$ corresponds to two real degrees of freedom of the complex scalar field b. The associated string state has a logarithmically normalizable wave function over the conifold in terms of the radial coordinate $\tilde{r}$ [10,39], see Eq. (2.22). On the Liouville side this corresponds to the borderline normalization of the massless state (3.16) with $j=-\frac{1}{2}$, $m=\pm\frac{1}{2}$ (see Ref. [13] for details).

The discrete spectrum (3.25) gives rise to physical hadron states in 4D SQCD. In particular, the mass spectrum of massive 4D states created by vertex operators (3.16) with $j=-\frac{1}{2}$ has the form [13],

$$\frac{M_{4D}^2}{8\pi T}=\Delta_{j,m}-\frac{1}{2}, \tag{3.28}$$

where $Q=\sqrt{2}$.

To conclude this subsection, let us make a comment on the principal continuous representation (3.26) of string states, which represents plane waves in the noncompact Liouville dimension. In [13,40] it was suggested an interpretation of these states: they correspond to multiparticle states associated with decay of normalizable 4D states. This interpretation is motivated by the observation that spectra of continuous states for half-integer m start from thresholds given by masses of (borderline) normalizable states. This issue however needs future clarification.

C. Mass deformation

Now consider $\mathbb{WCP}(N,N)$ model (2.7) with nonzero twisted masses starting with the large N approximation. Integrating out n^i and ρ^j fields at $\beta=0$ we, instead of (3.3), get

$$S_{\text{eff}}^{\sigma}=\frac{1}{4\pi}\int d^2x\sum_{A=1}^{2N}\frac{|\partial_\alpha\sigma|^2}{|\sqrt{2}\sigma+m_A|^2} = \frac{1}{4\pi}\int d^2x\frac{1}{2}\frac{|\partial_\alpha\sigma|^2}{|\sigma|^2}\sum_{A=1}^{2N}\frac{1}{|1+\frac{m_A}{\sqrt{2}\sigma}|^2} \tag{3.29}$$

for the effective action of the field σ.

Consider the choice of masses given by (2.4) and (2.5). K first n fields are massless, K last n fields are massive with the same mass M, $N=2K$, while masses of ρ fields are equal to masses of n fields, see Eq. (2.4). The action (3.29) takes the form,

$$S_{\text{eff}}=\frac{1}{4\pi}\int d^2x g_{cl}(\phi,Y)\left(\frac{1}{2}(\partial_\alpha\phi)^2+\frac{1}{2}(\partial_\alpha Y)^2\right), \tag{3.30}$$

where we use the parametrization (3.4), and the "classical" warp factor of the target space metric is[7]

$$g_{cl}(\phi,Y)=1+\frac{1}{\left|1+\frac{M}{\sqrt{2}}e^{\frac{\phi+iY}{Q}}\right|^2}, \tag{3.31}$$

while the radius of the compact dimension

$$Q\underset{K\to\infty}{\approx}\sqrt{2K}. \tag{3.32}$$

We also drop the dilaton term in (3.30) which we will restore later.

As we already mentioned nonzero twisted masses break conformal invariance in the world sheet model $\mathbb{WCP}(N,N)$. Therefore, we cannot use the mass-deformed model (3.30) for the string quantization. To find the true string vacuum we will solve the effective supergravity equations of motion in the next section. To find this solution we will use the expansion of the classical target space metric in (3.30) just as initial conditions at $\phi\to\infty$ where the deformation is small. Namely, expanding (3.30) at large ϕ we write the classical warp factor (3.31) as

$$g_{cl}(\phi,Y)\approx 1+\frac{2}{|M|^2}e^{-\frac{2\phi}{Q}}+\cdots, \tag{3.33}$$

and use this expression in the next section as initial conditions for the true quantum warp factor at $\phi\to\infty$. It just shows the initial "position" and the "velocity", in which the true quantum metric is pushed by the mass

[7]A comment on dimensions is in order. The canonical dimension of σ is unity. We can introduce dimensionless field $\sigma'=\sigma/\sqrt{4\pi T}$ and use parametrization (3.4) for σ'. Then dimensionless $M'=M/\sqrt{4\pi T}$ appears in (3.31). Below we consider dimensionless quantities dropping primes to simplify notations. We also remind that $2\pi T=1/\alpha'$.

deformation, while the true ”trajectory” should be found by solving the gravity equations of motion.

As initial conditions for the dilaton at large ϕ we use the linear dilaton in (3.9),

$$\Phi \approx -\frac{Q}{2}\phi + \cdots . \tag{3.34}$$

The starting point of our interpolation procedure is the limit $M \to \infty$ where the world sheet theory reduces to $\mathbb{WCP}(K, K)$ model and its Coulomb branch at $\beta = 0$ is given by the $\mathcal{N} = 2$ Liouville theory. As we mentioned above this equivalence is exact in K and we relax the large K condition using exact expression,

$$Q(K) = \sqrt{2(K-1)}, \tag{3.35}$$

see Eq. (3.14). For the case $K = 2$ when non-Abelian vortex string become a critical superstring this gives

$$Q = \sqrt{2}. \tag{3.36}$$

We also assume that boundary conditions (3.33) and (3.34) depend on K via $Q(K)$ (3.35) and extrapolate Eqs. (3.33) and (3.34) to $K = 2$.

IV. SOLUTIONS OF GRAVITY EQUATIONS

In this section we study effective gravity equations for the mass-deformed superstring background and find their solutions.

A. The setup

The bosonic part of the action of the type-II supergravity in the string frame is given by

$$S = \frac{1}{2\kappa^2}\int d^D x\sqrt{-G}e^{-2\Phi}\{R + 4G^{MN}\partial_M\Phi\partial_N\Phi_N + \cdots\}, \tag{4.1}$$

where G_{MN} is the D-dimensional metric and we keep in (4.1) only the metric and the dilaton terms, $M, N = 1, \ldots, D$. Here $2\kappa^2 = (2\pi)^{(\frac{D}{2}-2)} g_s^2/T^{\frac{D-2}{2}}$.

Einstein’s equations of motion following from the action (4.1) have the form

$$R_{MN} + 2D_M D_N\Phi = 0, \tag{4.2}$$

while the equation for the dilaton reads

$$R = 4G^{MN}\partial_M\Phi\partial_N\Phi - 4G^{MN}D_M D_N\Phi + p, \tag{4.3}$$

where $p = \frac{D-10}{2}$ (in dimensionless units) is included if $D \neq 10$.

We assume that our space-time is a direct product of the flat 4D Minkowski space and an internal space which has the nontrivial metric of the target space of the $\mathcal{N} = 2$ deformed Liouville theory. Thus, $D = 6$ and the ansatz for the internal metric is

$$ds^2_{\text{int}} = g(\phi, Y)\{d^2\phi + d^2Y\}. \tag{4.4}$$

It is inspired by the calculation in Sec. III C.

Let us note that in the limit $M \to \infty$ equations of motion are satisfied by the flat internal metric with $g = 1$ and the linear dilaton (3.9). Einstein’s equation is satisfied because the Ricci tensor is zero for the flat metric and covariant derivatives in (4.2) reduce to ordinary ones, so the second term in the lhs of (4.2) gives zero on the linear dilaton. Equation (4.3) is satisfied for $Q = \sqrt{2}$ and $p = -2$ [see Eq. (3.36)].

B. Solutions to the gravity equations

In this section we find a solution to the gravity Eqs. (4.2) and (4.3) which satisfy initial conditions (3.33) and (3.34). Flat Minkowski part of equations is trivial and decouples so we are left with gravity equations for the internal part. Due to $\mathcal{N} = (2, 2)$ supersymmetry the metric of the internal space is Kähler. Therefore, we introduce complex coordinates,

$$s = \phi + iY, \quad \bar{s} = \phi - iY. \tag{4.5}$$

In terms of these coordinates the metric (4.4) takes the form

$$ds^2_{\text{int}} = g(s, \bar{s})ds d\bar{s}, \quad g_{\bar{s}s} = g_{s\bar{s}} = \frac{1}{2}g(s, \bar{s}). \tag{4.6}$$

For this metric the only nonzero Christoffel symbols are

$$\Gamma^s_{ss} = \frac{1}{g}\partial_s g, \quad \Gamma^{\bar{s}}_{\bar{s}\bar{s}} = \frac{1}{g}\partial_{\bar{s}} g, \tag{4.7}$$

and nonzero Ricci tensor components take the form

$$R_{s\bar{s}} = R_{\bar{s}s} = -\partial_s\partial_{\bar{s}}\ln g, \tag{4.8}$$

while the Ricci scalar reads

$$R = -\frac{4}{g}\partial_s\partial_{\bar{s}}\ln g. \tag{4.9}$$

Then Einstein equations (4.2) with $s\bar{s}$ and ss indices reduce to

$$-\partial_s\partial_{\bar{s}}\ln g + 2\partial_s\partial_{\bar{s}}\Phi = 0 \tag{4.10}$$

and

$$\partial_s\partial_s\Phi - \frac{1}{g}\partial_s g\partial_s\Phi = 0, \tag{4.11}$$

respectively, while the equation with $\bar{s}\,\bar{s}$ indices is just a complex conjugate of (4.11).

The dilaton equation (4.3) reads

$$\frac{1}{g}\partial_s\partial_{\bar{s}}\ln g + \frac{4}{g}\partial_s\Phi\partial_{\bar{s}}\Phi - \frac{4}{g}\partial_s\partial_{\bar{s}}\Phi + \frac{p}{4} = 0. \tag{4.12}$$

The solution to Eq. (4.10) has the form,

$$\Phi = -\frac{Q}{4}(s+\bar{s}) + \frac{1}{2}\ln g, \tag{4.13}$$

where we use initial conditions (3.34). In principle, we can add to the rhs of (4.13) arbitrary holomorphic function of s plus its complex conjugate. However it is easy to see that this corresponds just to a reparametrization of the variable s. We fix the gauge assuming that this function is zero.

Substituting (4.13) in the Einstein equation (4.11) and the dilaton equation (4.12) gives

$$\partial_s^2 \ln g - (\partial_s \ln g)^2 + \frac{Q}{2}\partial_s \ln g = 0 \tag{4.14}$$

and

$$-\partial_{\bar{s}}\partial_s \ln g + \frac{Q^2}{4} + \frac{p}{4}g - \frac{Q}{2}(\partial_s \ln g + \partial_{\bar{s}} \ln g) + \partial_s \ln g \partial_{\bar{s}} \ln g = 0, \tag{4.15}$$

respectively.

Equation (4.14) is a first-order equation for the variable $\partial_s \ln g$ and admits separation of variables. One gets

$$\partial_s \ln g = \frac{Q}{2}\frac{1}{1-f(\bar{s})e^{\frac{Q}{2}s}}, \tag{4.16}$$

where $f(\bar{s})$ is a function of $\bar{s}$. To fix this function we use the complex conjugate of Eq. (4.14). This gives

$$\partial_s \ln g = \partial_{\bar{s}} \ln g = \frac{Q}{2}\frac{1}{1-Ae^{\frac{Q}{2}(s+\bar{s})}}, \tag{4.17}$$

where A is a constant. Integrating this equation we finally get

$$g = \frac{1}{1-\frac{1}{A}e^{-\frac{Q}{2}(s+\bar{s})}}, \tag{4.18}$$

where we put the integration constant to zero using the initial condition $g=1$ at $s\to\infty$. It is easy to check that this solution satisfies also the dilaton equation (4.15) for $Q=\sqrt{2}$ and $p=-2$.

The warp factor (4.18) can be written as

$$g(\phi) = \frac{1}{1-\frac{1}{A}e^{-Q\phi}} = \frac{1}{1-e^{-Q(\phi-\phi_0)}}, \tag{4.19}$$

while the solution for the dilaton takes the form,

$$\Phi(\phi) = -\frac{Q}{2}\phi - \frac{1}{2}\ln\left(1-\frac{1}{A}e^{-Q\phi}\right) = -\frac{Q}{2}\phi - \frac{1}{2}\ln\left[1-e^{-Q(\phi-\phi_0)}\right], \tag{4.20}$$

where we use (4.13). Here we introduce $\phi_0 = -\frac{1}{Q}\ln A$. We see that the warp factor of the metric and the dilaton are functions of the Liouville field ϕ and do not depend on Y.

Observe now that precisely for our case $Q=\sqrt{2}$, which corresponds to $K=2$ the warp factor (4.19) satisfy initial conditions (3.33) associated with the mass deformation. Now we can identify the parameter A in terms of the mass M. We have

$$A = \frac{M^2}{2}, \qquad \phi_0 = -\frac{1}{Q}\ln\left(\frac{M^2}{2}\right). \tag{4.21}$$

Note also that the first nontrivial term in the expansion of the warp factor (4.19) at large ϕ gives rise to the following deformation operator,

$$(\partial_z\phi - i\partial_z Y)(\partial_{\bar{z}}\phi + i\partial_{\bar{z}}Y)e^{-Q\phi}. \tag{4.22}$$

This operator has $j=-1$, $m=0$ and is marginal with conformal dimension $\Delta=(1,1)$. It is the bosonic part of so called nonchiral marginal deformation of $\mathcal{N}=2$ Lioville theory, see Ref. [45] for a review. We see that (4.19) and (4.20) represent exact solutions for the mass deformation which, infinitesimally, is associated with the nonchiral marginal operator (4.22).

Solutions (4.19) and (4.20) define the true quantum vacuum of the mass-deformed string theory. Namely, the bosonic part of the mass-deformed $\mathcal{N}=2$ Liouville world sheet theory takes the form,

$$S_{\rm ws} = \frac{1}{4\pi}\int d^2x\sqrt{h}\left\{g(\phi)\left[\frac{1}{2}(\partial_\alpha\phi)^2 + \frac{1}{2}(\partial_\alpha Y)^2\right] + \Phi(\phi)R^{(2)} + L_{\rm int}\right\}, \tag{4.23}$$

where the metric warp factor $g(\phi)$ and the dilaton $\Phi(\phi)$ are given by (4.19) and (4.20). Also we will show in the next section that the Liouville superpotential (3.11) is still a marginal deformation of the theory therefore, $L_{\rm int}$ in (4.23) is not modified and is still given by (3.11).

Thus, the action (4.23) defines a continues family of CFTs with the same central charge $\hat{c}_L = 3$ [see Eq. (3.13)] parametrized by the mass parameter M which we can use as world sheet theories for the string quantization.

C. Scales of the deformed Liouville theory

Let us discuss some properties of our solution. The metric warp factor (4.19) develop a naked singularity at $\phi = \phi_0$,

$$g|_{\phi\to\phi_0} \approx \frac{1}{Q(\phi-\phi_0)}, \tag{4.24}$$

with the Ricci tensor given by

$$R_{s\bar{s}} = -\frac{Q^2}{4}\frac{Ae^{\frac{Q}{2}(s+\bar{s})}}{\left[1-Ae^{\frac{Q}{2}(s+\bar{s})}\right]^2}\Bigg|_{\phi\to\phi_0} \approx \frac{1}{4}\frac{1}{(\phi-\phi_0)^2}. \tag{4.25}$$

Thus, the geometry is defined only at $\phi > \phi_0$.

Note that our exact solution for the metric warp factor (4.19) has some qualitative similarity with the "classical" warp factor (3.31) obtained by integration out massive n and ρ fields in the $\mathbb{WCP}(N,N)$ model. Namely, the classical warp factor (3.31) also has a singularity at $\phi = \phi_0$ if we take $Y = \pi Q$; however, the type of the singularity is different. Solving gravity equations of motion allows us to find a true quantum string vacuum which arises due to the mass deformation.

The deformed $\mathcal{N} = 2$ Liouville theory (4.23) has two scales. The first one is associated with the Liouville wall [the superpotential (3.11)] which prevents field ϕ from penetrating to the region of large negative values. The Liouville interaction becomes of order of unity at

$$\phi_{\text{wall}} \sim -Q \ln\frac{1}{|b|}, \tag{4.26}$$

where we used (3.15) for $K = 2$. The Liouville wall prevents ϕ from penetrating far below this value.

The second scale is associated with the singularity of the target space metric at $\phi = \phi_0$.

As we start our interpolating process at $M \to \infty$, $\phi_0 \to -\infty$ and is much smaller then ϕ_{wall}, $\phi_0 \ll \phi_{\text{wall}}$ so the geometry is almost flat in the allowed region of ϕ. The string spectrum associated with the Liouville world sheet theory found in [13] describes hadrons of $\mathcal{N} = 2$ SQCD with U(2) gauge group and $N_f = 4$ quark flavors. As the mass M reduces the geometry gets deformed and at $\phi_0 \sim \phi_{\text{wall}}$ ($M^2 \sim 1/|b|^2$) we expect a transition to the region of small M.

In the opposite limit $\phi_0 \gg \phi_{\text{wall}}$ in the region of small M the effect of the Liouville wall can be neglected and the string background given by (4.19) and (4.20) determines the string spectrum. In this limit our string theory is expected to describe hadrons of $\mathcal{N} = 2$ SQCD with U(4) gauge group and $N_f = 8$ quark flavors.

In the next section we take a first glance at the string spectrum leaving its detail study for a future work.

V. A FIRST GLANCE AT THE STRING SPECTRUM

In this section we develop an effective gravity approach which can be used to study the string spectrum associated with the mass-deformed $\mathcal{N} = 2$ Liouville world sheet theory (4.23). In particular, we show that massless 4D baryon b survives the mass deformation.

A. Tachyon equation

Primary tachyon vertex operators (3.16) can be described as scalar fields in the effective supergravity (4.1).[8] To take them into account we add the tachyonic term,

$$S_{\text{tachyon}} = \frac{1}{2\kappa^2}\int d^D x\sqrt{-G}e^{-2\Phi} \times \{-G^{MN}\partial_M \bar{T}_{j,m}\partial_N T_{j,m} + |T_{j,m}|^2\}, \tag{5.1}$$

to the gravity action (4.1), cf. [55]. This gives the equation for the tachyon field,

$$D_N D^N T_{j,m} - 2\partial_N \Phi \partial^N T_{j,m} + T_{j,m} = 0, \tag{5.2}$$

and we neglect the backreaction of tachyons on the metric and the dilaton.

Dressing the tachyon with the dependence on the 4D coordinates

$$e^{ip_\mu x^\mu} T_{j,m}, \tag{5.3}$$

we rewrite the tachyon equation (5.2) in the form,

$$\frac{4}{g}\{\partial_s\partial_{\bar{s}}T_{j,m} - \partial_s\Phi\partial_{\bar{s}}T_{j,m} - \partial_{\bar{s}}\Phi\partial_s T_{j,m}\} + \left(1+\frac{M_{4\text{D}}^2}{4\pi T}\right)T_{j,m} = 0, \tag{5.4}$$

where we used complex coordinates (4.5). Here the mass squared of the physical state in 4D is

$$M_{4\text{D}}^2 = -p_\mu p^\mu, \tag{5.5}$$

where the Minkowski 4D metric with the diagonal entries $(-1, 1, 1, 1)$ is used.

To conclude this subsection let us solve Eq. (5.4) in the limit $M \to \infty$ when $g = 1$ and the dilaton is given by (3.9). In this limit the solution of the equation (5.4) can be written in the form,

$$T_{j,m} = e^{Q[j\phi + imY]} = e^{Q[J_+ s + J_- \bar{s}]}, \tag{5.6}$$

where

[8] These states are, of course, not tachyonic in 4D, but we will use the standard terminology and refer to them as "tachyons".

$$J_+ = \frac{1}{2}(j+m), \quad J_- = \frac{1}{2}(j-m) \tag{5.7}$$

and we consider, say, the momentum modes along the compact dimension, $m \equiv m_L = -m_R$ for definiteness. Substituting this to Eq. (5.4) gives

$$\frac{M_{4D}^2}{4\pi T} = 2\Delta_{j,m} - 1, \tag{5.8}$$

where the conformal dimension $\Delta_{j,m}$ is given by (3.21). This coincides with the result in (3.28) obtained by world sheet methods, see Ref. [13] for details.

Note that the Liouville interaction (3.11) is not taken into account in the tachyon action (5.1). Therefore, we cannot find the spectrum of allowed values of j [see Eqs. (3.25) and (3.26)] from Eq. (5.4). This spectrum is determined by the reflection from the Liouville wall.

Note also that the result (5.8) together with the expression (3.21) for the conformal dimension of $T_{j,m}$ is exact. To see this consider the region of large s, $s \to \infty$ in Eq. (5.4) where $g \to 1$. In this region we can look for the solution for $T_{j,m}$ in the form (5.6) and repeat the same derivation as above to get (5.8). Of course, at finite ϕ, where g becomes nontrivial, the expression for $T_{j,m}$ gets modified. Moreover, we expect that the spectrum of allowed values of j is also modified in the mass-deformed theory and depends on $(\phi_{\rm wall} - \phi_0)$.

B. Massless *b*-baryon

In this subsection we show that the massless in 4D *b*-baryon associated with the complex structure modulus of the conifold survives the mass deformation. In terms of the Liouville theory it corresponds to the tachyon $T_{j,m}$ with $j = -\frac{1}{2}$ $m = \pm\frac{1}{2}$. Let us consider the case $m = -\frac{1}{2}$ for definiteness. The $J_- = 0$ for this state and it is described by a holomorphic function of s at least in the limit $M \to \infty$. In other words it is a chiral primary field.

Let us extrapolate this property to arbitrary M and look for the holomorphic solution $T_b(s) \equiv T_{j=m=-1/2}(s)$. Equation (5.4) reads in this case,

$$T_b - \frac{4}{g}\partial_{\bar{s}}\Phi\partial_s T_b = 0, \tag{5.9}$$

where we put the 4D mass M_{4D} to zero. Calculating $\partial_{\bar{s}}\Phi$ using (4.13) and (4.17) we get

$$\partial_{\bar{s}}\Phi = -\frac{Q}{4}g. \tag{5.10}$$

Then Eq. (5.9) takes the simple form,

$$T_b + Q\partial_s T_b = 0. \tag{5.11}$$

Observe now, that the metric warp factor disappeared from Eq. (5.11). Thus, its solution is the same as in the undeformed theory. Namely, we have

$$T_b = e^{-\frac{\phi+iY}{Q}}. \tag{5.12}$$

It coincides with the vertex $T_{j,m}$ in (5.6) with $j = m = -\frac{1}{2}$ for $Q = \sqrt{2}$. The case of $m = \frac{1}{2}$ corresponds to the complex conjugate of T_b.

Let us check the normalization of the *b*-baryon state over the Liouville dimension. Calculating its wave function (3.18) we get for $Q = \sqrt{2}$,

$$\Psi_b = e^{-\Phi}e^{-\frac{\phi+iY}{Q}} = \frac{1}{\sqrt{g}}e^{-\frac{iY}{Q}}, \quad |\Psi_b|^2 = \frac{1}{g}, \tag{5.13}$$

where we used (4.13). Then its norm is

$$\int d\phi dY g|\Psi|^2 = 2\pi Q\int d\phi, \tag{5.14}$$

where the factor g arises due to the square root of the determinant of the metric.

Thus, this state is on the borderline between normalizable and non-normalizable states much in the same way as in the undefomed theory, see Sec. III B. In terms of the conifold radial coordinate this corresponds to the logarithmically normalized state, see Eq. (2.22) and [13] for details.

We see that massless 4D baryon survives the mass deformation and is present in the 4D SQCD at all values of mass M. The (dressed) tachyon operator $T_{j=-\frac{1}{2},m=\pm\frac{1}{2}}$ describes two scalar components of the BPS hypermultiplet in the 4D $\mathcal{N} = 2$ SQCD [21]. This leads us to the conclusion that the transition between regions of large and small M is a smooth crossover rather then a sharp phase transition. The BPS state is not affected (so analyticity is preserved), while the spectrum of non-BPS states associated with $T_{j,m} \neq T_b$ is expected to change as a function of $(\phi_{\rm wall} - \phi_0)$.

Another related property of the solution (5.12) is that the conformal dimension of the operator in (5.12) is equal to $1/2$ and therefore, as we mentioned in the end of Sec. IV B, the Liouville superpotential (3.11) is not modified and remains a marginal deformation of the mass-deformed Liouville theory, see Eq. (4.23).

C. Schrödinger equation

In this section we rewrite the tachyon equation (5.4) in the form of the Schrödinger equation. Substituting (5.10) into (5.4) we get

$$\frac{4}{g}\partial_s\partial_{\bar{s}}T_{j,m} + Q(\partial_{\bar{s}}T_{j,m} + \partial_s T_{j,m}) + 2\Delta_{j,m}T_{j,m} = 0. \tag{5.15}$$

To get rid of terms with first derivatives we write,

$$T_{j,m} = e^{\Phi}\tilde{\Psi}_{j,m}, \tag{5.16}$$

where the dilaton Φ is given by (4.13). With this substitution the equation for $\tilde{\Psi}$ reads,

$$-(\partial_\phi^2+\partial_Y^2)\tilde{\Psi}_{j,m}+Q^2 g\left[\left(j+\frac{1}{2}\right)^2-m^2-\frac{1}{4}(g-1)\right]\tilde{\Psi}_{j,m}=0. \tag{5.17}$$

Since the warp factor g does not depend on Y we can look for solutions of (5.17) using the ansatz

$$\tilde{\Psi}_{j,m}(\phi,Y)=e^{iQmY}\Psi_{j,m}(\phi). \tag{5.18}$$

This gives the Schrödinger equation for the wave function $\Psi_{j,m}(\phi)$,

$$-\partial_\phi^2\Psi_{j,m}+V_{\rm eff}(\phi)\Psi_{j,m}=E_j\Psi_{j,m}, \tag{5.19}$$

where the potential is given by

$$V_{\rm eff}(\phi)=-Q^2(g-1)\left[m^2-\left(j+\frac{1}{2}\right)^2+\frac{g}{4}\right], \tag{5.20}$$

while energy levels are determined by j,

$$E_j=-Q^2\left(j+\frac{1}{2}\right)^2. \tag{5.21}$$

As we already mentioned in Sec. VA the Liouville interaction (3.11) is not taken into account in the tachyon action (5.1), therefore we cannot determine energy levels and the spectrum of allowed j from this equation in the region of large M, where the Liouville interaction is essential. Instead we can use it in the region of small M at $\phi_{\rm wall}\ll\phi_0$ or $M^2\ll 1/|b|^2$. In this region the Liouville interaction can be neglected and the energy levels and the spectrum of allowed j can be found solving Eq. (5.19). This will give the 4D mass spectrum via Eq. (5.8) which we interpret as a spectrum of hadrons in $\mathcal{N}=2$ SQCD with gauge group U(4) and $N_f=8$.

The potential (5.20) is attractive for $[m^2-(j+\frac{1}{2})^2+\frac{1}{4}]>0$ and tends to zero at $\phi\to\infty$. Therefore, one may expect the continues spectrum with $j=-\frac{1}{2}+is$ [see Eq. (3.26)] with positive E_j and the discrete spectrum with negative E_j.[9] However, the problem turns out to be more complicated because near the singularity at $\phi\to\phi_0$ the potential (5.20) is of the Calogero type [56] with the "falling to the center" behavior,

$$V_{\rm eff}(\phi)|_{\phi\to\phi_0}\approx\frac{\alpha}{(\phi-\phi_0)^2},\quad \alpha=-\frac{1}{4}, \tag{5.22}$$

where we used (4.24).

The Hamiltonian with this potential has the scale invariance and therefore seems to have no discrete spectrum. However, the accurate definition of what is the self-adjoint Hamiltonian leads to a well-defined setup of the Calogero problem [57]. It turns out that the spectrum crucially depends on the coefficient α in front of $1/(\phi-\phi_0)^2$. For example "falling to the center" occurs at $\alpha<-\frac{1}{4}$ when the discrete spectrum is not bounded from below. The coefficient $\alpha=-\frac{1}{4}$ [see Eq. (5.22)] represents a very special case. In this case there is only one discrete level [57].

Thus, we expect that our Schrödinger equation (5.19) has exactly one discrete level for each value of m allowed by the representation (3.17) and Gliozzi-Scherk-Olive projection. The detailed study of the string spectrum associated with this Calogero problem is left for a future work.

VI. CONCLUSIONS

In this paper we considered the mass deformation of the string theory for the critical non-Abelian vortex supported in $\mathcal{N}=2$ SQCD with gauge group U(2) and $N_f=4$ quark flavors. Our mass deformation interpolates in four dimensions between the above mentioned theory and $\mathcal{N}=2$ SQCD with gauge group U(4) and $N_f=8$ quark flavors. Building on previous results that the Coulomb branch of the world sheet theory for the critical non-Abelian string in $\mathcal{N}=2$ SQCD with gauge group U(2) and $N_f=4$ flavors is described by $\mathcal{N}=2$ Liouville theory we switch on the quark mass parameter M and study the mass deformation of the Liouville theory, which boils down to the M-dependent metric of its target space and the M-dependent dilaton.

To find the mass-deformed metric and the dilaton for the true string vacuum we solve the effective supergravity equations of motion. The solution shows the presence of the naked singularity of the metric. Nevertheless, we show that the massless b-baryon associated with the deformation of the complex structure of the conifold does not "feel" the metric deformation and remains massless in the mass-deformed theory.

Next we present the Schrödinger equation for tachyon vertex operators which at $j\leq-\frac{1}{2}$ describes normalizable and borderline normalizable string states. These states correspond to hadrons living in 4D $\mathcal{N}=2$ SQCD. We give a qualitative discussion of the structure of the mass spectrum of tachyon states. In particular, we show that in the region of small M finding the string spectrum is linked to the Calogero problem.

As a directions of future research we can mention the detail study of the string spectrum and its dependence on the mass parameter M. In particular, in the limit of small M

[9] Note that we are looking for the spectrum of normalizable and borderline normalizable states.

this spectrum gives the mass spectrum of hadrons in 4D $\mathcal{N} = 2$ SQCD with gauge group U(4) and $N_f = 8$ flavors of quarks.

Another challenging problem is to understand the physical nature of the naked singularity of the Liouville target space metric and its possible resolution.

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