Geometric perfect fluids and the dark side of the Universe

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Recently we showed that in Friedman-Lemaître-Robertson-Walker (FLRW) cosmology, the contribution from higher curvature terms in any generic metric gravity theory to the energy-momentum tensor is of the perfect fluid form. Such a geometric perfect fluid can be interpreted as a fluid remaining from the beginning of the Universe where the string theory is thought to be effective. Just a short time after the beginning of the Universe, it is known that the Einstein-Hilbert action is assumed to be modified by adding all possible curvature invariants. We propose that the observed late-time accelerating expansion of the Universe can be solely driven by this geometric fluid. To support our claim, we specifically study the quadratic gravity field equations in D dimensions. We show that the field equations of this theory for the FLRW metric possess a geometric perfect fluid source containing two critical parameters σ_1 and σ_2. To analyze this theory concerning its parameter space (σ_1, σ_2), we obtain the general second-order nonlinear differential equation governing the late-time dynamics of the deceleration parameter q. Hence, using some present-day cosmological data as our initial conditions, our findings for the $\sigma_2 = 0$ case are as follows: (i) To have a positive energy density for the geometric fluid ρ_g, the parameter σ_1 must be negative for all dimensions up to $D = 11$. (ii) For a suitable choice of σ_1, the deceleration parameter experiences signature changes in the past and future, and in the meantime it lies within a negative range, which means that the current observed accelerated expansion phase of the Universe can be driven solely by the curvature of the spacetime. (iii) q experiences a signature change, and as the dimension D of spacetime increases, this signature change happens at earlier and later times, in the past and future, respectively.

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I. INTRODUCTION

Although general relativity (GR) has been immensely successful in explaining a wide range of gravitational phenomena, there are certain observations that have motivated researchers to consider modifications to the theory. Two of these motivations are the following: (i) Theoretical consistency of GR: Modifications to gravity theories can arise from attempts to reconcile GR with other fundamental theories, such as quantum gravity or string theory. GR is assumed to be a low-energy approximation of a more complete theory where the effective action includes higher-curvature terms in addition to the usual Einstein-Hilbert term. Hence, modified gravity can be seen as an exploration of how gravity might behave at very low- or high-energy limits where the effects of quantum physics become significant. See, for instance, [1–5]. (ii) Dark matter and dark energy: The need for dark matter and dark energy to explain the observed motion of galaxies and the accelerated expansion of the universe, respectively, has led to questions about whether our understanding of gravity is complete. Modified gravity theories also seek to address these phenomena without introducing the need for dark matter or dark energy [6–19].

Higher-order curvature corrections to Einstein's field equations have been considered by many authors [20–29]. Recently we showed that, in Friedman-Lemaître-Robertson-Walker (FLRW) cosmology, the contribution of higher curvature terms in any generic theory of gravity to the energy-momentum tensor is of the perfect fluid form [30]. This is the reason that some authors have observed this fact [31–33] and verified it in some particular modified gravity theories such as $f(R)$ gravity [34,35], Gauss-Bonnet gravity [36], and other higher-order gravities [37,38]. In [30], the cases of general Lovelock and $F(R, G)$ theories are given as examples.

The FLRW metric is the most known and most studied metric in general relativity. This metric is mainly used to describe the Universe as a homogeneous, isotropic fluid distribution [39]. It is known that FLRW cosmology in Einstein's theory is not sufficient to explain the accelerating expansion of the universe. To explain this phenomena,

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staying in Einstein theory it is claimed that, in addition to the known ordinary matter distribution, the Universe should contain a dark component driving the accelerating expansion, the so called dark energy. If we consider the low-energy limit of the string theory, then the Einstein equations are modified by adding all possible curvature invariants. We call such a theory a generic gravity where the action takes the form

$$I = \int d^D x \sqrt{-g}\left[\frac{1}{\kappa}(R - 2\Lambda) + \mathcal{F}(g, \mathrm{Riem}, \nabla\mathrm{Riem}, \nabla\nabla\mathrm{Riem}, \ldots) + \mathcal{L}_M\right], \tag{1}$$

and here $\mathcal{F}$ is a function of all combinations of the metric tensor, curvature tensor, and the covariant derivatives of the curvature tensor of any order. Then the field equations take the form

$$\frac{1}{\kappa}(G_{\mu\nu} + \Lambda g_{\mu\nu}) + \mathcal{E}_{\mu\nu} = T^M_{\mu\nu}, \tag{2}$$

where $G_{\mu\nu}$ is the Einstein tensor, Λ is the cosmological constant, $T^M_{\mu\nu}$ is the matter energy-momentum tensor of perfect fluid distribution (with the energy density ρ and pressure p), and $\mathcal{E}_{\mu\nu}$ resulting from the higher-order curvature terms is any second rank tensor obtained from the metric tensor, Riemann tensor, Ricci tensor, Ricci scalar, and their covariant derivatives at any order. In [30], we have shown that $\mathcal{E}_{\mu\nu}$ can be written as the combination of the metric tensor $g_{\mu\nu}$ and $u_\mu u_\nu$; that is,

$$\mathcal{E}_{\mu\nu} = A g_{\mu\nu} + B u_\mu u_\nu, \tag{3}$$

where A and B are functions of scale factor $a(t)$ and its derivatives with respect to time t. This implies that in the Einstein field equations, in addition to the matter fluid energy-momentum tensor $T^M_{\mu\nu}$, there exists another fluid distribution which we call the geometric fluid distribution $T^g_{\mu\nu}$ with the energy density $\rho_g = A - B$ and pressure $p_g = -A$. Here, we adopt the idea that the source of the dark matter/energy is the geometrical fluid distribution. Hence, we conjecture that higher curvature modifications of the Einstein theory are complete in the sense that all cosmological observations can be explained by choosing appropriate modified theories studied by several authors in [20–29]. To support our claim, we specifically study the quadratic gravity field equations in D dimensions. We show that the field equations of this theory for the FLRW metric possess a geometric perfect fluid source containing two critical parameters σ_1 and σ_2. To analyze this theory concerning its parameter space (σ_1, σ_2), we obtain the general second-order nonlinear differential equation governing the late-time dynamics of the deceleration parameter q. Hence, using some present-day cosmological data as our initial conditions, our findings for the $\sigma_2 = 0$ case are as follows: (i) To have a positive energy density for the geometric fluid ρ_g, the parameter σ_1 must be negative for all dimensions up to $D = 11$. (ii) For a suitable choice of σ_1, the deceleration parameter experiences signature changes in the past and future, and in the meantime it lies within a negative range, which means that the current observed accelerated expansion phase of the Universe can be driven solely by the curvature of the spacetime. (iii) q experiences a signature change, and as the dimension D of spacetime increases, this signature change happens at earlier and later times, in the past and future, respectively. (iv) The geometric equation of state parameter w_g is negative for all integers $4 \leq D \leq 10$, specifically, for $D = 4$ and $w_g = -0.85$. For $\sigma_1 = 0$ (critical gravity), we find that there are two cases, both representing a possibility of having an accelerating expansion. Furthermore, we present some particular cosmological solutions in quadratic gravity depending upon the parameters σ_1 and σ_2.

It is well-known that linearized versions of most higher time-derivative theories suffer from the Ostragradsky instability (see, for instance, [40,41]). One way to avoid this instability is to consider such theories as low-energy approximations to a more fundamental theory, such as string theory. Namely, at the scales where negative norm states appear, the theory is expected to be replaced by a better-behaved UV theory. Another possibility is Weinberg's asymptotically safe gravity [42] where there are infinitely many powers of curvature and the negative norm states appear only in the truncated, perturbative version of the theory and disappear in the nonperturbative formulation where all the coupling constants, i.e., all higher derivative curvature terms, are taken into account. In [43], it is also noted that Ostrogradskian ghosts in higher-derivative gravity theories (generic gravity theories) are only apparent when one truncates the infinite series of curvature invariants, and hence, these ghosts can be removed by means of a suitable boundary condition. Furthermore, the absence of the Ostrogradsky instability manifests itself in theories with multiple fields; for example, in [44], the authors discuss that, in the extended-scalar-tensor class of theories for which the tensors are well-behaved and the scalar is free from gradient or ghost instabilities on FLRW spacetimes, one recovers the Horndeski theory up to field redefinitions. The general theorem introduced in the present paper addresses all theories that might be the low-energy limit of string theory, where the Einstein-Hilbert action and hence the field equations are modified by adding all possible curvature invariants. Based on this reason, we call such a theory as generic gravity. All our treatments such as proving our main theorem and all other derivations are nonperturbative. There is no truncation, and hence, the Ostragradsky instabilities are not in the scope of the present work.

In Sec. II, we summarize our theorem for the D-dimensional FLRW metrics. For illustration, we shall study the quadratic gravity theory in detail.

II. GENERIC GRAVITY FIELD EQUATIONS IN PERFECT FLUID FORM

Using the covariant decomposition, one can write the D-dimensional FLRW metric as

$$g_{\mu\nu} = -u_\mu u_\nu + a^2 h_{\mu\nu}, \tag{4}$$

where $\mu, \nu = 0 \cdots D-1$, $a = a(t)$, $u_\mu = \delta^0_\mu$, and $h_{\mu\nu}$ reads as

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & h_{ij} & \\ 0 & & & \end{pmatrix}, \tag{5}$$

where $h_{ij} = h_{ij}(x^a)$ is the metric of the spatial section of the spacetime possessing the constant curvature k, and $i, j = 1, \ldots, D-1$. One notes that

$$u^\mu h_{\mu\nu} = u_\mu h^{\mu\nu} = 0, \qquad h^\mu_\alpha = h^{\mu\alpha} h_{\alpha\nu} = \delta^\mu_\nu + u^\mu u_\nu. \tag{6}$$

The corresponding Christoffel symbols can be obtained as

$$\Gamma^\mu_{\alpha\beta} = \gamma^\mu_{\alpha\beta} - a\dot{a} u^\mu h_{\alpha\beta} + H(2u_\alpha u^\mu u_\beta + u_\beta \delta^\mu_\alpha + u_\alpha \delta^\mu_\beta), \tag{7}$$

where the dot sign represents the derivative with respect to time t, $H = \dot{a}/a$, and $\gamma^\mu_{\alpha\beta}$ is defined as

$$\gamma^\mu_{\alpha\beta} = \frac{1}{2} a^2 h^{\mu\gamma} (h_{\gamma\alpha,\beta} + h_{\gamma\beta,\alpha} - h_{\alpha\beta,\gamma}). \tag{8}$$

The Riemann curvature tensor can be written in the following linear form in terms of the metric $g_{\alpha\gamma}$ and the four vectors u_α and u^μ,

$$R^\mu_{\alpha\beta\gamma} = [\delta^\mu_\beta g_{\alpha\gamma} - \delta^\mu_\gamma g_{\alpha\beta}]\rho_1 + [u^\mu (g_{\alpha\gamma} u_\beta - g_{\alpha\beta} u_\gamma) - u_\alpha (\delta^\mu_\gamma u_\beta - \delta^\mu_\beta u_\gamma)]\rho_2, \tag{9}$$

where ρ_1 and ρ_2 are defined as

$$\rho_1 = H^2 + \frac{k}{a^2}, \tag{10}$$

$$\rho_2 = H^2 + \frac{k}{a^2} - \frac{\ddot{a}}{a} = \dot{H} + \frac{k}{a^2}. \tag{11}$$

The contractions of the Riemann tensor (9) gives the Ricci tensor and Ricci scalar, respectively, as

$$R_{\mu\nu} = P g_{\mu\nu} + Q u_\mu u_\nu, \qquad R = DP - Q, \tag{12}$$

where

$$P = (D-1)\rho_1 - \rho_2, \tag{13}$$

$$Q = (D-2)\rho_2. \tag{14}$$

One can also verify that the Weyl tensor

$$C^\mu_{\alpha\beta\gamma} = R^\mu_{\alpha\beta\gamma} + \frac{1}{D-2}(\delta^\mu_\gamma R_{\alpha\beta} - \delta^\mu_\beta R_{\alpha\gamma} + g_{\alpha\beta} R^\mu_\gamma - g_{\alpha\gamma} R^\mu_\beta) + \frac{1}{(D-1)(D-2)}(\delta^\mu_\beta g_{\alpha\gamma} - \delta^\mu_\gamma g_{\alpha\beta})R \tag{15}$$

vanishes for the metric (4). Since the conformal tensor is zero, the curvature tensor is expressed in terms of the Ricci tensor. This means that, for the FLRW spacetime, the basic geometrical tensors are the metric and Ricci tensors. The covariant derivatives of the Ricci tensor are given by[1]

$$R_{\mu\nu;\alpha} = \dot{P} u_\alpha g_{\mu\nu} - QH(u_\nu g_{\mu\alpha} + u_\mu g_{\nu\alpha}) + (\dot{Q} - 2QH) u_\mu u_\nu u_\alpha, \tag{16}$$

$$\Box R_{\mu\nu} = -[\ddot{P} + (D-1)H\dot{P} - 2QH^2] g_{\mu\nu} + [2DQH^2 - \ddot{Q} - (D-1)H\dot{Q}] u_\mu u_\nu, \tag{17}$$

$$\Box R = -D\ddot{P} - D(D-1)H\dot{P} + \ddot{Q} + (D-1)H\dot{Q}. \tag{18}$$

The field equations of any generic gravity theory in D dimensions with the action, together with the matter fields,

$$I = \int d^D x \sqrt{-g} \left[\frac{1}{\kappa}(R - 2\Lambda) + \mathcal{F}(g, \mathrm{Riem}, \nabla\mathrm{Riem}, \nabla\nabla\mathrm{Riem}, \ldots) + \mathcal{L}_M \right], \tag{19}$$

take the form

$$\frac{1}{\kappa}(G_{\mu\nu} + \Lambda g_{\mu\nu}) + \mathcal{E}_{\mu\nu} = T^M_{\mu\nu}, \tag{20}$$

[1]Equations (16)–(18) are from [30] [Eqs. (18) and (19)], here with corrected typos.

where $G_{\mu\nu}$ is the Einstein tensor, Λ is the cosmological constant, $T^M_{\mu\nu}$ is the energy-momentum tensor coming from the matter fields denoted by $\mathcal{L}_M$, and $\mathcal{E}_{\mu\nu}$ resulting from the higher-order curvature terms contained in the function $\mathcal{F}$ is any second rank tensor obtained from the metric tensor, Riemann tensor, Ricci tensor, and their covariant derivatives at any order. Hence, we arrive at the following theorem [30]:

Theorem 1. For the D-dimensional FLRW spacetimes given in (4), any second rank symmetric tensor obtained from the metric tensor, Riemann tensor, Ricci tensor, and their covariant derivatives at any order becomes a combination of the metric tensor $g_{\mu\nu}$ and $u_\mu u_\nu$; that is,

$$\mathcal{E}_{\mu\nu} = A g_{\mu\nu} + B u_\mu u_\nu, \tag{21}$$

where A and B are functions of the scale factor $a(t)$ and its time derivatives at any order and they depend on the underlying gravity theory.

And we have the following corollary of this theorem:

Corollary 2. The field equations of any generic gravity theory given in (20) take the form

$$\frac{1}{\kappa}(G_{\mu\nu} + \Lambda g_{\mu\nu}) = T^M_{\mu\nu} + T^g_{\mu\nu}, \tag{22}$$

where $T^M_{\mu\nu}$ is the energy-momentum tensor of perfect fluid distribution representing the baryonic matter fields,

$$T^M_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}, \tag{23}$$

with ρ and p being, respectively, the energy density and pressure of the fluid, and $T^g_{\mu\nu}$ is the tensor coming from the higher-order curvature terms in (19), which is also in the perfect fluid form,

$$T^g_{\mu\nu} = (\rho_g + p_g)u_\mu u_\nu + p_g g_{\mu\nu}, \tag{24}$$

with $\rho_g = A - B$ and $p_g = -A$, due to (21). Hence, defining an effective energy-momentum tensor as

$$T^{\text{eff}}_{\mu\nu} \equiv T^M_{\mu\nu} + T^g_{\mu\nu} = (\rho_{\text{eff}} + p_{\text{eff}})u_\mu u_\nu + p_{\text{eff}} g_{\mu\nu}, \tag{25}$$

the generic gravity field equations (22) for the FLRW spacetime with a perfect fluid source become

$$\frac{1}{\kappa}\left[\frac{(D-1)(D-2)}{2}\rho_1 - \Lambda\right] = \rho + A - B \equiv \rho_{\text{eff}}, \tag{26}$$

$$-\frac{1}{\kappa}\left[\frac{(D-1)(D-2)}{2}\rho_1 - (D-2)\rho_2 - \Lambda\right] = p - A \equiv p_{\text{eff}}. \tag{27}$$

Thus, the interpretations of the functions A and B appearing in the above formulation can be given as follows: the combination "$A - B$" is the energy density and "$-A$" is the pressure of a perfect fluid of purely geometric origin. The functions A and B differ in different modified theories. In each modification it is possible to arrange parameters of the theories to meet the observations. In particular, we shall analyze the quadratic gravity theory in D dimensions in Sec. IV and show that these purely geometric terms solely drive the late-time accelerating expansion of the Universe consistently with the current observations. In the next section, we shall give the cosmological parameters for generic gravity theories described by the action in (19) in D dimensions.

III. COSMOLOGICAL PARAMETERS IN GENERIC GRAVITY THEORIES

Using (10) and (26), one can obtain

$$\begin{aligned} H^2 &= \frac{2\kappa}{(D-1)(D-2)}\left[\rho + A - B + \frac{\Lambda}{\kappa}\right] - \frac{k}{a^2} \\ &= \frac{2\kappa}{(D-1)(D-2)}\sum_i \rho_i - \frac{k}{a^2}, \end{aligned} \tag{28}$$

where the label i denotes m, r, Λ, or g representing matter, radiation, cosmological constant (or dark energy), and dark geometric fluid, respectively. Defining the corresponding dimensionless density parameter for each of the mentioned components of the Universe as

$$\Omega_i = \frac{\rho_i}{\rho_{\text{crit}}} \quad \text{with} \quad \rho_{\text{crit}} = \frac{(D-1)(D-2)H^2}{2\kappa}, \tag{29}$$

we can write (28) as

$$1 = \Omega_m + \Omega_r + \Omega_\Lambda + \Omega_g + \Omega_k, \tag{30}$$

where

$$\begin{aligned} &\Omega_m = \frac{\rho_m}{\rho_{\text{crit}}}, \qquad \Omega_r = \frac{\rho_r}{\rho_{\text{crit}}}, \qquad \Omega_\Lambda = \frac{\rho_\Lambda}{\rho_{\text{crit}}}, \\ &\Omega_g = \frac{A-B}{\rho_{\text{crit}}}, \qquad \Omega_k = -\frac{k}{a^2H^2}, \end{aligned} \tag{31}$$

and note that $\rho_\Lambda = \Lambda/\kappa$. The first observation here is the contribution of the dark geometric fluid in determining the spatial curvature of the Universe. One observes that

$$\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_g < 1 \Leftrightarrow \text{open universe } (k = -1), \tag{32}$$

$$\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_g = 1 \Leftrightarrow \text{flat universe } (k = 0), \tag{33}$$

$$\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_g > 1 \Leftrightarrow \text{closed universe } (k = 1). \tag{34}$$

On the other hand, using (10), (11), and (28), Eq. (27) can be written equivalently as

$$-\frac{\ddot{a}}{aH^2}=\frac{\kappa}{(D-1)(D-2)H^2}\times\left[(D-3)\rho+(D-1)p-2A-B-2\frac{\Lambda}{\kappa}\right]. \quad (35)$$

Now, defining the left-hand side as the deceleration parameter q and considering the barotropic equation of state $p_j=\omega_j\rho_j$, where $j=(m,r)$, for the matter ($\omega_m=0$) and the radiation ($\omega_r=1/3$), Eq. (35) can be rewritten, with the help of (29), as

$$\begin{aligned}q&\equiv-\frac{\ddot{a}}{aH^2}\\&=\frac{1}{2}\left[\sum_j[(D-3)+(D-1)w_j]\Omega_j-2\Omega_\Lambda-\Omega_g^*\right]\\&=\frac{1}{2}\left[(D-3)\Omega_m+\frac{2}{3}(2D-5)\Omega_r-2\Omega_\Lambda-\Omega_g^*\right], \quad (36)\end{aligned}$$

where

$$\Omega_g^*=\frac{2A+B}{\rho_{\rm crit}}. \quad (37)$$

Here we see that the geometric fluid has a negative contribution to the deceleration (or, equivalently, positive contribution to the acceleration) of the Universe if $2A+B>0$. It can be observed that, setting $D=4$ and neglecting higher curvature modifications, one recovers the deceleration parameter in GR as

$$q=\frac{1}{2}(\Omega_m+2\Omega_r-2\Omega_\Lambda). \quad (38)$$

At this point, before proceeding further, it is appropriate to introduce the cosmological parameters

$$j(t)\equiv\frac{\dddot{a}}{aH^3}, \qquad s(t)\equiv\frac{\ddddot{a}}{aH^4}, \quad (39)$$

which are called "jerk" and "snap," respectively. These parameters, together with H and q, are defined by expanding the scale factor in a Taylor series in the vicinity of the current time t_0 [45]:

$$\frac{a(t)}{a_0}=1+H_0(t-t_0)-\frac{1}{2}q_0H_0^2(t-t_0)^2+\frac{1}{3!}j_0H_0^3(t-t_0)^3+\frac{1}{4!}s_0H_0^4(t-t_0)^4+O((t-t_0)^5), \quad (40)$$

where H_0, q_0, j_0, and s_0 are the present-day values of the Hubble, deceleration, jerk, and snap parameters and they can be used to determine the evolutionary behavior of the Universe.

IV. QUADRATIC GRAVITY AND CRITICALITY IN D DIMENSIONS

The action of the quadratic gravity theory [46,47] is

$$I=\int d^Dx\sqrt{-g}\left[\frac{1}{\kappa}(R-2\Lambda)+\alpha R^2+\beta R_{\mu\nu}^2+\gamma(R_{\mu\nu\sigma\rho}^2-4R_{\mu\nu}^2+R^2)+\mathcal{L}_M\right], \quad (41)$$

giving the field equations

$$\frac{1}{\kappa}\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\Lambda g_{\mu\nu}\right)+\mathcal{E}_{\mu\nu}=(\rho+p)u_\mu u_\nu+pg_{\mu\nu}, \quad (42)$$

where ρ and p are the energy density and pressure of the matter perfect fluid. Considering the FLRW metric (4), we find

$$\begin{aligned}\mathcal{E}_{\mu\nu}&\equiv2\alpha R\left(R_{\mu\nu}-\frac{1}{4}g_{\mu\nu}R\right)+(2\alpha+\beta)(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu)R\\&\quad+2\gamma\Big[RR_{\mu\nu}-2R_{\mu\sigma\nu\rho}R^{\sigma\rho}+R_{\mu\sigma\rho\tau}R_\nu{}^{\sigma\rho\tau}-2R_{\mu\sigma}R_\nu{}^\sigma\\&\quad-\frac{1}{4}g_{\mu\nu}(R_{\tau\lambda\sigma\rho}^2-4R_{\sigma\rho}^2+R^2)\Big]\\&\quad+\beta\Box\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)+2\beta\left(R_{\mu\sigma\nu\rho}-\frac{1}{4}g_{\mu\nu}R_{\sigma\rho}\right)R^{\sigma\rho}\\&=Ag_{\mu\nu}+Bu_\mu u_\nu, \quad (43)\end{aligned}$$

where A and B are given by

$$\begin{aligned}A&=\frac{\alpha}{2}[(D-1)(D\rho_1-2\rho_2)][-(D-1)(D-4)\rho_1+2(D-3)\rho_2]\\&\quad+(2\alpha+\beta)[(D-1)(D-2)H(-D\dot{\rho}_1+2\dot{\rho}_2)-D(D-1)\ddot{\rho}_1+2(D-1)\ddot{\rho}_2]\\&\quad+\frac{\gamma}{2}[(D-2)(D-3)(D-4)(-(D-1)\rho_1^2+4\rho_1\rho_2)]\\&\quad+\frac{\beta}{2}[(D-2)[(D-1)(\ddot{\rho}_1+(D-1)H\dot{\rho}_1)]-2(\ddot{\rho}_2+(D-1)H\dot{\rho}_2-2H^2\rho_2)]\\&\quad+\frac{\beta}{2}[-(D-1)(D-4)((D-1)\rho_1^2+\rho_2^2)+4(D^2-5D+5)\rho_1\rho_2], \quad (44)\end{aligned}$$

$$B = 2\alpha[(D-1)(D-2)(D\rho_1 - 2\rho_2)\rho_2] + (2\alpha+\beta)[(D-1)[H(-D\dot{\rho}_1 - 2\dot{\rho}_2) - (D\ddot{\rho}_1 - 2\ddot{\rho}_2)]] \\ + 2\gamma[(D-2)(D-3)(D-4)\rho_1\rho_2] - \beta[(D-2)[\ddot{\rho}_2 + (D-1)H\dot{\rho}_2 - 2DH^2\rho_2)]] \\ + 2\beta[(D-2)^2\rho_1\rho_2]. \tag{45}$$

Hence, we can write (42) as

$$\frac{1}{\kappa}\left[\frac{1}{2}(D-2)(D-1)\left(\frac{k}{a^2}+\frac{\dot{a}^2}{a^2}\right)-\Lambda\right] = \rho + \rho_{g_1} + \rho_{g_2} \equiv \rho_{\rm eff}, \tag{46}$$

$$-\frac{1}{\kappa}\left[\frac{(D-2)(D-3)}{2}\left(\frac{k}{a^2}+\frac{\dot{a}^2}{a^2}\right)+(D-2)\frac{\ddot{a}}{a}-\Lambda\right] = p + p_{g_1} + p_{g_2} \equiv p_{\rm eff}, \tag{47}$$

where $\rho_{g_1}, \rho_{g_2}, p_{g_1}$, and p_{g_2} have the geometric origin given by

$$\rho_{g_1} = -\frac{(D-1)\sigma_1}{2Da^4}\{k^2(D-2)^2 + \dot{a}^2[2D(D-3)a\ddot{a} - 4k(D-2)] - (D^2-4)\dot{a}^4 - Da^2\ddot{a}^2 + 2Da^2\dot{a}\,\dddot{a}\}, \tag{48}$$

$$\rho_{g_2} = -(D-1)\sigma_2\left(\frac{k}{a^2}+\frac{\dot{a}^2}{a^2}\right)^2, \tag{49}$$

$$p_{g_1} = -\frac{\sigma_1}{2Da^4}\{(D-2)(D-5)(D+2)\dot{a}^4 - k^2(D-2)^2(D-5) + a\ddot{a}(8k(D-2) - 3D(D-3)a\ddot{a}) \\ + 2\dot{a}^2[2k(D-2)(D-5) - ((D-9)D^2 + 12D + 8)a\ddot{a}] - 2D\ddddot{a}a^3 - 4(D-3)D\dddot{a}a^2\dot{a}\}, \tag{50}$$

$$p_{g_2} = \sigma_2\left(\frac{k}{a^2}+\frac{\dot{a}^2}{a^2}\right)\left[(D-5)\left(\frac{k}{a^2}+\frac{\dot{a}^2}{a^2}\right)+4\frac{\ddot{a}}{a}\right], \tag{51}$$

with

$$\sigma_1 = 4(D-1)\alpha + D\beta, \tag{52}$$

$$\sigma_2 = \frac{(D-2)(D-4)}{2D}[(D-1)(D-2)\alpha + D(D-3)\gamma]. \tag{53}$$

Remark 3. When the parameters σ_1 and σ_2 vanish together, the geometric contributions given in (48)–(51), resulting from the higher curvature terms in the action (42), vanish identically and Eqs. (46) and (47) reduce to the equations in pure Einstein's gravity. These so-called critical points, i.e., $\sigma_1 = 0$ and $\sigma_2 = 0$, were identified and studied in higher curvature gravity theories first in four dimensions (where σ_2 identically vanishes) in [48] and later in D dimensions in [49]. In these works, it is shown that the linearized excitations around these critical points have vanishing energies and the mass and corresponding entropy of the usual Schwarzschild-AdS black hole solution turn out to be zero at criticality.

Remark 4. It can also be observed that the above expressions are invariant under the scale transformations $a \to \eta a$ and $k \to \eta^2 k$, where η is a constant.

Remark 5. From the positiveness of ρ_{g_2} in (49) it follows that $\sigma_2 < 0$.

From Eqs. (46) and (47), we can also write

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{(D-1)(D-2)}\left[(D-3)\rho_{\rm eff} + (D-1)p_{\rm eff} - 2\frac{\Lambda}{\kappa}\right]. \tag{54}$$

Thus, when $\Lambda = 0$, to have a universe expanding in an accelerating fashion ($\ddot{a} > 0$), it must be that

$$w_{\rm eff} = \frac{p_{\rm eff}}{\rho_{\rm eff}} < -\frac{D-3}{D-1}. \tag{55}$$

Specifically, for $D = 4$, $w_{\rm eff} < -1/3$. Using (46), along with the definition of the Hubble parameter, $H = \dot{a}/a$, one can also write (54) as

$$\dot{H} = -\frac{\kappa}{D-2}(\rho_{\rm eff} + p_{\rm eff}) + \frac{k}{a^2}, \tag{56}$$

which is more convenient than (54) because it does not involve the cosmological constant Λ explicitly. This equation relates the acceleration of the Universe to its energy and momentum content. Together with the expressions (48)–(51), Eq. (56) becomes highly nonlinear, and therefore, it is not possible to give a compact analytical solution for $a(t)$. However, we can use (56) to investigate the late-time accelerated expansion of the Universe during which the higher curvature terms in the action (41) are assumed to be dominant. Indeed, recently it was observed that the Universe had entered into an accelerated expansion phase [50,51], and a possible cause of this late-time acceleration is the curvature of the spacetime itself related to the combination of the higher curvature terms that may appear in the action of a generic gravity theory. This observation implies that the acceleration, defined by $\ddot{a}(t)$, of the Universe has changed its sign from negative to positive, or, in other words, the deceleration parameter $q(t)$ has recently experienced a sign change from positive to negative. Therefore, for the purpose of investigating the late-time acceleration or deceleration behavior of the Universe, we shall work with the deceleration parameter $q(t)$, instead of $a(t)$, and convert (56) as a differential equation for $q(t)$ which can be solved numerically for given initial values of q and its derivatives. Let us first write $\rho_{\rm eff}$ and $p_{\rm eff}$ in terms of the Hubble (H), deceleration (q), jerk (j), and snap (s) parameters. From (48)–(51),

$$\rho_{\rm eff}=\rho-\frac{\sigma_1(D-1)}{2D}\left[\frac{k^2(D-2)^2}{a^4}-4\frac{k}{a^2}(D-2)H^2-2D(D-3)qH^4-(D^2-4)H^4-Dq^2H^4+2DjH^4\right]$$
$$-\sigma_2(D-1)\left(\frac{k}{a^2}+H^2\right)^2, \tag{57}$$

$$p_{\rm eff}=p-\frac{\sigma_1}{2D}\left[-\frac{k^2}{a^4}(D-2)^2(D-5)+4\frac{k}{a^2}(D-2)H^2[D-5-2q]+(D-2)(D+2)(D-5)H^4\right.$$
$$\left.-3D(D-3)q^2H^4+2(D^3-9D^2+12D+8)qH^4-2DsH^4-4(D-3)DjH^4\right]$$
$$+\sigma_2\left(\frac{k}{a^2}+H^2\right)\left[(D-5)\left(\frac{k}{a^2}+H^2\right)-4qH^2\right], \tag{58}$$

where

$$j=-\frac{\dot{q}}{H}+q(1+2q),$$
$$s=-\frac{\ddot{q}}{H^2}+2\frac{\dot{q}}{H}(1+3q)-q(1+2q)(1+3q). \tag{59}$$

In getting these expressions, use has been made of

$$q=-\frac{\ddot{a}}{aH^2}=-\frac{\dot{H}}{H^2}-1. \tag{60}$$

Now, we shall set the ordinary matter and curvature to zero in the formulation, i.e., $\rho=0=p$ and $k=0$, to both comply with the observations and investigate the late-time acceleration of the Universe resulting purely from the geometric terms related to curvature of the spacetime. Then, Eq. (56) becomes an equation involving H, q, and the first and second derivatives of q with respect to time; that is,

$$\left[1+\frac{4H^2\kappa\sigma_2}{D-2}\right](q+1)$$
$$+\frac{\kappa\sigma_1}{(D-2)D}\{-2H^2(D^2-4)+H(D-7)D\dot{q}+D\ddot{q}$$
$$+Hq[H(-6D^2+15D+8)-6D\dot{q}]-4H^2(D-5)Dq^2$$
$$+6H^2Dq^3\}=0. \tag{61}$$

However, this form is not appropriate for solving the equation numerically. To obtain an appropriate form, we will change the time derivatives to derivatives with respect to H, since both H and q are functions of time only and related to each other by (60). That is, using

$$\dot{q}=-H^2(1+q)q',$$
$$\ddot{q}=H^4(1+q)^2q''+2H^3(1+q)^2q'+H^4(1+q)q'^2, \tag{62}$$

where the prime denotes derivatives with respect to H, we can bring (61) into the following second-order nonlinear differential equation for q in H:

$$\left[1+\frac{4H^2\kappa\sigma_2}{D-2}\right](q+1)+\frac{H^2\kappa\sigma_1}{(D-2)D}\{8-2D^2+6Dq^3$$
$$-H(D-9)Dq'+H^2Dq'^2+H^2Dq''$$
$$+Dq^2(20-4D+8Hq'+H^2q'')$$
$$+q[8+15D-6D^2-H(D-17)Dq'+H^2Dq'^2$$
$$+2H^2Dq'']\}=0. \tag{63}$$

Once the initial values H_0, q_0, and q_0' are given, this equation can be numerically solved for the evolution of the deceleration parameter q in H. This kind of analysis was previously exploited in the context of $f(R)$ gravity in [15,25,26]. In the following analysis we will take the present-day values of the cosmological parameters H, q, j, and s as [52,53]

$$H_0 = 67.4\ \mathrm{km}\cdot\mathrm{s}^{-1}\cdot\mathrm{Mpc}^{-1}, \qquad q_0 = -0.71,$$
$$j_0 = 1.26, \qquad s_0 = 0.04. \tag{64}$$

Here, we shall consider two particular cases regarding our parameter space (σ_1, σ_2) corresponding to $\sigma_2 = 0$ (Case I) and $\sigma_1 = 0$ (Case II).

Case I. When $\sigma_2 = 0$, the deceleration equation (63) becomes

$$(q+1) + \frac{h^2\kappa\sigma_1}{(D-2)D}\{8 - 2D^2 + 6Dq^3 - h(D-9)Dq' + h^2Dq'^2 + h^2Dq'' + Dq^2(20 - 4D + 8hq' + h^2q'') + q[8 + 15D - 6D^2 - h(D-17)Dq' + h^2Dq'^2 + 2h^2Dq'']\} = 0. \tag{65}$$

Note that in writing this equation we have rescaled the Hubble parameter H in (63) as

$$H \equiv (100\ \mathrm{km}\cdot\mathrm{s}^{-1}\cdot\mathrm{Mpc}^{-1})h \tag{66}$$

to construct a dimensionless parameter h and assumed all the numerical constants are absorbed into the constant σ_1.

Remark 6. Before presenting the solution of (65), it would be useful to look at the energy density ρ_g related to the geometry: Since it must be positive, we can determine the sign of the constant σ_1 by evaluating the energy density at the present time. From (48), we can infer that the energy density at the present epoch is

$$\rho_g = \frac{(D-1)h_0^2\sigma_1}{2D}[2D(D-3)q_0 + (D^2-4) + Dq_0^2 - 2Dj_0]. \tag{67}$$

Remark 7. Inserting h_0, q_0, and j_0 from (64) into ρ_g and graphing with respect to σ_1 and D, we obtain the plot shown in Fig. 1. As is obvious from the figure, to have a positive ρ_g, the parameter σ_1 must be negative for all dimensions up to $D = 11$.

Now, we can solve Eq. (65) numerically: First, to observe the effect of the value of σ_1 on the solution, we plot the solution of $q(h)$ for $D = 4$ and for different values of σ_1, this is shown in Fig. 2.

Remark 8. Since h is related to the inverse of the cosmic time t, in Fig. 2, $h > h_0$ defines the past and $h < h_0$ defines the future in the cosmic evolution of the Universe. As is obvious from the figure, the deceleration parameter q experiences two signature changes: one is at the past and the other is at the future.

Remark 9. There is some time interval in which the deceleration parameter is negative (i.e., $q < 0$).

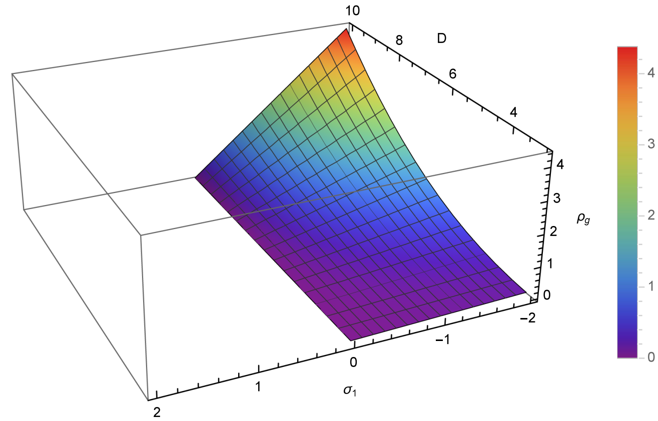

FIG. 1. The plot of ρ_g vs σ_1 and D for $h_0 = 0.674$, $q_0 = -0.71$, and $j_0 = 1.26$.

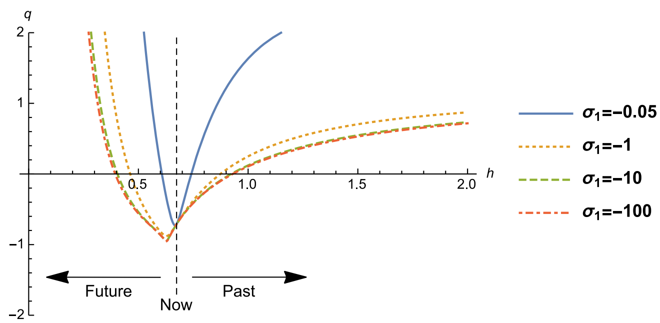

FIG. 2. The plot of q as a function of h for different values of σ_1 and for $D = 4$, $\kappa = 1$, $\sigma_2 = 0$, $h_0 = 0.674$, $q_0 = -0.71$, and $q_0' = 4.92$.

In particular, close to the present value of the Hubble parameter $h_0 = 0.674$, the deceleration parameter q is negative. This means that the observed accelerated expansion phase of the Universe can be driven solely by the curvature of the spacetime.

Remark 10. It can also be observed from Fig. 2 that, as the value of σ_1 decreases in negative (or increases in magnitude), the curves are opening out and approaching each other, and for very small values (or large magnitudes) they are becoming almost identical. This stems from the

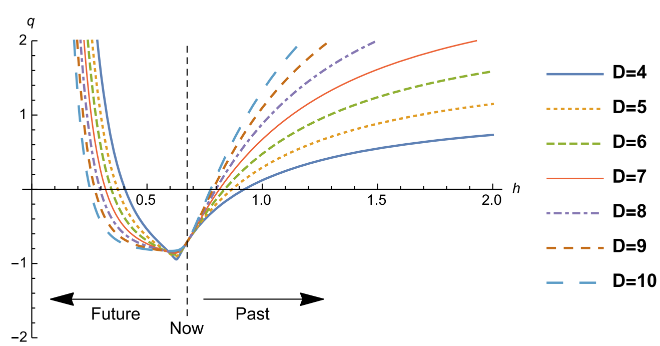

FIG. 3. Behavior of q for $4 \le D \le 10$, $\kappa = 1$, $\sigma_1 = -10$, $\sigma_2 = 0$, $h_0 = 0.674$, $q_0 = -0.71$, $q_0' = 4.92$.

fact that, as the magnitude of the parameter σ_1 increases in (65), the first term in the equation can be neglected and the equation is effectively reduced to the one in which the curly bracketed expression equals zero.

Remark 11. Additionally, we can also investigate the behavior of $q(h)$ with D. This is shown in Fig. 3. Here it is explicitly seen that, as D increases, the signature change of q occurs at later times.

On the other hand, one can also study the behavior of the equation of the state parameter of the dark fluid w_g stemmed from the terms proportional to σ_1 in (57) and (58); that is, since $w_g = p_g/\rho_g$, at the present time t_0 we obtain

$$w_g = \frac{(D-2)(D+2)(D-5) - 3D(D-3)q_0^2 + 2(D^3 - 9D^2 + 12D + 8)q_0 - 2Ds_0 - 4(D-3)Dj_0}{(D-1)[-2D(D-3)q_0 - (D^2-4) - Dq_0^2 + 2Dj_0]}, \tag{68}$$

the plot of which is given in Fig. 4 with respect to D, where the dotted lines represent the upper bounds in (55) decreasing in value as D increases.

Remark 12. As is obvious, w_g is negative for all integers $4 \leq D \leq 10$, consistently with Fig. 3, and satisfies the condition (55) for all dimensions. Specifically, for $D = 4$, $w_g = -0.85$.

Case II. When $\sigma_1 = 0$ and $D \neq 4$, the deceleration equation (63) becomes

$$\left[1 + \frac{4H^2\kappa\sigma_2}{D-2}\right](q+1) = 0. \tag{69}$$

Recalling that $\sigma_2 < 0$ follows from the positiveness of ρ_{g2}, there are two possibilities:

(i) $[1 + \frac{4H^2\kappa\sigma_2}{D-2}] \neq 0$ and $q = -1$, independently of the number of dimensions. Thus, the deceleration parameter is always negative, representing an accelerating Universe driven by the curvature of the spacetime. Also, one can show that for the equation of state of the geometric fluid w_g stemmed from the terms proportional to σ_2 in (57) and (58) is

$$w_g = -\frac{D-5-4q}{D-1} = -1, \tag{70}$$

for $q = -1$ independently of the number of dimensions. Hence, the geometric fluid can derive the accelerating expansion of the Universe playing the role of an effective cosmological constant.

(ii) When $H^2 = -\frac{(D-2)}{4\kappa\sigma_2}$, it follows that $q = -1$ automatically. This case represents an exact exponential solution for the scale factor, i.e., $a(t) = a_0 e^{\pm\lambda t}$ where $\lambda = \sqrt{-(D-2)/4\sigma_2}$.

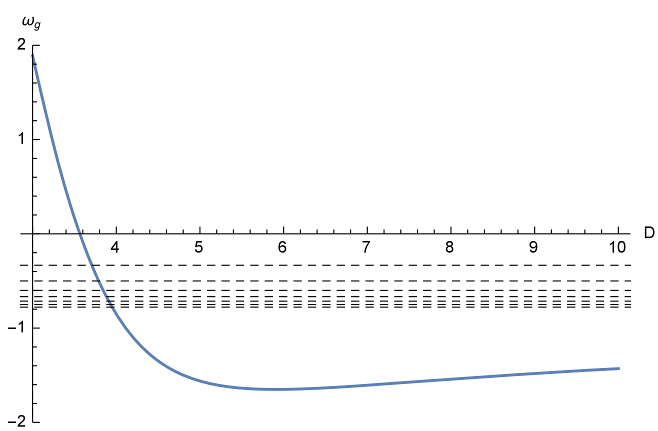

FIG. 4. Behavior of w_g for $\sigma_2 = 0$, $q_0 = -0.71$, $j_0 = 1.26$, $s_0 = 0.04$.

V. SOME PARTICULAR COSMOLOGICAL SOLUTIONS IN QUADRATIC GRAVITY

In this section, we shall investigate some particular solutions to the general equations presented in (46) and (47).

A. Solutions with $\sigma_2 = 0$

When σ_2 vanishes, the field equations (46) and (47) become

$$\begin{aligned}\frac{1}{\kappa}\left[\frac{(D-1)(D-2)}{2}\left(\frac{k}{a^2} + \frac{\dot{a}^2}{a^2}\right) - \Lambda\right] &= \rho - \frac{(D-1)\sigma_1}{2Da^4}[k^2(D-2)^2 \\ &\quad + \dot{a}^2(2D(D-3)a\ddot{a} - 4k(D-2)) \\ &\quad - (D^2-4)\dot{a}^4 - Da^2\ddot{a}^2 + 2Da^2\dot{a}\dddot{a}],\end{aligned} \tag{71}$$

$$\begin{aligned}-\frac{1}{\kappa}\left[\frac{(D-2)(D-3)}{2}\left(\frac{k}{a^2} + \frac{\dot{a}^2}{a^2}\right) + (D-2)\frac{\ddot{a}}{a} - \Lambda\right] &= p + \frac{\sigma_1}{2Da^4}\{(D-2)(D-5)(D+2)\dot{a}^4 \\ &\quad - k^2(D-2)^2(D-5) + a\ddot{a}[8k(D-2) - 3D(D-3)a\ddot{a}] \\ &\quad + 2\dot{a}^2[2k(D-2)(D-5) - (D((D-9)D+12)+8)a\ddot{a}] \\ &\quad - 2D\ddddot{a}a^3 - 4(D-3)D\dddot{a}a^2\dot{a}\}.\end{aligned} \tag{72}$$

It must be noted that these equations are valid when $D = 4$ and $(D-1)(D-2)\alpha + D(D-3)\gamma \neq 0$, or when $D \neq 4$ and $(D-1)(D-2)\alpha + D(D-3)\gamma = 0$, or when $D = 4$ and

$(D-1)(D-2)\alpha + D(D-3)\gamma = 0$. In four dimensions, these reduce to

$$\frac{1}{\kappa a^4}[3a^2(\dot{a})^2 + 3ka^2 - \Lambda a^4] = \rho - \frac{3\sigma_1}{2a^4}[2a^2\dot{a}\dddot{a} - a^2(\ddot{a})^2 + 2a(\dot{a})^2\ddot{a} - 3(\dot{a})^4 - 2k(\dot{a})^2 + k^2], \tag{73}$$

$$-\frac{1}{\kappa a^4}[2a^3\ddot{a} + a^2(\dot{a})^2 + ka^2 - \Lambda a^4] = p + \frac{\sigma_1}{2a^4}[2a^3\ddddot{a} + 4a^2\dot{a}\dddot{a} + 3a^2(\ddot{a})^2 - 12a(\dot{a})^2\ddot{a} + 3(\dot{a})^4 - 4ka\ddot{a} + 2k(\dot{a})^2 - k^2], \tag{74}$$

where $\sigma_1 = 4(3\alpha + \beta)$. This case is known as the critical gravity where (73) and (74) reduce to the ones in the usual Einstein's gravity when $\sigma_1 = 0$ [46,47]. Here we have the following immediate Remarks:

Remark 13. When the critical parameter $\sigma_1 = 0$ [48,49], it is quite interesting that for this special case the above energy density and pressure expressions for the FLRW metric reduce to the corresponding expressions in pure Einstein theory. This means that the highly nontrivial tensor field $\mathcal{E}_{\mu\nu}$ given in (21) reduces to

$$\mathcal{E}_{\mu\nu} = \alpha\left(2RR_{\mu\nu} - \frac{1}{2}R^2 g_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\Box R + \nabla_\mu\nabla_\nu R - 3\Box R_{\mu\nu} - 6R_{\mu\rho\nu\sigma}R^{\rho\sigma} + \frac{3}{2}g_{\mu\nu}R^{\rho\sigma}R_{\rho\sigma}\right), \tag{75}$$

which vanishes identically for $\alpha \neq 0$.

Remark 14. The case when $k = 0$ and vanishing of the coefficient of σ_1 in (73) and (74) corresponds to the work of Barrow and Hervik [23]. They found a power law solution for a and studied the stability of the solution.

1. Exponential solutions

Now, we shall study a special case that may correspond to the late-time accelerating era of the Universe. Let $a(t) = a_0 e^{H_0 t}$, where a_0 and H_0 are the scale factor and the Hubble constant at the time when the accelerating era begins, respectively. It is interesting that, for an exponentially expanding flat universe ($k = 0$), the contributions of the quadratic gravity terms related to σ_1 in (71) and (72) vanish identically. This means that the presence of the bare cosmological constant Λ is crucial for having exponential solutions in arbitrary D dimensions. When $k \neq 0$, the field equations (71) and (72) become

$$\rho = \frac{(D-1)(D-2)}{2D\kappa a_0^4}[D\beta_0 a_0^4 + ka_0^2\beta_2 e^{-2H_0 t} + (D-2)k^2\kappa\sigma_1 e^{-4H_0 t}], \tag{76}$$

$$p = -\frac{D-2}{2D\kappa a_0^4}[D\beta_0 a_0^4 + (D-3)ka_0^2\beta_2 e^{-2H_0 t} + (D-2)(D-5)k^2\kappa\sigma_1 e^{-4H_0 t}], \tag{77}$$

where $\beta_0 = H_0^2 - \frac{2\Lambda}{(D-1)(D-2)}$, $\beta_2 = D - 4\kappa\sigma_1 H_0^2$, and $\sigma_1 = 4(D-1)\alpha + D\beta$. We have the following consequences:

(a) If $\beta_0 > 0$ and $\beta_2 > 0$, the energy density remains positive for all t. At late times when $t \to \infty$, we have $\rho \to \frac{(D-1)(D-2)\beta_0}{2\kappa}$ and $p \to -\frac{(D-2)\beta_0}{2\kappa}$. Hence, the equation of state is of dark energy type, i.e., $p \to -\frac{\rho}{D-1}$.

(b) When $\beta_0 = 0$ or $H_0 = \sqrt{\frac{2\Lambda}{(D-1)(D-2)}}$ and as $t \to \infty$, then we obtain

$$\rho = \frac{(D-1)(D-2)}{2D\kappa a_0^4}[ka_0^2\beta_2 e^{-2H_0 t} + (D-2)k^2\kappa\sigma_1 e^{-4H_0 t}], \tag{78}$$

$$p = -\frac{D-2}{2D\kappa a_0^4}[(D-3)ka_0^2\beta_2 e^{-2H_0 t} + (D-2)(D-5)k^2\kappa\sigma_1 e^{-4H_0 t}]. \tag{79}$$

In these expressions, since the last terms decay faster than the first terms, one can deduce an equation of state at late times

$$p = -\frac{D-3}{D-1}\rho. \tag{80}$$

It can be seen that, for $D \geq 4$ and $\rho > 0$, the pressure is always negative. In $D = 4$, this gives $p = -\rho/3$, which corresponds to the equation of state of cosmic strings [54].

(c) To have positive pressure, we let $\beta_0 = 0$ and $\beta_2 = 0$ together so that $H_0 = \sqrt{\frac{2\Lambda}{(D-1)(D-2)}}$ and $4\kappa\sigma_1 H_0^2 = D$. We find

$$\rho = \frac{(D-1)(D-2)^2}{2Da_0^4}k^2\sigma_1 e^{-4H_0 t}, \tag{81}$$

$$p = -\frac{(D-2)^2(D-5)}{2Da_0^4}k^2\sigma_1 e^{-4H_0 t}. \tag{82}$$

This special solution can only be obtained when both k and σ_1 are nonvanishing. Hence, it can be obtained neither in Einstein theory nor in the work of Barrow and Hervik [23]. In other words, it is the effect of the quadratic gravity that predicts a de Sitter era at late times with the acceleration of the expansion being constant, i.e., the square of the Hubble constant ($H_0 = \frac{\dot{a}}{a}$) at the beginning of the accelerating phase. Furthermore, the above expressions for the energy density and pressure of the fluid provide an equation of state

$$p = -\frac{D-5}{D-1}\rho, \tag{83}$$

where, for $D \geq 6$ and $\rho > 0$, the pressure is always negative. It can also be seen that in $D = 5$ the pressure vanishes corresponding to the equation of state of dust matter, and in $D = 4$ it becomes $p = \frac{1}{3}\rho$ which corresponds to the equation of state of radiation. This solution is valid for both closed ($k = 1$) and open ($k = -1$) universes.

At this point, it should be stressed that all the above equations of state can mimic a variety of sources of "geometric" origin filling the Universe which accelerates like the de Sitter universe at late times.

2. A more general solution

Let us assume that $a(t)$ satisfies the following differential equation:

$$\dot{a}^2 = f(a), \tag{84}$$

where $f(a)$ is any arbitrary function of the scale factor $a(t)$. Hence, the energy density and pressure expressions in (71) and (72) become

$$\rho = \frac{(D-1)(D-2)}{2\kappa a^4}\Bigg\{(f+k)a^2 - \frac{2\Lambda}{(D-1)(D-2)}a^4 + \frac{\sigma_1\kappa}{D}\left[(D-2)k^2 - (D+2)f^2 - \frac{D}{4(D-2)}a^2 f'^2 - 4kf + \frac{D(D-3)}{(D-2)}aff' + \frac{D}{(D-2)}a^2 ff''\right]\Bigg\}, \tag{85}$$

$$p = -\frac{(D-1)(D-2)}{2\kappa a^4}\Bigg\{\frac{D-3}{D-1}(f+k)a^2 - \frac{2\Lambda}{(D-1)(D-2)}a^4 + \frac{1}{(D-1)}a^3 f' - \frac{\sigma_1\kappa}{D(D-1)}\bigg[(D+2)(D-5)f^2 - (D-2)(D-5)k^2 + 4kaf' - \frac{3D(D-3)}{4(D-2)}a^2 f'^2 + 4(D-5)kf - \frac{D^3 - 9D^2 + 12D + 8}{(D-2)}aff' - \frac{2D(D-3)}{(D-2)}a^2 ff'' - \frac{D}{2(D-2)}a^3(f'f'' + 2ff''')\bigg]\Bigg\}, \tag{86}$$

where $f' = \frac{df}{da}$. Now, we shall give some examples:

Example 1. Let $f = a_0 k + \frac{a_1}{a} + a_3 a^2$. For this choice, the acceleration of the Universe, $\ddot{a} = f'/2$, becomes positive when $a > (\frac{a_1}{2a_3})^{1/3}$. Taking $a_0 = -1$ and $a_3 = \frac{2\Lambda}{(D-1)(D-2)}$, we get

$$\rho = \frac{2(D-4)\sigma_1\Lambda^2}{D(D-2)^2} - \frac{a_1(D-1)}{2D\kappa a^6}\Bigg\{D(D-2)a^3 + \frac{\kappa\sigma_1}{4}\bigg[(8D^2 - 19D - 16)a_1 + 12D(D-3)ka - \frac{8(D+2)(D-4)\Lambda}{(D-1)(D-2)}a^3\bigg]\Bigg\}, \tag{87}$$

$$p = -\frac{2(D-4)\sigma_1\Lambda^2}{D(D-2)^2} - \frac{a_1}{2D\kappa a^6}\Bigg\{D(D-2)(D-4)a^3 + \frac{\kappa\sigma_1}{4}\bigg[(D-7)(8D^2 - 19D - 16)a_1 + 12D(D-3)(D-6)ka - \frac{8(D+2)(D-4)\Lambda}{(D-1)(D-2)}a^3\bigg]\Bigg\}, \tag{88}$$

where a_1 is an arbitrary constant. From these expressions, it can readily be observed that, when $a_1 = 0$, the equation of state of the fluid reduces to the form

$$p = -\rho, \tag{89}$$

which corresponds to a cosmological constant equation of state for $D \neq 4$. For $D = 4$ and $a_1 = 0$, the energy density and pressure vanish identically. This means that, for $f = -k + \frac{\Lambda}{3}R^2$, the FLRW metric is the vacuum solution of the quadratic gravity field equations. In particular, when $k = 0$, this is the usual de Sitter solution with the scale factor $a = a_0 e^{\sqrt{\Lambda/3}t}$.

Example 2. Let $f = a_0 k + \frac{a_2}{a^2} + a_3 a^2$. This time, the acceleration of the Universe, $\ddot{a} = f'/2$, becomes positive when $a > (\frac{a_2}{a_3})^{1/4}$. Taking $a_0 = -1$, we obtain

$$\rho = \frac{D-1}{2D\kappa}\left[(D-1)(D-4)\kappa\sigma_1 a_3^2 + D(D-2)\left(a_3 - \frac{2\Lambda}{(D-1)(D-2)}\right)\right] + \frac{a_2(D-1)}{2D\kappa a^8}\{D(D-2)a^4 - \kappa\sigma_1[(D-4)(1+3D)a_2 - 4D(D-4)ka^2 + 2(D^2 - 5D - 4)a_3 a^4]\}, \tag{90}$$

$$p=-\frac{D-1}{2D\kappa}\left[(D-1)(D-4)\kappa\sigma_1 a_3^2 +D(D-2)\left(a_3-\frac{2\Lambda}{(D-1)(D-2)}\right)\right] -\frac{a_2}{2D\kappa a^8}\{D(D-2)(D-5)a^4 -\kappa\sigma_1[(D-4)(D-9)(1+3D)a_2-4D(D-4)(D-7)ka^2 +2(D-5)(D^2-5D-4)a_3a^4]\}, \tag{91}$$

where a_2 and a_3 are arbitrary constants. Here we have the following special case: When $D=4$,

$$\rho=\frac{3a_3-\Lambda}{\kappa}+\frac{3a_2}{\kappa a^4}(1+2\kappa\sigma_1 a_3), \tag{92}$$

$$p=-\frac{3a_3-\Lambda}{\kappa}+\frac{a_2}{\kappa a^4}(1+2\kappa\sigma_1 a_3). \tag{93}$$

Now, choosing $a_3=\Lambda/3$, the equation of state becomes

$$p=\frac{1}{3}\rho, \tag{94}$$

which corresponds to a radiation equation of state for arbitrary a_2. This means that, for $f=-k+\frac{a_2}{a^2}+a_3a^2$, the spatially flat ($k=0$) FLRW metric solves the quadratic gravity field equations with the scale factor

$$a(t)=\frac{1}{\sqrt{2a_3}}e^{\sqrt{a_3}t}\sqrt{1-a_2a_3e^{-4\sqrt{a_3}t}}. \tag{95}$$

As it can be seen, $a(t)\to\frac{1}{\sqrt{2a_3}}e^{\sqrt{a_3}t}$ as $t\to\infty$, which means that, although the equation of state represents a radiation field, there is a de Sitter–like expansions at late times.

Example 3. Let $f=a_0e^{ma}$ where a_0 and m are two real arbitrary constants. For this function, the scale factor is

$$a=-\frac{2}{m}\log\left(1-\frac{m\sqrt{a_0}}{2}t\right), \tag{96}$$

where $a>0$ requires $m<0$. Thus, this corresponds to a decelerating universe. In this case, the energy density and pressure take the form

$$\rho=\frac{a_0^2(D-1)\sigma_1[3m^2Da^2+4m(D-3)Da-4(D^2-4)]}{8Da^4}e^{2ma}+\frac{a_0(D-1)(D-2)}{2\kappa a^2}e^{ma}-\frac{\Lambda}{\kappa}, \tag{97}$$

$$p=-\frac{a_0^2\sigma_1[6m^3Da^3+11m^2(D-3)Da^2+4m(D^3-9D^2+12D+8)a-4(D-5)(D-2)(D+2)]}{8Da^4}e^{2ma} -\frac{a_0(D-2)(ma+D-3)}{2\kappa a^2}e^{ma}+\frac{\Lambda}{\kappa}. \tag{98}$$

Example 4. Let $f=a_0a^m$ where a_0 and m are two arbitrary real constants. This function produces the scale factor as

$$a=\left[\left(\frac{2-m}{2}\right)\sqrt{a_0}\right]^{\frac{2}{2-m}}t^{\frac{2}{2-m}}, \tag{99}$$

where $a_0>0$ and $m\neq 2$. From this scale factor, one obtains the acceleration as $\ddot{a}=\frac{a_0m}{2}a^{m-1}$. To assure the positiveness of a and $\ddot{a}$, it must be that $0<m<2$. In this case, the energy density and pressure are

$$\rho=\frac{1}{\kappa}\left[\frac{(D-2)(D-1)(a_0a^m+k)}{2a^2}-\Lambda\right] +\frac{(D-1)\sigma_1}{8Da^4}\{a_0^2[3m^2D+4m(D-4)D-4(D-2)(D+2)]a^{2m}-16a_0k(D-2)a^m+4k^2(D-2)^2\}, \tag{100}$$

$$p=-\frac{1}{\kappa}\left[\frac{1}{2}a_0(D-2)(m+D-3)a^{m-2}+\frac{k(D-1)(D-5)}{2a^2}-\Lambda\right] -\frac{\sigma_1}{8Da^4}\{a_0^2[6m^3D+m^2D(11D-47)+4m(D^3-11D^2+20D+8)-4(D+2)(D-2)(D-5)]a^{2m} -16a_0k(D-2)(m+D-5)a^m+4k^2(D-5)(D-2)^2\}. \tag{101}$$

It can be observed that, when $t \to \infty$, the contributions of the higher curvature terms decay faster than the Einstein terms, and hence,

$$\rho \to \frac{1}{\kappa}\left[\frac{(D-2)(D-1)(a_0 a^m + k)}{2a^2} - \Lambda\right],$$
$$p \to -\frac{1}{\kappa}\left[\frac{1}{2}a_0(D-2)(m+D-3)a^{m-2} + \frac{k(D-1)(D-5)}{2a^2} - \Lambda\right].$$

Now, for the flat universe and $\Lambda = 0$, we can deduce that

$$w \equiv \frac{p}{\rho} = -\frac{D-3+m}{D-1}, \tag{102}$$

which reduces to $w = -\frac{1}{3}(1+m)$ in $D = 4$. Since $0 < m < 2$, it must be that $-1 < w < -\frac{D-3}{D-1}$. One can see that as D increases, the range squeezes and the upper bound approaches -1.

On the other hand, at early times (as $t \to 0$), the only conditions on the parameters of the scale factor are $a_0 > 0$ and $m < 2$. There are the following cases:

(i) For $m < 0$, we have

$$\rho \to \frac{(D-1)\sigma_1 a_0^2 a^{2m}}{8Da^4}[3m^2 D + 4m(D-4)D - 4(D-2)(D+2)], \tag{103}$$

$$p \to -\frac{(D-5+2m)\sigma_1 a_0^2 a^{2m}}{8Da^4}[3m^2 D + 4m(D-4)D - 4(D-2)(D+2)]. \tag{104}$$

Then one can deduce the following equation of state:

$$w \equiv \frac{p}{\rho} = -\frac{D-5+2m}{D-1}, \tag{105}$$

which becomes $w = \frac{1-2m}{3} > \frac{1}{3}$ in $D = 4$. For $m = -1$, it gives the stiff fluid equation of state, $w = 1$.

(ii) For $m = 0$, we have

$$\rho \to \frac{(D-1)(D-2)\sigma_1}{2Da^4}[k^2(D-2) - 4a_0 k - (D+2)a_0^2], \tag{106}$$

$$p \to -\frac{\sigma_1(D-2)(D-5)}{2Da^4}[k^2(D-2) - 4a_0 k - (D+2)a_0^2]. \tag{107}$$

Then

$$w \equiv \frac{p}{\rho} = -\frac{D-5}{D-1}. \tag{108}$$

This becomes $w = \frac{1}{3}$ in $D = 4$ and $w = 0$ in $D = 5$. The former represents the radiation and the latter represents the dust matter.

(iii) For $0 < m < 2$, we have

$$\rho \to \frac{(D-1)(D-2)^2\sigma_1 k^2}{2Da^4}, \tag{109}$$

$$p \to -\frac{(D-2)^2(D-5)\sigma_1 k^2}{2Da^4}. \tag{110}$$

Then

$$w \equiv \frac{p}{\rho} = -\frac{D-5}{D-1}. \tag{111}$$

Again this represents the radiation ($w = \frac{1}{3}$) in $D = 4$ and the dust matter ($w = 0$) in $D = 5$.

3. Approximate solutions near criticality with $\sigma_2 = 0$

Now, we assume that, in generic gravity theories in (19), the action contains some number of coupling constants, and hence, A and B in (26) and (27) are functions of these coupling constants and/or some combinations α_i of them. Assuming that these coupling constants are relatively smaller than the other parameters in these functions, we can expand the scale factor a in terms of these parameters

$$a(t, \alpha_i) = a_0(t) + \sum_i \alpha_i a_i(t) + \mathcal{O}(\alpha_i^2). \tag{112}$$

Following this approach, we obtain the functions A and B as

$$A = \sum_i \alpha_i A_i + \mathcal{O}(\alpha_i^2), \tag{113}$$

$$B = \sum_i \alpha_i B_i + \mathcal{O}(\alpha_i^2), \tag{114}$$

where A_i and B_i are functions depending on the explicit gravity theory. In what follows we shall keep only the terms linear in α_i. Hence, Eqs. (26) and (27) reduce to

$$\rho - \frac{1}{\kappa}\left[\frac{(D-1)(D-2)}{2}\rho_{10} - \Lambda\right] + \sum_i \alpha_i\left[A_i - B_i - \frac{(D-1)(D-2)}{2\kappa}\rho_{1i}\right] = 0, \tag{115}$$

$$p + \frac{1}{\kappa}\left[\frac{(D-1)(D-2)}{2}\rho_{10} - (D-2)\rho_{20} - \Lambda\right] - \sum_i \alpha_i\left[A_i - \frac{(D-2)}{\kappa}\left(\frac{D-1}{2}\rho_{1i} - \rho_{2i}\right)\right] = 0, \tag{116}$$

where

$$\rho_{10} = H_0^2 + \frac{k}{a_0^2}, \tag{117}$$

$$\rho_{20} = H_0^2 + \frac{k}{a_0^2} - \frac{\ddot{a}_0}{a_0} = \dot{H}_0 + \frac{k}{a_0^2}, \tag{118}$$

$$\rho_{1i} = \frac{6}{a_0}\left(\dot{a}_i H_0 - a_i H_0^2 - \frac{k a_i}{a_0^2}\right), \tag{119}$$

$$\rho_{2i} = \frac{2}{a_0}\left[\dot{a}_i H_0 - a_i H_0^2 - \frac{k a_i}{a_0^2} + \frac{1}{2}\left(\frac{\ddot{a}_0 a_i}{a_0} + \ddot{a}_i\right)\right], \tag{120}$$

where $H_0 = \dot{a}_0/a_0$. Now one can find a_i from (115) by assuming that the extra terms in α_i vanish, i.e.,

$$\sum_i \alpha_i \left[A_i - B_i - \frac{(D-1)(D-2)}{2\kappa}\rho_{1i}\right] = 0. \tag{121}$$

Remark 15. Then the expression for ρ becomes exactly the energy density without the modification of the generic gravity, but the expression for p has contributions from the generic gravity theory.

Now, we will specifically consider the critical quadratic gravity theory given by the action (41) with $\sigma_2 = 0$ in (53). Taking $i = 1$ and $\alpha_i = \sigma_1$ in the above formulation, from (112) the scale factor becomes

$$a(t, \sigma_1) = a_0(t) + \sigma_1 a_1(t) + \mathcal{O}(\sigma_1^2), \tag{122}$$

and the field equation (115) reads as

$$\begin{aligned}
&\rho - \frac{1}{\kappa}\left[\frac{(D-1)(D-2)}{2}\left(H_0^2 + \frac{k}{a_0^2}\right) - \Lambda\right] \\
&- \frac{(D-1)(D-2)\sigma_1}{\kappa a_0}\Bigg\{\dot{a}_1 H_0 - a_1 H_0^2 - \frac{k a_1}{a_0^2} \\
&+ \frac{\kappa}{2D(D-2)a_0^3}[k^2(D-2)^2 \\
&+ \dot{a}_0^2[2D(D-3)a_0\ddot{a}_0 - 4k(D-2)] \\
&- (D^2-4)\dot{a}_0^4 - D a_0^2 \ddot{a}_0^2 + 2D a_0^2 \dot{a}_0 \dddot{a}_0]\Bigg\} = 0.
\end{aligned} \tag{123}$$

To determine a_1, we assume that the coefficient terms of σ_1 vanish, i.e.,

$$\begin{aligned}
&\dot{a}_1 H_0 - a_1\left(H_0^2 + \frac{k}{a_0^2}\right) + \frac{\kappa}{2D(D-2)a_0^3}[k^2(D-2)^2 \\
&+ \dot{a}_0^2[2D(D-3)a_0\ddot{a}_0 - 4k(D-2)] \\
&- (D^2-4)\dot{a}_0^4 - D a_0^2 \ddot{a}_0^2 + 2D a_0^2 \dot{a}_0 \dddot{a}_0] = 0,
\end{aligned} \tag{124}$$

which can also be obtained from (121). This equation can be rewritten in the following first-order linear differential equation form

$$\dot{a}_1 + R(t) a_1 = S(t), \tag{125}$$

where

$$R(t) = -\left(H_0 + \frac{k}{a_0^2 H_0}\right), \tag{126}$$

$$\begin{aligned}
S(t) = &-\frac{\kappa}{2D(D-2)a_0^2 \dot{a}_0}\{k^2(D-2)^2 \\
&+ \dot{a}_0^2[2D(D-3)a_0\ddot{a}_0 - 4k(D-2)] \\
&- (D^2-4)\dot{a}_0^4 - D a_0^2 \ddot{a}_0^2 + 2D a_0^2 \dot{a}_0 \dddot{a}_0\}.
\end{aligned} \tag{127}$$

Equation (125) admits the general solution

$$a_1(t) = \frac{C}{\lambda(t)} + \frac{\mu(t)}{\lambda(t)}, \tag{128}$$

where C is an integration constant and

$$\lambda(t) = e^{\int R(t)dt}, \qquad \mu(t) = \int \lambda(t) S(t) dt. \tag{129}$$

Taking $D = 4$, we consider the following cases:

(1) Letting $a_0(t) = b_0 e^{c_0 t}$, where b_0 and c_0 are constants, we find $a_1(t)$ as

$$\begin{aligned}
a_1(t) &= e^{c_0 t}\left[C e^{-d_0 e^{-2c_0 t}} + \frac{\kappa}{4b_0}(k e^{-2c_0 t} - 4c_0^2 b_0^2)\right], \\
d_0 &= \frac{k}{2c_0^2 b_0^2}.
\end{aligned} \tag{130}$$

Hence, $a(t, \sigma_1)$ in (122) reads as

$$\begin{aligned}
a(t, \sigma_1) = b_0 e^{c_0 t} + \sigma_1 e^{c_0 t}\Bigg[&C e^{-d_0 e^{-2c_0 t}} \\
&+ \frac{\kappa}{4b_0}(k e^{-2c_0 t} - 4c_0^2 b_0^2)\Bigg].
\end{aligned} \tag{131}$$

For a flat universe ($k = 0$), it reduces to

$$a(t, \sigma_1) = \chi_0 e^{c_0 t}, \quad \chi_0 = b_0 + \sigma_1(C - \kappa c_0^2 b_0). \tag{132}$$

Then, the contribution of the higher curvature terms in the scale factor for a flat universe has the same exponential form $e^{c_0 t}$ as in GR. For $\chi_0 > 0$, the total acceleration, i.e., $\ddot{a}(t, \sigma_1)$, is positive.

(2) Letting $a_0(t) = m_0 t^n$, where m_0 and n are constants, we find $a_1(t)$ in (128) with

$$\lambda(t) = t^{-n} \cdot e^{b t^{2(1-n)}}, \tag{133}$$

$$\mu(t) = \int e^{bt^{2(1-n)}}\left(-\frac{3}{4}m_0\kappa n(1-2n)t^{-3} + \frac{\kappa nk}{2m_0}t^{-2n-1} - \frac{\kappa k^2}{4nm_0^3}t^{1-4n}\right)dt, \quad (134)$$

where $b = -\frac{k}{2nm_0^2(1-n)}$. Here we consider the following two cases:

(i) $k = 0$, $n = \frac{1}{2}$ corresponding to the radiation era in a flat universe in the context of GR. For this case, we obtain

$$a_1(t) = Ct^{\frac{1}{2}}, \quad (135)$$

and then

$$a(t, \sigma_1) = \chi_0 t^{\frac{1}{2}}, \qquad \chi_0 = m_0 + \sigma_1 C. \quad (136)$$

Then, the contribution of the higher curvature terms in the total scale factor for a flat universe has the same power law form $t^{\frac{1}{2}}$ as in GR. In contrast to the previous case of the scale factor with an exponential form, here we see that the total acceleration $\ddot{a}(t, \sigma_1) = -\frac{1}{4}\chi_0 t^{-\frac{3}{2}}$ is positive if $\chi_0 < 0$. For $\chi_0 > 0$, the higher curvature modifications cannot support the acceleration of the universe.

(ii) $k = 0$, $n = \frac{2}{3}$ corresponding to the dust matter era in a flat universe in the context of GR. For this case, we obtain

$$a_1(t) = Ct^{\frac{2}{3}} - \frac{1}{12}\kappa m_0 t^{-\frac{4}{3}}, \quad (137)$$

and then

$$a(t, \sigma_1) = \chi_0 t^{\frac{2}{3}} - \frac{1}{12}\kappa m_0 t^{-\frac{4}{3}}, \qquad \chi_0 = m_0 + \sigma_1 C. \quad (138)$$

In contrast to two previous cases, here we see that the contribution of the higher curvature terms in the scale factor are not of the same kind of GR, and there is an extra $t^{-\frac{4}{3}}$ type of contribution. The total acceleration has the form $\ddot{a}(t, \sigma_1) = -\frac{2}{9}\chi_0 t^{-\frac{4}{3}} - \frac{28}{108}\sigma_1\kappa m_0 t^{-\frac{10}{3}}$ which is negative for $(m_0, C, \sigma_1, \kappa) > 0$. Then, the higher curvature modifications cannot support the acceleration of the universe in a matter dominated era.

B. Solutions with $\sigma_1 = 0$

When $\sigma_1 = 0$ and $D \neq 4$, the field equations (46) and (47) become

$$\frac{1}{\kappa}\left[\frac{1}{2}(D-2)(D-1)\left(\frac{k}{a^2} + \frac{\dot{a}^2}{a^2}\right) - \Lambda\right] = \rho - (D-1)\sigma_2\left(\frac{k}{a^2} + \frac{\dot{a}^2}{a^2}\right)^2, \quad (139)$$

$$-\frac{1}{\kappa}\left[\frac{(D-2)(D-3)}{2}\left(\frac{k}{a^2} + \frac{\dot{a}^2}{a^2}\right) + (D-2)\frac{\ddot{a}}{a} - \Lambda\right] = p + \sigma_2\left(\frac{k}{a^2} + \frac{\dot{a}^2}{a^2}\right)\left[(D-5)\left(\frac{k}{a^2} + \frac{\dot{a}^2}{a^2}\right) + 4\frac{\ddot{a}}{a}\right]. \quad (140)$$

In this case, when $[(D-1)(D-2)\alpha + D(D-3)\gamma] = 0$ (i.e., $\sigma_2 = 0$), one again recovers the expressions in the Einstein gravity in $D \neq 4$ dimensions.

To see the effects of quadratic gravity terms on the expansion of the universe at late times, neglect ρ and p of baryonic matter in (139) and (140). Then, from (139), one can immediately obtain the following solutions for the scale factor:

$$a(t) = \begin{cases} \frac{\sinh[\sqrt{h_0}(t-t_0)]}{\sqrt{h_0}} & \text{for } k = -1, \\ e^{\sqrt{h_0}(t-t_0)} & \text{for } k = 0, \\ \frac{\cosh[\sqrt{h_0}(t-t_0)]}{\sqrt{h_0}} & \text{for } k = +1, \end{cases} \quad (141)$$

where t_0 is an arbitrary integration constant and

$$h_0 \equiv -\frac{D-2}{4\kappa\sigma_2}\left[1 \mp \sqrt{1 + \frac{16\sigma_2\Lambda}{(D-1)(D-2)^2}}\right] \quad (142)$$

with $\sigma_2\Lambda \geq -\frac{(D-1)(D-2)^2}{16}$. One can verify that, with this solution, Eq. (140) is identically satisfied. The energy density and pressure of the geometric fluid becomes

$$\rho_{g2} = -\sigma_2(D-1)h_0^2, \quad (143)$$

$$p_{g2} = \sigma_2(D-1)h_0^2. \quad (144)$$

Remark 16. These give an equation of state $p_{g2} = -\rho_{g2}$ that corresponds to the vacuum equation of state. It must be observed that the positiveness of the energy density requires $\sigma_2 < 0$. Also, it should be stated that in the absence of a cosmological constant ($\Lambda = 0$), the higher curvature terms in the theory behave as an effective cosmological constant driving the late-time exponentially accelerating expansion. These results are consistent with the ones discussed in Case II in Sec. IV.

VI. CONCLUSION

We have given all our findings both in the abstract and in the Introduction. Here we give a short summary of this work. We consider FLRW cosmology in the context of generic gravity theories in which the action includes all the combinations of the metric tensor, curvature tensor, and covariant derivatives of the curvature tensor of any order. Very recently we showed that in such theories with FLRW geometry, contributions of all higher-order terms reduce to a perfect fluid form which we now call the geometric fluid. Hence, all generic theories of gravity in FLRW geometry are equivalent to Einstein's theory of general relativity where the source term contains both matter and geometric fluids. We propose that the source of dark energy/matter is this geometrical fluid arising from higher-order gravity theories. Choosing any higher-order gravity, the parameters of the theory can be suitably arranged that the corresponding geometric fluid contributes to the accelerated expansion of the universe. We verified our assertion by taking the quadratic gravity as an example. Furthermore, we have given some particular exact cosmological solutions of quadratic gravity theory with matter and geometrical fluids.

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