Thermodynamics of frozen stars

Ram Brustein,[1,*] A. J. M. Medved,[2,3,†] and Tamar Simhon[1,‡]

[1]*Department of Physics, Ben-Gurion University, Beer-Sheva 84105, Israel*
[2]*Department of Physics & Electronics, Rhodes University, Grahamstown 6140, South Africa*
[3]*National Institute for Theoretical Physics (NITheP), Western Cape 7602, South Africa*

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The frozen star is a recent proposal for a nonsingular solution of Einstein's equations that describes an ultracompact object which closely resembles a black hole from an external perspective. The frozen star is also meant to be an alternative, classical description of an earlier proposal, the highly quantum polymer model. Here, we show that the thermodynamic properties of frozen stars closely resemble those of black holes: frozen stars radiate thermally, with a temperature and an entropy that are perturbatively close to those of black holes of the same mass. Their entropy is calculated using the Euclidean-action method of Gibbons and Hawking. We then discuss their dynamical formation by estimating the probability for a collapsing shell of "normal" matter to transition, quantum mechanically, into a frozen star. This calculation followed from a reinterpretation of a transitional region between the Euclidean frozen star and its Schwarzschild exterior as a Euclidean instanton that mediates a phase transition from the Minkowski interior of an incipient Schwarzschild black hole to a microstate of the frozen star interior. It is shown that, up to negligible corrections, the probability of this transition is $e^{-A/4}$, with A being the star's surface area. Taking into account that the dimension of the phase space is $e^{+A/4}$, we conclude that the total probability for the formation of the frozen star is of order unity. The duration of this transition is estimated, which we then use to argue, relying on an analogy to previous results, about the scaling of the magnitude of the off-diagonal corrections to the number operator for the Hawking-like particles. Such scaling was shown to imply that the corresponding Page curve indeed starts to go down at about the Page time, as required by unitarity.

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I. INTRODUCTION

The frozen star was originally meant to provide a classical, geometric alternative to the collapsed polymer model [1], which describes a highly excited configuration of long, closed, fundamental strings that is lacking a semiclassical geometric description. The polymer model is premised on the notion that the interior of any regular object which successfully mimics a black hole (BH) should be in a strongly nonclassical state [2], as follows from the idea that the uncertainty principle prevents the collapse of matter to a singularity, much in the same way that the quantum hydrogen atom is stable against collapse. A strongly nonclassical state is tantamount to having maximal entropy density, which in turn means maximally positive radial pressure for a state of fixed energy density. The frozen star model aims to mimic the properties of this highly quantum state in terms of a classical geometry, which amounts to "flipping" the radial pressure from maximally positive to maximally negative [3,4]. The name "frozen star" was first adopted in a later article [5] as an homage to classic literature [6]. The current version of the frozen star model, along with many technical details, can be found in [7] (see also [8]).

The relevant aspects of the model will be reviewed in the next section, but let us take immediate note of some of the important features of the frozen star. It has been known for quite a while that large negative pressure provides a way for stabilizing ultracompact objects against further collapse [9–12] and for evading singularity theorems [13,14]. Lesser known is that a star with maximally negative radial pressure will be ultrastable against perturbations [3,5,8]. Another feature is that the modifications from general relativity extend over horizon-sized scales. This happens to be essential for the self-consistency of the model once quantum-induced evaporation is included; otherwise, the radiated energy will far exceed the original mass of the object [15,16].

As much of the focus of the current paper is on the thermodynamics of a frozen star, let us recall some of the lore pertaining to "conventional" BHs. Bekenstein [17] first proposed that BHs possess an entropy $S_{\rm BH}$ that is proportional to the surface area of the horizon A, and Hawking [18] later fixed the constant of proportionality according to

*Contact author: ramyb@bgu.ac.il
†Contact author: j.medved@ru.ac.za
‡Contact author: simhot@post.bgu.ac.il

$S_{\rm BH} = \frac{A}{4}$. The latter contribution came about when Hawking discovered that a BH of mass M, emits thermal radiation with a temperature of $T_H = \frac{1}{8\pi M}$. Hawking was able to subsequently show that the density matrix of the emitted radiation is exactly thermal, leading to what is famously known as the information-loss problem [19]. Closely related to this paradox is the no-hair theorem [20], as well as the trans-Planckian problem [21], and less closely related is the BH species problem [22]. Although the mathematics is consensually beyond dispute (however, see Ref. [23]), interpreting the thermodynamic properties of BHs and resolving the associated problems remains a challenge in spite of intense efforts.

As the BH entropy is a critical element of the thermodynamic paradigm, any viable BH mimicker should be able to account for its exceptionally large value. We will be able to show that the frozen star can do just that. This is a highly nontrivial challenge, as even the most compact of stars, when composed of conventional matter, would have an associated entropy S that is many orders of magnitude smaller than that determined by the Bekenstein–Hawking area law. Indeed, out of the many proposals for ultracompact, almost-black objects (See, for example, [24]), none of these, as far as we know, can replicate the entropy–area law—*including the factor of* $1/4$—in an unambiguous way. To understand the difficulty, suppose that we replace the infinite redshift of the BH horizon with a finite but large redshift $1/\epsilon^2$, then what is the line of demarcation between a(n almost) BH and a very compact star? Meanwhile, the BH entropy can be viewed as a geometric construct that is associated with (perfectly black) horizons and cannot be assigned generically to regular stars, no matter how compact they might be. This argument is presented at the end of Gibbons and Hawking's paper on BH thermodynamics via Euclidean actions and partition functions [25]. As we will show, Gibbons and Hawking made a couple of strong assumptions regarding the composition of the stellar material that are not valid for the frozen star; thus our results are not in contradiction with theirs.

An outstanding question about the frozen star is how a collapsing shell of matter might dynamically transition into a geometry that deviates so dramatically from the Schwarzschild geometry over a horizon-sized length scale. If this transition does transpire, such a scenario is rather suggestive of a quantum-induced phase transition. Here, we take a first step in support of this idea by applying Mathur's argument [26] that such a transition is indeed possible because the large entropy means there is an exponentially large number of microstates associated with the final-state configuration. And so we evaluate the probability of transition Γ, from the Minkowski interior of collapsing shell of matter, or incipient BH, geometry to that of a frozen star and then, following Mathur, view this as the quantum-transition probability from the initial state to a single microstate of the frozen star. Our calculation requires us to interpret a transitional layer straddling the outer surface of the star as a Euclidean instanton that mediates the stated transition. We find that the action of this instanton is equal to the BH entropy up to perturbatively small corrections and, by the above reasoning, conclude that the probability to transition to a single microstate is $\Gamma \sim e^{-S_{\rm BH}}$. Finally, applying Mathur's argument about multiplying the single-state probability by the large degeneracy of states, we are then led to a total probability of order unity, suggesting that such a transition is not only possible but likely. To our knowledge, this is the first time such an estimate of the transition probability has been successfully carried out for an ultracompact, horizonless object.

The subsequent sections are organized as follows: The next section provides a very brief review on the frozen star. Longer discussions on its conceptual origins and important physical features can be found in [3–5,7,8]. In Sec. III, we explain why the temperature of emitted radiation from the star would be perturbatively close to the Hawking value [18]. Included is a heuristic calculation that is premised on the Schwinger effect [27], which was originally presented in [28]. See Ref. [29] for a recent discussion on the temperature of compact objects that is similarly based on the Schwinger effect. Additionally, we show that a recent discussion, based on the properties of the thermal atmosphere [30], applies to our model and can be used to show that the frozen star radiates at exactly the Hawking temperature. Section IV begins with a general discussion on Euclidean BH thermodynamics, following the Euclidean-action method of Gibbons and Hawking [25]. We go on to confirm that, at leading order, $S = A/4$ for the frozen star by adopting their method, albeit in an unorthodox way as the outer surface of the star is no longer a boundary for the Euclidean spacetime like it is in the Schwarzschild case. This calculation of the entropy is novel and one of the main results in the paper. We continue on, in Sec. V, to reinterpret a narrow interpolating layer—smoothly connecting the Schwarzschild exterior to the frozen star interior—as a Euclidean instanton which facilitates the transition from the Minkowski interior of an incipient BH to a single microstate of the frozen star geometry. We are able to confirm that the probability of this transition is precisely $e^{-A/4}$, at leading order in our perturbative parameters, just as advertised. The duration of this transition is also estimated, and this result is used in Sec. VI to argue that the number operator and thus also the density matrix for the frozen star radiation could have nontrivial off-diagonal elements. We further argue, following an earlier treatment [31], that the magnitude of the off-diagonal elements scales so as to reproduce Page's famous curve describing a unitary evaporation process [32]. Finally, a brief overview is presented in Sec. VII. There, we point out that the Euclidean picture of the frozen star closely resembles a BH sourced by a condensate of stringy winding modes [33] (see also, [34]) and remark on the significance of this "coincidence."

II. THE FROZEN STAR GEOMETRY

Here, we will be reviewing the relevant aspects of the frozen star geometry, especially as it pertains to the following analysis. See [5,7] for formal details.

Let us first discuss the frozen star metric. We assume that it is static and spherically symmetric, $ds^2 = f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$, where the equality $g_{tt} = -g^{rr}$ follows from the star's characteristic equation of state $\rho + p_r = 0$.

As common to textbook discussions, one defines a mass function,

$$m(r) = 4\pi \int_0^r dx\, x^2 \rho(x) \quad \text{for } r \le R, \tag{1}$$

leading to a Schwarzschild-like form for the t–r sector,

$$f(r) = 1 - \frac{2m(r)}{r}. \tag{2}$$

If R is the radial size of the star, as implied in Eq. (1), then the total mass is defined as $M = m(R)$.

We now deviate from Schwarzschild by setting $f(r) = \varepsilon^2$, leading to the following line element,

$$ds^2 = -\varepsilon^2 dt^2 + \frac{1}{\varepsilon^2} dr^2 + r^2 d\Omega^2. \tag{3}$$

The dimensionless parameter ε^2 should be regarded as a very small number[1] (recently estimated to be smaller than 10^{-22} [35]) but always finite. With this choice, the mass function is

$$m(r) = \frac{r}{2}(1 - \varepsilon^2), \tag{4}$$

and so

$$f(r) = \varepsilon^2. \tag{5}$$

To understand the significance of the perturbative parameter ε^2, it is useful to consider the limit $\varepsilon^2 \to 0$. In this limiting case, $g_{rr} \to \infty$ at the surface of the star, so that we can identify this limit with the classical one $l_P \to 0$ (l_P is the Planck length). In other words, $\varepsilon^2 \sim \frac{l_P}{R}$.

The matter densities, or diagonal stress-tensor components, then take the forms

$$8\pi\rho = \frac{1 - (rf)'}{r^2} = \frac{1 - \varepsilon^2}{r^2}, \tag{6}$$

$$8\pi p_r = -\frac{1 - (rf)'}{r^2} = -\frac{1 - \varepsilon^2}{r^2}, \tag{7}$$

$$8\pi p_\perp = \frac{(rf)''}{2r} = 0, \tag{8}$$

where a prime represents a radial differentiation and note that all of the off-diagonal components are vanishing.

To ensure that the metric of the frozen star interior joins smoothly to the external Schwarzschild metric, we have added an outer layer T_{out} of width $2\lambda R$, with $\lambda \ll 1$, that allows for a continuous transition from one geometry to the other [5]. As the density in the bulk of the frozen star goes as $\sim \frac{1}{r^2}$ and so is formally divergent at the star's center, there is also a need to smooth out the density in the central region [5]. We have accomplished this by adding a second transitional layer T_{in} of width ηR that connects the bulk geometry to a regularized core of width ηR, where $\eta \ll 1$. The geometry of the frozen star is depicted in Fig. 1.

To understand the significance of the perturbative parameter λ, we call upon the polymer model and its string-theoretical perspective. In this picture, the string energy density is decreasing in the transitional layer from its finite value in the bulk to zero at the outer surface. This transitional layer needs to still be introduced in the limit $\varepsilon^2 \to 0$, indicating that λ and ε^2 are independent perturbative parameters. Since we have already identified small values of ε^2 with the classical limit, it follows that λ controls the strength of classical stringy effects; that is, the effects of α' or the inverse of the string tension. It can then be concluded that $\lambda^2 \sim \frac{\alpha'}{R^2}$.

The nonregularized density leads to small corrections of order η, when integrated over the entirety of this central

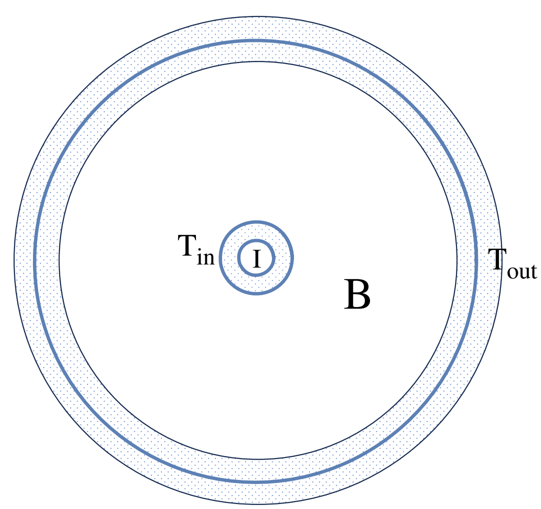

FIG. 1. The frozen star geometry. The core $r \le \eta R$ is denoted by I, the transitional layer $\eta R < r < 2\eta R$ by T_{in}, the frozen star bulk $2\eta R \le r \le R(1-\lambda)$ by B and the outer transitional layer connecting the star to the Schwarzschild region $R(1-\lambda) \le r \le R(1+\lambda)$ by T_{out}. Both transitional layers and the core are not drawn to scale, so they can be distinguished in the figure. The actual relative scale is smaller by many orders of magnitude.

[1] What we now call ε^2 had, until recently, been called ε. The change was made to emphasize its positivity.

region; meaning that the distinction between the bulk and regularized expressions is irrelevant to our leading-order calculations in Sec. IV. Further note that, in both layers, the condition $\rho + p_r = 0$ is maintained and so it is true throughout the entirety of the star's interior. As an immediate consequence, the stress-tensor conservation equation reduces from its general static form $p_r' + \frac{1}{2}(\ln f)'(\rho + p_r) + \frac{2}{r}(p_r - p_\perp) = 0$ to

$$p_r' + \frac{2}{r}(p_r - p_\perp) = 0 \tag{9}$$

or, equivalently,

$$p_\perp = \frac{1}{2r}\partial_r(r^2 p_r). \tag{10}$$

We define the interior bulk of the frozen star as the region that ends at $r = R(1-\lambda) \approx 2M(1-\lambda+\varepsilon^2)$, as can be verified by evaluating $M = m(R)$ via Eqs. (1) and (6). The exterior geometry formally begins at $r = R(1+\lambda) \approx 2M(1+\lambda+\varepsilon^2)$. The outer transitional layer is interpolating between the exterior geometry and the interior bulk.

For future reference, one can view the leading-order scaling relation $m = \frac{r}{2}$ as meaning that each spherical slice of the interior is, up to small corrections, like a BH horizon.

III. HAWKING RADIATION AND TEMPERATURE

The thermal distribution of outgoing particle emissions from a BH and their associated temperature [18] is traditionally calculated under two assumptions: (1) A future, eternal event horizon forms and (2) the exterior geometry can be treated as static. However, it turns out that these assumptions can be relaxed; in which case, compact objects emit Hawking-like thermal radiation provided that (a) they are compact enough so that the redshift near their surface is sufficiently large or, equivalently, the gravitational force $F_G = \frac{G_N M E}{r^2}$ near their surface is strong enough to produce particles with energy $E \sim 1/M$ in Planck units and (b) a suitable adiabatic condition is satisfied.

There are two separate arguments showing a Hawking flux is generated if an object has a large enough redshift at the outer surface so that it is externally indistinguishable from a schwarzschild black hole for all practical purposes.

(1) The Schwinger pair production argument, discussed in Sec. III.A and presented recently in [29], does not depend in any way on knowledge of the interior of the ultracompact object. It depends only on having a large redshift. The value of the temperature can only be approximately estimated using this method, but the fact that the object radiates is clearly established. The backreaction and the extremely weak time-dependence which is induced by it, is inconsequential to this conclusion.

(2) Mathur and Mehta in [30], argue convincingly that the properties of the object's thermal atmosphere can be used to fix the temperature of radiated modes at precisely the Hawking value. Again, establishing that a static configuration (ignoring the backreaction) can radiate, provided that it satisfies the following four conditions:

(1) The mass as seen from infinity is M.
(2) Standard semiclassical physics is a good approximation to the dynamics at $r \geq R$.
(3) The redshift in the semiclassical region reaches a a high value $O(M/m_P)$. The mass contained in the region $r \leq R$ should be $M(R) = M - o(M)$.
(4) Causality holds.

These conditions are all satisfied by the frozen star model as discussed in Sec. II.

In [36], the two properties (a) Having a compact enough object and (b) That a suitable adiabatic condition is satisfied, were formally defined, showing that horizonless objects will indeed emit thermal radiation but with two provisions: (i) There is an approximately exponential relation between the affine parameters for the null generators on past and future null infinity and (ii) the surface gravity of the object changes at a slow-enough rate. The frozen star obeys both requirements and is, therefore, expected to emit radiation with properties that are similar to those of BH-emitted Hawking radiation.

In the following, we first recall a heuristic argument from [28] (see also [29]) that relies on a gravitational Schwinger process and then apply explicitly the more formal ideas of [36] to the frozen star.

A. Gravitational particle production analog of the Schwinger effect

Let us first consider a pair-production event in the atmosphere of a massive, compact, horizonless object, which is induced by the gravitational analog of the Schwinger effect [28]. To recall, the Schwinger equation [27] predicts the rate per unit volume $\mathcal{R}^{E\ell}_{PP}$ of electron–positron pair production in an electric field $\mathcal{E}$, such that $\mathcal{R}^{E\ell}_{PP} = \frac{\alpha^2}{\pi^2}\mathcal{E}^2 e^{-\frac{\pi m_e^2}{e\mathcal{E}}}$, where $\alpha = e^2/(4\pi)$, e is the electron's charge and m_e is its mass. Now suppose that one exchanges the electric force $F_E = \mathcal{E}e$ with the gravitational force F_G in Schwinger's equation. If M is the mass of the compact object and E is the relativistic energy of the positive-energy particle, the gravitational force is approximately given by $F_G = \frac{G_N M E}{r^2}$, where we restored the dependence on Newton's constant G_N. If the object is compact enough so that its radius satisfies $R \approx 2MG_N$, then $F_G \sim \frac{1}{2}\frac{E}{R}$. Substituting $m_e^2 \to E^2$ and $e\mathcal{E} \to \frac{1}{2}\frac{E}{R}$ into $\mathcal{R}^{E\ell}_{PP}$, one finds that the resulting rate of the gravitational pair production per unit volume $\mathcal{R}^G_{PP}$ in the proximity of the object's outer surface is

$$\mathcal{R}^G_{PP} \sim \frac{\hbar}{R^4}\left(\frac{RE}{2\pi\hbar}\right)^2 e^{-\frac{2\pi RE}{\hbar}}. \tag{11}$$

This rate is maximized when $E = \frac{\hbar}{\pi R}$, which is of the order of the Hawking temperature $T_H = \frac{\hbar}{4\pi R}$ of a Schwarzschild BH; in which case, $\mathcal{R}^{\mathrm{G}}_{PP} \sim \frac{\hbar}{R^4}$. This estimate suggests that one Hawking-like pair of particles, each with energy $E \sim T_H$, is produced per light-crossing time R from a volume R^3, which is the standard Hawking emission result. Meanwhile, away from the outer surface, the rate of pair production becomes exponentially small; indicating that Hawking-like radiation can originate near the outer surface of a compact object, even if it is not formally a horizon.

B. Temperature and spectrum of emitted radiation

1. Formal conditions for the validity of the calculation

For a more formal derivation, we recall the results of [36], which reveals that horizonless objects will indeed emit thermal radiation under certain conditions.

Following [36], let us first consider null curves starting from past infinity $\mathcal{J}^-$ and arriving at future infinity $\mathcal{J}^+$. There will be some invertible relation between their respective affine parameters U and u; namely, $U = p(u)$ and $u = p^{-1}(U)$. The surface gravity can then be defined formally as $\kappa(u) = -\frac{\ddot{p}(u)}{\dot{p}(u)}$, where dots denote derivatives with respect to u. Next, picking one particular null curve and labeling it by U_* on $\mathcal{J}^-$ and u_* on $\mathcal{J}^+$, one can then write

$$U = U_* + C_* \int_{u_*}^{u} e^{-\int_{u_*}^{\bar{u}} \kappa(\tilde{u}) d\tilde{u}} d\bar{u}, \tag{12}$$

for some dimensionless constant C_*.

Assuming that $\kappa(u)$ satisfies the following adiabatic condition:

$$\frac{|\dot{\kappa}(u_*)|}{\kappa^2(u_*)} \ll 1, \tag{13}$$

one can further show that outgoing particles reaching $\mathcal{J}^+$ at u_* obey a Planckian distribution. In which case, Hawking-like radiation exists with a temperature of

$$T(u_*) = \frac{\kappa(u_*)}{2\pi}. \tag{14}$$

These ideas can be applied explicitly to the frozen star because they rely solely on the nature of spacetime outside the star. Importantly, the exterior region for $r > R_{\mathrm{star}} \equiv R(1+\lambda) \approx 2M(1+\lambda+\varepsilon^2)$ is described by the Schwarzschild metric; thus the affine parameters are the Eddington-Finkelstein outgoing coordinate $u = t - r_*$, where r_* is the usual tortoise coordinate that is defined by $dr_* = \frac{dr}{\sqrt{-g_{tt}g^{rr}}}$, and U is a Kruskal-like coordinate that is related to u in the standard way. One can write r_* for both the interior and exterior regions and, since r^* is continuous across the inner and outer surfaces of the translational layer (and throughout the layer itself), u and U are continuous as well.

The surface gravity for the Schwarzschild metric, when defined at the surface of a star of radius R_{star}, is

$$\kappa^2 = -\frac{1}{2}(\nabla_\mu \xi_\nu)(\nabla^\mu \xi^\nu)\Big|_{r=R_{\mathrm{star}}}, \tag{15}$$

where ξ is the timelike Killing vector. As in the case of a Schwarzschild BH, κ can be expressed as $\kappa(r) = \frac{M}{r^2}$, so that, on the surface $r = R_{\mathrm{star}}$ of a frozen star, the leading-order result is simply $\kappa = \frac{M}{R^2}$.

With the above relations in hand, we can now define the affine parameters U on $\mathcal{J}^-$ and u on $\mathcal{J}^+$ for the frozen star,

$$U \approx U_H^* - A_* e^{-\kappa_* u}, \tag{16}$$

$$u = -\frac{1}{\kappa_*}\ln\left(\frac{U_H^* - U}{A_*}\right), \tag{17}$$

where A_* is a dimensional constant, the best estimate for the would-be horizon is at $U_H^* = R_{star} \approx R$ and notice that κ_* satisfies $\kappa_* = -\ddot{U}/\dot{U}$.

The surface gravity naively vanishes in the interior of the frozen star because of the constancy of g_{tt} and g^{rr}. However, as we will argue later, each spherical slice of the interior maintains the same properties that one would attribute to a would-be horizon of that radial size. The physical reason that these inner slices do not directly contribute to the radiation is not that their Killing vector vanishes but because of the large interior redshift greatly suppressing the propagation of particles in the bulk of the frozen star. So that, even if such particles are produced, they are effectively trapped within the star.

The adiabatic condition (13) for the frozen star model can be verified by evaluating the derivatives of the surface gravity. In general, for a star of radius R_{star},

$$\dot{\kappa} = \frac{d\kappa}{du} = \frac{\partial\kappa}{\partial t} - f(r)\frac{\partial\kappa}{\partial r}\Big|_{r=R_{star}}. \tag{18}$$

In the frozen star case, $\frac{\partial\kappa}{\partial t} = 0$ and $\kappa(r) = \frac{M}{r^2}$, and so

$$\dot{\kappa} = \frac{2M}{r^3} f(r)\Big|_{r=R_{star}}, \tag{19}$$

which at leading order becomes

$$\dot{\kappa} = \frac{\varepsilon^2 + \lambda}{R^2}. \tag{20}$$

On the other hand, at $r = R_{\mathrm{star}}$, another leading-order result is

$$\kappa^2 = \frac{1}{4R^2}, \tag{21}$$

so that the adiabatic condition (13) translates into

$$\frac{|\dot{\kappa}|}{\kappa^2} = \frac{1}{4}(\epsilon^2 + \lambda) \ll 1, \tag{22}$$

meaning that the adiabatic condition for the frozen star is easily satisfied.

2. Hawking-like radiation from a frozen star

We are now well equipped to derive the thermal spectrum of the emitted radiation for a frozen star by suitably adapting the derivation in [36].

Defining the modes on $\mathcal{J}^-$ as

$$\xi(u) = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega u} \tag{23}$$

and those on $\mathcal{J}^+$ as

$$\Phi(U) = \frac{1}{\sqrt{4\pi\Omega}} e^{-i\Omega U}, \tag{24}$$

we can express the Bogoliubov coefficient

$$\alpha_{\omega\Omega} = i \int dU[\partial_U \Phi^* \xi - \Phi^* \partial_U \xi] \tag{25}$$

as

$$\alpha_{\omega\Omega} = -\frac{1}{4\pi\sqrt{\omega\Omega}} \int_{-\infty}^{U_H^*} dU e^{i\Omega U}\left[\Omega\left(\frac{U_H^* - U}{A_*}\right)^{\frac{i\omega}{\kappa_*}} + \frac{\omega}{\kappa_*}\left(\frac{U_H^* - U}{A_*}\right)^{\frac{i\omega}{\kappa_*}-1}\right]. \tag{26}$$

Recalling that $\beta_{\omega\Omega} \approx -i\alpha_{\omega\Omega}$ and $i = e^{i\frac{\pi}{2}}$, we then find that

$$\beta_{\omega\Omega} = -i\frac{1}{2\pi}\sqrt{\frac{\omega}{\Omega}}\frac{(\Omega)^{-\frac{i\omega}{\kappa_*}}}{\kappa_*} e^{\frac{\pi\omega}{2\kappa_*}}\Gamma\left(\frac{i\omega}{\kappa_*}\right), \tag{27}$$

which allows us to calculate the expectation value of the number operator for the outgoing particles via the usual definition $\langle 0|b^\dagger b|0\rangle = \int_0^\infty d\Omega \beta_{\omega\Omega}\beta^*_{\omega'\Omega}$. The result is

$$|\beta_{\omega\Omega}|^2 = \frac{1}{e^{\frac{2\pi\omega}{\kappa_*}} - 1}\delta(\omega - \Omega). \tag{28}$$

The frozen star temperature can now be readily identified as

$$T_{\rm FS} = \frac{\kappa_*}{2\pi} = \frac{1}{8\pi M} = T_H, \tag{29}$$

to leading order in ϵ^2 and λ.

One might be concerned with the effects of the backreaction on the geometry due to the emission of Hawking-like modes, as such effects could modify the position of the star's outer surface with respect to its Schwarzschild radius (or would-be horizon). However, if the surface starts out having an exponentially large relation between the affine parameters as in our Eq. (16), it will remain so for a long time, while the coefficient κ_* can be treated as a constant. This is essentially the same argument that Hawking had made in Ref. [19] to justify using the eternal black hole geometry for the purposes of computing the spectrum of radiation. The geometry can be treated as static at each instant of time, with the frozen star solution then parametrized by the appropriate mass and corresponding radius. None of this is specific to the existence or not of a formal horizon, and so is equally applicable to our model or any other one that starts out with an exponentially large relation between the affine parameters as in Eq. (16). As such, our derivation of the temperature of emission and the upcoming one for the entropy of the star are justified.

In summary, the temperature of the emitted radiation from a frozen star is the same as Hawking temperature up to a perturbatively small correction.

IV. ENTROPY

We start here with a calculation of the entropy of the frozen star by integrating the first law, as was first done by Bekenstein and Hawking [17,18]. Following this, the entropy of the frozen star will be calculated using the Euclidean-action method as pioneered by Gibbons and Hawking [25].

A. Entropy from the first law

The first law states, in the current context, that

$$dS = \beta(M)dM, \tag{30}$$

where $\beta(M)$ is the inverse of the temperature.

To derive the entropy of a frozen star of mass M, it is helpful to first consider a star of much-smaller mass $m_0 \ll M$. From the calculation of the temperature in the previous section, the inverse temperature at leading order is

$$\beta(m_0) = 8\pi m_0. \tag{31}$$

One can now imagine increasing the mass of the frozen star by a small amount Δm, so that $m_0 + \Delta m$ is the new mass and $8\pi(m_0 + \Delta m)$ is the new inverse temperature. It follows that

$$S_{FS}(m_0 + \Delta m) = S_{FS}(m_0) + 8\pi m_0 \Delta m + \mathcal{O}(\Delta m^2). \quad (32)$$

Integrating the previous equation, with the key observation that $\beta(m)$ grows linearly with m,[2]

$$\beta(m) = 8\pi m, \quad (33)$$

one obtains

$$\begin{aligned} S_{\rm FS}(M) &= S_{\rm FS}(m_0) + \int_{m_0}^{M} dm\, 8\pi m \\ &= S_{\rm FS}(m_0) + 4\pi(M^2 - m_0^2) \\ &= S_{\rm FS}(m_0) + S_{\rm BH}(M) - S_{\rm BH}(m_0). \end{aligned} \quad (34)$$

If we start out with the frozen star's seed radius being close to its core radius, then $S_{\rm FS}(m_0)$ and $S_{\rm BH}(m_0)$ are negligible in comparison to $S_{\rm BH}(M)$. The conclusion is that, to leading perturbative order, the frozen star entropy is equal to the Bekenstein–Hawking entropy of a same-mass BH,

$$S_{\rm FS}(M) = S_{\rm BH}(M). \quad (35)$$

B. Entropy from Euclidean action

Next is the calculation of the entropy of the frozen star using the method of Gibbons and Hawking [25]. We start by calculating the action of the bulk of the frozen star, which extends from the center up to the beginning of the outer transitional layer. We then calculate the entropy of the bulk and show that it is equal, at leading order in ϵ^2 and λ, to the entropy of a same-mass BH. The action of the outer transitional layer will then be calculated separately. In the next section, this outer layer will be interpreted as a Euclidean instanton that mediates the transition between a collapsing shell of ordinary matter with a Minkowski interior and the bulk of a frozen star. The current section ends with a calculation of the entropy for the total spacetime, which again agrees at leading order with that of a BH of the same mass.

The Euclidean-action method then provides us with two ways of determining the frozen star entropy; either by calculating the action of the bulk of the interior or by evaluating the action for the total spacetime. This duality then provides us, in turn, a means for realizing both of the competing perspectives on the entropy, matter based and geometric, but with no ambiguity in the final answer. Note that the geometric point of view follows automatically if the limits $\epsilon^2 \to 0$ and $\lambda \to 0$ are imposed; in which case, the outer surface of the star tends to that of a true horizon. However, it is more interesting and realistic to consider what happens when ϵ^2 and λ differ from zero.

[2] A similar scaling relation is used in [37] to define an effective surface gravity.

Let us now recall how the calculation in [25] is performed. Gibbons and Hawking started out by evaluating the action of the Euclidean BH solution $I[g]$ and applying the identification

$$I[g] = \ln \mathcal{Z}[\beta] = -\beta F = S - \beta M, \quad (36)$$

where $\ln \mathcal{Z}[\beta]$ is the logarithm of the thermal partition function and F is the Helmholtz free energy.

The Euclidean Einstein action consists of both a volume and a surface contribution,

$$I[g] = -\frac{1}{16\pi}\int_{\mathcal{M}} \sqrt{g}\mathcal{R} - \frac{1}{8\pi}\int_{\partial\mathcal{M}} \sqrt{h}[K]. \quad (37)$$

The integrand of the volume portion trivially vanishes for the Schwarzschild solution, and one is left only with the boundary contributions from the horizon and timelike infinity, each of which is an integral of the trace of the extrinsic curvature K (or second fundamental form), where $[K]$ indicates that the contribution from flat space has been subtracted off. The Euclidean geometry has zero area at the horizon and, once the conical singularity at the horizon has been resolved by assigning the Euclidean time τ with its appropriate periodicity, $\Delta\tau = \beta = 8\pi M = 4\pi R$, the contribution from the horizon vanishes.

The only surviving contribution is then that of the asymptotic boundary I_∞. The Schwarzschild result goes as

$$I_\infty = -\frac{1}{8\pi}\oint d\tau \sqrt{g^{rr}}\partial_r(4\pi r^2\sqrt{g_{\tau\tau}})_{|r\gg 2M}. \quad (38)$$

After appropriately subtracting the divergent flat-space contribution and taking the $r \to \infty$ limit at the end, one finds that

$$I = -\frac{1}{2}\beta M. \quad (39)$$

Now recalling Eq. (36), one arrives at an entropy of $S = \beta M/2$ or, since $\beta = 8\pi M$, $S = 4\pi M^2$. This is simply $S = S_{\rm BH} = A/4$ and thus the Bekenstein–Hawking area law [17,18]. For future reference, one should notice that $I_\infty = -S_{\rm BH}$.

The Gibbons-Hawking calculation relies on the fact that the Euclidean action is a boundary term on the solution of the Einstein equations of motion. Clearly, if the sole contribution to the Euclidean action comes from the boundary term at infinity, one is guaranteed to get exactly the BH entropy. In the original calculation, the interior of the BH is excised, the horizon shrinks to a point and the conical singularity is removed. Therefore, the sole contribution does indeed come from the asymptotic boundary term. However, in Sec. 5 of [38], it was pointed out that the same result is still obtained for a horizonless and nonsingular spherical geometry for which the volume of the sphere at the origin shrinks to zero. In this case, there is no contribution from the inner boundary and the entropy is exactly equal to that of a BH. We will encounter a similar situation in what follows.

1. Entropy of the frozen star bulk

To calculate the entropy of the bulk (again to leading order in ε^2 and λ), it is important to realize that β depends on r. From Eq. (33) and from the relation $m(r) = \frac{1}{2}r$ in Eq. (4), it follows that

$$\beta(r) = 4\pi r, \tag{40}$$

which is obviously different than the constant $\beta_H = 8\pi M$ in [25]. A radially dependent compactification scale may seem unusual, but then so is the frozen star, which features large deviations from Schwarzschild over the entirety of its interior. The Euclidean geometry of the frozen star is depicted in Fig. 2. A similar geometry was found in [34].

It is possible to understand this peculiar dependence by considering the near-horizon geometry of the Euclidean Schwarzschild BH, for which the relevant part of the metric can be expressed as $\frac{4\pi^2}{\beta_H^2}\tilde{r}^2 d\tau^2 + d\tilde{r}^2$. Now, replacing the constant β_H by $\beta \sim r$ and taking into account that $\tilde{r} \sim r$ for the frozen star interior, one finds a constant $g_{\tau\tau}$. In any case, such radial dependence inside of the frozen star does not affect the validity of the method. As we have already observed in Sec. III, in the exterior of the frozen star, the temperature is a constant and its inverse is equal at leading order to the standard value of β_H.

The Euclidean action of the bulk does not receive any contribution from the extrinsic-curvature boundary terms because, as argued above, it vanishes at $r = 0$ due to the shrinking of the volume of the sphere to zero at the origin and the outermost surface of the bulk is not a true boundary of spacetime. The remaining volume integral is

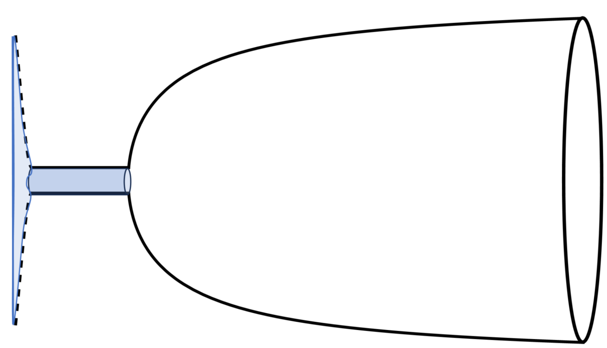

FIG. 2. The frozen star Euclidean geometry. The exterior geometry is essentially the same as the Euclidean Schwarzschild geometry. The bulk has the geometry of a cylinder of radius $\sqrt{\varepsilon^2}$ and the geometry of the inner transitional layer, drawn here not to scale, expands back to the asymptotic radius.

$$I_{\rm Bulk} = -\frac{1}{16\pi}\int_{\rm Bulk} d^4x\sqrt{g}\mathcal{R}, \tag{41}$$

and using Eq. (40) for the compactification scale of the Euclidean time direction, one can rewrite this as

$$I_{\rm Bulk} = -\frac{1}{4}\int_0^R dr\, 4\pi r^2 r\mathcal{R}. \tag{42}$$

Let us next use the trace of Einstein's equations, while recalling the form of the stress tensor $\mathcal{T}^a_b$ from Sec. II, to recast the scalar curvature in terms of the energy density and pressure in the frozen star's bulk, $\mathcal{R} = -8\pi\mathcal{T} = -8\pi(-\rho + p_r + 2p_\perp) = 16\pi\rho = \frac{2}{r^2}$. So that

$$I_{\rm Bulk} = -2\pi\int_0^R dr\, r = -\pi R^2 \tag{43}$$

and, therefore,

$$I_{\rm Bulk} = -\frac{1}{2}\beta(R)M. \tag{44}$$

Using the relationship (36) between the action and the entropy, we conclude that $S_{\rm Bulk} = S_{\rm BH}$.

It is worthwhile to understand how we managed to evade Gibbons and Hawking's conclusion that a normal star or, for that matter, any horizonless object could not have a geometric entropy, regardless of how compact it may be [25]. Looking at the discussion leading up to their Eq. (3.16), one can identify two critical assumptions: (1) The star is composed of "normal" matter in thermal equilibrium and (2) its radius R and temperature T satisfy $R^3 \gg T^{-3}$. Note that T here is not the Hawking temperature but rather the actual thermodynamic temperature of the matter.

The condition ($VT^3 \gg 1$), as well as an approximately flat spacetime. Both of these conditions are violated in a major way by the frozen star model. The same is true of Eq. (3.17), which generalizes (3.16) to curved space provided that $1/T$ is much smaller than gravitational length scales. In our case, these length scales are parametrically equal. The former condition does not apply to our model, as the source of the frozen star is exotic matter at zero temperature.[3] We would like to emphasize that although we eventually express our results in terms of the energy density and pressure, this is done only for convenience, as our own calculation of the thermodynamic potential is based on the Euclidean Einstein action and certain components of the Einstein (geometric) tensor. In other words, in spite of some appearances to the contrary, our calculation

[3] Fundamentally, the frozen star would consist of a state of strings close to Hagedorn temperature [1], but this matter is no less exotic.

of the entropy is best viewed as being primarily a geometric one.

2. Action of the outer transitional layer

It is straightforward to evaluate the action of the outer transitional layer I_{TL} using the metric and stress-tensor components as presented in Sec. II. Recall that this transitional layer is a thin shell of width $2\lambda R$, with $\lambda \ll 1$. The metric of this layer smoothly connects the bulk metric to the exterior Schwarzschild metric. For simplicity, we evaluate the action to leading order in ε^2 and λ and similarly for the rest of the calculations. Here, we only need to evaluate the volume term in the action because the boundary terms are supported only on the true boundaries at $r = 0$ and $r \to \infty$, which are outside the domain of integration. In this layer, it remains true that $p = -\rho$, but $p_\perp$ is nonvanishing and large, so that now $\mathcal{R} = -8\pi\mathcal{T} = 16\pi(\rho - p_\perp)$. It follows that

$$I_{\mathrm{TL}} = -4\pi\beta \int_{\mathrm{TL}} dr\, r^2(\rho - p_\perp), \tag{45}$$

where, in principle, β could be a function of r; however, since the width of the boundary layer scales as $\sim\lambda$ and the value of β at the outer edge of the layer is $\beta = 4\pi R$, we can ignore any possible r dependence at leading order.

We now recall our first form of the frozen star conservation equation (9) and also consider that $p_r' \sim \frac{|p_r|}{\lambda} \gg |p_r|, \rho$, from which it can be deduced that $p_\perp \sim \frac{p_r}{\lambda} \gg |p_r|, \rho$. Hence, we can ignore the contribution from ρ in Eq. (45) at leading order. Also recalling our second form of the conservation equation (10) and observing that the factors of r appearing in Eq. (45) can be approximated as R's, we can reexpress the action I_{TL} as

$$\begin{aligned} I_{\mathrm{TL}} &= 16\pi^2 R \int_{R(1+\varepsilon^2-\lambda)}^{R(1+\varepsilon^2+\lambda)} dr \frac{R}{2} \partial_r(r^2 p) \\ &= 8\pi^2 R^2 (r^2 p)\Big|_{R(1+\varepsilon^2-\lambda)}^{R(1+\varepsilon^2+\lambda)}. \end{aligned} \tag{46}$$

The value of p at the upper end is its Schwarzschild value of $p = 0$, while the value of p at the lower end is its frozen star bulk value of $r^2 p = -\frac{1}{8\pi}$, meaning that

$$I_{\mathrm{TL}} = +\pi R^2 = S_{\mathrm{BH}}. \tag{47}$$

3. Total entropy of the frozen star

The action of the Schwarzschild exterior is given by

$$I_{\mathrm{EXT}} = -\frac{1}{16\pi} \int_{r>R(1+\varepsilon^2+\lambda)} dx^4 \sqrt{g}\mathcal{R} - \frac{1}{8\pi} \int_{\partial\mathcal{M}} \sqrt{h}[K]. \tag{48}$$

The volume term vanishes for the exterior Schwarzschild metric and the surface contribution comes only from the boundary at infinity, as the boundary at $r = 0$ is outside the domain of integration.

The contribution of the boundary at infinity is given in Eq. (39) and is equal to $-S_{\mathrm{BH}}$, so that

$$I_{\mathrm{EXT}} = -\frac{1}{2}\beta M = -S_{\mathrm{BH}}, \tag{49}$$

which can be combined with our previous calculations to give

$$\begin{aligned} I[\text{frozen star}] &= I_{\mathrm{Bulk}} + I_{\mathrm{TL}} + I_{\mathrm{EXT}} \\ &= -S_{\mathrm{BH}} + S_{\mathrm{BH}} - S_{\mathrm{BH}} = -S_{\mathrm{BH}}. \end{aligned} \tag{50}$$

It follows that the frozen star has a leading-order entropy that is equal to that of a BH with the same mass,

$$S_{\mathrm{FS}} = S_{\mathrm{BH}}. \tag{51}$$

Let us comment that, because the outer surface becomes a true horizon in the $\varepsilon^2, \lambda \to 0$ limits, it must be true that the zeroth-order contributions to I_{Bulk} and I_{TL} exactly cancel, just as was found above. This tells us, unequivocally, that the scaling relation $\beta = 4\pi r$ is the correct choice as anything else would have offset the required cancellation.

It should be noted that, as long as the temperature is known, a high redshift is not required for evaluating the Euclidean path integral, which can be understood in two ways. First, one can integrate over the whole space and then notice, as in Ref. [38], that the action is a total derivative and that there are two boundary terms, one at $r \to \infty$ and the other at $r = 0$. The contribution from infinity gives the same entropy as a black hole of the same mass, so that, for the action to give the correct value of the entropy, the contribution from $r = 0$ has to vanish, which it does. This calculation is formally valid regardless of the value of the redshift at the surface of the frozen star. The other way is to calculate the bulk action of the frozen star by identifying the inverse temperature, as we did in Eq. (40), and then calculate the action of the transition layer and the exterior. Here, again the calculation is formally valid for any redshift. The fact that the redshift is high determines how well the leading term in ε^2 and λ estimates the action.

V. DYNAMICAL COLLAPSE OF MATTER TO A FROZEN STAR

As suggested in the Introduction, it is unreasonable to expect a "normal" system of collapsing matter to slowly evolve into an ultracompact object whose geometry deviates from the Schwarzschild geometry on horizon-order length scales. What is rather needed is something akin to a quantum-induced phase transition, where we have implicitly recalled that the final system of matter is, in its fundamental description, a highly quantum state of strings at a finite temperature. It is then natural, in the Euclidean picture, to regard the transitional layer as a gravitational instanton that mediates the transition from an infalling shell of matter with a Schwarzschild exterior and an empty interior to a frozen star.

There is, indeed, a long tradition of using Euclidean instantons in just this way, whether it has been to facilitate bounces between two phases of de Sitter (e.g., [39]), to induce a quantum-tunneling process from nothingness to a Lorentzian inflationary cosmology (e.g., [40]) or, in the guise of a domain wall, to trigger the creation of a pair of BHs (e.g., [41]). For a general discussion, see Refs. [42,43]. The theme in all these works, in analogy to similar ideas in quantum mechanics, is that the probability of transition is given (up to a numerical prefactor) by $\Gamma = e^{-I_{\rm Inst}}$, where the instanton action $I_{\rm Inst}$ is typically evaluated using either the on-shell solution or at a saddle point of an appropriate partition function. The instanton interpolates between an initial spacelike section of some specific solution, which is typically the false vacuum solution, and a final spacelike section of another solution, so that the action $I_{\rm Inst}$ can be defined by a subtraction procedure of the respective Euclidean actions of these two solutions.

In our case, the initial-state geometry is defined on the outermost slice of the transitional layer and corresponds to the solution when the collapsing shell of matter, which can also be viewed as a domain wall, is just on the brink of passing through its would-be horizon. The interior of the shell at this initial time contains the false vacuum, which is simply flat Minkowski space. All this is in agreement with our calculation at the end of the last section, which makes it clear that the Euclidean action vanishes on the initial-state spacelike slice, since a Schwarzschild exterior enclosing normal matter will have a negligible free energy until a horizon forms.

Meanwhile, the final-state geometry is defined on the innermost slice of the transitional layer, at a radius just inside the would-be horizon.[4] Once this time has been reached, the shell and its once-empty interior has transitioned into a fully formed frozen star. From this perspective, the inner-most slice represents a bubble enclosing the true vacuum, as the frozen star represents a state of maximal entropy or minimal free energy.

This picture depends on having, from an external perspective, a finite initial thickness for the matter shell and a slow velocity for its descent. Given that the shell's width far exceeds that of the transitional layer, $2\lambda R$, and that its collapse is slow, $\frac{dr}{dt} \sim \lambda$, the phase transition can be triggered long before the entirety of the shell passes through its Schwarzschild radius.

The standard prescription in the case of a bubble or bounce (B) mediated transition at finite temperature from a false vacuum (FV) to the true one is an exponent of the form [42,43]

$$\ln \Gamma = I_B - I_{\rm FV}, \tag{52}$$

which is, in our case,

$$\ln \Gamma = I[R(1+\varepsilon^2-\lambda)] - I[R(1+\varepsilon^2+\lambda)] = -I_{\rm TL}, \tag{53}$$

and so

$$\Gamma \sim e^{-S_{\rm BH}}, \tag{54}$$

as previously advertised. Formally, in the classical limit of $l_P \to 0$, the transition probability vanishes and the frozen star never forms; the shell rather continues on to form a singular BH. As will soon be shown, there is evidence that, if a frozen star does form, its evaporation is unitary, painting a nicely consistent picture.

We now follow Mathur (e.g., [26]) and identify this as the probability for transition to any one of the $e^{S_{\rm FS}} = e^{S_{\rm BH}}$ frozen star microstates, so that the total probability is

$$\Gamma_{\rm Tot} \sim e^{+S_{\rm BH}}\Gamma \sim 1. \tag{55}$$

That is to say, the transition from an incipient BH to any frozen star configuration of equal mass is a likely, if not certain, outcome.

A. How fast?

As for the duration of the transition from an external observer's perspective, this is simply the width of the transitional layer $2\lambda R$, divided by the shell's approximate radial velocity in the outer layer $\lambda \pm \varepsilon^2 \sim \lambda$. In other words, a timescale of order R, the light-crossing time of the star. Consider though that the analog calculation for a BH is exponentially large. However, from the perspective of a particle that exits the matter shell just before the frozen star has fully formed, the relevant timescale is the proper-time equivalent or, quite simply, the width of the transitional layer divided by a speed of order unity, $T_{\rm trans} \sim \lambda R \sim \sqrt{\alpha'} \sim l_s$, where we have recalled a discussion from Sec. II relating λ to the inverse of the string tension and l_s is the fundamental string length, which in weakly coupled string theory satisfies $l_P = l_s g_s$ with g_s being the string-coupling strength.

The scale $T_{\rm trans}$ can be compared to the analog timescale in general relativity that is used in the calculation of the Hawking effect, where the transition (or horizon formation) is assumed to be instantaneous. This string-scale fuzziness in the transition time—or, equivalently, in the thickness of the transitional layer—will be important in what follows.

VI. RADIATION REVISITED

In this section, we argue on the basis of previous discussions in [31,44] that, when the entropy $S_{\rm BH}$ is finite, the evaporation of the radiation is consistent with unitarity. The key observation is that this finiteness implies that the

[4]One of our working assumptions is that $\varepsilon^2 < \lambda$ (see Sec. II), and so $R + \varepsilon^2 - \lambda < R$.

would-be horizon is not a surface of infinite redshift but rather a surface of large, finite redshift and, consequently, has some quantum width. As a result, the number operator of the Hawking particles is no longer diagonal and likewise for the density matrix of the emitted radiation.

It proves to be a worthwhile exercise to think about the radiation process of the frozen star in the context of Hawking's seminal calculation of the thermal spectrum [18]. In Hawking's original model, one considers a collapsing shell of matter, viewed as an incipient BH, and focuses on a certain class of null rays. Namely, those rays which enter the interior of the shell when its radius $R_{\rm shell}$ is substantially larger than its Schwarzschild radius R but exit the interior at a time close to that of horizon formation, so that $(R_{\rm shell} - R)_{\rm in} \gg (R_{\rm shell} - R)_{\rm out}$. Because the ray exits when $(R_{\rm shell} - R)_{\rm out}/R \ll 1$, it undergoes an exponentially large redshift after passing through the interior region. From the perspective of an asymptotic observer, it is this strong time-dependent effect that leads to the creation of Hawking particles. Hawking was able, of course, to make this description of particle emission from an incipient BH quantitatively precise.

The analogous situation for the frozen star is that the null ray must enter the collapsing matter shell before the transition (as described in Sec. V) takes place but then exit the shell only after the transition has transpired. This is the only way in which the ray can experience the effect of a suitably large enough redshift. It can then be deduced that, if the transition occurs at an advanced time of $v = v_{\rm trans}$, the prospective Hawking mode should have entered the matter distribution no later than this time. What should now be clear is that $v_{\rm trans}$ is the analog of Hawking's v_0, which is defined by the last incoming ray that can exit before the horizon finally forms. For now on, we will denote both of these cutoff times by $v_\star$.

If the transition from ordinary matter to a frozen star was an instantaneous process, this would be the end of the story, as the mathematics would basically replicate those of the Hawking calculation. However, as discussed in the previous section, the timescale of the transition is, locally, given by $T_{\rm trans} \sim \lambda R \sim l_s$[5]; meaning that $v_\star$ in the frozen star model should not be regarded as precise but should rather be smeared over this scale. Alternatively, one could view the smearing as an uncertainty in the location of the would-be horizon. Either way, this smearing modifies the Hawking calculation in a subtle but important manner. Fortunately, we are well positioned to account for just such an effect, as will now be explained.

Let us recall an earlier discussion [44], where the effects of a quantum-uncertain horizon position were incorporated into the otherwise standard Hawking calculation. Here, we will just sketch the argument, leaving a precise calculation to a future publication.

The key revision of Hawking's calculation in [44] is based on an integral that appears in Hawking's original calculation of the number operator for the outgoing particles,

$$I_C = \frac{1}{2\pi}\int_{-\infty}^{\infty} dv\, e^{iv(\omega' - \omega'')} = \delta(\omega' - \omega''), \tag{56}$$

where ω' and ω'' are incoming-particle frequencies and the advanced time is defined as $v = t + r_*$. For a formal definition of the number operator, see Sec. III. This integral is to be adapted so that it includes the effects of semiclassical physics. In [44], these effects resulted from the quantum blurring of the horizon position; while here, they result from the string-scale blurring in the transition time. The resulting modified integral depends on an additional parameter $v_{\rm shell}$,

$$I_{\rm SC}(\omega' - \omega''; v_{\rm shell}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dv\, e^{i(v - v_{\rm shell})(\omega' - \omega'')}, \tag{57}$$

where $v_{\rm shell}$ is to be regarded as a variable that is to be integrated against a normalized Gaussian probability distribution $P(v_{\rm shell})$ with width σ. In the case of the frozen star, $\sigma \propto T_{\rm trans}$.

The diagonal and off-diagonal contributions are then separated,

$$\begin{aligned}&\langle I_{\rm SC}(\omega' - \omega''; v_{\rm shell})\rangle\\ &\quad = \delta(\omega' - \omega'') + \left\langle \frac{1}{2\pi}\int_0^{v_{\rm shell}} dv'\, e^{iv'(\omega' - \omega'')}\right\rangle,\end{aligned} \tag{58}$$

where $\langle f(v)\rangle = \int dv P(v) f(v)$.

Having mapped out a formally similar framework to that of [44], we can now simply recall the end result. After an extremely long calculation, it was revealed that the number operator for outgoing particles picks up an off-diagonal part[6] that is a rather complicated combination of gamma functions and exponential factors whose arguments contain the two out-frequencies ω, $\tilde{\omega}$ and their difference $\Delta\omega = \omega - \tilde{\omega}$. Most important for current considerations is that the off-diagonal contribution scales with σ/R, which, on general grounds, should scale as $1/\sqrt{S_{\rm BH}}$ [31].

Although the formal steps has been glossed over, it is useful to consider the actual mechanism that is responsible for the off-diagonal terms in the number operator. First, in the Hawking calculation, the number operator is measuring the overlap of two incoming waves at the same advanced time v. Such waves form an orthogonal basis in

[5]We use the local timescale $R\lambda$ and not the asymptotic scale R as what is required is the analog of Hawking's instantaneous-collapse model.

[6]It is explained in [44] as to why the other two types of two-point functions, that of ingoing particles and mixed, are unaffected by the inclusion of quantum blurring.

frequency-space, leading to adiagonal expression in frequency space $I_C \sim \delta(\omega' - \omega)$. This diagonal character is then preserved by the outgoing waves in the number operator after performing a subsequent Fourier transform. On the other hand, if these modes become wavepackets that are smeared over a finite timescale, then the orthogonality condition is modified in such a way that the would-be delta function becomes a Gaussian and the strength of the off-diagonal elements in the number operator scales linearly with the duration of the smearing scale. This happens in part because there are two subsequent Fourier transforms that are each weighted by a Gaussian. The first is the "expectation value" appearing in Eq. (58) and the second transforms a function of $\omega' - \omega''$ into a function of $\Delta\omega = \omega - \tilde{\omega}$, namely, the number operator.

Another long calculation, this time in [31], reveals that an off-diagonal correction of this strength in the number operator translates into off-diagonal elements in the multiparticle density matrix that are suppressed by a factor of $(\sigma/R)^2$ with respect to the diagonal elements. Consequently, the off-diagonal contributions are exponentially suppressed at early times but will become significant and eventually dominate, once the number of emitted particles $N \lesssim S_{\mathrm{BH}}$ is enough to compensate for the suppression factor $(\sigma/R)^2 \gtrsim 1/S_{\mathrm{BH}}$. This tipping point occurs near the midway point of evaporation, the so-called Page time [32]. Moreover, this type of relation between diagonal and off-diagonal elements is exactly what is needed to reproduce the famous Page curve [32], which can be viewed as evidence of a unitary evaporation process. That this is so has already been verified in [31]. Here, we recall our previous form of the entropy-versus-time plot, reproduced in Fig. 3 and refer back to the original source for mathematical details.

Tracking this flow of information over the lifetime of the star is operationally unfeasible. Hence, this form of density matrix is in agreement with a previous argument that a

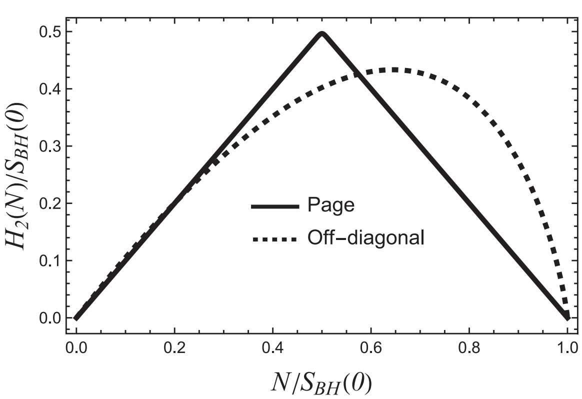

FIG. 3. Dependence of the Rényi entropy H_2 of the BH radiation on the number of emitted particles N for the case of off-diagonal elements discussed in the text (dashed). For comparison, the solid line depicts the prediction of the Page model. Figure reproduced from [31].

standard BH and a BH mimicker, such as the frozen star, should be observationally indistinguishable when in equilibrium [45]. Observational differences would rather require an out-of-equilibrium event such as a BH merger in which the BH is emitting a macroscopic amount of energy in the form of gravitational waves.

VII. CONCLUSION

Our main lesson is that the frozen star may be unique among ultracompact but horizonless objects in that it is able to reproduce, up to perturbative corrections, the same entropy–area law as that of a conventional BH. We have also shown that there is a natural choice for a Euclidean instanton which mediates the phase transition between the interior geometries of an incipient BH and a frozen star. Standard techniques were used to reveal a probability of transition that goes as e^{-S}, just as expected via statistical reasoning. Moreover, we have argued, following Mathur, that the large degeneracy of the frozen star predicts a total transition probability of order unity. Meaning that the transition from a collapsing matter system to a frozen star would be commonplace if it is at all possible.

An unexpected consequence of the probability-of-transition calculation is that it yields an extremely large action for the intervening layer, being of the same order as the bulk action of the frozen star itself. The actual matter content, however, is suppressed by a factor of λ, the shell width in Schwarzschild units, as a rather simple calculation reveals. A possible interpretation is that this matter might be in a highly excited state, which is suggestive of a BH "firewall" [46]. If this interpretation of the large action is indeed correct, it would be interesting to understand why a firewall-like structure is associated with a horizonless object.

It should be reemphasized that our calculation of the entropy is fundamentally geometric in that it is based directly on the Euclidean-action method. The energy density appears but has no obvious connection to matter fields in the context of the frozen star model. The microscopic origin of the matter densities must rather be gleaned from the polymer model which is a string-theoretic description of the same compact object. The frozen star and collapsed polymer models are supposed to be two different perspectives of the same physical object. That the two models are describing the same object is supported by the similarities of this Euclidean geometry with that of the punctured cigar which is discussed in [33]. In the punctured cigar, the Euclidean time circles are decreasing with r from the outer surface to the origin and are cut off, or "punctured," close to the $r = 0$ tip. The frozen star geometry puncture comes about because of its small, central, regularized zone. This similarity is relevant because the polymer model, once Euclideanized, can also be viewed as a condensate for the winding modes of the thermal scalar [47–49]. We hope to solidify this connection in the future, as BH physics—which is becoming more and more accessible as gravitational-wave data collection and analysis improves—could then provide a window into fundamental string theory.

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