Generating Kerr–anti–de Sitter thermodynamics

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In the present work we study the construction of different thermodynamic descriptions for the Kerr-anti–de Sitter (KadS) black holes. The early versions of the KadS thermodynamics are briefly discussed, highlighting some of its strong points and shortcomings. Isohomogeneous transformations, a procedure for generating new thermodynamics, are presented and geometrically interpreted for KadS. This tool is used to determine possible KadS thermodynamics that can be constructed to satisfy a Smarr formula, and the validity of the first law in the generated thermodynamics. The connection between new thermodynamic theories and early Hawking's approach is considered. In this new framework, the usual KadS thermodynamics is complemented with its geometric construction, and Hawking's proposal, which does not satisfy the first law, is improved to an alternative thermodynamic theory. With the quantum statistical relation, Hawking's and this alternative KadS thermodynamics are also generalized from four to higher dimensions.

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I. INTRODUCTION

Anti–de Sitter/field theory correspondences [1–3] play an important role in the investigation of fundamental aspects of the quantum properties of gravity. It is also a practical approach to gravity problems, providing computational tools in regimes that would be otherwise inaccessible with more traditional methods. In particular, rotating black holes have been studied in this context [4,5]. One area where anti–de Sitter/field theory correspondences are being used is in the study of thermodynamic properties of black holes. From a theoretical point of view, black-hole thermodynamics is very interesting because it connects different branches of physics—general relativity, quantum mechanics and thermodynamics. Since thermodynamics is an effective theory of quantum statistics, it could serve as a guide to a deeper understanding of the quantum mechanical properties of gravity.

When the black hole is not asymptotically flat, additional difficulties are present. For instance, the Komar integral used to define the notion of energy diverges, leading to a generalized Komar mass. For Schwarzschild-anti–de Sitter, several routes have been used to study the associated thermodynamics [6–10]. The addition of rotation to this picture, taking to the Kerr-anti–de Sitter (KadS) geometry, represents a further complication and it has been proposed in several works [11–18].

In contrast to what occurs in traditional black-hole thermodynamics, several approaches to KadS can be found in the literature [15,16,19]. On the other hand, the thermodynamics presented in [11] is the most explored and it is often regarded as the thermodynamic theory for this spacetime. More recently, this field has gained interest due to Dolan's works [13,14,20,21]. For instance, this author shows that this thermodynamics behaves similarly to a van der Waals gas. However, despite the extensive investigation of thermodynamic properties for this specific thermodynamic description of KadS, a proper geometric construction for this theory has not been sufficiently developed.

Another proposal for the thermodynamics of KadS can be extracted from [16]. Unlike the previous case, this is a geometric-oriented work, where thermodynamic quantities were defined via (renormalized) Komar integrals. However, as noted in [15], Hawking's proposal [16] does not satisfy a first law, and therefore it fails as a proper thermodynamic description. Alternatively, it was shown in [19] that a generalization of the Iyer-Wald formalism [22], which includes the cosmological constant, can lead to a different thermodynamics for KadS black holes. The goal of the present work is to discuss how these thermodynamic theories, and infinitely many others, can be obtained.

A key feature of a thermodynamic theory is scale invariance, which is implemented by requiring that all

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equations of state are given by homogeneous functions. Despite the peculiarities in black-hole thermodynamics, where extensive and intensive properties are lost, homogeneity still plays an important role [9,23,24]. In the present work, explicit forms for transformations that preserve homogeneity and the first law are given and multiple thermodynamics can be obtained when they are applied to a given thermodynamic description.

From a geometric point of view, although different Smarr formulas can be obtained depending on a choice of a Killing field, there is no guarantee that these thermodynamic variables will satisfy a first law. Nevertheless, the isohomogeneous transformations introduced here have a geometric counterpart, which is implemented in the Killing vector fields that define the Killing horizon. Thus, in this framework, we have the convergence of thermodynamic and geometric ideas. More precisely, the general thermodynamic approach involving isohomogeneous transformations is used to create Smarr formulas whose quantities, defined from Komar integrals, satisfy a first law.

With this toolkit, a geometric construction for the usual KadS thermodynamics of [11] can be given. Of all the possible KadS thermodynamics that can be generated from our procedure, one of them can be interpreted as the thermodynamic version of Hawking's proposal. Moreover, by combining this framework with the quantum statistical relation, Hawking's KadS thermodynamics is generalized to arbitrary dimensions.

The structure of this paper is presented as follows. In Sec. II, some main features of early treatments of the Kerr-anti–de Sitter thermodynamics are reviewed, focusing on Hawking's and the usual proposals. In Secs. III and IV, isohomogeneous transformations are presented and interpreted geometrically. These are the main tools developed in this paper for the thermodynamic analysis of Kerr-anti–de Sitter black holes. Applications of the isohomogeneous transformations are presented in Sec. V. In particular, Hawking's and the usual models are discussed and an alternative Kerr-anti–de Sitter thermodynamics is constructed. In Sec. VI, extensions of the presented results to arbitrary dimensions are derived. Final comments are presented in Sec. VII. In this paper, we use the geometric unit system and signature $(-,+,+,+)$.

II. OVERVIEW OF KADS THERMODYNAMICS

A. Four-dimensional Kerr-anti–de Sitter spacetime

Kerr-anti–de Sitter spacetime is a stationary and axisymmetric spacetime which models an asymptotically anti–de Sitter spinning black hole. A given KadS black hole is specified by a choice of the mass parameter m, rotation parameter a and (negative) cosmological constant Λ. The set $\{(m,a,\Lambda)\}$ parametrizes all possible four-dimensional KadS solutions. In Boyer-Lindquist-like coordinates, the line element for this geometry is written as [25]

$$
\mathrm{d}s^2 = -\frac{\Delta_r}{\rho^2}\left(\mathrm{d}t - \frac{a\sin^2\theta\,\mathrm{d}\phi}{\Xi}\right)^2 + \frac{\Delta_\theta\sin^2\theta}{\rho^2}\left[a\mathrm{d}t - \frac{(r^2+a^2)\mathrm{d}\phi}{\Xi}\right]^2 + \frac{\rho^2}{\Delta_r}\mathrm{d}r^2 + \frac{\rho^2}{\Delta_\theta}\mathrm{d}\theta^2, \tag{1}
$$

with

$$
\Delta_r \equiv \frac{L^2+r^2}{L^2}(r^2+a^2) - 2mr, \quad \Delta_\theta \equiv 1 - \frac{a^2}{L^2}\cos^2\theta,
$$
$$
\rho^2 \equiv r^2 + a^2\cos^2\theta, \quad \Xi \equiv 1 - \frac{a^2}{L^2}, \quad L^2 \equiv -\frac{3}{\Lambda}. \tag{2}
$$

In this coordinate system, the coordinate t was chosen such that stationarity is implemented by the Killing vector field ∂_t, and axial symmetry is expressed by the Killing vector field ∂_ϕ. We are interested in the nonextreme regime, where the spacetime has two Killing horizons with topology $S^2 \times \mathbb{R}$. The external horizon is located at $r = r_+$, where r_+ is the largest positive real root of the function Δ_r. This horizon is considered as the boundary of the KadS black hole. To enforce the Lorentzian character of the metric, a necessary condition for the validity of the chart (t,r,θ,ϕ) is $\Xi > 0$.

The line element (1) can have its components rearranged to the following useful form, from which the angular velocity of the black hole can be easily extracted,

$$
\mathrm{d}s^2 = -N^2\,\mathrm{d}t^2 + \frac{\rho^2}{\Delta_r}\mathrm{d}r^2 + \frac{\rho^2}{\Delta_\theta}\mathrm{d}\theta^2 + \frac{\Sigma^2\sin^2\theta}{\rho^2\Xi^2}(\mathrm{d}\phi - \omega\mathrm{d}t)^2. \tag{3}
$$

The functions Σ, N, and ω are defined as

$$
\Sigma^2 \equiv (r^2+a^2)^2\Delta_\theta - a^2\Delta_r\sin^2\theta, \qquad N^2 \equiv \frac{\rho^2\Delta_r\Delta_\theta}{\Sigma^2},
$$
$$
\omega \equiv \frac{a\Xi}{\Sigma^2}[\Delta_\theta(r^2+a^2) - \Delta_r]. \tag{4}
$$

The quantity ω can be interpreted as the angular velocity for the so-called zero-angular-momentum observer with respect to the coordinate system (t,r,θ,ϕ). Moreover, it has the following limits:

$$
\lim_{r\to r_+}\omega = \Omega_H, \qquad \lim_{r\to\infty}\omega = -\frac{a}{L^2}, \tag{5}
$$

where

$$
\Omega_H \equiv \frac{a\Xi}{a^2+r_+^2} \tag{6}
$$

is interpreted as the angular velocity of the black hole and $-\frac{a}{L^2}$ is the angular velocity of the spacetime at infinity, with respect to the chart (t,r,θ,ϕ) [11].

The surface $r = r_+$ is a Killing horizon for a Killing field of form

$$K^\mu \equiv \xi^\mu + \Omega\varphi^\mu, \quad (7)$$

where ξ^μ and φ^μ express stationarity and axial symmetry, respectively, and Ω is a scalar. The simplest case is

$$\xi^\mu = \partial_t, \qquad \varphi^\mu = \partial_\phi, \qquad \Omega = \Omega_H. \quad (8)$$

With these choices, the surface gravity associated to this horizon is given by

$$\kappa_+ = \frac{(L^2 + 3r_+^2)r_+^2 - a^2(L^2 - r_+^2)}{2L^2 r_+(r_+^2 + a^2)}, \quad (9)$$

and the area of its two-dimensional spatial sections is

$$A = \frac{4\pi(r_+^2 + a^2)}{\Xi}. \quad (10)$$

Moreover, it is useful to express the mass parameter as

$$m = \frac{(r_+^2 + a^2)(L^2 + r_+^2)}{2r_+ L^2}. \quad (11)$$

B. Early KadS thermodynamics

One of the first works to consider the thermodynamics of the KadS geometry is due to Hawking *et al.* [16]. In the thermodynamic treatment of KadS black holes, a nonextreme KadS spacetime (for a given choice of the parameters m, a, and Λ) is associated with a thermal-equilibrium state. The set of possible (nonextreme) KadS spacetimes forms the thermodynamic ensemble. It is assumed in [16] that the mass (M_H) and the angular momentum (J),

$$M_H \equiv \frac{m}{\Xi}, \qquad J \equiv \frac{am}{\Xi^2}, \quad (12)$$

are given by generalized Komar integrals associated, respectively, to the Killing vector fields in Eq. (8). In the present work, the subscript H refers to Hawking's proposal.

One important characteristic of the treatment based on the quantities M_H and J is that there is an associated Smarr formula:

$$M_H = 2TS + 2\Omega_H J - 2V_H P, \quad (13)$$

where

$$T \equiv \frac{\kappa_+}{2\pi} = \frac{(L^2 + 3r_+^2)r_+^2 - a^2(L^2 - r_+^2)}{4\pi L^2 r_+(r_+^2 + a^2)},$$
$$P \equiv -\frac{\Lambda}{8\pi} = \frac{3}{8\pi L^2}, \qquad S \equiv \frac{A}{4} = \pi\frac{r_+^2 + a^2}{\Xi},$$
$$V_H = \frac{4\pi}{3}\frac{r_+(r_+^2 + a^2)}{\Xi}. \quad (14)$$

The notions of temperature and entropy in Eq. (14) coincide with the standard view of usual black-hole thermodynamics [26]. The pressure term P can be understood by interpreting the cosmological constant as a (dark-) energy contribution in the Universe [27], with a constant energy density $\rho \equiv \frac{\Lambda}{8\pi}$ and equation of state $\rho = -P$. This is a typical viewpoint in cosmology. The volume term is better understood in the pseudo-Cartesian coordinates of the Kerr-Schild form for the metric. Surfaces of constant r ($r = r_+$) are ellipsoids with Euclidean volume V_H in Eq. (14) [28].

Relation (13) is a version of the Smarr formula for nonzero cosmological constant. Considering a Killing field normal to the horizon expressed as Eq. (7), we propose writing the generalized Smarr formula as

$$\underbrace{-\frac{1}{8\pi}\int \nabla_\mu\xi_\nu \mathrm{d}A^{\mu\nu} + \frac{\Lambda}{4\pi}\int g_{\mu\nu}\xi^\mu \mathrm{d}\Sigma^\nu}_{M}$$
$$\underbrace{-2\Omega\frac{1}{16\pi}\int \nabla_\mu\varphi_\nu \mathrm{d}A^{\mu\nu}}_{2\Omega J}$$
$$= \underbrace{-\frac{1}{8\pi}\int \nabla_\mu K_\nu \mathrm{d}A^{\mu\nu}}_{2TS} + \underbrace{\frac{\Lambda}{4\pi}\int g_{\mu\nu}\xi^\mu \mathrm{d}\Sigma^\nu}_{-2VP}, \quad (15)$$

where A is a spatial two-surface at the horizon, and the vector volume is used to integrate over a hypersurface Σ extending from the singularity to $r = r_+$. The vector volume is the integral over a hypersurface of a divergence-free vector field ξ^μ.[1] Adopting (8), Eq. (15) reproduces Hawking's Smarr formula (13).

In Eq. (15), the quantity M is a (modified) Komar mass,

$$M = -\frac{1}{8\pi}\int \nabla_\mu\xi_\nu \mathrm{d}A^{\mu\nu} + \frac{\Lambda}{4\pi}\int g_{\mu\nu}\xi^\mu \mathrm{d}\Sigma^\nu. \quad (16)$$

The second term on the right-hand side of Eq. (16) can be interpreted as a contribution of the cosmological constant to M. If the integration were to be extended from the horizon to spatial infinity, as is usually performed for asymptotically flat spacetimes [29], the quantity M would be the usual Komar mass, which would diverge for $\Lambda \neq 0$ [9,16]. In this case, a possible approach would be to renormalize

[1]This is an operational definition, which is formalized in [28].

the Λ divergence, by subtracting an appropriately chosen background. However, such a procedure is generally not unique, since the choice of the reference background may be ambiguous [11].

In the present work, a quasilocal approach was implemented in (15) using the concept of vector volume. With this method, the generalized Smarr formula emerges in a straightforward way. More precisely, the traditional Smarr formula is based on the fact that, for a Killing vector K^μ, the term $\nabla_\mu \nabla_\nu K^\mu$ vanishes for vacuum solutions of the Einstein gravitational field equation. This result is used to generate conserved charges from Stokes' theorem. However, in general cases, the vanishing object is $\nabla_\mu \nabla_\nu K^\mu - R_{\nu\mu} K^\mu$. In the absence of a conventional energy-momentum tensor but with a cosmological constant, $R_{\mu\nu} = \Lambda g_{\mu\nu}$. This justifies the extra terms that appear in Eq. (15).[2]

It should be stressed that, in Hawking's work [16], no consideration about the first law was made. However, as noted in [15], it can be verified that[3]

$$\mathrm{d}M_H \neq T\,\mathrm{d}S + \Omega_H\,\mathrm{d}J. \tag{17}$$

From Eq. (17), we come to an important conclusion: in Hawking's proposal for a KadS thermodynamics, the quantity M_H can not be associated to a well-defined notion of free energy.

In order to construct a thermodynamic theory based on KadS spacetime which is consistent with the first law, Caldarelli *et al.* [11] and Gibbons *et al.* [15] considered modified definitions for energy and angular momentum,

$$M_U \equiv \frac{m}{\Xi^2}, \qquad J \equiv \frac{am}{\Xi^2}, \tag{18}$$

connected to generalized Komar integrals associated to the Killing fields

$$\frac{\partial_t}{\Xi} \quad \text{and} \quad \partial_\phi, \tag{19}$$

respectively. In the present work, the subscript U refers to usual thermodynamic theory, abbreviated UTT.

This approach attracted further attention with the contribution of Dolan [13,14]. In those works, an association of Λ with pressure is made and the first law is written as

$$\mathrm{d}M_U = T\mathrm{d}S + \Omega_U \mathrm{d}J + V_U \mathrm{d}P, \tag{20}$$

where

$$\Omega_U \equiv \Omega_H + \frac{a}{L^2} = \frac{a}{a^2 + r_+^2}\left(1 + \frac{r_+^2}{L^2}\right), \tag{21}$$

$$\begin{aligned} V_U &\equiv V_H + \frac{4\pi}{3} a^2 M_U \\ &= \frac{2\pi}{3} \frac{(r_+^2 + a^2)(2r_+^2 L^2 + a^2 L^2 - r_+^2 a^2)}{L^2 \Xi^2 r_+}. \end{aligned} \tag{22}$$

It can be checked that, with these choices for the thermodynamic variables, Smarr formula is also satisfied. However, a relevant point to be addressed in the present work is that, with $\Omega = \Omega_U$, it is obtained from Eq. (7)

$$\xi^\mu \neq \frac{\partial_t}{\Xi}, \qquad \varphi^\mu = \partial_\phi, \tag{23}$$

which is different from what one would expect. The correct form for K^μ will be obtained from our formalism and presented in Sec. V.

C. Homogeneity and Euler relation

Homogeneity of the equations of state is a crucial feature in black-hole thermodynamics [9,24]. A function $M(X_1, X_2, X_3, \ldots)$ is said to be a homogeneous function of degree r, on the variables X_i of degree α_i, if

$$M(\lambda^{\alpha_1} X_1, \lambda^{\alpha_2} X_2, \lambda^{\alpha_3} X_3, \ldots) = \lambda^r M_0(X_1, X_2, X_3, \ldots). \tag{24}$$

For such functions, Euler's theorem for homogeneous functions states that

$$rM = \alpha_1 X_1 \frac{\partial M}{\partial X_1} + \alpha_2 X_2 \frac{\partial M}{\partial X_2} + \alpha_3 X_3 \frac{\partial M}{\partial X_3} + \ldots. \tag{25}$$

In the thermodynamic-oriented literature, Eq. (25) is known as Euler relation. In early versions of KadS thermodynamics (discussed in Sec. II B), the independent variables are $\{S, J, P\}$, function M denotes the black-hole mass, and we have $\alpha_1 = \alpha_2 = -\alpha_3 = 2$ and $r = 1$. One can read from the dimensions of these quantities:

$$[S] \sim [J] \sim [\text{length}]^2, \quad [P] \sim [\text{length}]^{-2}, \quad [M] \sim [\text{length}]. \tag{26}$$

A scaling argument, based on the relationship between dimension and homogeneity, is used in [9] to justify the inclusion of $P \equiv -\frac{\Lambda}{8\pi}$ as a pressure in the thermodynamic description. This relation can be employed, by directed comparison of Smarr and Euler relations, to determine the equations of states of the thermodynamic description [30,31].

However, this identification method should be applied with caution. The connection between dimensional analysis and homogeneity can lead to the conclusion that the Smarr

[2] It was also used the property that φ^μ does not contribute to the vector volume (the integral vanishes for it) and, thus, the vector volume for K^μ is the same as the one for ξ^μ.

[3] In [15], the authors do not treat the cosmological constant as a thermodynamic variable. However, even if Λ is considered a thermodynamic pressure, the inequality is not corrected. In this case, one would still obtain $\mathrm{d}M_H \neq T\,\mathrm{d}S + \Omega_H\,\mathrm{d}J + V_H\,\mathrm{d}P$.

relation is always the geometric counterpart of the Euler relation. But not all Smarr relations can be identified as Euler relations. A concrete example is given by Eq. (13). Although it is a proper Smarr relation, it is not an Euler relation because it does not provide a first law. This means that this procedure is not always reliable to furnish proper thermodynamic descriptions. Tools for a more careful analysis will be developed in the next section.

III. ISOHOMOGENEOUS TRANSFORMATIONS

In the present work we study the construction of different thermodynamic descriptions for the KadS black hole. Let us motivate the main ideas used in this section. It is believed that the Smarr relation represents the geometric side of the Euler relation for homogeneous functions [9]. However, as discussed in Sec. II B, Hawking's proposal shows that this is not true in general. Indeed, from Eq. (13),

$$M_H = 2TS + 2\Omega_H J - 2V_H P \neq 2\frac{\partial M_H}{\partial S}S + 2\frac{\partial M_H}{\partial J}J - 2\frac{\partial M_H}{\partial P}P. \tag{27}$$

In this section we analyze under which circumstances a Smarr relation does not represent an Euler relation. We develop tools that allow to transform this Smarr relation into a valid Euler relation that gives a proper first law. Furthermore, the geometrical significance of this thermodynamic development is clarified in the next section.

Let us consider a thermodynamic potential $M_0 = M_0(S, X_2, X_3, \ldots)$, that is, a homogeneous function of degree r such that

$$M_0(\lambda^{\alpha_1}S, \lambda^{\alpha_2}X_2, \lambda^{\alpha_3}X_3, \ldots) = \lambda^r M_0(S, X_2, X_3, \ldots), \tag{28}$$

$$\mathrm{d}M_0 = T_0\,\mathrm{d}S + \sum_i Y_0^i\,\mathrm{d}X_i, \tag{29}$$

where α_i is the degree of ith independent variable. Since M_0 is a homogeneous function, it obeys Euler relation:

$$rM_0 = \alpha_1 S T_0 + \alpha_2 X_2 Y_0^2 + \alpha_3 X_3 Y_0^3 + \cdots. \tag{30}$$

Our goal is to analyze the transformations on thermodynamic potential $M_0 = M_0(S, X_2, X_3, \cdots)$ that preserve homogeneity. Consider a homogeneous function $g = g(S, X_2, X_3, \cdots)$ of degree zero,

$$\alpha_1 S\frac{\partial g}{\partial S} + \alpha_2 X_2\frac{\partial g}{\partial X_2} + \alpha_3 X_3\frac{\partial g}{\partial X_3} + \cdots = 0. \tag{31}$$

Let us also assume that g is positive definite. Multiplying rM_0 by g and using Eq. (30),

$$rgM_0 = \alpha_1 gST_0 + \alpha_2 gX_2Y_0^2 + \alpha_3 gX_3Y_0^3 + \cdots. \tag{32}$$

Adding expression (31) to Eq. (32),

$$rgM_0 = \alpha_1 gST_0 + \alpha_2 gX_2Y_0^2 + \cdots + h\left(\alpha_1 S\frac{\partial g}{\partial S} + \alpha_2 X_2\frac{\partial g}{\partial X_2} + \cdots\right), \tag{33}$$

for an arbitrary function h.

To preserve homogeneity, we impose that h is a general homogeneous function with the same degree as M_0. Thus, from the original Euler relation (30), it follows that

$$rM_1 = \alpha_1 ST_1 + \alpha_2 X_2 Y_1^2 + \alpha_3 X_3 Y_1^3 + \cdots, \tag{34}$$

where

$$M_1 \equiv gM_0, \qquad T_1 \equiv gT_0 + h\frac{\partial g}{\partial S}, \qquad Y_1^i \equiv gY_0^i + h\frac{\partial g}{\partial X_i}. \tag{35}$$

Relation (34) is equivalent to Eq. (30).

For any positive definite homogeneous function g of degree zero and for any homogeneous function h of degree r, we call Eq. (35) an isohomogeneous transformation. The next step is to determine when Eq. (34) is also an Euler relation.

But while relation (34) enforces homogeneity, not all functions M_1 can be identified as a legitimate thermodynamic potential. Specifically, from Eq. (35),

$$T_1\,\mathrm{d}S + \sum_i Y_1^i\,\mathrm{d}X_i = g\,\mathrm{d}M_0 + h\,\mathrm{d}g = \mathrm{d}M_1 + \left(h - \frac{M_1}{g}\right)\mathrm{d}g. \tag{36}$$

It follows from Eq. (36) that, for a nonconstant g, M_1 will be a thermodynamic potential only if $h \equiv \frac{M_1}{g} = M_0$. In this case, the right-hand side is an exact differential,

$$h = M_0 \Rightarrow \mathrm{d}M_1 = T_1\,\mathrm{d}S + \sum_i Y_1^i\,\mathrm{d}X_i, \tag{37}$$

and Eq. (34) is a valid Euler relation. However, as we will see, the thermodynamic descriptions associated to M_0 and M_1 are different.

When Eq. (34) is an Euler relation, satisfying a first law of thermodynamics, we call this relation a thermodynamic Smarr formula. An example of a thermodynamic Smarr formula is the one from the UTT [11]. The other cases will be classified as nonthermodynamic Smarr formulas. For instance, Hawking's proposal [16] (discussed in Sec. II B) involves a nonthermodynamic Smarr formula. Given a thermodynamic Smarr formula with a thermodynamic potential M_0, the transformation to another thermodynamic Smarr formula, which follows from choosing a homogeneous function of degree zero g and fixing $h = M_0$, will be called a exact isohomogeneous transformation.[4]

One remark is in order. A given thermodynamic description is characterized by a set of equations of state $(T, \{Y^i\})$. In principle more than one description can share some specific equations of state. For instance, we can search for all thermodynamics that share the same temperature. In this example, given a legitimate thermodynamic potential M_0 characterized by a temperature T_0, other legitimate potentials $\{\tilde{M}\}$ are obtained setting $h = M_0$. In fact, imposing that they are also characterized by the same T_0,

$$\tilde{T} = T_0 \Rightarrow (\tilde{g} - 1)T_0 + M_0 \frac{\partial \tilde{g}}{\partial S} = 0. \tag{38}$$

To clarify, tilted versions are used to explicit the fact that the temperature ($\tilde{T}$) and function ($\tilde{g}$) are not being fixed in the development.

Multiple solutions of Eq. (38) can be derived. Indeed, for any arbitrary function f (with degree r) that does not depends on S, we obtain

$$\tilde{g} = 1 - \frac{f(X_i)}{M_0}. \tag{39}$$

The presented development can be extended for several equations of state. That is, given a thermodynamics with N variables, it is possible to construct another one preserving $(N - 1)$ equations of state. In particular, multiple thermodynamic descriptions (for adS black holes) can have the same definition of temperature that is used by Hawking in [16].

IV. GEOMETRY OF ISOHOMOGENEOUS TRANSFORMATIONS ON KADS

In the previous section, general ideas involving isohomogeneous transformations in a thermodynamic context were presented. In the present section, this development will be discussed geometrically, considering the KadS spacetime. Adapting the nomenclature of Sec. III to the four-dimensional KadS black holes, we have

$$\begin{gathered} r = 1, \qquad \alpha_1 = 2, \qquad \alpha_2 = 2, \qquad \alpha_3 = -2, \\ X_2 = J, \qquad X_3 = P, \\ Y_0^2 = \Omega_0, \qquad Y_0^3 = V_0, \qquad Y_1^2 = \Omega_1, \qquad Y_1^3 = V_1. \end{gathered} \tag{40}$$

The general expressions (34) and (35) imply that, for the KadS case,

$$M_1 = 2ST_1 + 2J\Omega_1 - 2PV_1, \tag{41}$$

where

$$\begin{gathered} M_1 = gM_0, \qquad T_1 = gT_0 + h\frac{\partial g}{\partial S}, \\ \Omega_1 = g\Omega_0 + h\frac{\partial g}{\partial J}, \qquad V_1 = gV_0 + h\frac{\partial g}{\partial P}. \end{gathered} \tag{42}$$

The geometric counterpart to the isohomogeneous transformations is implemented on the Killing fields that define the Killing horizon. The resulting transformed object must also be a Killing field normal to the horizon, allowing its use as a generator of a new Smarr formula. By adopting this geometric perspective, the distinct thermodynamic theories for KadS can be understood within the framework of rotating reference frames.

In accordance with previous section, we introduce a positive function g of the thermodynamic variables $\{S, J, P\}$. With g, other Smarr formulas can be obtained, rescaling the Killing field in Eq. (7) as

$$K^\mu \longrightarrow gK^\mu = g\xi^\mu + g\Omega_0\varphi^\mu. \tag{43}$$

In stationary asymptotically flat spacetimes, there is a choice of g which normalizes ξ^μ at infinity [26]. However, a different g can be used to normalize ξ^μ at a finite distance. This gives the Tolman redshift factor for the temperature [32]. Therefore, keeping g arbitrary is useful, especially when dealing with a nonasymptotically flat geometry.

On the other hand, this does not represent the entirety of possible transformations in a spacetime with axial symmetry. We can also consider combinations between the Killing fields themselves,

$$K^\mu \longrightarrow gK^\mu = \left(g\xi^\mu - h\frac{\partial g}{\partial J}\varphi^\mu\right) + \left(g\Omega_0 + h\frac{\partial g}{\partial J}\right)\varphi^\mu, \tag{44}$$

where h is an arbitrary function of $\{S, J, P\}$. Equation (44) has been structured in a way that is consistent with our developments in the previous section. Notably, the resultant angular velocity is expressed in the format found in Eq. (42). Moreover, although the quantities proportional to h sum to zero on the right-hand side of Eq. (44), in this form, h is associated with the change to a new frame that rotates with angular velocity

$$\frac{\mathrm{d}\phi_1}{\mathrm{d}t} = \frac{\mathrm{d}\phi}{\mathrm{d}t} + \frac{h}{g}\frac{\partial g}{\partial J}, \tag{45}$$

where (t, r, θ, ϕ) is the original coordinate system. Furthermore, g represents a choice of normalization for the Killing vector normal to the horizon, which, as we will see, rescales some thermodynamic quantities.

[4]Exact because the left-hand side of Eq. (36) is an exact differential.

The transformation on the Killing field (44) propagates to the Komar formulas, allowing a reinterpretation of the thermodynamic quantities:

$$\underbrace{-\frac{g}{8\pi}\int\nabla_\mu\xi_\nu \mathrm{d}A^{\mu\nu}+\frac{g\Lambda}{4\pi}\int g_{\mu\nu}\xi^\mu \mathrm{d}\Sigma^\nu}_{M_1}+\underbrace{h\frac{\partial g}{\partial J}\frac{1}{8\pi}\int\nabla_\mu\varphi_\nu \mathrm{d}A^{\mu\nu}}_{2h\frac{\partial g}{\partial J}J}$$
$$\underbrace{-2\left(g\Omega_0+h\frac{\partial g}{\partial J}\right)\frac{1}{16\pi}\int\nabla_\mu\varphi_\nu \mathrm{d}A^{\mu\nu}}_{2\Omega_1 J}$$
$$=\underbrace{-\frac{g}{8\pi}\int\nabla_\mu K_\nu \mathrm{d}A^{\mu\nu}}_{2gT_0S}+\underbrace{\frac{g\Lambda}{4\pi}\int g_{\mu\nu}\xi^\mu \mathrm{d}\Sigma^\nu}_{-2gV_0P}, \tag{46}$$

where M_1 and Ω_1 are presented in Eq. (42).

To identify Eq. (46) as a Smarr relation, the term $2h\frac{\partial g}{\partial J}J$ must be eliminated. This can be done by imposing that g is a homogeneous function of degree zero. Therefore, using Eq. (31),

$$J\frac{\partial g}{\partial J}=P\frac{\partial g}{\partial P}-S\frac{\partial g}{\partial S}, \tag{47}$$

and Eq. (46) furnishes

$$M_1+2h\left(P\frac{\partial g}{\partial P}-S\frac{\partial g}{\partial S}\right)-2\Omega_1 J=2gT_0S-2gV_0P. \tag{48}$$

Factorizing S and P, we see that the temperature and volume change according to Eq. (42) and Smarr relation (41) is recovered. From a geometrical point of view, expression (47) links surface and hypersurface integrals:

$$\int\nabla_\mu\varphi_\nu \mathrm{d}A^{\mu\nu},\qquad \int\nabla_\mu K_\nu \mathrm{d}A^{\mu\nu},\qquad \int g_{\mu\nu}\xi^\mu \mathrm{d}\Sigma^\nu. \tag{49}$$

The terms multiplying h have a zero net contribution to the change of the Killing field in Eq. (44), but, from Eq. (42), it is observed that this rotating frame changes how the energy is distributed into heat ($T\mathrm{d}S$) and mechanical work ($\Omega\mathrm{d}J+V\mathrm{d}P$). In the geometric language, the change in the angular velocity could be interpreted as a rotation between frames adapted to different thermodynamic descriptions.

It is worth emphasizing that what defines distinct thermodynamics are not different choices of coordinates, but rather different Killing fields and their contribution to the Komar integrals. Nevertheless, adapted coordinates facilitate geometric and physical interpretations behind each theory and also how they relate to each other. For instance, transformation (44) implies that, for a coordinate system (t,r,ϕ,θ), where ξ^μ and φ^μ are vectors from the coordinate basis (∂_t and ∂_ϕ), other coordinates t_1 and ϕ_1 can be chosen so that

$$\partial_{t_1}=g\partial_t-h\frac{\partial g}{\partial J}\partial_\phi,\qquad \partial_{\phi_1}=\partial_\phi. \tag{50}$$

The respective coordinate change is

$$t=gt_1,\qquad \phi=\phi_1-h\frac{\partial g}{\partial J}t_1. \tag{51}$$

In the adapted coordinates, the Killing field becomes

$$gK=\partial_{t_1}+\Omega_1\partial_{\phi_1}, \tag{52}$$

and Eq. (45) is immediately recovered.

The discussion involving thermodynamic and nonthermodynamic Smarr formulas, in the context of the isohomogeneous transformations, reappears with the geometric formalism. Indeed, although the transformations in the Killing field (44) transforms a Smarr formula into an equally valid one, it may destroy the thermodynamic description of the system, generating thermodynamic variables that do not satisfy the first law. More precisely, even if the description with the Killing field (7) is related to a legitimate thermodynamic theory,

$$\mathrm{d}M_0=T_0\mathrm{d}S+\Omega_0\mathrm{d}J+V_0\mathrm{d}P, \tag{53}$$

it is not always true that its transformed version also satisfies the first law. That is, it may be the case that the thermodynamic description is invalidated:

$$\mathrm{d}M_1\neq T_1\mathrm{d}S+\Omega_1\mathrm{d}J+V_1\mathrm{d}P. \tag{54}$$

Nonetheless, from Sec. III, we know that the first law is preserved if the transformed Smarr formula is obtained by an exact isohomogeous transformation, i.e., taking $h=M_0$.

V. APPLYING THE ISOHOMOGENEOUS TRANSFORMATIONS

A. Generating thermodynamic theories from Hawking's proposal

In Hawking's proposal, discussed in Sec. II B, the internal energy M_H is given by

$$M_1=M_H=\frac{m}{\Xi}. \tag{55}$$

It is an example of a nonthermodynamic theory, in the sense that it does not have a well-defined first law:

$$dM_H\neq T\mathrm{d}S+\Omega_H\mathrm{d}J+V_H\mathrm{d}P. \tag{56}$$

It is possible to turn the inequality (56) into an equality by adding an extra term on the left side of Eq. (56):

$$T\mathrm{d}S + \Omega_H \mathrm{d}J + V_H \mathrm{d}P = dM_H - \frac{M_H}{2\Xi}\mathrm{d}\Xi. \tag{57}$$

Comparing Eq. (57) with Eq. (36), we have

$$\frac{M_H}{2\Xi} = \left(\frac{M_H}{g} - h\right)\frac{\mathrm{d}g}{\mathrm{d}\Xi}, \tag{58}$$

where g is written as a function of Ξ. The development (31) only required that g be a function of S, J, and P, so the representation $g = g(\Xi)$ is a constraint on the functional form of g.

We emphasize that there are an infinite number of possible proper thermodynamic theories related to Hawking's proposal. In fact, consider the following choice for g and the associated function h:

$$g = \Xi^n \Rightarrow h = \frac{M_H}{\Xi^n}\left(1 - \frac{1}{2n}\right), \tag{59}$$

with $n \neq 0$. The (multiple) proper thermodynamics generated (with subindex 0 following the notation of Sec. III), related to Hawking's thermodynamics, are

$$T = \xi T_0, \qquad \Omega_H = \xi\Omega_0 - (\xi - \Xi^n)\frac{M_H}{J\Xi^n},$$
$$V_H = \xi V_0 - (\xi - \Xi^n)\frac{M_H}{2P\Xi^n}, \tag{60}$$

where

$$\xi = \frac{\Xi^n}{1-(2n-1)(1-\Xi)},$$
$$\Xi(J,S,P) = 1 + \frac{3}{8PS+3} - \frac{3(4\pi^2 J^2 + S^2)}{12\pi^2 J^2 + S^2(8PS+3)}. \tag{61}$$

Note from Eq. (42) that the temperature for a theory will be proportional to its surface gravity only when $h = 0$. This corresponds to $\xi = g = \sqrt{\Xi}$ in Eq. (61). Given a proper thermodynamic description where the temperature coincides with its surface gravity, characterized by energy M_0, all other cases are obtained by taking $h = M_0$. Thus, with a suitable normalization, there is a unique thermodynamic theory for which the temperature coincides with its surface gravity.

Of all the possible KadS thermodynamics that can be generated with the formalism in this work, two will receive special attention in the remainder of this section. The first is the UTT, which has been extensively studied in the literature. The second is the one that minimally modifies Hawking's proposal, which will be analyzed on a deeper level. The latter is, in a sense, the natural extension of Hawking's proposal to a KadS thermodynamics.

B. From Hawking's proposal to the usual thermodynamic theory

Using result (59), consider the following configuration:

$$n = 1 \Rightarrow g = \Xi, \qquad h = \frac{M_H}{2\Xi}, \qquad \xi = 1. \tag{62}$$

In this case, a valid thermodynamic theory connected to Hawking's proposal is the one with energy

$$M_0 = \frac{M_1}{g} = \frac{M_H}{\Xi} = \frac{m}{\Xi^2} = M_U, \tag{63}$$

which is the UTT (discussed in Sec. II B).

Note that Eq. (62) is not an exact isohomogeneous transformation because it links nonthermodynamic and thermodynamic Smarr formulas. For the remaining thermodynamic variables, from Eq. (42),

$$T_H = \Xi T_U + \frac{M_H}{2\Xi}\frac{\partial\Xi}{\partial S} \Rightarrow T_U = T_H = T, \tag{64}$$

$$\Omega_H = \Xi\Omega_U + \frac{M_H}{2\Xi}\frac{\partial\Xi}{\partial J} \Rightarrow \Omega_U = \Omega_H + \frac{a}{L^2}, \tag{65}$$

$$V_H = \Xi V_U + \frac{M_H}{2\Xi}\frac{\partial\Xi}{\partial P} \Rightarrow V_U = V_H + \frac{4\pi}{3}a^2 M_U. \tag{66}$$

Since

$$T_H = \Xi T_U + \frac{M_H}{2\Xi}\frac{\partial\Xi}{\partial S}, \tag{67}$$

it is significant that the temperature from the UTT coincides with the temperature from Hawking's proposal, that is, $T_U = T_H = T$. Moreover, new thermodynamic descriptions (with the same temperature) can be constructed using the exact isohomogeneous transformation in Eq. (39), showing that the UTT is not the only one that shares the same definition of temperature with Hawking's proposal.

It is also a feature of the UTT that the volume term deviates from the geometric volume, V_H, by the addition of an extra term [as it is in Eq. (66)]. The origin of this term has been revealed by the isohomogeneous transformations developed in the present work.

Another point of interest is that the Killing field that generates the UTT can be determined by the functions g and h from Eq. (44),

$$\frac{1}{\Xi}K = \left[\frac{1}{\Xi}\partial_t - \frac{ar_+^2}{L^2(a^2+r_+^2)}\partial_\phi\right] + \underbrace{\left[\frac{\Omega_H}{\Xi} + \frac{ar_+^2}{L^2(a^2+r_+^2)}\right]}_{\Omega_U}\partial_\phi. \tag{68}$$

This Killing field is null and normal to the horizon, as it should be. Result (68) emphasizes the geometric interpretation of the UTT.

Still on the geometric approach to the UTT, an observation is in order. It is commonly stated that the temperature from the UTT coincides with the black-hole surface gravity. However, this is not an accurate statement. In the geometric construction of a thermodynamic theory, the surface gravity is specified by the normalization of the Killing field normal to the horizon. While it is true to state the equality of the temperature from the UTT with the surface gravity from Hawking's proposal, notice that the Killing field associated with the UTT is normalized by a factor of $\frac{1}{\Xi}$. Therefore, its surface gravity is multiplied by the same factor when compared to the surface gravity of Hawking's proposal. In other words, the temperature T does not match the surface gravity of the UTT. This should be the case, since this thermodynamics is obtained from Hawking's proposal by setting $h \neq 0$.

C. From Hawking's proposal to an alternative thermodynamic theory

We can look for a theory that minimally modifies Hawking's proposal. From Eq. (60), this can be done by setting $h = 0$. In this case, all thermodynamic quantities differ only by one (and the same) factor $g = \Xi^n$,

$$h = 0 \Rightarrow T = \Xi^n T_0, \quad \Omega_H = \Xi^n \Omega_0, \quad V_H = \Xi^n V_0. \tag{69}$$

Moreover, from Eq. (59), it is straightforward to check that this case comes from

$$n = \frac{1}{2} \Rightarrow g = \sqrt{\Xi}, \qquad h = 0, \qquad \xi = \sqrt{\Xi}. \tag{70}$$

In this configuration, the proper thermodynamic theory associated with Hawking's is the one with energy

$$M_A \equiv M_0 = \frac{M_1}{g} = \frac{M_H}{\sqrt{\Xi}} = \frac{m}{\Xi^{\frac{3}{2}}}. \tag{71}$$

The subindex A stands for alternative thermodynamic theory, abbreviated ATT.

The remaining thermodynamic variables are

$$T_A = \frac{T}{\sqrt{\Xi}}, \qquad \Omega_A = \frac{\Omega_H}{\sqrt{\Xi}}, \qquad V_A = \frac{V_H}{\sqrt{\Xi}}. \tag{72}$$

This thermodynamic version of Hawking's proposal is the one where the temperature coincides with the surface gravity. In this case, we have $\frac{\kappa_+}{\sqrt{\Xi}}$, where κ_+ is given by Eq. (9). Since this property is not shared by any other construction, this theory is the closest one to the usual thermodynamics of asymptotically flat black holes.

The formalism from the previous section guarantees that the first law is satisfied:

$$\mathrm{d}M_A = T_A \mathrm{d}S + \Omega_A \mathrm{d}J + V_A \mathrm{d}P. \tag{73}$$

Furthermore, any thermodynamics for Kerr-anti–de Sitter black holes is connected to the ATT by

$$h = M_A, \qquad g = g(J, P, N). \tag{74}$$

Therefore, the ATT is the only theory which corrects Hawking's proposal by only a multiplicative factor.

The Killing field related to the ATT is

$$K = \frac{\partial_t}{\sqrt{\Xi}} + \frac{\Omega_H}{\sqrt{\Xi}} \partial_\phi, \tag{75}$$

which is null and normal to the horizon, as expected. Working with adapted coordinates (t', r, θ, ϕ), where

$$t' = \sqrt{\Xi} t, \tag{76}$$

the Killing vector field K is written as

$$K = \partial_{t'} + \Omega_A \partial_\phi. \tag{77}$$

With the chart (t', r, θ, ϕ), KadS metric in Eq. (3) assumes the form

$$ds^2 = -\frac{N^2}{\Xi} dt'^2 + \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2 + \frac{\Sigma^2 \sin^2\theta}{\rho^2 \Xi^2} (d\phi - \omega' dt')^2, \tag{78}$$

with

$$\omega' = \frac{\omega}{\sqrt{\Xi}}. \tag{79}$$

These considerations show that the angular velocity of the black hole and of the asymptotic limit are, measured with the new time coordinate t',

$$\lim_{r\to r_+} \omega' = \frac{\Omega_H}{\sqrt{\Xi}} = \Omega_A, \qquad \lim_{r\to\infty} \omega' = -\frac{a}{L^2 \sqrt{\Xi}}. \tag{80}$$

Thus, Hawking's proposal is improved (i.e., it satisfies the first law) by using the coordinate time t'. As can be seen from Eq. (77), the thermodynamics is now constructed from the Killing vectors written with a coordinate basis, a convenient tool that is used extensively in the analysis of black-hole thermodynamics.

VI. EXTENSION TO ARBITRARY DIMENSIONS

A. D-dimensional KadS spacetime and the quantum statistical relation

The formalism developed in the context of the four-dimensional KadS geometry can be generalized to higher-dimensional black holes. According to the results of Sec. V A, a large set of KadS thermodynamics can be obtained by applying a (nonexact) isohomogeneous transformation to the nonthermodynamic Smarr relation proposed by Hawking. To construct a D-dimensional version for the ATT, presented in Sec. V C, we extend Hawking's nonthermodynamic Smarr relation to higher dimensions and apply a nonexact isohomogeneous transformation to it. For this purpose, we will use the generalization to higher dimensions of the thermodynamic relations obtained in [15]. In this extended scenario, the D-dimensional Kerr-anti–de Sitter spacetime models a spinning black hole characterized by N independent rotation parameters $\{a_i\}$ with respect to the azimuthal angles $\{\varphi_i\}$, where

$$N \equiv \begin{cases} (D-1)/2, & \text{odd } D \\ (D-2)/2, & \text{even } D \end{cases}, \tag{81}$$

and $D \geq 4$. The latitudinal angular coordinates μ_i satisfy the constraint

$$\sum_{i=1}^{D-N-1} \mu_i^2 = 1. \tag{82}$$

The D-dimensional KadS metric is [15,33]

$$\begin{aligned} \mathrm{d}s^2 = &-W\left(1+\frac{r^2}{L^2}\right)\mathrm{d}\tau^2 + \frac{2m}{U}\left(W\mathrm{d}\tau - \sum_{i=1}^{N}\frac{a_i\mu_i^2\mathrm{d}\varphi_i}{\Xi_i}\right)^2 \\ &+ \sum_{i=1}^{N}\frac{r^2+a_i^2}{\Xi_i}\mu_i^2\mathrm{d}\varphi_i^2 + \frac{U}{X-2m}\mathrm{d}r^2 \\ &+ \sum_{i=1}^{D-N-1}\frac{r^2+a_i^2}{\Xi_i}\mathrm{d}\mu_i^2 - \frac{1}{W(L^2+r^2)} \\ &\times\left(\sum_{i=1}^{D-N-1}\frac{r^2+a_i^2}{\Xi_i}\mu_i\mathrm{d}\mu_i\right)^2. \end{aligned} \tag{83}$$

To maintain consistency with the four-dimensional notation, the mass parameter and the (negative) cosmological constant are denoted by m and Λ, respectively. The parameters $\{\Xi_i\}$ and the functions W, U, and X are given by

$$\begin{aligned} &\Xi_i \equiv 1-\frac{a_i^2}{L^2}, \qquad W \equiv \sum_{i=1}^{D-N-1}\frac{\mu_i^2}{\Xi_i}, \\ &U \equiv r^{D-2N-1}\sum_{i=1}^{D-N-1}\frac{\mu_i^2}{r^2+a_i^2}\prod_{j=1}^{N}(r^2+a_j^2), \\ &X \equiv r^{D-2N-3}\left(1+\frac{r^2}{L^2}\right)\prod_{i=1}^{N}(r^2+a_i^2). \end{aligned} \tag{84}$$

The N independent rotational parameters are associated with N angular momentum variables $\{J_i\}$ [11]. Thus, considering also entropy and pressure, there are a total of $(N+2)$ independent variables for the thermodynamic description. Isohomogeneous transformations can be applied to the D-dimensional KadS scenario. In this case, the quantities defined in Sec. III are

$$\begin{aligned} &r = D-3, \quad \alpha_i = D-2, \quad \alpha_{N+2} = -2, \quad i = 1,\ldots,N+1, \\ &X_k = J_k, \quad X_{N+2} = P, \quad k = 2,\ldots,N+1, \\ &Y_0^k = \Omega_0^k, \quad Y_0^{N+2} = V_0, \quad Y_1^k = \Omega_1^k, \quad Y_1^{N+2} = V_1. \end{aligned} \tag{85}$$

From the Euclidean action formalism [34], the thermodynamic quantities obey the so-called quantum statistical relation [15],

$$E_0^{(D)} - T_0 S - \sum_i \Omega_0^i J_i = T_0 I_D, \tag{86}$$

where I_D is the Euclidean action that, for the KadS geometry, takes the form

$$I_D = \frac{1}{T_0}\frac{A_{D-2}}{8\pi\prod_{j=1}^{N}\Xi_j}\left[m - \frac{(r_+)^c}{L^2}\prod_{i=1}^{N}(r_+^2+a_i^2)\right]. \tag{87}$$

In Eq. (87), A_{D-2} is the volume of the unit $(D-2)$-sphere used to construct the area A of the event horizon,

$$A_{D-2} = \frac{2\pi^{(D-1)/2}}{\Gamma(\frac{D-1}{2})}, \qquad A = \frac{A_{D-2}}{(r_+)^c}\prod_{i=1}^{N}\frac{r_+^2+a_i^2}{\Xi_i}. \tag{88}$$

The symbol Γ denotes the usual gamma function, and the constant c is defined as

$$c \equiv \begin{cases} 0, & \text{odd } D \\ 1, & \text{even } D \end{cases}. \tag{89}$$

The thermodynamic quantities are given by

$$\begin{aligned} &S \equiv \frac{A}{4}, \qquad \Omega_0^i \equiv a_i\frac{(1+r_+^2L^{-2})}{r_+^2+a_i^2} = \Omega_H^i + \frac{a_i}{L^2}, \\ &\Omega_H^i \equiv \frac{a_i\Xi_i}{r_+^2+a_i^2}, \qquad E_0^{(D)} \equiv \frac{mA_{D-2}}{4\pi\Pi_j\Xi_j}\left(\sum_{i=1}^{N}\frac{1}{\Xi_i} - \frac{1-c}{2}\right), \\ &J_i \equiv \frac{ma_iA_{D-2}}{4\pi\Xi_i\Pi_j\Xi_j}. \end{aligned} \tag{90}$$

These quantities satisfy the relation

$$\mathrm{d}E_0^{(D)} = T_0\mathrm{d}S + \sum_i \Omega_0^i \mathrm{d}J_i, \tag{91}$$

from where the temperature T_0 can be computed.

It should be noticed that, since L is not considered a thermodynamic variable in [15], the pressure and volume terms are missing in this construction and quantities in Eq. (86) do not obey the Smarr relation. The addition of these quantities to a proper KadS thermodynamics is our goal in the next subsection.

B. Thermodynamic volume and the Smarr relation

To derive a Smarr relation from Eq. (86), it is convenient to rewrite the energy $E_0^{(D)}$ as

$$E_0^{(D)} = (D-2)\frac{mA_{D-2}}{8\pi\Pi_j\Xi_j} + L^{-2}\sum_{i=1}^{N} J_i a_i. \tag{92}$$

We recognize Eq. (92) as a thermodynamic Smarr relation by introducing

$$P \equiv -\frac{\Lambda}{8\pi} = \frac{(D-2)(D-1)}{16\pi L^2} \tag{93}$$

as a thermodynamic variable and defining a geometric volume [35]

$$V = \frac{r_+}{D-1}A = \frac{(r_+)^c A_{D-2}}{D-1}\prod_i \frac{r_+^2 + a_i^2}{\Xi_i}. \tag{94}$$

Using theses quantities, Eq. (87) is written as

$$I_D = \frac{1}{T_0}\left(\frac{mA_{D-2}}{8\pi\Pi_j\Xi_j} - \frac{2}{D-2}PV\right). \tag{95}$$

From Eqs. (86) and (92), it is straightforward to verify that the theory now obeys the Smarr relation and the first law:

$$(D-3)E_0^{(D)} = (D-2)T_0 S + (D-2)\Omega_0^i J_i - 2V_0 P, \tag{96}$$

$$\mathrm{d}E_0^{(D)} = T_0 \mathrm{d}S + \Omega_0^i \mathrm{d}J_i + V_0 \mathrm{d}P, \tag{97}$$

with

$$V_0 = V + \frac{8\pi}{(D-2)(D-1)}\sum_{i=1}^{N} J_i a_i. \tag{98}$$

The thermodynamic volume V_0 in Eq. (98), obtained here using the quantum statistical relation (86), agrees with that obtained in [35] using a different approach.

C. Extension of Hawking's KadS thermodynamics to D dimensions

Based on the four-dimensional case, we can reinterpret Hawking's proposal as the theory for which the Smarr relation involves the same temperature T_0 presented in Eq. (96), but with the angular velocity of the black hole Ω_H^i and the geometric volume V. Considering now the D-dimensional expression of Eq. (96), and rearranging the terms, we can write

$$(D-3)E_H^{(D)} = (D-2)T_0 S + (D-2)\Omega_H^i J_i - 2PV, \tag{99}$$

where

$$E_H^{(D)} \equiv (D-2)\frac{mA_{D-2}}{8\pi\Pi_j\Xi_j}. \tag{100}$$

We define the quantity E_H as the D-dimensional version of Hawking's energy. This energy gives a nonthermodynamic Smarr relation. Explicitly,

$$T_0 \mathrm{d}S + \Omega_H^i \mathrm{d}J_i + V\mathrm{d}P = \mathrm{d}E_H^{(D)} - \frac{E_H^{(D)}}{D-2}\sum_{i=1}^{N}\frac{1}{\Xi_i}\mathrm{d}\Xi_i. \tag{101}$$

Following the strategy used in the four-dimensional case, it is straightforward to obtain other thermodynamic descriptions comparing Eqs. (101) and (36).

The extension to higher dimensions of Hawking's proposal presented here differs from the five-dimensional case in [16]. This is because, in the present work, the angular momentum is given by a Komar integral in the frame in which the black hole spins with an angular velocity Ω_H^i. Moreover, our generation satisfies a first law and the quantum statistical relation, while the one in [16] does not.

D. The alternative thermodynamic theory in D dimensions

The ATT, presented in Sec. V C, can be reinterpreted as the theory obtained from Hawking's thermodynamics with an isohomogeneous transformation characterized by $h = 0$. Extending this idea to D dimensions,

$$g = \exp\left(\sum_{i=1}^{N}\frac{\ln\Xi_i}{D-2}\right) = \prod_{i=1}^{N}\Xi_i^{\frac{1}{D-2}}. \tag{102}$$

The function g in Eq. (102) transforms the D-dimensional version of Hawking's proposal into a D-dimensional generalization for the ATT.

The energy associated to the D-dimensional ATT is given by

$$E_A^{(D)} \equiv \frac{E_H^{(D)}}{g} = \frac{D-2}{8\pi}mA_{D-2}\prod_{i=1}^{N}\Xi_i^{-\frac{D-1}{D-2}}. \tag{103}$$

The above construction gives a theory valid for arbitrary dimensions, but different from the one presented in [15]. In addition to providing a proper Smarr relation, our construction can be seen as the thermodynamic version for a generalization of Hawking's proposal. Also, while [15] states the failure of the Smarr-Gibbs-Duhem relation for KadS, we have shown that this relation holds if the cosmological constant is interpreted as a pressure term alongside with a thermodynamic volume. Nevertheless, to our knowledge, the expression for the energy in Eq. (103) is the first one to satisfy the first law of thermodynamics and the Smarr relation by construction, valid for $D \geq 4$.

VII. FINAL REMARKS

In this work, we propose a procedure for constructing thermodynamic descriptions for Kerr-anti–de Sitter black holes that are compatible with a Smarr formula. In order to consider asymptotically anti–de Sitter spacetimes, the Smarr formula is generalized from its standard version. For this, a vector volume is naturally related to geometries with a nonzero Λ. The derivation of the Smarr formula requires a Killing field K normal to the horizon. However, this Killing field is not unique, since it can be multiplied by any function g of the parameters m, a, and Λ [or, equivalently by a function $g = g(S, J, P)$]. Furthermore, there are infinitely many ways to write K as a combination of a timelike and a rotating Killing field. As a result, different choices for the Killing fields can alter the Komar integrals and thus the thermodynamic variables. While a Smarr formula gives the geometric side for the black-hole thermodynamics, the first law relates variations in the Komar integrals. Contrary to what might be expected, these integrals may not carry any information about the scale invariance of the system.

Our method follows from finding out which transformations can be performed in these thermodynamic variables that preserve homogeneity. We show that these isohomogeneous transformations have a geometric counterpart as a corresponding transformation in the Killing vectors. Specifically, it reduces g to a homogeneous function of degree zero and restricts the viable combination between the timelike and angular Killing vectors. With these isohomogeneous transformations, and given a thermodynamics associated with a physical system, other descriptions can be obtained. Among all the possible KadS thermodynamics that can be generated with our formalism, two receive a special attention: the UTT and Hawking's proposal. The explicit construction of a thermodynamic version of Hawking's approach in four dimensions and its generalization to arbitrary dimensions are also highlighted.

Although the usual thermodynamics is widely explored in the literature, a proper geometric construction for the theory is still undeveloped. One of our contributions in this article is to fill this gap, presenting the Killing vector associated with this theory. Moreover, our formalism clarifies why the thermodynamic and geometric volumes in the UTT do not coincide. In contrast, Hawking's proposal is not a proper thermodynamics, despite having an associated Smarr formula. Within our formalism, there is a wide set of isohomogeneous transformations that relate Hawking's proposal to proper thermodynamic theories, where the first law is satisfied.

In addition, there is a specific theory that can be considered, in a precise sense, as the thermodynamic version of Hawking's proposal. This ATT for KadS is the closest to the standard asymptotically flat black-hole thermodynamic, since it is the one in which the temperature coincides with its own surface gravity. That is, considering a coherent normalization for the Killing fields. This claim can only be made because the present work provides a geometric construction for the theories. We show that, contrary to what has been previously asserted, the UTT has a surface gravity that does not agrees with its temperature.

With the isohomogeneous transformations, new proposals for the thermodynamic description of KadS black holes can now be made, furnishing different KadS thermodynamics. In anticipation of applications of the KadS thermodynamics in the context of the adS/CFT correspondence, the ATT is generalized from four to arbitrary dimensions. In the process, the extension of Hawking's proposal is also explicitly constructed. Our generalization differs from the five-dimensional case presented in Hawking's work, satisfying a first law and the quantum statistical relation.

The results of this work allow for further investigations. We anticipate a connection between the thermodynamic results presented in this study and the Hamiltonian dynamics of KadS black holes. In [36], the expression for the energy of the ATT also appears in this different context. Furthermore, we speculate that an analysis of our results from the perspective of concrete observers may provide a deeper understanding of the variety of thermodynamic descriptions for the black hole. Semiclassical treatments could also provide new insights in this area. Research along these lines is ongoing.

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[1] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Gauge theory correlators from non-critical string theory, Phys. Lett. B **428**, 105 (1998).
[2] J. M. Maldacena, The large N limit of superconformal field theories and supergravity, Adv. Theor. Math. Phys. **2**, 231 (1998).
[3] E. Witten, Anti-de Sitter space and holography, Adv. Theor. Math. Phys. **2**, 253 (1998).
[4] J. B. Amado, B. C. da Cunha, and E. Pallante, On the Kerr-AdS/CFT correspondence, J. High Energy Phys. 08 (2017) 094.
[5] M. Guica, T. Hartman, W. Song, and A. Strominger, The Kerr/CFT correspondence, Phys. Rev. D **80**, 124008 (2009).
[6] J. D. Brown, J. Creighton, and R. B. Mann, Temperature, energy and, heat capacity of asymptotically anti-de Sitter black holes, Phys. Rev. D **50**, 6394 (1994).
[7] W. S. Elias, C. Molina, and M. C. Baldiotti. Thermodynamics of bosonic systems in anti-de Sitter spacetime, Phys. Rev. D **99**, 084028 (2019).
[8] S. Hemming and L. Thorlacius, Thermodynamics of large AdS black holes, J. High Energy Phys. 11 (2007) 086.
[9] D. Kastor, S. Ray, and J. Traschen, Enthalpy and the mechanics of adS black holes, Classical Quantum Gravity **26**, 195011 (2009).
[10] J. Louko and S. N. Winters-Hilt, Hamiltonian thermodynamics of the Reissner-Nordstrom anti-de Sitter black hole, Phys. Rev. D **54**, 2647 (1996).
[11] M. M. Caldarelli, G. Cognola, and D. Klemm, Thermodynamics of Kerr-Newmann-AdS black holes and conformal field theories, Classical Quantum Gravity **17**, 399 (2000).
[12] V. Cardoso, Ó. J. C. Dias, G. S. Hartnett, L. Lehner, and J. E. Santos, Holographic thermalization, quasinormal modes and superradiance in Kerr-AdS, J. High Energy Phys. 04 (2014) 183.
[13] B. P. Dolan, Pressure and volume in the first law of black hole thermodynamics, Classical Quantum Gravity **28**, 235017 (2011).
[14] B. P. Dolan, Where is the PdV in the first law of black hole thermodynamics?, Open Questions in Cosmology (IntechOpen, 2012), arXiv:1209.1272.
[15] G. W. Gibbons, M. J. Perry, and C. N. Pope, The first law of thermodynamics for Kerr-anti de Sitter black holes, Classical Quantum Gravity **22**, 1503 (2005).
[16] S. W. Hawking, C. J. Hunter, and M. M. Taylor-Robinson, Rotation and the AdS/CFT correspondence, Phys. Rev. D **59**, 064005 (1999).
[17] J. P. S. Lemos, F. J. Lopes, M. Minamitsuji, and J. V. Rocha, Thermodynamics of rotating thin shells in the BTZ spacetime, Phys. Rev. D **92**, 064012 (2015).
[18] D. Kubizňák, R. B. Mann, and M. Teo, Black hole chemistry: Thermodynamics with Lambda, Classical Quantum Gravity **34**, 063001 (2017).
[19] Y. Gao, Z. Di, and S. Gao, General mass formulas for charged Kerr-adS black holes, arXiv:2304.10290.
[20] B. P. Dolan, The cosmological constant and black-hole thermodynamic potentials, Classical Quantum Gravity **28**, 125020 (2011).
[21] B. P. Dolan, Compressibility of rotating black holes, Phys. Rev. D **84**, 127503 (2011).
[22] V. Iyer and R. W. Wald, Some properties of the Noether charge and a proposal for dynamical black hole entropy, Phys. Rev. D **50**, 846 (1994).
[23] W. B. Fontana, M. C. Baldiotti, R. Fresneda, and C. Molina, Extended quasilocal thermodynamics of Schwarzschild-anti de Sitter black holes, Ann. Phys. (Amsterdam) **411**, 167954 (2019).
[24] M. C. Baldiotti, R. Fresneda, and C. Molina, A Hamiltonian approach for the thermodynamics of AdS black holes, Ann. Phys. (Amsterdam) **382**, 22 (2017).
[25] J. B. Griffiths and J. Podolský, *Exact Space-Times in Einstein's General Relativity* (Cambridge University Press, Cambridge, England, 2009).
[26] J. M. Bardeen, B. Carter, and S. W. Hawking, The four laws of black hole mechanics, Commun. Math. Phys. **31**, 161 (1973).
[27] B. Ryden, *Introduction to Cosmology* (Cambridge University Press, Cambridge, England, 2017).
[28] W. Ballik and K. Lake, Vector volume and black holes, Phys. Rev. D **88**, 104038 (2013).
[29] E. Poisson, *A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics* (Cambridge University Press, Cambridge, England, 2004).
[30] J. P. Gauntlett, R. C. Myers, and P. K. Townsend, Black holes of $D = 5$ supergravity, Classical Quantum Gravity **16**, 1 (1999).
[31] P. K. Townsend and M. Zamaklar, The first law of black brane mechanics, Classical Quantum Gravity **18**, 5269 (2001).
[32] J. Santiago and M. Visser, Tolman-like temperature gradients in stationary spacetimes, Phys. Rev. D **98**, 064001 (2018).
[33] G. W. Gibbons, H. Lü, D. N. Page, and C. N. Pope, The general Kerr-de Sitter metrics in all dimensions, J. Geom. Phys. **53**, 49 (2005).
[34] G. W. Gibbons and S. W. Hawking, Action integrals and partition functions in quantum gravity, Phys. Rev. D **15**, 2752 (1977).
[35] M. Cvetič, G. W. Gibbons, D. Kubizňák, and C. N. Pope, Black hole enthalpy and an entropy inequality for the thermodynamic volume, Phys. Rev. D **84**, 024037 (2011).
[36] P. T. Chruściel, J. Jezierski, and J. Kijowski, Hamiltonian dynamics in the space of asymptotically Kerr-de Sitter spacetimes, Phys. Rev. D **92**, 084030 (2015).