Horizon-penetrating form of parametrized metrics for static and stationary black holes

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The Rezzolla-Zhidenko (RZ) and Konoplya-Rezzolla-Zhidenko (KRZ) frameworks provide an efficient approach to characterize agnostically spherically symmetric or stationary black-hole spacetimes in arbitrary metric theories. In their original construction, these metrics were defined only in the spacetime region outside of the event horizon, where they can reproduce any black-hole metric with percent precision and a few parameters only. At the same time, numerical simulations of accreting black holes often require metric functions that are regular across the horizon, so that the inner boundary of the computational domain can be placed in a region that is causally disconnected from the exterior. We present a novel formulation of the RZ/KRZ parametrized metrics in coordinate systems that are regular at the horizon and defined everywhere in the interior. We compare the horizon-penetrating form of the KRZ and RZ metrics with the corresponding forms of the Kerr metric in Kerr-Schild coordinates and of the Schwarzschild metric in Eddington-Finkelstein coordinates, noting the similarities and differences. We expect the horizon-penetrating formulations of the RZ/KRZ metrics to represent new tools to study via simulations the physical processes that occur near the horizon of an arbitrary black hole.

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I. INTRODUCTION

The past few years have provided compelling evidence that black holes as predicted by Einstein's general relativity are perfectly compatible with gravitational-wave [1] and electromagnetic [2,3] observations. Yet, because of the uncertainties accompanying these observations, there is still plenty of room for alternative interpretations within other theories of gravity (see, e.g., Refs. [4–8]).

Because of the wide variety of existing alternative theories of gravity, and to avoid the impractical approach in which a validation of observations is made on a case-by-case manner for every single theory, model-independent representations of generic black-hole spacetime have been proposed to measure the deviation from general relativity of alternative theories of gravity. In this way, it is in principle possible to invoke astronomical observations to constrain possible deviations between different black-hole geometries [9]. A first attempt in this direction is the parametrization of rotating black holes by Johannsen and Psaltis [10] and Johannsen [11], who expanded the deviation from the Kerr metric in terms of a Taylor series of the dimensionless compactness parameter M/r, where M is the black-hole mass and r is a generic radial coordinate. Despite some of its expansion coefficients being observationally constrained, such an approach requires an infinite number of parameters with equal importance and is only able to reproduce small deviations from general relativity [12].

These shortcomings were addressed first for nonrotating black holes by the Rezzolla-Zhidenko (RZ) parametrization [13], which expresses the deviation of a generic spherically symmetric metric from the Schwarzschild metric in terms of a Padé expansion of a compactified radial coordinate.[1] The superior convergence properties of the continued-fraction expansion allows one to approximate arbitrary black holes in alternative theories reaching a percent precision with only a few expansion coefficients [14,15]. The extension of this approach to stationary black-hole spacetimes was later obtained with the Konoplya-Rezzolla-Zhidenko (KRZ) parametrization [16]. The KRZ metric adopts the same continued fraction expansion in the radial direction, and a Taylor expansion in the polar direction, providing excellent convergence to various black-hole metrics [17]. Since these parametrized metrics are not the result of a generic parametrized action, no field equations can be associated with the RZ/KRZ metrics, which thus cannot be employed in scenarios or simulations where the spacetime is dynamical.

Although the RZ/KRZ parametrizations have been successful in describing an arbitrary black-hole metric in

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[1]The general formulation of the parametrized metric makes it applicable also to nonvacuum spacetimes, such as those involving a compact star or a boson star [14].

a theory-independent manner, they are constructed only for the exterior portion of the spacetime, that is, for the part of the spacetime between the event horizon and spatial infinity in the case of vacuum spacetimes, or for the part of the spacetime between the surface of the compact star and spatial infinity in the case of nonvacuum spacetimes. The interior region, despite not being necessarily undefined, is disconnected from the exterior by virtue of the coordinate singularity at the horizon. Numerical simulations, on the other hand, traditionally make use of "horizon-penetrating" (HP) coordinates, that is, coordinate systems that are regular across the horizon, so that the inner boundary of the computational domain can be placed in a region that is causally disconnected from the exterior and hence does not require special or sophisticated boundary conditions. In the case of Kerr spacetimes, the most commonly simulated spacetimes for rotating black holes [18], this is accomplished by expressing the Kerr metric not in Boyer-Lindquist coordinates (that are singular at the horizon), but in terms of Kerr-Schild coordinates, hence obtaining a coordinate mapping that is regular everywhere in the interior with the obvious exception of the ring singularity. Clearly, it would be useful to have RZ/KRZ parametrizations that have similar features, that is, that are regular at the event horizon and in the interior.

A first attempt to derive a version of the KRZ parametrization that is regular on the event horizon was proposed by Konoplya, Kunz, and Zhidenko [19]. Although HP in principle, the KRZ formulation in Ref. [19] is not useful in practice. This is because they used a coordinate transformation that alters the curvature invariants, leading to a Kerr reduction that is not Ricci flat. The metric form in Ref. [19] also has a zero g_{rr} component and hence a zero determinant of the three-metric, incompatible with a $3+1$ split of spacetime normally employed in numerical simulation codes. We here present a different HP formulation of the RZ/KRZ parametrizations with an invariant Ricci scalar and nonzero three-metric determinant. As such, they can be used in numerical simulations modeling; for instance, the accretion flows onto arbitrary black holes in alternative theories of gravity. In such scenarios, in fact, the gravitational mass of the accreting material is many orders of magnitude smaller than that of the black hole, and the spacetime is therefore determined by the black hole to a very good approximation.[2] Hence, given a specific physical scenario where the background spacetime is not influenced by the dynamics of matter or fields to be evolved (baryonic matter, electromagnetic fields, radiation fields, scalar fields, etc.), the use of our HP KRZ metric allows one to simulate black-hole accretion processes by evolving the corresponding conservation equations of energy, momentum, and rest-mass from large distances down to the black-hole interior without encountering any coordinate singularity.

The structure of the paper is as follows. In Sec. II we review the basic aspects of the KRZ parametrization for general rotating black holes. Section III reports instead the main steps needed for the derivation of our HP versions of the KRZ parametrization in spherical coordinates. We also offer an alternative Cartesian formulation of the HP KRZ metric in Sec. IV. Section V applies our HP coordinates to a couple of examples of rotating black holes. Section VI provides a reduction of the HP KRZ metric to nonrotating black holes. Finally, a brief discussion of our results is presented in Sec. VII. Hereafter, we adopt a set of units in which $c = 1 = G$, with c and G being the speed of light and the gravitational constant, respectively. Furthermore, as usual, we adopt Greek letters for indices running from 0 to 3 and Latin letters for indices running from 1 to 3.

II. GENERAL STATIONARY BLACK-HOLE METRICS

In our construction of HP coordinates of parametrized black-hole metrics, it is more convenient to start from the case of stationary black holes and reduce the resulting expressions to the nonrotating case. Hence, we start from the standard KRZ parametrization [16], where the spacetime around a general rotating black hole in an arbitrary metric theory of gravity is stationary and axisymmetric, described by a metric in the form

$$ds^2 = -\frac{N^2 - W^2\sin^2\theta}{K^2}dt^2 - 2Wr\sin^2\theta dt d\phi + \Sigma\left(\frac{B^2}{N^2}dr^2 + r^2 d\theta^2\right) + K^2 r^2 \sin^2\theta d\phi^2, \quad (1)$$

where the coordinates t and ϕ are associated with the timelike and spacelike (azimuthal) Killing vectors marking the spacetime symmetry, while r and θ are along the radial and angular directions perpendicular to ϕ, so that (t, r, θ, ϕ) form a set of coordinates that are spherical asymptotically.[3] In the metric (1), the functions $B = B(r,\theta)$, $N = N(r,\theta)$, $K = K(r,\theta)$, $W = W(r,\theta)$, and $\Sigma = \Sigma(r,\theta)$ are functions of r and θ only, with the latter being defined as

$$\Sigma \coloneqq 1 + \frac{a_*^2\cos^2\theta}{r^2}, \quad (2)$$

with $a_* \coloneqq J/M$ being the rotation parameter of the black hole, while M and J are the black-hole mass and angular momentum, respectively.

[2] Taking for example the case of the accretion flows onto the supermassive black holes M87* [2] or Sgr A* [3], the mass of a typical accretion disk with the external edge at 10^4 gravitational radii is only 10^{-12} (10^{-6}) that of the central black-hole M87* [20] (Sgr A* [21]). Similar small ratios can be computed in the case of accretion onto x-ray binaries [22].

[3] In the case of the Kerr metric, such coordinates are represented by the Boyer-Lindquist coordinates.

As a result, the determinant of the KRZ metric (1) is then given by

$$g \coloneqq \det(g_{\mu\nu}) = -\Sigma^2 B^2 r^4 \sin^2\theta, \tag{3}$$

while the inverse metric can be found by using the identity $\delta^\mu_\nu = g^{\mu\lambda} g_{\lambda\nu}$, with the nonzero components being

$$g^{tt} = -\frac{g_{\phi\phi}}{g_{t\phi}^2 - g_{tt}g_{\phi\phi}} = -\frac{K^2}{N^2}, \tag{4}$$

$$g^{t\phi} = \frac{g_{t\phi}}{g_{t\phi}^2 - g_{tt}g_{\phi\phi}} = -\frac{W}{N^2 r}, \tag{5}$$

$$g^{rr} = \frac{1}{g_{rr}} = \frac{N^2}{\Sigma B^2}, \tag{6}$$

$$g^{\theta\theta} = \frac{1}{g_{\theta\theta}} = \frac{1}{\Sigma r^2}, \tag{7}$$

$$g^{\phi\phi} = -\frac{g_{tt}}{g_{t\phi}^2 - g_{tt}g_{\phi\phi}} = \frac{N^2 - W^2\sin^2\theta}{N^2K^2r^2\sin^2\theta}. \tag{8}$$

The event horizon of the black hole in the KRZ metric is located at a surface where the metric component g_{rr} diverges (or g^{rr} vanishes), that is,

$$N^2(r,\theta) = 0. \tag{9}$$

In particular, r_0 is the horizon radius in the equatorial plane (i.e., at $\theta = \pi/2$), so that $N^2(r_0, \pi/2) = 0$. The other important surface of a stationary black hole is represented by the "static limit" and is defined as the surface at which the metric component g_{tt} vanishes:

$$N^2 = W^2 \sin^2\theta. \tag{10}$$

The region between these two surfaces is the ergosphere, and hence it is characterized by the condition

$$0 < N^2 < W^2 \sin^2\theta, \tag{11}$$

and, within the Kerr metric, it represents the region where energy and angular momentum can be extracted from the black hole via the Penrose process (see, e.g., Ref. [23] for a very comprehensive overview).

The geometry of such axisymmetric spacetime depends on five quantities: one constant parameter a_*, and four functions B, N, K, W, that can be parametrized in terms of expansions in the r and θ directions. An essential aspect of the RZ, and therefore of the KRZ framework, is the introduction of a compactified radial coordinate

$$\tilde{x} \coloneqq 1 - \frac{r_0}{r}, \tag{12}$$

which maps the black-hole exterior $r \in [r_0, \infty)$ to $\tilde{x} \in [0, 1)$ and hence allows one to impose rather trivially the asymptotic properties of the spacetime. Similarly, one can introduce the new angular coordinate

$$\tilde{y} \coloneqq \cos\theta, \tag{13}$$

which maps the polar angle $\theta \in [0, \pi/2]$ to $\tilde{y} \in [1, 0]$. Adopting these new variables, the metric functions in terms of $(\tilde{x}, \tilde{y})$ of the KRZ metric are expressed after a variable separation in a product of functions of $\tilde{x}$ and a Taylor series of $\tilde{y}$

$$N^2(\tilde{x},\tilde{y}) = \tilde{x}A_0(\tilde{x}) + \sum_{i=1}^{\infty} A_i(\tilde{x})\tilde{y}^i, \tag{14}$$

$$B(\tilde{x},\tilde{y}) = 1 + \sum_{i=0}^{\infty} B_i(\tilde{x})\tilde{y}^i, \tag{15}$$

$$W(\tilde{x},\tilde{y}) = \frac{1}{\Sigma}\sum_{i=0}^{\infty} W_i(\tilde{x})\tilde{y}^i, \tag{16}$$

$$K^2(\tilde{x},\tilde{y}) = 1 + \frac{a_* W}{r} + \frac{1}{\Sigma}\sum_{i=0}^{\infty} K_i(\tilde{x})\tilde{y}^i, \tag{17}$$

and then the functions of only $\tilde{x}$ are expressed as

$$B_i(\tilde{x}) \coloneqq b_{i0}(1-\tilde{x}) + \tilde{B}_i(\tilde{x})(1-\tilde{x})^2, \tag{18}$$

$$W_i(\tilde{x}) \coloneqq w_{i0}(1-\tilde{x})^2 + \tilde{W}_i(\tilde{x})(1-\tilde{x})^3, \tag{19}$$

$$K_i(\tilde{x}) \coloneqq k_{i0}(1-\tilde{x})^2 + \tilde{K}_i(\tilde{x})(1-\tilde{x})^3, \tag{20}$$

$$A_0(\tilde{x}) \coloneqq 1 - \epsilon_0(1-\tilde{x}) + (a_{00} + k_{00} - \epsilon_0)(1-\tilde{x})^2 + \tilde{A}_0(\tilde{x})(1-\tilde{x})^3, \tag{21}$$

$$A_{i\geq 1}(\tilde{x}) \coloneqq K_i(\tilde{x}) + \epsilon_i(1-\tilde{x})^2 + a_{i0}(1-\tilde{x})^3 + \tilde{A}_i(\tilde{x})(1-\tilde{x})^4, \tag{22}$$

where the tilded functions $\tilde{A}_i$, $\tilde{B}_i$, $\tilde{K}_i$, and $\tilde{W}_i$ are expressed as Padé series in terms of continued fractions of $\tilde{x}$

$$\tilde{A}_i(\tilde{x}) \coloneqq \frac{a_{i1}}{1 + \frac{a_{i2}\tilde{x}}{1 + \frac{a_{i3}\tilde{x}}{1+\cdots}}}, \tag{23}$$

$$\tilde{B}_i(\tilde{x}) \coloneqq \frac{b_{i1}}{1 + \frac{b_{i2}\tilde{x}}{1 + \frac{b_{i3}\tilde{x}}{1+\cdots}}}, \tag{24}$$

$$\tilde{K}_i(\tilde{x}) \coloneqq \frac{k_{i1}}{1 + \frac{k_{i2}\tilde{x}}{1 + \frac{k_{i3}\tilde{x}}{1+\cdots}}}, \tag{25}$$

$$\tilde{W}_i(\tilde{x}) \coloneqq \frac{w_{i1}}{1 + \frac{w_{i2}\tilde{x}}{1 + \frac{w_{i3}\tilde{x}}{1+\cdots}}}. \tag{26}$$

Thanks to the superior convergence properties offered by continued fractions, the expanded metric can approximate deviations from a Schwarzschild spacetime with the same mass or a Kerr spacetime with the same spin in terms of a few parameters only. A detailed discussion of the properties of the expansion in a variety of spacetimes can be found in a number of recent works [5,14,15,24–28].

III. HORIZON-PENETRATING COORDINATES

Since the rr component of the metric (1) diverges at the event horizon, we need to find a coordinate system that is HP, namely, where this function is regular at the event horizon and hence allows one to smoothly join the interior with the exterior.

A. A first Ansatz

As mentioned in the Introduction, as a first attempt to derive a version of the KRZ parametrization that is regular on the event horizon, Konoplya, Kunz, and Zhidenko [19] introduced a new time and azimuthal variable in an Eddington-Finkelstein–like form

$$d\hat{t} = dt + C(r,\theta)dr, \tag{27}$$

$$d\hat{\phi} = d\phi + D(r,\theta)dr, \tag{28}$$

where C and D are smooth functions of r and θ. The new coordinates $\hat{t}$ and $\hat{\phi}$ defined have now a dependence on the coordinates r and θ, so that the KRZ metric (1) in these coordinates becomes

$$\begin{aligned} ds^2 = &-\frac{N^2 - W^2\sin^2\theta}{K^2}d\hat{t}^2 - 2Wr\sin^2\theta d\hat{t}d\hat{\phi} \\ &+ K^2r^2\sin^2\theta d\hat{\phi}^2 + \Sigma r^2 d\theta^2 + g_{rr}dr^2 \\ &+ 2g_{r\hat{t}}drd\hat{t} + 2g_{r\hat{\phi}}drd\hat{\phi}, \end{aligned} \tag{29}$$

where

$$g_{rr} = \left(\frac{\Sigma B^2}{N^2} - \frac{N^2}{K^2}C^2\right) + \left(\frac{W}{K^2 r}C - D\right)^2 K^2 r^2 \sin^2\theta, \tag{30}$$

$$g_{r\hat{t}} = \frac{N^2}{K^2}C - \left(\frac{W}{K^2 r}C - D\right)Wr\sin^2\theta, \tag{31}$$

$$g_{r\hat{\phi}} = \left(\frac{W}{K^2 r}C - D\right)K^2 r^2 \sin^2\theta. \tag{32}$$

Since the functions C and D are arbitrary, they minimized the number of new terms by imposing a relation

$$D = \frac{W}{K^2 r}C. \tag{33}$$

As a result, all the associated brackets in Eqs. (30)–(32) vanish. To specify the transformation function C, those authors proposed

$$C = \frac{\sqrt{\Sigma B^2 K^2}}{N^2}, \tag{34}$$

so that the two terms in g_{rr} cancel out exactly

$$g_{rr} = \frac{\Sigma B^2}{N^2} - \frac{N^2}{K^2}C^2 = 0, \tag{35}$$

and thus the only new metric component is

$$g_{r\hat{t}} = \frac{N^2}{K^2}C = \sqrt{\frac{\Sigma B^2}{K^2}}. \tag{36}$$

In summary, adopting this new set of variables, and dropping the hat symbol for the $\hat{t}$ and $\hat{\phi}$ coordinates, the KRZ metric (1) is transformed into

$$\begin{aligned} ds^2 = &-\frac{N^2 - W^2\sin^2\theta}{K^2}dt^2 + 2\sqrt{\frac{\Sigma B^2}{K^2}}dtdr \\ &- 2Wr\sin^2\theta dtd\phi + \Sigma r^2 d\theta^2 + K^2 r^2 \sin^2\theta d\phi^2. \end{aligned} \tag{37}$$

Despite taking a simple form, with five nonzero components as in the original Boyer-Lindquist–like coordinates and as is regular at the horizon, the KRZ metric in these coordinates (37) is problematic. It is easy to realize this by considering the metric (37) when reduced to the case of a Kerr spacetime. We recall that the Kerr metric in Boyer-Lindquist coordinates takes the form

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2Mr}{\Sigma_K}\right)dt^2 - \frac{4Mr}{\Sigma_K}a_*\sin^2\theta dtd\phi + \frac{\Sigma_K}{\Delta}dr^2 \\ &+ \Sigma_K d\theta^2 + \frac{A}{\Sigma_K}\sin^2\theta d\phi^2, \end{aligned} \tag{38}$$

where the functions

$$\Sigma_K \coloneqq r^2 + a_*^2\cos^2\theta, \tag{39}$$

$$\Delta \coloneqq r^2 - 2Mr + a_*^2, \tag{40}$$

$$A \coloneqq (r^2 + a_*^2)^2 - a_*^2\Delta\sin^2\theta, \tag{41}$$

and where it should be noted that the function Σ_K differs by a quadratic term in the radial coordinate from the corresponding KRZ function, i.e., $\Sigma_K = \Sigma r^2$ (Σ is dimensionless while Σ_K has the dimensions of a squared length).

A direct comparison between the Kerr metric (38) and the original KRZ metric presented in Eq. (1) implies that the free metric functions in the metric (37) for the case of a Kerr spacetime are given by

$$B^2 = 1, \tag{42}$$

$$N^2 = \frac{\Delta}{r^2}, \tag{43}$$

$$K^2 = \frac{A}{\Sigma_K r^2}, \tag{44}$$

$$W = \frac{2Ma_*}{\Sigma_K}. \tag{45}$$

Substituting the relations (42)–(45) in the transformed HP metric (37) yields the KRZ metric reduced to a Kerr spacetime in the HP coordinates

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma_K}\right)dt^2 + \frac{2\Sigma_K}{\sqrt{A}}dtdr - \frac{4Mr}{\Sigma_K}a_*\sin^2\theta dtd\phi + \Sigma_K d\theta^2 + \frac{A}{\Sigma_K}\sin^2\theta d\phi^2. \tag{46}$$

The Ricci scalar of such a metric does not vanish everywhere, but is instead given by

$$R \coloneqq g^{\mu\nu}R_{\mu\nu} = \frac{a_*^4\Delta\sin^2\theta\cos^2\theta}{\Sigma_K^3 A^2}F_1(r,\theta), \tag{47}$$

where the function F_1 has the radial dependence $F_1(r,\theta) \propto r^n$ with $n \leq 6$. In other words, the metric is flat only at the horizon, the polar axis, and the equatorial plane. Because a well-posed coordinate transformation cannot change the Ricci-flatness property of the Kerr solution, this is an indication that the initial coordinate transformations (27) and (28) are not adequate and alternative approaches need to be found.

Besides leading to a Kerr reduction that is not Ricci flat, the metric in Eq. (37) also suffers from an additional drawback when it needs to be implemented in numerical simulations (see Refs. [4,29] for some examples of numerical simulations of accretion onto black holes in different theories of gravity). This is because numerical codes solving the equations of general-relativistic hydrodynamics or magnetohydrodynamics systematically adopt a $3+1$ split of spacetime where the metric and its inverse have components given by (see, e.g., [30,31])

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta^k\beta_k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}, \tag{48}$$

$$g^{\mu\nu} = \begin{pmatrix} -1/\alpha^2 & \beta^j/\alpha^2 \\ \beta^i/\alpha^2 & \gamma^{ij} - \beta^i\beta^j/\alpha^2 \end{pmatrix}, \tag{49}$$

where α is the (scalar) lapse function, $\boldsymbol{\beta}$ is the shift vector, and $\boldsymbol{\gamma}$ is the spatial three-metric. In the $3+1$ split, the determinant of the four-metric $g \coloneqq \det(g_{\mu\nu})$ and that of the three-metric $\gamma \coloneqq \det(\gamma_{ij})$ are related by the simple expression

$$\sqrt{-g} = \alpha\sqrt{\gamma}. \tag{50}$$

Since the metric in Eq. (37) has $g_{rr} = \gamma_{rr} = 0$, the determinant of the three-metric is zero ($\gamma = 0$), but that of the four-metric is nonzero as in Eq. (3), thus leaving the lapse function divergent. Overall, therefore, the coordinate transformation in Eqs. (27) and (28) leads to a KRZ form (37) that is not useful in practice. In view of these drawbacks, in the following section we present a KRZ form in HP coordinates that is regular at the horizon and that can be implemented in numerical simulations. [See Appendix B for a discussion about the coordinate transformation leading to the KRZ formulation in Eq. (37).]

B. A new Ansatz for a subclass of the KRZ metric

As anticipated in the previous section, a different approach to find a form of the KRZ metric that is HP and leads to a Ricci-flat Kerr reduction is to start from a formulation of the KRZ expansion where the metric functions that are themselves separable. Fortunately, this problem has already been solved by Konoplya, Stuchlík, and Zhidenko [32], who have derived a subclass of the KRZ metric that allows for separation of variables. In such a subclass, the KRZ functions can be written as

$$B(r,\theta) \eqqcolon R_B(r), \tag{51}$$

$$N^2(r,\theta) \eqqcolon 1 - \frac{R_M(r)}{r} + \frac{a_*^2}{r^2}, \tag{52}$$

$$W(r,\theta) = \frac{1}{\Sigma(r,\theta)}\frac{a_* R_M(r)}{r^2}, \tag{53}$$

$$\begin{aligned} K^2(r,\theta) &= \frac{1}{\Sigma(r,\theta)}\left(1 + \frac{a_*^2}{r^2} + \frac{a_*^2\cos^2\theta}{r^2}N^2(r)\right) + \frac{a_* W(r,\theta)}{r} \\ &= \frac{1}{\Sigma(r,\theta)}\left[1 + \frac{a_*^2}{r^2} + \frac{a_*^2}{r^2}\frac{R_M(r)}{r} + \frac{a_*^2\cos^2\theta}{r^2} \times \left(1 - \frac{R_M(r)}{r} + \frac{a_*^2}{r^2}\right)\right] \\ &= \Sigma(r,\theta) + \left(1 + \frac{R_M(r)}{\Sigma(r,\theta)r}\right)\frac{a_*^2\sin^2\theta}{r^2}, \end{aligned} \tag{54}$$

where Eq. (52) can be taken as the implicit definition of the function $R_M(r)$. Note that in this way, four free KRZ functions: B and N depend on r alone, while the functions W and K have also a θ-dependence, but only in terms of $\Sigma(r,\theta)$ and $\sin^2\theta$.

We can now proceed with a coordinate transformation that will guarantee regularity across the horizon by requiring that the transformation functions $\hat{C}$ and $\hat{D}$ are independent of θ, i.e., $\hat{C} = \hat{C}(r)$, $\hat{D} = \hat{D}(r)$. The differential form is then reduced to

$$d\hat{t} = dt + C(r)dr, \tag{55}$$

$$d\hat{\phi} = d\phi + D(r)dr, \tag{56}$$

so that the transformed metric remains as in Eqs. (29)–(32). Note that when choosing the transformation functions $C(r)$ and $D(r)$ to regularize the metric components in Eqs. (30)–(32), we can no longer force the relation (33), because the factor $W/(K^2 r)$ is dependent on θ. Thus, none of the brackets in Eqs. (30)–(32) vanish, but each of them has to be regular at the horizon; i.e., they cannot contain a factor N^2 in the denominator. To this scope we set

$$C(r) = \frac{R_B R_M}{N^2 r}, \tag{57}$$

$$D(r) = \frac{R_B a_*}{N^2 r^2}, \tag{58}$$

which results in the condition

$$\begin{aligned} \frac{W}{K^2 r} C - D &= \frac{R_B a_*}{N^2 r^2}\left(\frac{1}{\Sigma K^2}\frac{R_M^2}{r^2} - 1\right) \\ &= -\frac{R_B a_*}{K^2 r^2}\left(1 + \frac{R_M}{\Sigma r}\right). \end{aligned} \tag{59}$$

The corresponding modified metric components are then simplified as

$$\begin{aligned} g_{rr} &= \frac{R_B^2}{N^2 K^2}\left(\Sigma K^2 - \frac{R_M^2}{r^2}\right) + \frac{R_B^2 a_*^2}{K^2 r^2}\left(1 + \frac{R_M}{\Sigma r}\right)^2 \sin^2\theta \\ &= \frac{\Sigma R_B^2}{K^2}\left(1 + \frac{R_M}{\Sigma r}\right) + \frac{R_B^2}{K^2}\left(1 + \frac{R_M}{\Sigma r}\right)^2 \frac{a_*^2 \sin^2\theta}{r^2} \\ &= \left(1 + \frac{R_M}{\Sigma r}\right) R_B^2, \end{aligned} \tag{60}$$

$$g_{r\hat{t}} = \frac{R_B R_M}{K^2 r} + \frac{R_B R_M}{K^2 r}\left(1 + \frac{R_M}{\Sigma r}\right)\frac{a_*^2 \sin^2\theta}{\Sigma r^2} = \frac{R_B R_M}{\Sigma r}, \tag{61}$$

$$g_{r\hat{\phi}} = -\left(1 + \frac{R_M}{\Sigma r}\right) R_B a_* \sin^2\theta, \tag{62}$$

leading to the following form of the separable KRZ metric in HP coordinates (where we drop the hat on the t and ϕ coordinates)

$$\begin{aligned} ds^2 = &-\left(1 - \frac{R_M}{\Sigma r}\right)dt^2 + 2\frac{R_M}{\Sigma r} R_B dt dr - 2\frac{R_M}{\Sigma r} a_* \sin^2\theta dt d\phi \\ &+ \left(1 + \frac{R_M}{\Sigma r}\right) R_B^2 dr^2 - 2\left(1 + \frac{R_M}{\Sigma r}\right) R_B a_* \sin^2\theta dr d\phi \\ &+ \Sigma r^2 d\theta^2 + K^2 r^2 \sin^2\theta d\phi^2. \end{aligned} \tag{63}$$

We note that, as already remarked in Ref. [32], it is in principle possible to introduce a new radial coordinate defined through the differential $d\hat{r} = R_B dr$, such that the function $R_B(r)$ is absorbed into the differential and hence the function $B(\hat{r}) = 1$. While this choice does produce an apparent simplification of the metric form (63), it does at the cost of introducing more complex expressions for the radial functions Σ, N^2, and K^2, which have to adopt a definition different from those given in Eqs. (2), (52), and (54), and hence containing an additional radial function $R_\Sigma(\hat{r})$. In practice, therefore, the new radial coordinate $\hat{r}$ does not lead to a significant (or effective) simplification to the metric (63); for this reason, we will not consider it further in the following sections.

To calculate the inverse of the metric in Eq. (63), it is useful to first use the following identity:

$$g_{tt} g_{r\phi}^2 - g_{tr} g_{t\phi} g_{r\phi} + g_{tr}^2 g_{\phi\phi} - g_{tt} g_{rr} g_{\phi\phi} = \Sigma R_B^2 r^2 \sin^2\theta, \tag{64}$$

which, in turn, simplifies the expressions of the inverse metric, whose nonzero components are

$$g^{tt} = \frac{g_{r\phi}^2 - g_{rr} g_{\phi\phi}}{g_{tt} g_{r\phi}^2 - g_{tr} g_{t\phi} g_{r\phi} + g_{tr}^2 g_{\phi\phi} - g_{tt} g_{rr} g_{\phi\phi}} = -\left(1 + \frac{R_M}{\Sigma r}\right), \tag{65}$$

$$g^{tr} = \frac{g_{tr} g_{\phi\phi} - g_{t\phi} g_{r\phi}}{g_{tt} g_{r\phi}^2 - g_{tr} g_{t\phi} g_{r\phi} + g_{tr}^2 g_{\phi\phi} - g_{tt} g_{rr} g_{\phi\phi}} = \frac{R_M}{\Sigma R_B r}, \tag{66}$$

$$g^{rr} = \frac{g_{t\phi}^2 - g_{tt} g_{\phi\phi}}{g_{tt} g_{r\phi}^2 - g_{tr} g_{t\phi} g_{r\phi} + g_{tr}^2 g_{\phi\phi} - g_{tt} g_{rr} g_{\phi\phi}} = \frac{N^2}{\Sigma R_B^2}, \tag{67}$$

$$g^{r\phi} = \frac{g_{tt} g_{r\phi} - g_{tr} g_{t\phi}}{g_{tt} g_{r\phi}^2 - g_{tr} g_{t\phi} g_{r\phi} + g_{tr}^2 g_{\phi\phi} - g_{tt} g_{rr} g_{\phi\phi}} = \frac{a_*}{\Sigma R_B r^2}, \tag{68}$$

$$g^{\phi\phi} = \frac{g_{tr}^2 - g_{tt} g_{rr}}{g_{tt} g_{r\phi}^2 - g_{tr} g_{t\phi} g_{r\phi} + g_{tr}^2 g_{\phi\phi} - g_{tt} g_{rr} g_{\phi\phi}} = \frac{1}{\Sigma r^2 \sin^2\theta}, \tag{69}$$

$$g^{\theta\theta} = \frac{1}{g_{\theta\theta}} = \frac{1}{\Sigma r^2}. \tag{70}$$

Note that none of the metric components in Eqs. (63) and (65)–(70) contain the singular term $1/N^2$, thus manifesting the HP nature of such a form of the separable KRZ metric.

With the HP KRZ metric (63), it is important to check whether it yields a Ricci-flat Kerr reduction. We manage to obtain an expression for the Ricci scalar while keeping $R_B(r)$ and $R_M(r)$ unspecified,

$$R \coloneqq g^{\mu\nu}R_{\mu\nu} = \frac{1}{\Sigma^3 R_B^3 r^6}[R_B(R_B^2-1)F_2(r,\theta) + R_B(2R_M' + R_M'' r)F_3(r,\theta) + R_B' r F_4(r,\theta)], \quad (71)$$

where the radial dependence in the functions F_i with $i = 2$, 3, 4 is given by $F_i(r,\theta) \propto r^n$ with $n \leq 4$ and a prime indicates a radial derivative. Clearly, under the conditions that $R_B = 1$ and $R_M = \text{const}$ (see Sec. V), the Ricci scalar vanishes, as expected for the Kerr metric. Moreover, expression (71) does not change if evaluated in the original coordinates, i.e., the standard KRZ metric (1) with separability constraints in Eqs. (51)–(54), thus proving that the coordinate transformation chosen in Eqs. (55) and (56) has the desired properties of providing an HP form of the KRZ metric that is Ricci flat when reduced to a Kerr black hole.

To facilitate a direct implementation in numerical codes, we report here explicitly the $3+1$ metric components of (63) and (65)–(70):

$$\gamma_{rr} = \left(1 + \frac{R_M}{\Sigma r}\right)R_B^2, \quad (72)$$

$$\gamma_{r\phi} = -\left(1 + \frac{R_M}{\Sigma r}\right)R_B a_* \sin^2\theta, \quad (73)$$

$$\gamma_{\theta\theta} = \Sigma r^2, \quad (74)$$

$$\gamma_{\phi\phi} = K^2 r^2 \sin^2\theta, \quad (75)$$

$$\beta_r = \frac{R_B R_M}{\Sigma r}, \quad (76)$$

$$\beta_\phi = -\frac{R_M}{\Sigma r} a_* \sin^2\theta, \quad (77)$$

$$\alpha^2 = \frac{1}{1 + R_M/(\Sigma r)}, \quad (78)$$

while the corresponding inverse is

$$\gamma^{rr} = \frac{K^2}{\Sigma R_B^2}\frac{1}{1 + R_M/(\Sigma r)}, \quad (79)$$

$$\gamma^{r\phi} = \frac{a_*}{\Sigma R_B r^2}, \quad (80)$$

$$\gamma^{\theta\theta} = \frac{1}{\Sigma r^2}, \quad (81)$$

$$\gamma^{\phi\phi} = \frac{1}{\Sigma r^2 \sin^2\theta}, \quad (82)$$

$$\beta^r = \frac{R_M}{\Sigma R_B r}\frac{1}{1 + R_M/(\Sigma r)}, \quad (83)$$

$$\beta^\phi = 0, \quad (84)$$

$$\beta^k\beta_k = \frac{R_M^2/(\Sigma^2 r^2)}{1 + R_M/(\Sigma r)}. \quad (85)$$

IV. CARTESIAN FORM OF THE KRZ METRIC IN HP COORDINATES

It is not unusual that general-relativistic codes implement black-hole metrics in Cartesian coordinates as these, by construction, remove possible coordinate singularities at the polar axis and are generally easier to handle in fully numerical-relativity codes (see, e.g., [18] for a comparison of different codes). Hence, it is convenient to derive expressions of the KRZ metric in HP coordinates also in Cartesian coordinates, and to this scope, we take inspiration by the mathematical path followed when expressing the Kerr metric in Cartesian Kerr-Schild coordinates. More specifically, we start with a "Kerr-Schild decomposition" where the metric is split into two parts,

$$g_{\mu\nu} = \mathring{g}_{\mu\nu} + f\ell_\mu\ell_\nu, \quad (86)$$

where $\mathring{g}_{\mu\nu}$ is a reference metric with a particularly simple form, f is a scalar function, and ℓ_μ is a null vector with respect to the full metric, i.e.,

$$g^{\mu\nu}\ell_\mu\ell_\nu = 0. \quad (87)$$

The decomposition (86) is totally generic and hence employable in any coordinate system. However, it is easier for us to try a Kerr-Schild form starting from spherical coordinates and then transform over to Cartesian ones. Inspired by what was done for the Kerr solution in Kerr-Schild coordinates, we fix the scalar function f in Eq. (86) to be

$$f \coloneqq \frac{R_M}{\Sigma r}. \quad (88)$$

From the relevant metric functions

$$g_{rr} = \left(1 + \frac{R_M}{\Sigma r}\right)R_B^2, \quad (89)$$

$$g_{r\theta} = 0, \quad (90)$$

$$g_{r\phi} = -\left(1 + \frac{R_M}{\Sigma r}\right)R_B a_* \sin^2\theta, \quad (91)$$

the null vector is assumed to have components

$$\ell_r = R_B, \quad \ell_\theta = 0, \quad \ell_\phi = -a_* \sin^2\theta. \tag{92}$$

As a result, we can solve Eq. (87) to obtain the component ℓ_t, namely,

$$\begin{aligned} g^{\mu\nu}\ell_\mu\ell_\nu &= g^{tt}\ell_t\ell_t + 2g^{tr}\ell_t\ell_r + g^{rr}\ell_r\ell_r \\ &\quad + 2g^{r\phi}\ell_r\ell_\phi + g^{\phi\phi}\ell_\phi\ell_\phi \\ &= -\left(1 + \frac{R_M}{\Sigma r}\right)\ell_t\ell_t + \frac{2R_M}{\Sigma r}\ell_t + \left(1 - \frac{R_M}{\Sigma r}\right) = 0, \end{aligned} \tag{93}$$

with the solutions being

$$\ell_t = 1 \quad \text{and} \quad \ell_t = -\frac{\Sigma r - R_M}{\Sigma r + R_M}. \tag{94}$$

For simplicity, hereafter we will consider the simpler solution $\ell_t = 1$. The remaining part of the metric has then the following form:

$$\mathring{g}_{tt} = -1, \tag{95}$$

$$\mathring{g}_{tr} = 0, \tag{96}$$

$$\mathring{g}_{t\phi} = 0, \tag{97}$$

$$\mathring{g}_{rr} = R_B^2, \tag{98}$$

$$\mathring{g}_{r\phi} = -R_B a_* \sin^2\theta, \tag{99}$$

$$\mathring{g}_{\theta\theta} = \Sigma r^2, \tag{100}$$

$$\mathring{g}_{\phi\phi} = (r^2 + a_*^2)\sin^2\theta. \tag{101}$$

Next, when wishing to express the Kerr metric in Cartesian Kerr-Schild coordinates we can adopt a set of coordinates defined as

$$x \coloneqq \sqrt{r^2 + a_*^2}\sin\theta\cos\left(\phi + \arctan\frac{a_*}{r}\right), \tag{102}$$

$$y \coloneqq \sqrt{r^2 + a_*^2}\sin\theta\sin\left(\phi + \arctan\frac{a_*}{r}\right), \tag{103}$$

$$z \coloneqq r\cos\theta. \tag{104}$$

Employing the Kerr-Schild decomposition (86) and the Cartesian coordinates (102)–(104), we can obtain the separable KRZ metric in Cartesian HP coordinates. In particular, the corresponding null vector ℓ_μ will have spatial components

$$\ell_x = \frac{x}{\Sigma r}\left[R_B - \frac{a_*^2(x^2+y^2)}{(r^2+a_*^2)^2}\right] + \frac{a_* y}{r^2 + a_*^2}, \tag{105}$$

$$\ell_y = \frac{y}{\Sigma r}\left[R_B - \frac{a_*^2(x^2+y^2)}{(r^2+a_*^2)^2}\right] - \frac{a_* x}{r^2 + a_*^2}, \tag{106}$$

$$\ell_z = \frac{z}{\Sigma r}\left[\frac{r^2 + a_*^2}{r^2}R_B - \frac{a_*^2(x^2+y^2)}{r^2(r^2+a_*^2)}\right], \tag{107}$$

where Σ is now expressed as

$$\Sigma = 1 + \frac{a_*^2 z^2}{r^4} \tag{108}$$

and r is implicitly determined by the relation

$$\frac{x^2+y^2}{r^2+a_*^2} + \frac{z^2}{r^2} = 1, \tag{109}$$

which marks the ring singularity when $z = 0$.

Similarly, the components of the reference metric $\mathring{g}_{\mu\nu}$ will be given by

$$\mathring{g}_{tx} = \mathring{g}_{ty} = \mathring{g}_{tz} = 0, \tag{110}$$

$$\begin{aligned} \mathring{g}_{xx} = &\frac{x^2}{\Sigma^2 r^2}\left[R_B^2 - \frac{a_*^2(x^2+y^2)}{(r^2+a_*^2)^2}(2R_B - 1) + \Sigma\frac{z^2(r^2+a_*^2)}{r^2(x^2+y^2)}\right] \\ &+ \frac{y^2}{x^2+y^2} - \frac{a_*}{\Sigma r}\frac{2xy}{r^2+a_*^2}(R_B - 1), \end{aligned} \tag{111}$$

$$\begin{aligned} \mathring{g}_{yy} = &\frac{y^2}{\Sigma^2 r^2}\left[R_B^2 - \frac{a_*^2(x^2+y^2)}{(r^2+a_*^2)^2}(2R_B - 1) + \Sigma\frac{z^2(r^2+a_*^2)}{r^2(x^2+y^2)}\right] \\ &+ \frac{x^2}{x^2+y^2} - \frac{a_*}{\Sigma r}\frac{2xy}{r^2+a_*^2}(R_B - 1), \end{aligned} \tag{112}$$

$$\begin{aligned} \mathring{g}_{zz} = &\frac{z^2}{\Sigma^2 r^2}\left[\left(\frac{r^2+a_*^2}{r^2}\right)^2 R_B^2 - \frac{a_*^2(x^2+y^2)}{r^4}(2R_B - 1)\right. \\ &\left.+ \Sigma\frac{r^2(x^2+y^2)}{z^2(r^2+a_*^2)}\right], \end{aligned} \tag{113}$$

$$\begin{aligned} \mathring{g}_{xy} = &\frac{xy}{\Sigma^2 r^2}\left[R_B^2 - \frac{a_*^2(x^2+y^2)}{(r^2+a_*^2)^2}(2R_B - 1) + \Sigma\frac{z^2(r^2+a_*^2)}{r^2(x^2+y^2)}\right] \\ &- \frac{xy}{x^2+y^2} - \frac{a_*}{\Sigma r}\frac{x^2-y^2}{r^2+a_*^2}(R_B - 1), \end{aligned} \tag{114}$$

$$\begin{aligned} \mathring{g}_{xz} = &\frac{xz}{\Sigma^2 r^2}\left[\frac{r^2+a_*^2}{r^2}R_B^2 - \frac{a_*^2(x^2+y^2)}{r^2(r^2+a_*^2)}(2R_B - 1) - \Sigma\right] \\ &+ \frac{a_*}{\Sigma r}\frac{yz}{r^2}(R_B - 1), \end{aligned} \tag{115}$$

$$\mathring{g}_{yz} = \frac{yz}{\Sigma^2 r^2}\left[\frac{r^2+a_*^2}{r^2}R_B^2 - \frac{a_*^2(x^2+y^2)}{r^2(r^2+a_*^2)}(2R_B-1) - \Sigma\right] - \frac{a_*}{\Sigma r}\frac{xz}{r^2}(R_B-1). \tag{116}$$

It is then not difficult to verify that the null condition (87) still holds in terms of the full metric (86). In particular, in the case of $R_B(r) = 1$, the null vector is reduced to

$$\ell_x = \frac{rx + a_* y}{r^2 + a_*^2}, \tag{117}$$

$$\ell_y = \frac{ry - a_* x}{r^2 + a_*^2}, \tag{118}$$

$$\ell_z = \frac{z}{r}, \tag{119}$$

and the reference metric can be dramatically simplified since $\mathring{g}_{\mu\nu} = \eta_{\mu\nu}$, the Minkowski metric for flat spacetime. For compactness, we will not present here the $3+1$ expressions for the KRZ metric in Cartesian HP coordinates, which we, however, report in Appendix A.

V. REDUCTION TO KNOWN STATIONARY BLACK HOLES

We next show how various known metrics describing stationary black holes can be represented within the framework of the KRZ metric in HP coordinates by suitably choosing specific expressions for the metric functions $R_B(r)$ and $R_M(r)$.

A. Kerr metric

Starting from the generic metric (63), the reduction to the Kerr spacetime is obtained when fixing

$$R_B(r) = 1, \tag{120}$$

$$R_M(r) = 2M, \tag{121}$$

so that Eqs. (51)–(54) reduce to Eqs. (42)–(45). Under the conditions (120) and (121), the HP KRZ metric (63) becomes

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma_K}\right)dt^2 + \frac{4Mr}{\Sigma_K}dtdr - \frac{4Mr}{\Sigma_K}a_* \sin^2\theta dt d\phi + \left(1 + \frac{2Mr}{\Sigma_K}\right)dr^2 - 2\left(1 + \frac{2Mr}{\Sigma_K}\right)a_* \sin^2\theta dr d\phi + \Sigma_K d\theta^2 + \frac{A}{\Sigma_K}\sin^2\theta d\phi^2, \tag{122}$$

which is precisely the Kerr-Schild form of the Kerr metric and hence Ricci flat. The fact that the reduction of the KRZ metric in HP coordinates (63) leads to the Kerr-Schild metric in the case of the Kerr solution represents a very important feature, as it considerably simplifies the comparison between Kerr black holes and other, non-Kerr but stationary, black holes described by the KRZ metric (63).

The Cartesian HP form of the metric satisfies the relation

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{2Mr}{\Sigma_K}\ell_\mu \ell_\nu, \tag{123}$$

with the null four-vector having components

$$\ell_\mu = \left(1, \frac{rx + a_* y}{r^2 + a_*^2}, \frac{ry - a_* x}{r^2 + a_*^2}, \frac{z}{r}\right). \tag{124}$$

B. Kerr-Newman metric

Similarly, the Kerr-Newman metric in HP coordinates can be obtained from the generic HP KRZ metric (63) after setting

$$R_B(r) = 1, \tag{125}$$

$$R_M(r) = 2M - \frac{Q^2}{r}, \tag{126}$$

where Q is the electric charge of the black hole. The KRZ functions are then modified as

$$B^2 = 1, \tag{127}$$

$$N^2 = \frac{\Delta}{r^2} + \frac{Q^2}{r^2}, \tag{128}$$

$$K^2 = \frac{A}{\Sigma_K r^2} - \frac{Q^2 a_*^2}{\Sigma_K r^4}, \tag{129}$$

$$W = \frac{2Ma_*}{\Sigma_K} - \frac{Q^2 a_*}{\Sigma_K r}. \tag{130}$$

As a result, the reduction of the HP KRZ metric (63) in the case of the Kerr-Newman solution is

$$ds^2 = -\left(1 - \frac{2Mr - Q^2}{\Sigma_K}\right)dt^2 + 2\frac{2Mr - Q^2}{\Sigma_K}dtdr - 2\frac{2Mr - Q^2}{\Sigma_K}a_* \sin^2\theta dt d\phi + \left(1 + \frac{2Mr - Q^2}{\Sigma_K}\right)dr^2 - 2\left(1 + \frac{2Mr - Q^2}{\Sigma_K}\right) \times a_* \sin^2\theta dr d\phi + \Sigma_K d\theta^2 + \frac{Ar^2 - Q^2 a_*^2}{\Sigma_K r^2}\sin^2\theta d\phi^2, \tag{131}$$

which, to the best of our knowledge, has not been presented before in the literature (an HP formulation of the Kerr-Newmann solution in Cartesian coordinates can be found in Ref. [33], while a version in null coordinates has been presented in Ref. [34]). The corresponding Cartesian HP form is then

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{2Mr - Q^2}{\Sigma_{\rm K}} \ell_\mu \ell_\nu, \tag{132}$$

with the null vector (124) unchanged.

C. Rotating dilaton metric

For a rotating dilaton black hole characterized by rotation parameter a_* and dilaton parameter b_*, the KRZ functions have the following form:

$$B^2 = \frac{r^2}{r^2 + b_*^2}, \tag{133}$$

$$N^2 = 1 - \frac{2M\rho}{r^2} + \frac{a_*^2}{r^2} =: \frac{\Delta_{\rm d}}{r^2}, \tag{134}$$

$$K^2 = \frac{\Sigma_{\rm K}}{r^2} + \left(1 + \frac{2M\rho}{\Sigma_{\rm K}}\right) \frac{a_*^2 \sin^2\theta}{r^2} =: \frac{A_{\rm d}}{\Sigma_{\rm K} r^2}, \tag{135}$$

$$W = \frac{2M\rho a_*}{\Sigma_{\rm K} r}, \tag{136}$$

which imply

$$R_B(r) = \sqrt{\frac{r^2}{r^2 + b_*^2}}, \tag{137}$$

$$R_M(r) = \frac{2M\rho}{r}, \tag{138}$$

where

$$\rho := \sqrt{r^2 + b_*^2} - b_*. \tag{139}$$

Therefore, the corresponding HP KRZ metric (63) describing a rotating dilaton black hole is

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2M\rho}{\Sigma_{\rm K}}\right) dt^2 + \frac{4M\rho}{\Sigma_{\rm K}} \sqrt{\frac{r^2}{r^2 + b_*^2}} dt dr \\ &- \frac{4M\rho}{\Sigma_{\rm K}} a_* \sin^2\theta dt d\phi + \left(1 + \frac{2M\rho}{\Sigma_{\rm K}}\right) \frac{r^2}{r^2 + b_*^2} dr^2 \\ &- 2\left(1 + \frac{2M\rho}{\Sigma_{\rm K}}\right) \sqrt{\frac{r^2}{r^2 + b_*^2}} a_* \sin^2\theta dr d\phi \\ &+ \Sigma_{\rm K} d\theta^2 + \frac{A_{\rm d}}{\Sigma_{\rm K}} \sin^2\theta d\phi^2. \end{aligned} \tag{140}$$

The Cartesian HP form again satisfies the decomposition

$$g_{\mu\nu} = \mathring{g}_{\mu\nu} + \frac{2M\rho}{\Sigma_{\rm K}} \ell_\mu \ell_\nu, \tag{141}$$

with the null vector and reference metric as expressed in Eqs. (105)–(107) and Eqs. (110)–(116).

VI. REDUCTION TO STATIC BLACK HOLES

The KRZ framework describing generic stationary black holes [16] inherited much of the mathematical properties that make it so efficient from the previous RZ approach describing generic static black holes [13]. We complete our treatment of the HP formulation of the KRZ metric by considering also the simpler, but often more transparent, case of nonrotating spacetimes. To this scope, we recall that the RZ metric, to which the KRZ metric reduces in the case of spherically symmetric spacetimes, takes the form [13]

$$ds^2 = -N^2 dt^2 + \frac{B^2}{N^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{142}$$

A rapid comparison with the KRZ metric (1) indicates that the two are equivalent if

$$a_* = 0, \tag{143}$$

$$\Sigma = 1, \tag{144}$$

$$W = 0, \tag{145}$$

$$K^2 = 1, \tag{146}$$

and thus the remaining metric functions become

$$B(r) =: R_B(r), \tag{147}$$

$$N^2(r) =: 1 - \frac{R_M(r)}{r}, \tag{148}$$

so that the HP KRZ metric (63) reduces to the RZ metric in HP coordinates

$$\begin{aligned} ds^2 = &-\left(1 - \frac{R_M}{r}\right) dt^2 + \frac{2R_B R_M}{r} dt dr + \left(1 + \frac{R_M}{r}\right) R_B^2 dr^2 \\ &+ r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \\ = &-N^2 dt^2 + 2(1 - N^2) B dt dr + (2 - N^2) B^2 dr^2 \\ &+ r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \end{aligned} \tag{149}$$

Note that the HP RZ metric (149) has nontrivial inverse metric components given by

$$g^{tt} = -(2 - N^2), \tag{150}$$

$$g^{tr} = \frac{1-N^2}{B}, \tag{151}$$

$$g^{rr} = \frac{N^2}{B^2}, \tag{152}$$

$$g^{\theta\theta} = \frac{1}{r^2}, \tag{153}$$

$$g^{\phi\phi} = \frac{1}{r^2 \sin^2\theta}. \tag{154}$$

It is interesting to note that the metric (149) effectively represents the generalization to arbitrary static spacetimes of the well-known Eddington-Finkelstein coordinates employed for a Schwarzschild black hole [30,35,36]. In the case of a Schwarzschild black hole, in fact, the RZ metric functions are fixed to be

$$B^2 = 1, \tag{155}$$

$$N^2 = 1 - \frac{2M}{r}, \tag{156}$$

so that the metric (149) takes the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{4M}{r}dtdr + \left(1 + \frac{2M}{r}\right)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{157}$$

which indeed corresponds to the Schwarzschild solution in (ingoing) Eddington-Finkelstein coordinates.

Finally, and in analogy with what was done with the KRZ metric in HP coordinates, we provide below explicit expressions for the corresponding components when the metric is written in a $3+1$ split:

$$\gamma_{ij} = \mathrm{diag}(B^2(2-N^2), r^2, r^2\sin^2\theta), \tag{158}$$

$$\gamma^{ij} = \mathrm{diag}\left(\frac{1}{B^2(2-N^2)}, \frac{1}{r^2}, \frac{1}{r^2\sin^2\theta}\right), \tag{159}$$

$$\beta_r = B(1-N^2), \tag{160}$$

$$\beta^r = \frac{1-N^2}{B(2-N^2)}, \tag{161}$$

$$\beta^k\beta_k = \frac{(1-N^2)^2}{2-N^2}, \tag{162}$$

$$\alpha^2 = \frac{1}{2-N^2}. \tag{163}$$

VII. CONCLUSION

As new and unprecedented observations of stellar-mass and supermassive black holes are now becoming available, and as the number of new alternative theories of gravity is increasing steadily, it is clear that parametrized approaches to describe the metric of static and stationary black holes represent a very effective approach to extract agnostically information from such observations.

In this spirit, we have here considered the family of the Rezzolla-Zhidenko and Konoplya-Rezzolla-Zhidenko parametrizations of black-hole spacetimes—which are able to reproduce arbitrary black-hole spacetimes with percent precision and only a few coefficients—as a reference approach to study the phenomenology of matter near black holes. These parametrizations, however, have been constructed only for the exterior portion of the spacetime that, in the case of black holes, spans the region between the event horizon and spatial infinity. For this reason, they are not optimal for actual numerical simulations as those studying the accretion onto supermassive black holes, which instead make use of horizon-penetrating coordinates that are well defined in the black-hole interior and up to the physical singularity.

We have therefore discussed two different Ansaetze for the derivation of an HP form of the KRZ metric. The first approach leads to a regular version of the KRZ metric, but suffers from having a nonzero Ricci scalar when reduced to the Kerr limit. The violation of this constraint is to be found in the coordinate transformations employed that are too general and unrestricted to provide at the same time regularity and a Ricci-flat Kerr reduction. To compensate for these shortcomings, we have considered an alternative Ansatz which starts from a subclass of the KRZ metric that already allows for separation of variables of the Hamilton-Jacobi equations. We then show that in such a subclass, the KRZ metric can be written in an HP form that shares many similarities with the Kerr-Schild decomposition. To facilitate the use of this HP form of the KRZ metric, we have derived the corresponding expressions in $3+1$ decompositions of the spacetime when employing either spherical or Cartesian coordinates. Finally, we have shown that when reduced to the case of Kerr and Schwarzschild black holes, the HP form of the KRZ metric reduces, respectively, to the Kerr-Schild representation of the Kerr spacetime and to the Eddington-Finkelstein formulation of the Schwarzschild spacetime. This highlights that in the relevant limit, the HP KRZ formulation leads to Ricci-flat spacetimes.

Because these parametrized metrics will allow one to model accretion flows onto arbitrary black holes in alternative theories of gravity, they have the potential of being a very effective tool to extract important information from the astronomical observations of black holes.

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APPENDIX A: THE 3+1 FORM OF THE KRZ METRIC IN CARTESIAN HP COORDINATES

Here we report the $3+1$ expressions for the KRZ metric in Cartesian HP coordinates. According to Sec. IV and Eqs. (48) and (49), the three-metric in the Cartesian HP coordinates has components

$$\gamma_{xx} = \frac{x^2}{\Sigma^2 r^2}\left[R_B^2 - \frac{a_*^2(x^2+y^2)}{(r^2+a_*^2)^2}(2R_B-1) + \Sigma\frac{z^2(r^2+a_*^2)}{r^2(x^2+y^2)}\right] + \frac{y^2}{x^2+y^2} - \frac{a_*}{\Sigma r}\frac{2xy}{r^2+a_*^2}(R_B-1) + \frac{R_M}{\Sigma r}\ell_x^2, \tag{A1}$$

$$\gamma_{yy} = \frac{y^2}{\Sigma^2 r^2}\left[R_B^2 - \frac{a_*^2(x^2+y^2)}{(r^2+a_*^2)^2}(2R_B-1) + \Sigma\frac{z^2(r^2+a_*^2)}{r^2(x^2+y^2)}\right] + \frac{x^2}{x^2+y^2} - \frac{a_*}{\Sigma r}\frac{2xy}{r^2+a_*^2}(R_B-1) + \frac{R_M}{\Sigma r}\ell_y^2, \tag{A2}$$

$$\gamma_{zz} = \frac{z^2}{\Sigma^2 r^2}\left[\left(\frac{r^2+a_*^2}{r^2}\right)^2 R_B^2 - \frac{a_*^2(x^2+y^2)}{r^4}(2R_B-1) + \Sigma\frac{r^2(x^2+y^2)}{z^2(r^2+a_*^2)}\right] + \frac{R_M}{\Sigma r}\ell_z^2, \tag{A3}$$

$$\gamma_{xy} = \frac{xy}{\Sigma^2 r^2}\left[R_B^2 - \frac{a_*^2(x^2+y^2)}{(r^2+a_*^2)^2}(2R_B-1) + \Sigma\frac{z^2(r^2+a_*^2)}{r^2(x^2+y^2)}\right] - \frac{xy}{x^2+y^2} - \frac{a_*}{\Sigma r}\frac{x^2-y^2}{r^2+a_*^2}(R_B-1) + \frac{R_M}{\Sigma r}\ell_x\ell_y, \tag{A4}$$

$$\gamma_{xz} = \frac{xz}{\Sigma^2 r^2}\left[\frac{r^2+a_*^2}{r^2}R_B^2 - \frac{a_*^2(x^2+y^2)}{r^2(r^2+a_*^2)}(2R_B-1) - \Sigma\right] + \frac{a_*}{\Sigma r}\frac{yz}{r^2}(R_B-1) + \frac{R_M}{\Sigma r}\ell_x\ell_z, \tag{A5}$$

$$\gamma_{yz} = \frac{yz}{\Sigma^2 r^2}\left[\frac{r^2+a_*^2}{r^2}R_B^2 - \frac{a_*^2(x^2+y^2)}{r^2(r^2+a_*^2)}(2R_B-1) - \Sigma\right] - \frac{a_*}{\Sigma r}\frac{xz}{r^2}(R_B-1) + \frac{R_M}{\Sigma r}\ell_y\ell_z; \tag{A6}$$

the shift vector β_i is proportional to ℓ_i, the spatial part of the null vector

$$\beta_x = \frac{R_M}{\Sigma r}\ell_x = \frac{R_M x}{\Sigma^2 r^2}\left[R_B - \frac{a_*^2(x^2+y^2)}{(r^2+a_*^2)^2}\right] + \frac{R_M}{\Sigma r}\frac{a_* y}{r^2+a_*^2}, \tag{A7}$$

$$\beta_y = \frac{R_M}{\Sigma r}\ell_y = \frac{R_M y}{\Sigma^2 r^2}\left[R_B - \frac{a_*^2(x^2+y^2)}{(r^2+a_*^2)^2}\right] - \frac{R_M}{\Sigma r}\frac{a_* x}{r^2+a_*^2}, \tag{A8}$$

$$\beta_z = \frac{R_M}{\Sigma r}\ell_z = \frac{R_M z}{\Sigma^2 r^2}\left[\frac{r^2+a_*^2}{r^2}R_B - \frac{a_*^2(x^2+y^2)}{r^2(r^2+a_*^2)}\right]; \tag{A9}$$

the lapse function remains as

$$\alpha^2 = \frac{1}{1+R_M/(\Sigma r)}. \tag{A10}$$

APPENDIX B: A DIFFERENT ANSATZ

For completeness, and setting aside the problematic aspects related to the $3+1$ decomposition of the KRZ form (37), we here provide a potential explanation about why it leads to a Kerr reduction that is not Ricci flat. We believe the origin of this behavior is to be found in the incomplete coordinate transformation in Eqs. (27) and (28). To see this, it is sufficient to integrate them and obtain

$$\hat{t} = t + \hat{C}(r,\theta), \tag{B1}$$

$$\hat{\phi} = \phi + \hat{D}(r,\theta), \tag{B2}$$

where

$$\hat{C}(r,\theta) \coloneqq \int C(r,\theta)dr, \tag{B3}$$

$$\hat{D}(r,\theta) \coloneqq \int D(r,\theta)dr. \tag{B4}$$

Taking again the differential of Eqs. (B3) and (B4) we obtain

$$d\hat{t} = dt + C_r(r,\theta)dr + C_\theta(r,\theta)d\theta, \tag{B5}$$

$$d\hat{\phi} = d\phi + D_r(r,\theta)dr + D_\theta(r,\theta)d\theta, \tag{B6}$$

where

$$C_r \coloneqq \frac{\partial\hat{C}}{\partial r}, \qquad C_\theta \coloneqq \frac{\partial\hat{C}}{\partial\theta}, \tag{B7}$$

$$D_r \coloneqq \frac{\partial\hat{D}}{\partial r}, \qquad D_\theta \coloneqq \frac{\partial\hat{D}}{\partial\theta}. \tag{B8}$$

Clearly, Eqs. (B5) and (B6) are different from the original ansatz (27) and (28) because of the additional

$d\theta$ term, thus indicating that such an ansatz is not general enough. This consideration motivates us to use the full transformation in Eqs. (B5) and (B6) to obtain a novel form, to the best of our knowledge, of the KRZ metric in HP coordinates. After some algebra, the transformed KRZ metric has the form

$$ds^2 = -\frac{N^2 - W^2\sin^2\theta}{K^2}d\hat{t}^2 - 2Wr\sin^2\theta d\hat{t}d\hat{\phi} + K^2r^2\sin^2\theta d\hat{\phi}^2 + g_{rr}dr^2 + 2g_{r\hat{t}}drd\hat{t} + 2g_{r\hat{\phi}}drd\hat{\phi} + 2g_{r\theta}drd\theta + g_{\theta\theta}d\theta^2 + 2g_{\theta\hat{t}}d\theta d\hat{t} + 2g_{\theta\hat{\phi}}d\theta d\hat{\phi}, \quad \text{(B9)}$$

where

$$g_{rr} = \left(\frac{\Sigma B^2}{N^2} - \frac{N^2}{K^2}C_r^2\right) + \left(\frac{W}{K^2 r}C_r - D_r\right)^2 K^2r^2\sin^2\theta, \quad \text{(B10)}$$

$$g_{r\hat{t}} = \frac{N^2}{K^2}C_r - \left(\frac{W}{K^2 r}C_r - D_r\right)Wr\sin^2\theta, \quad \text{(B11)}$$

$$g_{r\hat{\phi}} = \left(\frac{W}{K^2 r}C_r - D_r\right)K^2r^2\sin^2\theta, \quad \text{(B12)}$$

$$g_{r\theta} = -\frac{N^2}{K^2}C_rC_\theta + \left(\frac{W}{K^2 r}C_r - D_r\right) \times \left(\frac{W}{K^2 r}C_\theta - D_\theta\right)K^2r^2\sin^2\theta, \quad \text{(B13)}$$

$$g_{\theta\theta} = \left(\Sigma r^2 - \frac{N^2}{K^2}C_\theta^2\right) + \left(\frac{W}{K^2 r}C_\theta - D_\theta\right)^2 K^2r^2\sin^2\theta, \quad \text{(B14)}$$

$$g_{\theta\hat{t}} = \frac{N^2}{K^2}C_\theta - \left(\frac{W}{K^2 r}C_\theta - D_\theta\right)Wr\sin^2\theta, \quad \text{(B15)}$$

$$g_{\theta\hat{\phi}} = \left(\frac{W}{K^2 r}C_\theta - D_\theta\right)K^2r^2\sin^2\theta. \quad \text{(B16)}$$

Using now the same relation between C_r and D_r adopted in Eq. (33), the transformed metric components (B10)–(B16) become

$$g_{rr} = \frac{\Sigma B^2}{N^2} - \frac{N^2}{K^2}C_r^2, \quad \text{(B17)}$$

$$g_{r\hat{t}} = \frac{N^2}{K^2}C_r, \quad \text{(B18)}$$

$$g_{r\hat{\phi}} = 0, \quad \text{(B19)}$$

$$g_{r\theta} = -\frac{N^2}{K^2}C_rC_\theta, \quad \text{(B20)}$$

$$g_{\theta\theta} = \left(\Sigma r^2 - \frac{N^2}{K^2}C_\theta^2\right) + \left(\frac{W}{K^2 r}C_\theta - D_\theta\right)^2 K^2r^2\sin^2\theta, \quad \text{(B21)}$$

$$g_{\theta\hat{t}} = \frac{N^2}{K^2}C_\theta - \left(\frac{W}{K^2 r}C_\theta - D_\theta\right)Wr\sin^2\theta, \quad \text{(B22)}$$

$$g_{\theta\hat{\phi}} = \left(\frac{W}{K^2 r}C_\theta - D_\theta\right)K^2r^2\sin^2\theta. \quad \text{(B23)}$$

The exact form of the functions C_θ and D_θ is still unknown, but could be uniquely determined by the choices made for the functions C_r and D_r to guarantee the regularity of the metric at the event horizon. For instance, we could fix the function C_r, integrate it in the radial direction to obtain $\hat{C}(r,\theta)$, and then get C_θ after differentiating in the polar direction. However, this is possible in practice only for the special class of functions for which $\hat{C}(r,\theta)$ is separable, i.e., for $\hat{C}(r,\theta) = \Phi(r)\Psi(\theta)$. While it is in principle possible to proceed in this manner, and this may be explored in the future, the Ansatz made in Sec. III B to restrict the KRZ to metric functions that are themselves separable has turned out to be the simplest and most effective in practice.

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