

Scalar probes on wormholes in Lovelock theories with unique vacuum

Octavio Fierro¹, Daniela Narbona,² Julio Oliva², Constanza Quijada,² and Guillermo Rubilar²

¹*Facultad de Ingeniería, Arquitectura y Diseño, Universidad San Sebastián,
Lientur 1457, Concepción 4080871, Chile*

²*Departamento de Física, Universidad de Concepción, Casilla 160-C, Concepción, Chile*



(Received 22 April 2024; accepted 8 June 2024; published 12 July 2024)

In this paper we construct new wormhole solutions of Lovelock theories in vacuum, when the coupling constants are such that all the maximally symmetric solutions coincide, extending to arbitrary dimensions wormhole solutions previously known in the Chern-Simons case. Like the latter, the wormholes are characterized by an integration constant ρ_0 that controls the contribution to the energy content from one of the boundaries. Then, we study the effects of the constant ρ_0 on the spectrum of a massive, (non)minimally coupled scalar probes, with Dirichlet boundary conditions at both asymptotic regions. As a result, a deformed Breitenlohner-Freedman bound emerges, which is sensitive to the value of ρ_0 . The scalar spectra are numerically obtained in detail in dimension five, and in such dimension we also present a new family of wormhole geometries for the Einstein-Gauss-Bonnet theory with a unique vacuum. The new geometries are constructed via a double analytic continuation of a wormhole previously reported in the literature, but now the constant ρ_0 appears in the centrifugal terms of the equations for the geodesic and scalar probes. The mass of these configurations vanishes nontrivially, since the contributions to the mass integral from each boundary are nonvanishing, but only differ in sign, providing a new example of a spacetime having “mass without mass.”

DOI: [10.1103/PhysRevD.110.024027](https://doi.org/10.1103/PhysRevD.110.024027)

I. INTRODUCTION

In the realm of four dimensional general relativity (GR), it is a difficult task to construct asymptotically flat, spherically symmetric wormhole geometries since they generally require violations of different energy conditions [1]. In particular the averaged null energy condition must be circumvented for a wormhole to exist if the throat is to provide a shorter path connecting points of each asymptotic region through the bulk. The obstructions can be avoided, for instance, by going beyond GR (see, e.g., [2,3] and references therein), by the inclusion of a NUT charge [4], or by constructing wormholes with long throats supported by Casimir energy [5]. The inclusion of a negative cosmological constant allowed to construct asymptotically AdS wormholes which in the radial direction are foliated by warped AdS spacetimes and are devoid of closed timelike curves [6], which possess a noncontractible S^1 , and contain a family of BPS configurations [7].

The general features that a static metric must fulfill in order to describe a traversable, asymptotically flat wormhole

were studied in the seminal papers [8,9], and [10], and the metrics considered in such references have been used as toy models to study the propagation of probe fields on spacetimes with wormhole topology. For example, the propagation of scalar and electromagnetic waves may correctly lead to the interpretation of the wormhole geometry as an extended particle [11]. Transmission and reflection coefficients for asymptotically flat, ultrastatic wormholes in $2+1$ and $3+1$ dimensions have been studied in Refs. [12,13], and [14], featuring resonances for particular values of the wormhole parameters, and leading to almost reflectionless effective potentials.¹ In Ref. [14] it was also shown that a nonminimally coupled scalar field may lead to an instability since the effective Schrödinger potential for the perturbation turns out to be negative definite. Even more, when the wormhole is asymptotically flat in both asymptotic regions, outgoing boundary conditions at both infinities lead to a quasinormal spectrum which can mimic that of a black hole of mass M for a given wormhole mass M^{-1} [14] (see also the recent works [19,20] and references therein). By imposing

¹There is also an extensive literature on Euclidean wormholes (see, e.g., [15]) leading to instantons and on the use of scalar probes to test the stability of the configuration (see, e.g., [16,17] and references therein). It has been recently established that geometries with such properties can be embedded in M-theory [18]. In this paper we will be interested in traversable, Lorentzian wormholes.

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International license](https://creativecommons.org/licenses/by/4.0/). Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

boundary conditions on the asymptotic AdS boundary as well as at the throat, a particular family of smooth wormholes in Einstein's theory coupled to a nonlinear electrodynamics has also been shown to be stable provided an effective Breitenlohner-Freedman bound [21–23] is fulfilled for the probe scalar field [24].

Asymptotically AdS wormholes in theories with quadratic terms in the curvature do exist in five dimensions [25], in vacuum. The action for the Einstein-Gauss-Bonnet theory in five dimensions reads

$$I = \frac{1}{16\pi G} \int \sqrt{-g} (R - 2\Lambda + \alpha(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta})) d^5x, \quad (1)$$

where α is a coupling constant with mass dimension equal to -2 . When formulated in first order, the local Lorentz invariance of the theory is enlarged to a local (A)dS group at the point $\Lambda\alpha = -3/4$ (see, e.g., [26] and references therein). In this case, the Lagrangian can be written as a Chern-Simons form where the spin connection as well as the vielbein transform as components of an (A)dS gauge connection. This structure can be extended to higher odd-dimensions as well as to dimension three, but in contrast to the latter, the higher dimensional case does describe a theory with local degrees of freedom in the bulk. As shown in [27], all these theories contain generalizations of the static $2+1$ -dimensional BTZ black hole [28,29], characterized by an integration constant which can be identified with the mass. In [25] it was shown that this theory actually admits a larger family of static solutions, including wormhole geometries in vacuum with two asymptotically locally AdS $_{2n+1}$ regions for $n \geq 2$. In five dimensions, the line element of these solutions is given by

$$ds^2 = l^2 [-\cosh^2(\rho - \rho_0) dt^2 + d\rho^2 + \cosh^2 \rho (d\phi^2 + d\Sigma_2^2)], \quad (2)$$

where $-\infty < t < \infty$, $-\infty < \rho < \infty$, $0 \leq \phi < 2\pi$, identified, and $d\Sigma_2$ stands for the line-element of an Euclidean 2-dimensional manifold, whose local geometry is that of the hyperbolic space with radius $3^{-1/2}$, globally equivalent to the quotient of the hyperbolic space by a Fuchsian group, i.e., Σ_2 is homeomorphic to the quotient of H_2 by a freely acting, discrete subgroup of $SO(2,1)$. For any local purpose one can consider complex projective coordinates such that

$$d\Sigma_2^2 = \frac{1}{3} \left(1 - \frac{z\bar{z}}{4} \right)^{-2} dzd\bar{z}. \quad (3)$$

The spacetime (2) represents a wormhole geometry with a throat located at $\rho = 0$, connecting two asymptotically locally AdS spacetimes with curvature radius $l^2 = 4\alpha = -3/\Lambda$. The constant ρ_0 is an arbitrary integration

constant and determines the apparent mass of the wormhole as seen by an asymptotic observer at a given asymptotic region [25–30]. The presence of $\rho_0 \neq 0$, creates a region which has interesting properties for particles with angular momentum, probing these geometries. As shown in [30], the effective potential for the geodesics has two contributions, one of which is due to the angular momentum of the particle around the S^1 factor parametrized by the coordinate ϕ in (2). This centrifugal contribution always points outward the throat located at $\rho = 0$, while the remaining contribution flips its direction at the surface $\rho = \rho_0$ and always points toward it. Therefore, the gravitational pull acting on a particle can be balanced by the centrifugal contribution only if the particle is in one of the regions $-\infty < \rho < 0$ or $\rho_0 < \rho < +\infty$, for positive ρ_0 . Geodesic probes turn out to be expelled from the region $0 < \rho < \rho_0$. And for probe strings propagating in this background the surface $\rho = \rho_0/2$ defines the turning point of strings with both ends attached to the same asymptotic region [31].

In arbitrary odd dimensions, $d = 2n + 1$, these wormholes can be embedded in Lovelock theories at the Chern-Simons point, at which the Lagrangian can be written as a Chern-Simons form for the $SO(2n, 2)$ group. In this work, first we show that in even dimension, $d = 2n$, the wormhole geometry can also be embedded in Lovelock theory with a unique vacuum, when the maximum power in the curvature is present. Therefore we would have identified a sensible gravity theory in every dimension for which the wormhole geometry is a solution. Then we will study a minimally coupled massive scalar field on the wormhole geometry (2), revisiting the case with $\rho_0 = 0$ which can be solved analytically (see Ref. [32]). Following a complementary approach to that in Ref. [32], by formulating the problem in a Schrödinger form, we show that the effective potential for the scalar probe corresponds to a Rosen-Morse potential, which explains the integrability of the spectrum [33]. Remarkably, we find that also in the limit $\rho_0 \rightarrow \infty$ the spectrum can be obtained in a closed form as well, which would be particularly useful for obtaining the normal frequencies of the scalar on wormholes with large ρ_0 . Then, we will numerically solve the spectrum for normal frequencies of the scalar field for finite, nonvanishing values of ρ_0 in dimension five, connecting the two exactly solvable models. We also show that there is an effective Breitenlohner-Freedman mass for the scalar probe, which depends on ρ_0 . In the last section, via a double Wick rotation, we construct a new family of wormholes, which maintain the properties of the asymptotic regions. We also show that the propagation of a scalar probe on the new family of wormholes, is equivalent to the propagation on the former geometries, by performing the double Wick rotation at the level of the quantum numbers, namely by setting $\omega \rightarrow i\eta$ and $n \rightarrow i\omega$. Finally we obtain the mass of this new wormhole geometry.

II. EMBEDDING THE WORMHOLE IN LOVELOCK THEORIES IN EVEN DIMENSIONS

Before analysing the propagation of a scalar probe on the wormhole geometry in arbitrary dimensions, below we identify a Lovelock theory which admits the wormhole as a vacuum solution, in even dimension. As mentioned before, in odd dimensions it is enough to consider Lovelock theory with the couplings related in such a manner that the action can be written as a Chern-Simons form for the AdS group. Since Lovelock theories with generic couplings fulfill a Birkhoff's theorem (see Refs. [34,35]), we will have to consider some relation between the couplings in order to by-pass such uniqueness result. For concreteness, let us focus on the Lovelock theory that has a unique maximally symmetric solution and contains all the possible powers of the curvature allowed in dimension d , i.e., we consider all the Lovelock terms of the form \mathcal{R}^k with $k \leq [(d-1)/2]$. The field equations of such theory can be written in a very compact manner

$$E_B^A := \delta_{BD_1 \dots D_{2k}}^{AC_1 \dots C_{2k}} \bar{R}_{C_1 C_2}^{D_1 D_2} \dots \bar{R}_{C_{2k-1} C_{2k}}^{D_{2k-1} D_{2k}} = 0, \quad (4)$$

where $\bar{R}_{C_1 C_2}^{D_1 D_2} := R_{C_1 C_2}^{D_1 D_2} + l^{-2} \delta_{C_1 C_2}^{D_1 D_2}$. Here l is the curvature radius of the unique, maximally symmetric, AdS solution of the theory. Notice that all allowed powers of the curvature appear in the field equations. It is easy to see that for the wormhole geometry

$$ds^2 = l^2 [-\cosh^2(\rho - \rho_0) dt^2 + d\rho^2 + \cosh^2 \rho d\Sigma_{d-2}^2], \quad (5)$$

the components of the shifted curvature \bar{R}_{CD}^{AB} are

$$\bar{R}^{i\rho}{}_{i\rho} = \bar{R}^{\rho i}{}_{\rho i} = 0, \quad \bar{R}^{ij}{}_{ij} = \frac{(1 - \tanh \rho \tanh(\rho - \rho_0))}{l^2} \delta_j^i, \quad (6)$$

$$\bar{R}^{ij}{}_{kl} = \tilde{R}^{ij}{}_{kl} + \delta_{kl}^{ij}, \quad (7)$$

where Latin indices $\{i, j, k, l\}$ are coordinate indices on the manifold Σ_{d-2} , which has an intrinsic Riemann tensor $\tilde{R}^{ij}{}_{kl}$. As explained in [25], in odd dimension $d = 2k + 1$ the field equations imply a single scalar constraint on the Euclidean manifold Σ_{d-2}

$$\delta_{j_1 \dots j_{2k-2}}^{i_1 \dots i_{2k-2}} \left(\tilde{R}^{j_1 j_2}{}_{i_1 i_2} + \delta_{i_1 i_2}^{j_1 j_2} \right) \dots \left(\tilde{R}^{j_{2k-3} j_{2k-2}}{}_{i_{2k-3} i_{2k-2}} + \delta_{i_{2k-3} i_{2k-2}}^{j_{2k-3} j_{2k-2}} \right) = 0, \quad (8)$$

which is solved for example by the manifold $S^1 \times H_{d-3}$, with H_{d-3} with a suitable radius.

Here we are interested in the embedding of the wormhole geometry in a Lovelock theory with unique vacuum, in

even dimension with $d = 2k + 2$. In this case the field equations (4) reduce to a tensor and a scalar constraint on the manifold Σ_{d-2} , which respectively read

$$\delta_{l j_1 \dots j_{2k}}^{k i_1 \dots i_{2k}} \left(\tilde{R}^{j_1 j_2}{}_{i_1 i_2} + \delta_{i_1 i_2}^{j_1 j_2} \right) \dots \left(\tilde{R}^{j_{2k-3} j_{2k-2}}{}_{i_{2k-3} i_{2k-2}} + \delta_{i_{2k-3} i_{2k-2}}^{j_{2k-3} j_{2k-2}} \right) = 0, \quad (9)$$

$$\delta_{j_1 \dots j_{2k}}^{i_1 \dots i_{2k}} \left(\tilde{R}^{j_1 j_2}{}_{i_1 i_2} + \delta_{i_1 i_2}^{j_1 j_2} \right) \dots \left(\tilde{R}^{j_{2k-1} j_{2k}}{}_{i_{2k-1} i_{2k}} + \delta_{i_{2k-1} i_{2k}}^{j_{2k-1} j_{2k}} \right) = 0. \quad (10)$$

To fix ideas, if one considers these constraints in dimension six, they respectively reduce to that of an Einstein manifold $\tilde{R}^i{}_j = -3\delta_j^i$ with a constant Kretschman scalar $\tilde{R}_{ijkl} \tilde{R}^{ijkl} = 36$. In consequence, provided we fulfill these conditions, we would have found a new wormhole solution of Lovelock theory with a unique vacuum in even dimensions. A simple inspection of these equations show that one can solve them by considering suitable products of constant curvature spacetimes, or even products of homogeneous geometries (see Refs. [36,37]).

III. NORMAL MODES OF THE SCALAR PROBE

The scalar probe

$$(\square - m^2)\Phi(x^\mu) = 0, \quad (11)$$

on the wormhole geometry

$$ds^2 = l^2 [-\cosh^2(\rho - \rho_0) dt^2 + d\rho^2 + \cosh^2 \rho (d\phi^2 + d\Sigma_{d-3}^2)], \quad (12)$$

can be separated as

$$\Phi(x) = \mathcal{R} \left(\sum_{n=-\infty}^{\infty} \int d\omega R_{\omega,n}(\rho) e^{-i\omega t + in\phi} \right), \quad (13)$$

and hereafter we will drop the dependence of $R_{\omega,n}(\rho)$ on ω and n . Notice that we have assumed the scalar probe to be independent of the coordinates parametrizing the manifold Σ_{d-3} . Since the equation is linear, one finds decoupled, second order, linear ODEs for each mode, leading to

$$0 = \partial_\rho [\cosh(\rho - \rho_0) \cosh^{d-2}(\rho) \partial_\rho R(\rho)] + \cosh(\rho - \rho_0) \cosh^{d-2}(\rho) \times \left[\frac{\omega^2}{\cosh^2(\rho - \rho_0)} - \frac{n^2}{\cosh^2(\rho - \rho_0)} - m^2 l^2 \right] R(\rho) \quad (14)$$

Had we considered dependence on the coordinates of Σ_{d-3} , by including in (13) an eigenfunction of the Laplace

operator on Σ_{d-3} denoted by $Y_k(\sigma_{d-3})$, the Eq. (14) would have acquired a modification of the form $n^2 \rightarrow n^2 + k^2$, where $-k^2$ is the eigenvalue of the Laplace operator on Σ_{d-3} .

Introducing the inversion $z = (1 - \tanh \rho)/2$, we map $-\infty < \rho < +\infty$ to $1 > z > 0$, and obtain the asymptotic behavior for the radial dependence of the scalar field:

$$R(z) \rightarrow c_1 z^{\Delta_+} [1 + \mathcal{O}(z)] + c_2 z^{\Delta_-} [1 + \mathcal{O}(z)] \quad \text{as } z \rightarrow 0, \quad (15)$$

and

$$R(z) \rightarrow d_1 (1-z)^{\Delta_+} [1 + \mathcal{O}(1-z)] + d_2 (1-z)^{\Delta_-} [1 + \mathcal{O}(1-z)] \quad \text{as } z \rightarrow 1, \quad (16)$$

with

$$\Delta_{\pm} = \left(\frac{d-1}{4} \right) \pm \frac{1}{2} \sqrt{m^2 + \left(\frac{d-1}{2} \right)^2}, \quad (17)$$

$c_{1,2}$ and $d_{1,2}$ being integration constants.

Even though scalar fields on asymptotically AdS spacetimes can have negative m^2 , we will focus on the case $m^2 \geq 0$ (for a thorough analysis of the properties of a scalar field on the particular case of the wormhole with $\rho_0 = 0$ and the deformed Breitenlohner-Freedman bound, see Ref. [32]). Reflective boundary conditions require the field to vanish at infinity and therefore only the Δ_+ branch in (15) and (16) is allowed. Under these boundary conditions the equation (14) defines a Sturm-Liouville problem which can be manifestly seen by transforming the radial problem (14) into a Schrödinger-like equation.

In order to simplify the presentation, here after, let us fix the dimension to $d = 5$. Equation (14) can be written as

$$-\frac{d^2 u}{d\bar{\rho}^2} + U(\bar{\rho})u = \omega^2 u, \quad (18)$$

where

$$u = R \cosh^{3/2}(\rho) \quad \text{and} \quad \rho = \rho_0 + \ln \left(\tan \left(\frac{\bar{\rho}}{2} \right) \right). \quad (19)$$

The radial coordinate $\bar{\rho}$ works as a ‘‘tortoise’’-like coordinate in the sense that in terms of $(t, \bar{\rho})$ the two-dimensional part of the metric (2) is manifestly conformally flat. The coordinate $\bar{\rho}$ connects both asymptotically AdS regions $0 < \bar{\rho} < \pi$. The effective Schrödinger potential takes the form

$$U(\bar{\rho}) = \frac{(4n^2 - 3)}{4} \left(\frac{\cos(\bar{\rho}) \cosh(\rho_0) - \sinh(\rho_0)}{\cos(\bar{\rho}) - \coth(\rho_0)} \right)^2 - (2n^2 - 3) \left(\frac{\cos(\bar{\rho}) \cosh(\rho_0) - \sinh(\rho_0)}{\cos(\bar{\rho}) \operatorname{sech}(\rho_0) - \operatorname{csch}(\rho_0)} \right) + \frac{\cosh^2(\rho_0)(4n^2 - 9)}{4} + \frac{(4m^2 + 15)}{4 \sinh^2(\bar{\rho})}. \quad (20)$$

This potential $U(\bar{\rho})$ (depicted in Fig. 1), parametrically depends on the wormhole integration constant ρ_0 , and for arbitrary values of the latter, the equation cannot be solved analytically. Even though we will solve the equation numerically to find the normal modes of the scalar probe on the wormhole, it is interesting to note that there are two particular values of ρ_0 for which the potential is shape invariant [33] and (18) can be solved analytically. Those values are $\rho_0 = 0$ and $\rho_0 \rightarrow +\infty$, and the corresponding

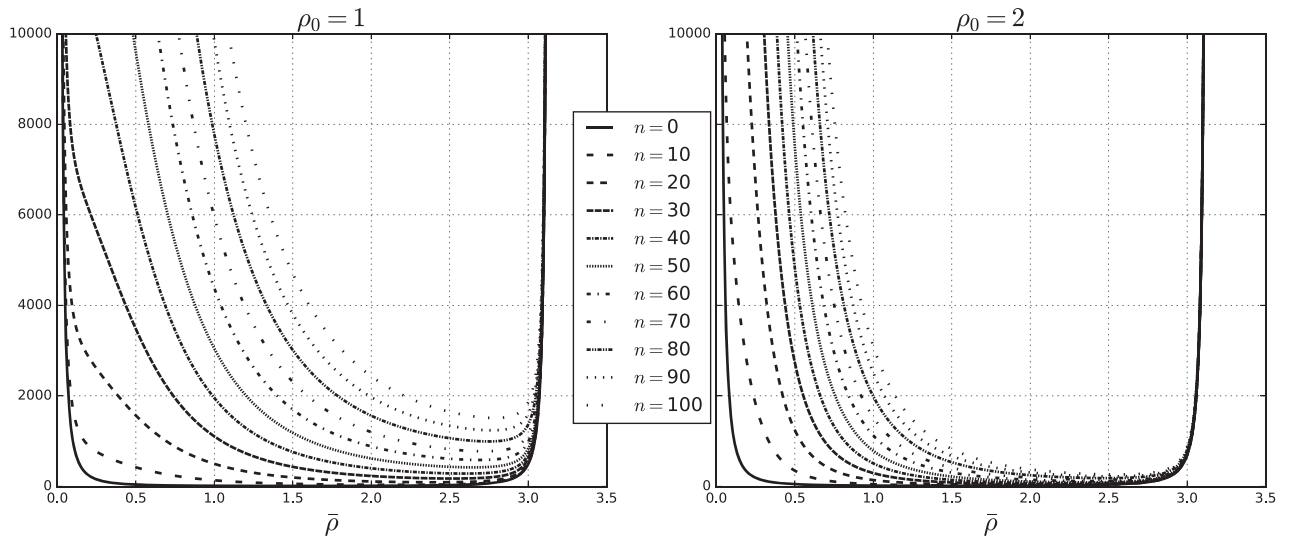


FIG. 1. Effective potential for the radial dependence of the scalar probe, for different values of the parameters. As expected in the presence of a negative cosmological constant, the potential diverges at the boundaries $\bar{\rho} = 0, \pi$.

potentials U are given by

$$U^{(0)}(\bar{\rho}) := \frac{15}{4} \frac{1}{\sin^2 \bar{\rho}} + \frac{m^2}{\sin^2 \bar{\rho}} - \frac{9}{4} + n^2 + \mathcal{O}(\rho_0), \quad (21)$$

$$U^{(\infty)}(\bar{\rho}) := -\frac{1}{4} \frac{6 \cos \bar{\rho} - 4m^2 - 9}{\sin^2 \bar{\rho}} + \mathcal{O}(e^{-2\rho_0}). \quad (22)$$

The leading term of $U^{(0)}$ leads to an effective quantum mechanical problem with a Rosen-Morse potential, which can be solved analytically and with energies and bound states given by

$$\omega_{(0),p}^2 = \left(\frac{1}{2} + \sqrt{4 + m^2 + p} \right)^2 + n^2 - \frac{9}{4}, \quad (23)$$

$$u_p^{(0)}(\bar{\rho}) = A_p^{(0)} (\sin \bar{\rho})^{s+p} P_p^{(-s-p, -s-p)}(i \cot(\bar{\rho})), \quad (24)$$

with $s = \sqrt{4 + m^2} + 1/2$, $A_p^{(0)}$ an arbitrary integration constant that can be fixed by normalization, and $p = 0, 1, 2, 3, \dots$ as the mode number. For the particular value $\rho_0 = 0$ the wormhole acquires a reflection symmetry with respect to the throat. P_a^b are the corresponding Jacobi polynomials.

Remarkably, for $\rho_0 \rightarrow \infty$, the Schrödinger problem in (22) can also be integrated analytically since it corresponds to a Scarf potential, and leads to the following frequencies and eigenfunctions

$$\omega_{(\infty),p}^2 = (A + p)^2, \quad (25)$$

$$u_p^{(\infty)}(\bar{\rho}) = A_p^{(\infty)} (1 - \cos(\bar{\rho}))^{\frac{A-B}{2}} (\cos(\bar{\rho}) + 1)^{\frac{A+B}{2}} \times P_p^{(A-B-\frac{1}{2}, A+B-\frac{1}{2})}(\cos(\bar{\rho})), \quad (26)$$

where the constant $A > B$ are given by

$$2A = 1 + \sqrt{5 + 2m^2 + 2\sqrt{(m^2 + 1)(m^2 + 4)}}, \quad (27)$$

$$6B = \left(5 + 2m^2 - 2\sqrt{(m^2 + 1)(m^2 + 4)} \right) \times \sqrt{5 + 2m^2 + 2\sqrt{(m^2 + 1)(m^2 + 4)}} \quad (28)$$

$A_p^{(\infty)}$ is an integration constant and $p = 0, 1, 2, \dots$, is again the mode number. It is worth to mention that in this case the frequencies $\omega_p^{(\infty)}$ lead to an equispaced, fully resonant spectrum. This result may be particularly relevant for obtaining the spectrum of the scalar field on the wormhole, for large ρ_0 , by perturbative methods. The strict limit ρ_0 to infinity can be taken after suitable regularization of the

geometry [30], leading to a wormhole that connects two different asymptotic regions. Since such wormhole is not asymptotically locally AdS we left its analysis for the Appendix.

Here we are interested in the effects of a finite value of ρ_0 on the propagation of a minimally coupled scalar field. Therefore, we are obligated to integrate equation (18) numerically, with reflective boundary conditions at both infinities which can be achieved only for a countably infinity number of frequencies ω_p . This is done by using a variation of the so-called shooting method (see, for instance, [38]), which in our case consists in varying the value of ω in (18) and seeking those values which fulfill the boundary condition at $\bar{\rho} = \pi$. More specifically, the Schrödinger equation is recast as a first order ODE system, which is then integrated “from left to right”, i.e. from $\bar{\rho} = 0$ to $\bar{\rho} = \pi$ for each value of ω . The integration is performed using the standard 4th order Runge-Kutta method [38], which thus determines the value of the solution at the right boundary, i.e. $u(\bar{\rho} = \pi, \omega)$. The later can be considered as a function of ω , for which the sought after values ω_p are zeros of, in virtue of the right boundary condition $u(\bar{\rho} = \pi, \omega_p)$. After a solution for ω_p is found, the value of p is determined by counting the number of nodes of the function $u(\bar{\rho}, \omega_p)$.

IV. SPECTRA FOR THE MINIMALLY COUPLED SCALAR PROBE

Since the problem for the normal modes with $\rho_0 \neq 0$ is a Sturm-Liouville problem, with potential fixed by the angular momentum of the field n , as well as ρ_0 and m^2 , the eigenfunctions and eigenfrequencies will both be labeled by an integer p . In all the plots shown below, we have included small black bars near $\rho_0 = 0$ and ρ_0 large, marking the analytic values for the frequencies obtained in those cases. In what follows we plot some of the numerically obtained spectra, which present interesting features as ρ_0 varies. The frequencies of the fundamental mode ($p = 0$) are shown on the left panel of Fig. 2, while the right panel shows a set of frequencies for the tenth overtone ($p = 10$). While the fundamental mode is a monotonic function of ρ_0 , we observe that the excited modes have a maximum frequency for a critical value of ρ_0 and then decay to a given value. As expected, the frequencies increase with the angular momentum of the field, and consistently with the asymptotic expression for $U^{(\infty)}$ in Eq. (22), the frequencies merge regardless the value of n .

Figure 3 shows the fundamental and ten excited modes for the “s-wave” of the scalar ($n = 0$) on the left panel, while the frequencies for a spinning scalar probe with $n = 10$ have been plotted on the right-panel, both as a function of ρ_0 . It is particularly interesting to notice that the “s-wave” fundamental and excited frequencies remain almost constant regardless the value of the integration constant ρ_0 .

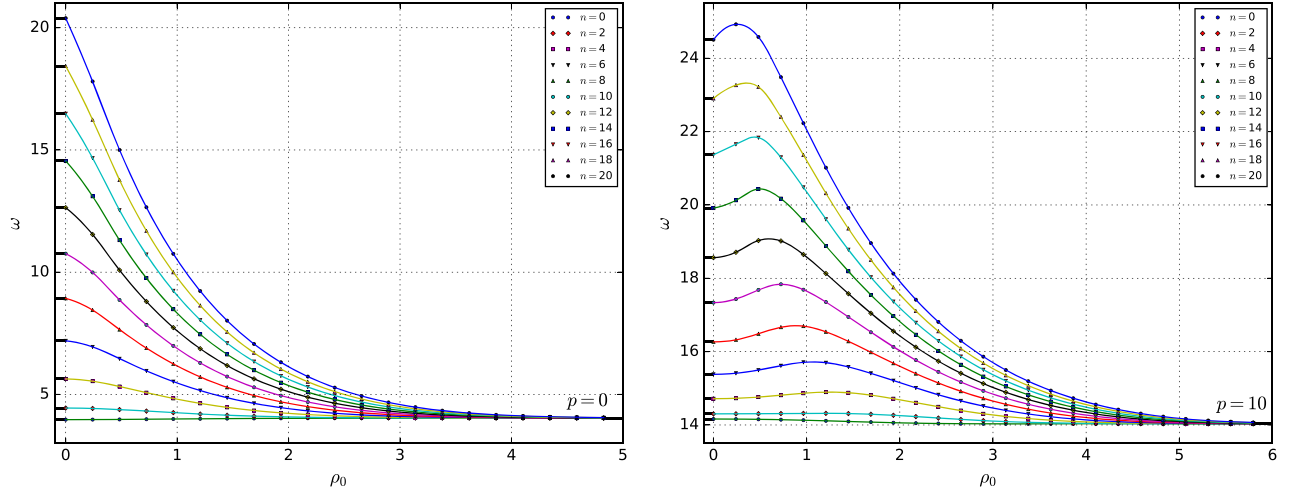


FIG. 2. Spectra for the fundamental mode $p = 0$ (left-panel) and tenth overtone $p = 10$ (right-panel) as a function of ρ_0 for different even values of the angular momentum of the scalar probe $n = 0, 2, 4, \dots, 10$, for $m^2 = 10$. The fundamental mode has a monotonic behavior with ρ_0 while excited frequencies present a maximum for a given critical value of ρ_0 which depends on the angular momentum of the field.

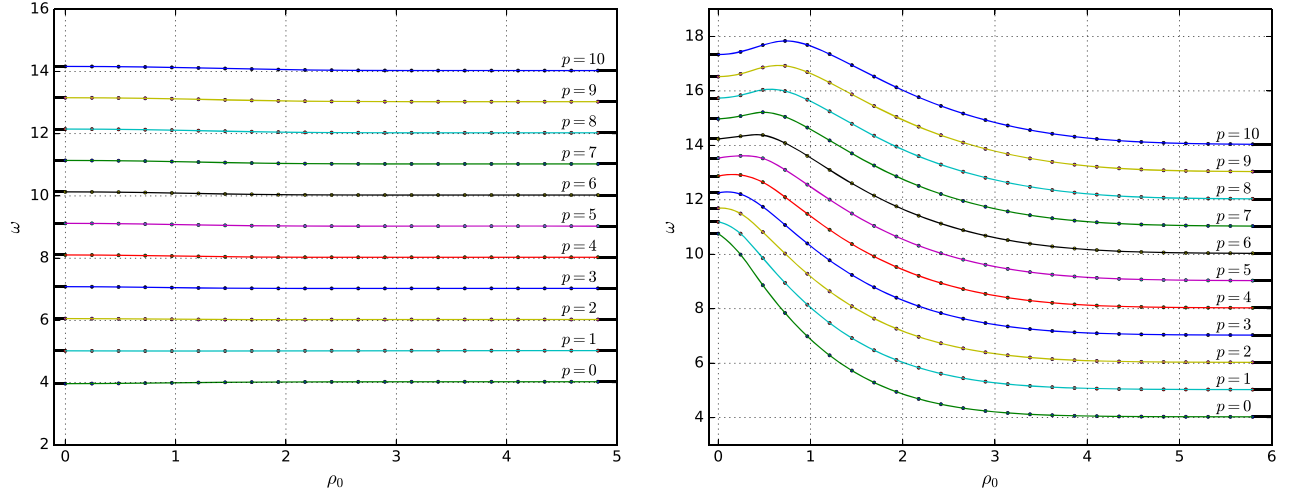


FIG. 3. Spectra for vanishing angular momentum $n = 0$ (left-panel) and a spinning scalar probe with $n = 10$ (right-panel) as a function of ρ_0 for the fundamental mode and the first 10 excited states ($m^2 = 10$).

Finally, Fig. 4 shows the fundamental and first overtones for different values of the angular momentum and the mass of the scalar. Even though the effective Schrödinger problem is one-dimensional, there are degeneracies due to the fact the effective potential depends on n , therefore different values of n lead to different one-dimensional Sturm-Liouville problems.

V. SCALARS PROBES WITH NONMINIMAL COUPLING

As shown in [32], for the case $\rho_0 = 0$ the equation for a scalar nonminimally coupled with the scalar curvature can also be solved in an analytic manner. This is remarkable

since the Ricci scalar of the wormhole background is a nontrivial function of the radial coordinate ρ , therefore including a nonminimal coupling with the scalar curvature do not stand for a shift in the mass. Here, we explore the effect of a nonminimal coupling on the spectrum of the scalar for $\rho_0 \neq 0$, still within the context of dimension five. The equation for the scalar in this case is given by

$$(\square - m^2 - \xi R)\Phi(x^\mu) = 0. \quad (29)$$

This scalar probe is invariant under local Weyl rescalings for $\xi = 3/16$. When ρ_0 vanishes, the equation for the radial dependence of the nonminimally coupled scalar can be obtained from that of the minimally coupled one by shifting

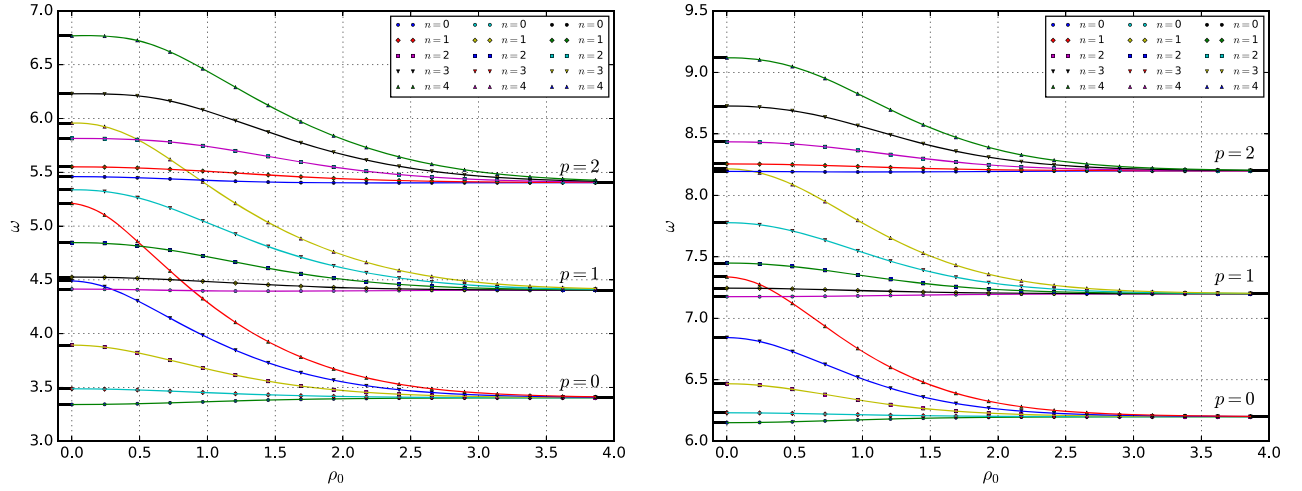


FIG. 4. Fundamental and first two overtones for the scalar probe with $n = 0, 1, 2, 3, 4$, and $m^2 = 6$ (left-panel) and $m^2 = 30$ (right-panel). For a given value of ρ_0 the same frequency can be obtained for different modes since the effective potential depends explicitly on n .

the mass as well as the eigenvalues of the Laplace operator on the manifold $\Sigma_3 = S^1 \times H_2/\Gamma$. For nonvanishing ρ_0 , one cannot rely on this shift, and one is forced to numerically find the spectrum of normal modes. Using the separation (13) and the change of variables (19), one leads to a Schrödinger-like equation of the form (18) with an intricate potential given by

$$U_\xi(\bar{\rho}) = \frac{(4n^2 - 3)}{4} \left(\frac{\cos(\bar{\rho}) \cosh(\rho_0) - \sinh(\rho_0)}{\cos(\bar{\rho}) - \coth(\rho_0)} \right)^2 - (2n^2 - 3 + 6\xi) \left(\frac{\cos(\bar{\rho}) \cosh(\rho_0) - \sinh(\rho_0)}{\cos(\bar{\rho}) \operatorname{sech}(\rho_0) - \operatorname{csch}(\rho_0)} \right) + \frac{\cosh^2(\rho_0)(4n^2 - 9 + 24\xi)}{4} + \frac{(4m^2 + 15)}{4 \sinh^2(\bar{\rho})}. \quad (30)$$

Even with the nonminimal coupling with the scalar curvature, in the extremal values $\rho_0 = 0$ and $\rho_0 \rightarrow \infty$ one also recovers shape invariant potentials, since

$$U_\xi^{(0)}(\bar{\rho}) = \left(\frac{15}{4} + m^2 - 20\xi \right) \frac{1}{\sin^2 \bar{\rho}} + n^2 - \frac{9}{4} + 6\xi + O(\rho_0) \quad (31)$$

$$U_\xi^{(\infty)}(\bar{\rho}) = \frac{1}{4} \frac{6(4\xi - 1) \cos \bar{\rho} + 4m^2 + 9 - 56\xi}{\sin^2 \bar{\rho}} + O(e^{-2\rho_0}), \quad (32)$$

corresponding to a Rosen-Morse potential and a Scarf potential, respectively.

It is interesting to note that for modes without angular momentum ($n = 0$) and $\xi = 3/8$ both potentials lead to a fully resonant, equispaced spectra, which might enhance the energy transfer between modes when nonlinearities are

included [39], as it happens in AdS. Note that in a truly quantum-mechanical problem nor the Rosen-Morse, neither the Scarf potential lead to equispaced energies since in that case the eigenvalue is quadratic in the principal quantum number, and actually the harmonic oscillator is the unique potential with such property. Nevertheless, in our relativistic theory the eigenvalue is quadratic in the frequencies, therefore all the potentials which are quadratic in the principal quantum number, may lead to a equispaced set of ω_n (see Ref. [40]).

The asymptotic behaviors for the function $R(z)$ with $z = (1 - \tanh(\rho))/2$ that defined the radial dependence of the field (13), is the same as that given in equations (15) and (16), replacing the, conformal weights by

$$\Delta_\pm^\xi = 1 \pm \frac{1}{2} \sqrt{m^2 - 20\xi + 4}. \quad (33)$$

Regarding the asymptotic behavior one can define an effective mass $m_{\text{eff}}^2 := m^2 - 20\xi$ and we will be focused in the region $m_{\text{eff}}^2 \geq 0$, in which the reflective boundary conditions lead to a unique set of normal frequencies. Below, in Figs. 5 and 6, we present the spectra for different values of the parameters characterizing the potential, i.e., (ξ, ρ_0, m^2, n) .

VI. A NEW WORMHOLE SOLUTION OF EINSTEIN-GAUSS-BONNET AND LOVELOCK THEORIES

Before finishing, let us report on a new, wormhole solution of Einstein-Gauss-Bonnet or even Lovelock theories with a unique vacuum, provided (12), solve the corresponding field equations. The new wormhole geometry is given by

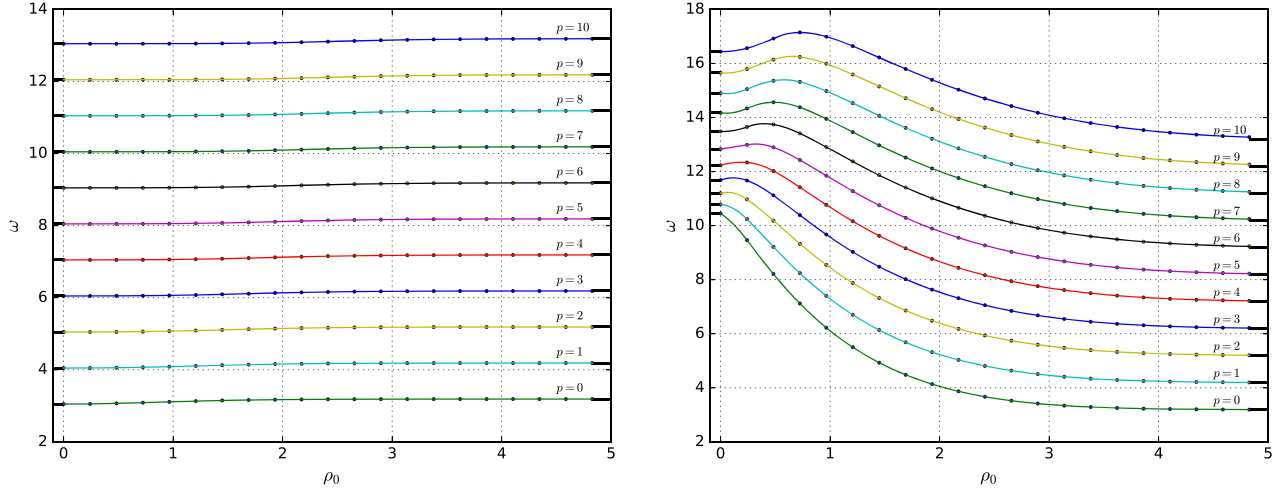


FIG. 5. Fundamental and first overtones for the nonminimally coupled scalar probe without angular momentum ($n = 0$, left), and $n = 10$ (right), for $m^2 = 10$ and $\xi = 3/8$. The figure in the left smoothly connects two fully resonant spectra for $\rho_0 = 0$ and $\rho_0 \rightarrow \infty$.

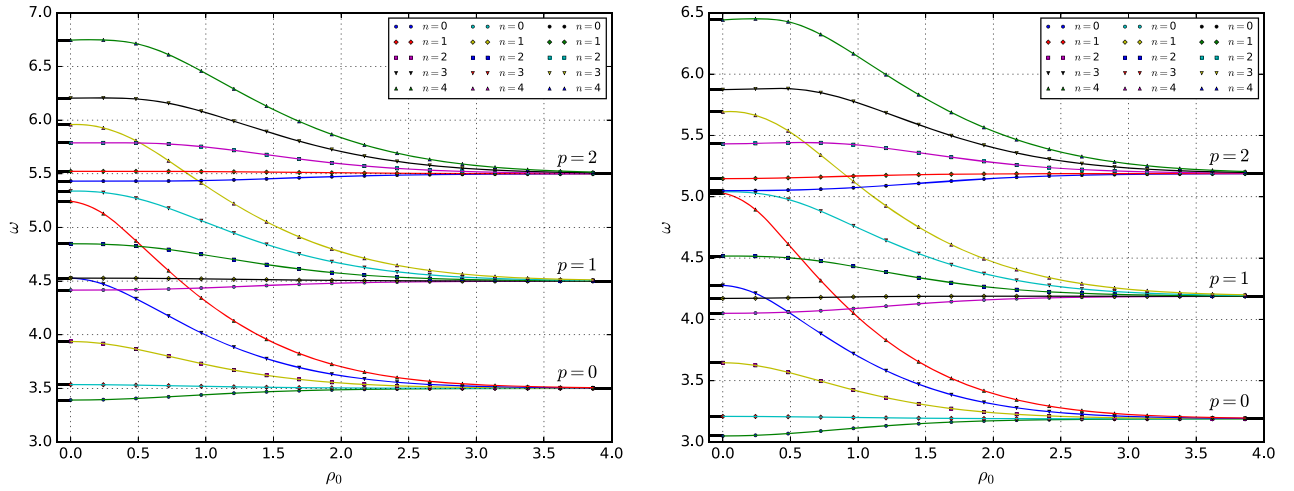


FIG. 6. First three modes spectra for $\xi = 1/4$ (left) and $\xi = 3/8$ (right) for $m^2 = 10$ for $n = 0, 1, 2, 3, 4$.

$$ds^2 = l^2 [-\cosh^2 \rho dt^2 + d\rho^2 + \cosh^2(\rho - \rho_0) d\phi^2 + \cosh^2 \rho d\Sigma_2^2], \quad (34)$$

that can be constructed from (12) via the double Wick rotation $t \rightarrow i\phi$ and $\phi \rightarrow it$. Notice that the double Wick rotation produces a different spacetime only when $\rho_0 \neq 0$, which is exactly the case we are considering in the present work. The geodesic equation on this background, freezing the dynamics along the coordinates of Σ_2 , which must be of constant Ricci scalar curvature equal to -6 , leads to

$$\dot{t} = \frac{E}{l^2 \cosh^2 \rho}, \quad \dot{\phi} = \frac{L}{l^2 \cosh^2 \rho - \rho_0} \quad (35)$$

$$l^2 \dot{\rho}^2 = -b + \frac{E^2}{l^2 \cosh^2 \rho} - \frac{L^2}{\cosh^2(\rho - \rho_0)} \quad (36)$$

where L and E , are the angular momentum and the energy of the particle and $b = +1, 0, -1$ for timelike, null or spacelike geodesics, respectively. From the geodesic radial equation (36), one can see that the existence of a value of the radial coordinate $\rho = \rho_c$, for which the noncentrifugal (proportional to E^2) and the centrifugal (proportional to L^2) contributions balance, is nontrivial. In this case, the noncentrifugal contribution pushes toward the surface $\rho = 0$, while the centrifugal contribution pushes particles away from the surface $\rho = \rho_0$, in consequence, in the region $0 < \rho < \rho_0$, it is impossible to balance both contributions. A similar mechanism precludes the existence of orbits within the same region for the wormhole (12) (see Ref. [25]). On the other hand, the equation for a scalar field probe (11), (13), on the new geometry (34), leads to the following equation for the radial dependence

$$\begin{aligned}
0 = & \partial_\rho [\cosh(\rho - \rho_0) \cosh^3(\rho) \partial_\rho R(\rho)] \\
& + \cosh(\rho - \rho_0) \cosh^3(\rho) \left[-\frac{n^2}{\cosh^2(\rho - \rho_0)} \right. \\
& \left. + \frac{\omega^2}{\cosh^2(\rho - \rho_0)} - m^2 l^2 \right] R(\rho) \quad (37)
\end{aligned}$$

which exactly corresponds to (14) setting $d = 5$, $\omega \rightarrow in$ and $n \rightarrow i\omega$.

Before finishing this section, it is interesting to consider the energy content of the new wormhole geometry (34). For Einstein-Gauss-Bonnet, in five dimensions, at the Chern-Simons point, the regularized action principle of [41] leads, via Noether theorem, to a definition for the energy of an asymptotically locally AdS spacetime. The mass of the new wormhole (34) receives contributions from both boundaries $\rho \rightarrow \pm\infty$, which as for the wormhole (12) (see Ref. [25]), vanishes, since the contributions from both boundaries cancel each other, and in this case are given by

$$M_{\pm\infty}^{\text{new}} = \mp \frac{5\sigma_2}{8\pi G} \sinh \rho_0, \quad (38)$$

where $\sigma_2 = 2\pi \text{Vol}(\Sigma_2)$. In this normalization, the contribution to the mass coming from each boundary on the original wormhole geometry (2) in dimension five reads $M_{\pm\infty}^{\text{old}} = \pm \frac{3\sigma_3}{8\pi G} \sinh \rho_0$. As mentioned above, in both cases the total mass vanishes.

VII. CONCLUSIONS

In this work we have embedded in Lovelock theory in arbitrary dimensions, the $d = 2n + 1$ -dimensional, wormhole geometry originally constructed in [25]. The selected theories in even dimension are characterized by possessing a unique, maximally symmetric AdS vacuum, and the geometry of the wormhole throat is restricted to fulfill a tensor (9) and a scalar (10) constraint, containing one curvature less than the original theory. We have also obtained the spectrum of normal modes for a scalar probe on the asymptotically locally AdS wormhole in dimension five, complementing the work of [32], where the case $\rho_0 = 0$ was considered, a case that can be solved in an analytic manner. Here we have shown that the particular case with $\rho_0 = 0$ leads to a problem for the radial dependence of the scalar probe of the Schrödinger type, in the well-known Rosen-Morse potential. Remarkably, we have shown that when $\rho_0 \rightarrow \infty$, the spectrum can be also obtained in an analytic manner and it does not depend on the angular momentum of the scalar. This exact result can be useful to obtain the spectrum on a wormhole with a large, finite ρ_0 by perturbation theory. As discussed in the Appendix the exact limit $\rho_0 \rightarrow \infty$ can be taken after suitable regularizations leading to a geometry that also describes a wormhole. We left some of the details of such case to the Appendix, since

only one of the asymptotic regions is locally AdS in that setup. It is important to mention that since the g_{tt} component of the metric is not a constant, the wormhole is not ultrastatic. Even more, for $\rho_0 \neq 0$ the minimum of the g_{tt} component of the metric does not coincide with the throat.

In [24], the authors considered a scalar probe propagating on the so-called natural wormholes that can be constructed by smoothly matching spherically symmetric solutions of GR, coupled to a Born-Infeld electrodynamics. In the effective radial eigenvalue problem, they imposed reflective boundary condition at one of the asymptotic regions and ingoing boundary conditions at the throat, leading to complex quasinormal frequencies. In our case, we have defined our eigenvalue problem by ensuring that the field is nondivergent at both asymptotic regions, requiring reflective boundary conditions when $\rho \rightarrow \pm\infty$.

The frequencies of the scalar field are given with respect to the coordinate time t , which in our case coincides with the proper time of a static observer located at $\rho = \rho_0$. Note that such observer is a geodesic one. In an asymptotically flat spacetime, as for example in the Schwarzschild black hole, the frequencies are referred to an observer located at the asymptotic region, which is independent of the value of the black hole mass. Nevertheless, when a negative cosmological constant is included, as for example in the four-dimensional Schwarzschild-AdS black hole, in Schwarzschild-like coordinates, a time dependence of the form $e^{-i\omega t}$ in a scalar probe implies that the frequencies correspond to those measured by an observer located at $r = (2Ml^2)^{1/3}$. Note that such observer is nongeodesic, in contraposition to our geodesic observer measuring frequencies in the wormhole. From the holographic viewpoint this coordinate time is actually the time in the dual CFT.

We have also included a nonminimal coupling between the scalar probe and the scalar curvature, which allows, for some particular values, to connect two fully resonant, equispaced spectra. It is worth to mention here also that fully resonant, equispaced spectra play a central role in the turbulent energy transfer that leads to nonperturbative AdS instability, see, e.g., [42–47]. This interesting phenomenology has also been observed in other nonlinear models as gravitating scalars on a spherical cavity in $3 + 1$ [48], on systems governed by the Gross-Pitaevskii equation [49], on conformal dynamics on the Einstein Universe [50] and on vortex precession in Bose-Einstein condensates [51].

Finally, we have also introduced a new family of wormhole geometries, that are obtained from the first set via a double Wick rotation. The propagation of a test particle on this new geometry, unveils a region where no geodesic, circular orbit exist, and the spectrum of a scalar field probe can also be obtained from the one on the seed geometries by a suitable Wick rotation in frequency/momentum domain.

It would be interesting to explore the sector with negative squared masses. Since there is an asymptotically locally AdS

asymptotic behavior, it is natural to expect the existence of an effective Breitenlohner-Freedman bound that may depend on ρ_0 and that would have to be obtained numerically. We expect to report on this problem in the future.

ACKNOWLEDGMENTS

The authors want to thank Eloy-Ayon Beato, Francisco Correa, Danilo Diaz, Oleg Evnin and Nicolas Grandi for enlightening comments. This work was partially supported by CONICYT Fellowships 21161099 and 22171243, FONDECYT Grants No. 1181047, No. 11191175, No. 1221504, No. 1200022, No. 1200293, No. 1210500, No. 1210635.

APPENDIX

Here we discuss some features of the scalar probe on the spacetime obtained after a suitable regularization in the limit $\rho_0 \rightarrow +\infty$. It is useful to return to Schwarzschild-like coordinates, that cover part of the wormhole spacetime (2). In such coordinates the metric reads [30]

$$ds^2 = -\left(\frac{r}{l^2} + a\sqrt{\frac{r^2}{l^2} - 1}\right)^2 d\bar{t}^2 + \left(\frac{r^2}{l^2} - 1\right)^{-1} dr^2 + r^2(d\phi^2 + d\Sigma_2^2), \quad (\text{A1})$$

where $r = l \cosh(\rho)$, $t = \bar{t}/(l \cosh(\rho_0))$ and the integration constant ρ_0 relates to a by $\rho_0 := -\tanh^{-1}(a)$. Now we consider that $\rho_0 \rightarrow \infty$. In this case $a = -1$ and returning to the proper radial coordinate ρ one obtains

$$ds^2 = l^2[-e^{-2\rho} dt^2 + d\rho^2 + \cosh^2(\rho)(d\phi^2 + d\Sigma_2^2)], \quad (\text{A2})$$

where $t = \bar{t}/l$. This solution also describes a wormhole geometry with a traversable throat located at $\rho = 0$. Note that this spacetime is asymptotically locally AdS only when $\rho \rightarrow -\infty$. Hereafter we set $l = 1$.

The radial equation for the nonminimally coupled scalar probe on this wormhole, for the ansatz $\Phi = e^{-i\omega t} e^{in\phi} R(\rho)$ reduces to

$$\frac{d^2}{d\rho^2} R(\rho) - (1 - 3 \tanh(\rho)) \frac{d}{d\rho} R(\rho) + \left(e^{2\rho} \omega^2 - m^2 + (14 - 6 \tanh(\rho)) \xi - \frac{n^2}{\cosh^2(\rho)} \right) R(\rho) = 0. \quad (\text{A3})$$

The asymptotic behavior of the solution is given by

$$R(\rho) \stackrel{\rho \rightarrow -\infty}{\sim} A_1 e^{(2 - \sqrt{4+m^2-20\xi})\rho} + A_2 e^{(2 + \sqrt{4+m^2-20\xi})\rho} + \mathcal{O}(e^{2\rho}), \quad (\text{A4})$$

$$R(\rho) \stackrel{\rho \rightarrow \infty}{\sim} B_1 e^{-3\rho/2 - i\omega e^\rho} + B_2 e^{-3\rho/2 + i\omega e^\rho} + \mathcal{O}(e^{-\rho}). \quad (\text{A5})$$

As expected the behavior of the scalar when $\rho \rightarrow -\infty$ is a power law behavior on the areal coordinate $r \sim e^\rho$, typical of scalars on asymptotically AdS regions. While the behavior at the other non-AdS asymptotic region corresponds to that of an ingoing and outgoing wave. The equation can be solved analytically in terms of confluent Heun functions [52]. The equation can be recast in a Schrödinger form by scaling the radial function and considering the ansatz $\Phi = e^{-i\omega t} e^{in\phi} u(\rho) / \cosh^{3/2}(\rho)$ and using the new radial coordinate $\bar{\rho} = e^\rho$, with $\bar{\rho} \in (0, \infty)$, leading to

$$-\frac{d^2}{d\bar{\rho}^2} u(\bar{\rho}) + U u(\bar{\rho}) = \omega^2 u(\bar{\rho}), \quad (\text{A6})$$

with the effective potential explicitly given in terms of the coordinate $\bar{\rho}$ by

$$U := \frac{m^2}{\bar{\rho}^2} - \frac{4(2\bar{\rho}^2 + 5)}{\bar{\rho}^2(\bar{\rho}^2 + 1)} \xi + \frac{4n^2}{(\bar{\rho}^2 + 1)^2} + \frac{3\bar{\rho}^4 + 2\bar{\rho}^2 + 5}{4(\bar{\rho}^2 + 1)^2 \bar{\rho}^2}. \quad (\text{A7})$$

Note that this potential depends explicitly on the angular momentum of the scalar n while $U_\xi^{(\infty)}$ defined in (32), does not. This confirms what we previously discuss in the paper regarding the fact that the exactly solvable potential obtained for the leading term of (32), has only to be thought as a tool to perturbatively obtain the frequencies for large ρ_0 .

Note that here, at least on one side of the wormhole it is clear how to recognize ingoing and outgoing modes. It would be interesting in this case to compute the transmission and reflection coefficients of the wormhole, along the lines of the computation of graybody factors in asymptotically AdS black holes [53].

- [1] M. Visser, *Lorentzian Wormholes: From Einstein to Hawking* (AIP, Woodbury, USA, 1995), p. 412.
- [2] F. S. N. Lobo, *EPJ Web Conf.* **58**, 01006 (2013).
- [3] P. Bueno, P. A. Cano, R. A. Hennigar, and R. B. Mann, *J. High Energy Phys.* **10** (2018) 095.
- [4] E. Ayon-Beato, F. Canfora, and J. Zanelli, *Phys. Lett. B* **752**, 201 (2016).
- [5] J. Maldacena, A. Milekhin, and F. Popov, *Classical Quantum Gravity* **40**, 155016 (2023).
- [6] A. Anabalón and J. Oliva, *J. High Energy Phys.* **04** (2019) 106.
- [7] A. Anabalón, B. de Wit, and J. Oliva, *J. High Energy Phys.* **09** (2020) 109.
- [8] H. G. Ellis, *J. Math. Phys. (N.Y.)* **14**, 104 (1973).
- [9] M. S. Morris and K. S. Thorne, *Am. J. Phys.* **56**, 395 (1988).
- [10] M. S. Morris, K. S. Thorne, and U. Yurtsever, *Phys. Rev. Lett.* **61**, 1446 (1988).
- [11] G. Clement, *Int. J. Theor. Phys.* **23**, 335 (1984).
- [12] S. Kar, D. Sahdev, and B. Bhawal, *Phys. Rev. D* **49**, 853 (1994).
- [13] S. Kar, S. Minwalla, D. Mishra, and D. Sahdev, *Phys. Rev. D* **51**, 1632 (1995).
- [14] P. Taylor, *Phys. Rev. D* **90**, 024057 (2014); **95**, 109904(E) (2017).
- [15] A. Hebecker, T. Mikhail, and P. Soler, *Front. Astron. Space Sci.* **5**, 35 (2018).
- [16] P. Betzios, N. Gaddam, and O. Papadoulaki, *Phys. Rev. D* **97**, 126006 (2018).
- [17] C. Jonas, G. Lavrelashvili, and J. L. Lehners, *Phys. Rev. D* **109**, 086022 (2024).
- [18] A. Anabalón, Á. Arboleya, and A. Guarino, *Phys. Rev. D* **109**, 106007 (2024).
- [19] F. S. Khoo, B. Azad, J. L. Blázquez-Salcedo, L. M. González-Romero, B. Kleihaus, J. Kunz, and F. Navarro-Lérida, *Phys. Rev. D* **109**, 084013 (2024).
- [20] S. Alfaro, P. A. González, D. Olmos, E. Papantonopoulos, and Y. Vásquez, *Phys. Rev. D* **109**, 104009 (2024).
- [21] P. Breitenlohner and D. Z. Freedman, *Phys. Lett.* **115B**, 197 (1982).
- [22] P. Breitenlohner and D. Z. Freedman, *Ann. Phys. (N.Y.)* **144**, 249 (1982).
- [23] L. Mezincescu and P. K. Townsend, *Ann. Phys. (N.Y.)* **160**, 406 (1985).
- [24] J. Y. Kim, C. O. Lee, and M. I. Park, *Eur. Phys. J. C* **78**, 990 (2018).
- [25] G. Dotti, J. Oliva, and R. Troncoso, *Phys. Rev. D* **75**, 024002 (2007).
- [26] M. Hassaine and J. Zanelli, *Chern-Simons (Super)Gravity (100 Years of General Relativity)* (World Scientific, Singapore, 2016).
- [27] M. Banados, C. Teitelboim, and J. Zanelli, *Phys. Rev. D* **49**, 975 (1994).
- [28] M. Banados, C. Teitelboim, and J. Zanelli, *Phys. Rev. Lett.* **69**, 1849 (1992).
- [29] M. Banados, M. Henneaux, C. Teitelboim, and J. Zanelli, *Phys. Rev. D* **48**, 1506 (1993); **88**, 069902(E) (2013).
- [30] G. Dotti, J. Oliva, and R. Troncoso, *Phys. Rev. D* **76**, 064038 (2007).
- [31] M. Ali, F. Ruiz, C. Saint-Victor, and J. F. Vazquez-Poritz, *Phys. Rev. D* **80**, 046002 (2009).
- [32] D. H. Correa, J. Oliva, and R. Troncoso, *J. High Energy Phys.* **08** (2008) 081.
- [33] F. Cooper, A. Khare, and U. Sukhatme, *Phys. Rep.* **251**, 267 (1995).
- [34] R. Zegers, *J. Math. Phys. (N.Y.)* **46**, 072502 (2005).
- [35] S. Ray, *Classical Quantum Gravity* **32**, 195022 (2015).
- [36] A. Anabalón, F. Canfora, A. Giacomini, and J. Oliva, *Phys. Rev. D* **84**, 084015 (2011).
- [37] J. Matulich and R. Troncoso, *J. High Energy Phys.* **10** (2011) 118.
- [38] M. Newmann, *Computational Physics* (CreateSpace Independent Publishing Platform, 2013).
- [39] A. Anabalón, J. Oliva, and C. Quijada, *Phys. Rev. D* **99**, 104022 (2019).
- [40] O. Evnin, H. Demirchian, and A. Nersessian, *Phys. Rev. D* **97**, 025014 (2018).
- [41] P. Mora, R. Olea, R. Troncoso, and J. Zanelli, *J. High Energy Phys.* **06** (2004) 036.
- [42] P. Bizon and A. Rostworowski, *Phys. Rev. Lett.* **107**, 031102 (2011).
- [43] O. J. C. Dias, G. T. Horowitz, and J. E. Santos, *Classical Quantum Gravity* **29**, 194002 (2012).
- [44] V. Balasubramanian, A. Buchel, S. R. Green, L. Lehner, and S. L. Liebling, *Phys. Rev. Lett.* **113**, 071601 (2014).
- [45] B. Craps, O. Evnin, and J. Vanhoof, *J. High Energy Phys.* **10** (2014) 048.
- [46] B. Craps, O. Evnin, and J. Vanhoof, *J. High Energy Phys.* **01** (2015) 108.
- [47] P. Bizoń, M. Maliborski, and A. Rostworowski, *Phys. Rev. Lett.* **115**, 081103 (2015).
- [48] M. Maliborski, *Phys. Rev. Lett.* **109**, 221101 (2012).
- [49] A. Biasi, P. Bizon, B. Craps, and O. Evnin, *Phys. Rev. E* **98**, 032222 (2018).
- [50] P. Bizoń, B. Craps, O. Evnin, D. Hunik, V. Luyten, and M. Maliborski, *Commun. Math. Phys.* **353**, 1179 (2017).
- [51] A. Biasi, P. Bizon, B. Craps, and O. Evnin, *Phys. Rev. A* **96**, 053615 (2017).
- [52] *Heun's Differential Equations*, edited by A. Ronveaux (Oxford University Press, New York, 1995).
- [53] S. Fernando, *Gen. Relativ. Gravit.* **37**, 461 (2005); G. Panotopoulos and Á. Rincán, *Phys. Lett. B* **772**, 523 (2017).