Regular black holes as an alternative to black bounce

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The so-called black-bounce mechanism of singularity suppression, proposed by Simpson and Visser, consists of replacing the spherical radius r in the metric tensor with $\sqrt{r^2+a^2}$, $a = \mathrm{const} > 0$. This removes a singularity at $r = 0$ and its neighborhood from space-time and there emerges a regular minimum of the spherical radius that can be a wormhole throat or a regular bounce (if located inside a black hole). Instead, it is proposed here to make $r = 0$ a regular center by proper (Bardeen type) replacements in the metric, preserving its form at large r. Such replacements are applied to a class of metrics satisfying the condition $R^t_t = R^r_r$ for their Ricci tensor, in particular, to the Schwarzschild, Reissner-Nordström, and Einstein-Born-Infeld solutions. A simpler version of nonlinear electrodynamics (NED) is considered, for which a black hole solution is similar to the Einstein-Born-Infeld one but is simpler expressed analytically. All new regular metrics can be presented as solutions to NED-Einstein equations with radial magnetic fields.

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I. INTRODUCTION

In general relativity (GR) and its classical extensions, the presence of singularities is quite a common though undesirable phenomenon, where a theory itself shows the boundaries of its applicability. In general, the researchers probably do not believe in the existence of singularities in nature and hope that they must be somehow suppressed by effects of quantum gravity. However, the numerous models and approaches in quantum gravity, being translated to the language of classical physics, produce quite different results, see, e.g., [1–8] and a discussion in [9]. Thus, when applied to black holes, some models predict black hole–white hole transitions [2–5], while others describe scenarios to a nonzero constant value of the spherical radius at late times of the evolution [6], there also emerge configurations without any horizons [8], etc. Different ways of quantum gravity regularization of black hole singularities are also discussed in the recent papers [10–12]. One can conclude that quantum gravity at its present stage of development is not yet ready to produce clear and unique predictions.

Therefore, it looks natural that the proposal made by Simpson and Visser (SV) [13] to obtain a regular static, spherically symmetric metric from a singular one by replacing the spherical radius r with the expression $r(u) = \sqrt{u^2+a^2}$, thus removing a singularity at $r = 0$, has caused significant interest and was followed by a number of extensions and discussions.[1] This simple trick may be an easy way to simulate possible quantum gravity effects in the framework of classical gravity, leaving aside any details of quantization methods. Additionally, it turned out that new geometries that emerge in this way can have their own features of interest.

This proposal, being applied to the Schwarzschild solution, results in the globally regular metric [13]

$$ds^2 = \left(1 - \frac{2M}{\sqrt{u^2+a^2}}\right)dt^2 - \left(1 - \frac{2M}{\sqrt{u^2+a^2}}\right)^{-1}du^2 - (u^2+a^2)(d\theta^2 + \sin^2\theta d\varphi^2). \tag{1}$$

At small values of the regularization constant a relative to the Schwarzschild mass M ($a < 2M$) (its smallness looks most natural), the metric (1) describes a black hole with two horizons at $u = \pm\sqrt{4M^2 - a^2}$. Larger values of a lead to an extremal regular black hole with a horizon at $u = 0$ (if $a = 2M$) and a wormhole with a throat at $u = 0$ at still larger a. In the black hole case $a < 2M$, the minimum of $r(u)$, observed at $u = 0$, is located in a Kantowski-Sachs anisotropic cosmological region between two horizons, where u is a temporal coordinate. Thus at $u = 0$ happens a

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[1]Here, $u \in \mathbb{R}$ is a new radial coordinate instead of r. The notation r is here kept for the quantity of evident geometric meaning, the spherical radius, $r = \sqrt{-g_{\theta\theta}}$, other radial coordinates are denoted by other letters to avoid confusion.

bounce in one of the two scale factors, $r(u)$, of this cosmology, called a "black bounce" as suggested in [13]. (The other scale factor in this cosmology is $2M/\sqrt{u^2+a^2}-1$, and it takes a maximum value at the same instant $u=0$.) With a discussion of other families of space-time metrics obtained with the same trick, all of them received the slang name of "black-bounce space-times" or simply black bounces. One can also recall that black bounces in the precise meaning of this term appear as a common feature in many solutions of GR and its extensions in the presence of phantom scalar fields. Such geometries have been named "black universes;" these are black hole space-times containing beyond their horizon an expanding universe that becomes isotropic at late times, see, e.g., [14–19].

A black-bounce regularization of the Reissner-Nordström solution of GR was constructed in [20]. The same approach was used to obtain a wide class of regular black hole and wormhole space-times in [21]. The diverse geometries found in this manner have attracted much attention, and further studies involved their rotating counterparts [22–24], quasinormal modes, gravitational wave echoes at possible black hole/wormhole transitions, and gravitational lensing parameters [25–39].

A separate issue is to present SV-like space-times as possible solutions to the equations of GR with different field sources. For static, spherically symmetric space-times, such representations were obtained in [40] and [41], and it was shown [40] that a large class of such space-times are obtainable as solutions to the Einstein equations with a combined source consisting of a minimally coupled phantom scalar field with a self-interaction potential and an electromagnetic field within nonlinear electrodynamics (NED), whereas NED alone or a scalar field alone are unable to form a necessary source. A phantom field is necessary for the existence of a minimum of the spherical radius r, while a NED source is required for adjusting the total stress-energy tensor (SET) T^ν_μ. The explicit forms of scalar and NED sources of SV-regularized Schwarzschild and Reissner-Nordström metrics were obtained in [40], along with their global structure diagrams, including metrics with three and four horizons. A similar method was applied to some cosmological space-times in [42,43].

SV-like regularizations for other two families of singular solutions of GR were constructed in [44]: these were Fisher's solution with a massless canonical scalar field [45] and a subset of dilatonic black hole solutions with interacting massless scalar and electromagnetic fields [46–49]. In both cases, the SV substitution was applied in the simplest possible way ($x \mapsto \sqrt{u^2+a^2}$) to the factor x that produced a space-time singularity at its zero value. Scalar-NED sources for the regularized versions of these space-times were also found, and it turned out that such a scalar field cannot be only canonical (with positive kinetic energy) or only phantom (with negative kinetic energy), but has to change its nature from one region to another, in other words, demonstrated what had been previously called a "trapped ghost" behavior [50,51]. The possible role of such fields in the stability properties of black hole and wormhole space-times was discussed in [19,52]. More generally [44], a combination of NED and a minimally coupled scalar field (in general, of trapped ghost nature) in GR is able to provide a source for *any* static, spherically symmetric metric, while, according to [53], any such metric may be produced (though only piecewise) with a nonminimally coupled scalar field as the only source.

A general feature of the SV proposal and its extensions is that the singularity in a geometry under study is simply removed from space-time together with its neighborhood, being replaced by a throat or a black bounce. It leads to more complex geometries and causal structures, which may be considered, from different viewpoints, both as an advantage and a shortcoming. There is, however, a natural alternative to this approach: to try, instead of removing the singularity location $r=0$, to convert it to a regular center. There are a great number of stellar and field models with regular centers, in particular, with NED sources (see, e.g., [54,55] and references therein) and those whose origin is ascribed to vacuum properties of various quantum fields including the gravitational one, see, e.g., [56,57] and references therein. However, our goal here is not to construct nonsingular models from the outset but to try to cure the already existing singularities at $r=0$ by introducing small regularizing parameters which may be hopefully ascribed to quantum gravity effects. As such examples, we will consider metrics obeying the condition $R^t_t = R^r_r$ for their Ricci tensors since it is the property of many of the most important solutions of GR (which previously received regularization by the SV method): the Schwarzschild, Reissner-Nordström solutions, their extensions with a cosmological constant, and some others. Moreover, if we construct a regular metric that preserves the property $R^t_t = R^r_r$, its source can be constructed with NED alone, with no need for other kinds of matter.

The paper is organized as follows. In Sec. II we make some preparations, recalling the regular center conditions and the way to obtain NED sources for the metrics under consideration. Section III is devoted to regular versions of the Schwarzschild and Reissner-Nordström solutions and finding their pure NED sources. In Sec. IV we discuss a possible regularization of the Einstein-Born-Infeld space-time and one more solution of GR with a new NED resembling the Born-Infeld one but leading to simple analytical expressions. Section V contains some concluding remarks. The metric signature $(+---)$ is adopted, along with geometrized units such that $8\pi G = c = 1$.

II. PRELIMINARIES

A. Regularity conditions

In this subsection we recall some well-known facts to be used in what follows. Consider a pseudo-Riemannian space-time $\mathbb{M}$ with an arbitrary static, spherically symmetric metric

$$ds^2 = A(x)dt^2 - \frac{dx^2}{A(x)} - r^2(x)d\Omega^2,$$
$$d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2, \tag{2}$$

written here in terms of the so-called quasiglobal radial coordinate x [58]. This choice of the radial coordinate is well suited for the description of any static, spherically symmetric space-times including black holes [where horizons appear as regular zeros of $A(x)$ provided $r(x)$ is finite] and wormholes [where throats appear as regular minima of $r(x)$ provided $A(x) > 0$].

If our space-time contains a location where $r \to 0$ under the condition $A > 0$, this location is called a center, and it is indeed a center of symmetry in spatial sections of $\mathbb{M}$. If $r \to 0$ in a region where $A < 0$ (as happens inside black holes), it is a cosmological-type singularity instead of a center since x can there be used as a time coordinate.

A center is regular if all algebraic curvature invariants are there finite and smooth, which includes, in particular, the existence of a tangent flat space-time at this point. For the metric (2) it implies that, at some $x \to x_0$,

$$A(x) = A_0 + \mathcal{O}(r^2), \qquad A(x)r'^2(x) = 1 + \mathcal{O}(r^2),$$
$$A_0 = \text{const} > 0, \tag{3}$$

where the prime denotes d/dx, and the symbol $\mathcal{O}(r^2)$ means a quantity of the same order as r^2 or smaller. The second condition provides a correct circumference to radius ratio for small circles around the center.

B. Space-times with $R^t_t = R^x_x$ and their NED sources

It makes sense to single out the important case of space-times where the Ricci tensor satisfies the condition $R^t_t = R^x_x$. Then, by the Einstein equations

$$G^\nu_\mu \equiv R^\nu_\mu - \frac{1}{2}\delta^\nu_\mu R = -T^\nu_\mu, \tag{4}$$

the SET of matter satisfies the same condition, and it holds not only for vacuum and a cosmological constant but also for NED under spherical symmetry. Moreover, with the metric (2), the equality $R^t_t = R^x_x$ leads to the condition $r''(x) = 0$, and almost without loss of generality we can put $r(x) \equiv x$ (we thus only reject "flux tubes" with $r = \text{const}$), so the quasiglobal coordinate x coincides with the more frequently used Schwarzschild coordinate r, and we are dealing with the metric

$$ds^2 = A(r)dt^2 - \frac{dr^2}{A(r)} - r^2 d\Omega^2. \tag{5}$$

Since now $r' \equiv 1$, in the regularity conditions (3) we must put $A_0 = 1$.

Now, the only two nontrivial components of the Einstein equations read (the prime denotes d/dr)

$$G^t_t = G^r_r = \frac{1}{r^2}[-1 + A + rA'] = -T^t_t, \tag{6}$$

$$G^\theta_\theta = G^\varphi_\varphi = \frac{1}{2r}[rA'' + 2r'A'] = -T^\theta_\theta. \tag{7}$$

This structure of the Ricci and Einstein tensors, hence the SET, can be represented by spherically symmetric NED fields. In particular, if we consider the NED Lagrangian in the form $-\mathcal{L}(\mathcal{F})$, where $\mathcal{F} = F_{\mu\nu}F^{\mu\nu}$, and $F_{\mu\nu}$ is the electromagnetic field tensor, the SET is in general

$$T^\nu_\mu[F] = -2\mathcal{L}_\mathcal{F} F_{\mu\sigma}F^{\nu\sigma} + \frac{1}{2}\delta^\nu_\mu \mathcal{L}(\mathcal{F}), \tag{8}$$

with $\mathcal{L}_\mathcal{F} = d\mathcal{L}/d\mathcal{F}$, and the electromagnetic field equations are

$$\nabla_\mu(\mathcal{L}_\mathcal{F} F^{\mu\nu}) = 0. \tag{9}$$

With the present space-time symmetry, we may consider only radial electric and magnetic fields, the only nonzero components of $F_{\mu\nu}$ being $F_{rt} = -F_{tr}$ and $F_{\theta\varphi} = -F_{\varphi\theta}$. Let us suppose the existence of only the magnetic components, such that

$$F_{\theta\varphi} = -F_{\varphi\theta} = q\sin\theta, \tag{10}$$

where q is a monopole magnetic charge. Then Eq. (9) is trivially satisfied, while the invariant $\mathcal{F}$ is expressed as $\mathcal{F} = 2q^2/r^4$, independent of the choice of $\mathcal{L}(\mathcal{F})$. The electromagnetic SET takes the form

$$T^\nu_\mu[F] = \frac{1}{2}\text{diag}\left(\mathcal{L}, \mathcal{L}, \mathcal{L} - \frac{4q^2}{r^4}\mathcal{L}_\mathcal{F}, \mathcal{L} - \frac{4q^2}{r^4}\mathcal{L}_\mathcal{F}\right). \tag{11}$$

The way of obtaining the corresponding solutions with magnetic fields is described in a number of papers devoted to GR-NED regular black holes, e.g., [54,55] and references therein. Thus, Eq. (6) may be presented in the integral form

$$A(r) = 1 - \frac{2M(r)}{r}, \qquad M(r) = \frac{1}{2}\int \rho(r)r^2 dr, \tag{12}$$

where $M(r)$ is called the mass function, $\rho(r) = T^t_t$ is the matter density, and thus $\mathcal{L}(\mathcal{F}) = 2\rho$ can be found as a function of r as follows:

$$\mathcal{L}(\mathcal{F}(r)) = \frac{2}{r^2}(1 - A - rA'), \tag{13}$$

and it can be verified that the derivative $\mathcal{L}_\mathcal{F} = \mathcal{L}'/\mathcal{F}'$ calculated from (13) coincides with $\mathcal{L}_\mathcal{F}$ determined using

Eq. (7), as must be the case. Moreover, in such solutions, as should happen in regular magnetic black holes [54], $\mathcal{L}(\mathcal{F})$ is finite at the center. Let us remark that to obtain a solution regular at $r = 0$, the integration in (12) must be carried out from 0 to r, which leads to a total mass of purely electromagnetic origin.

One might consider NED sources with a radial electric field instead of a magnetic one, as is done, in particular, in [59] for a number of black-bounce space-times. This may be implemented using, in addition to $\mathcal{F}$, the auxiliary electromagnetic invariant $\mathcal{P} = \mathcal{F}\mathcal{L}_{\mathcal{F}}^2$, and the whole construction does not look much more complicated than with magnetic fields. It is, however, necessary to mention that electric NED solutions generically involve different Lagrangians $\mathcal{L}(\mathcal{F})$ in different regions of space, and this happens each time when $\mathcal{P}(\mathcal{F})$, or conversely, $\mathcal{F}(\mathcal{P})$ is not a monotonic function. As was observed in [59], in electric sources of black-bounce space-times this ambiguity does emerge, but not in every solution. Unlike that, for space-times with a regular center we can predict that the multivaluedness of $\mathcal{L}(\mathcal{F})$ will necessarily emerge, in full analogy with the reasoning used in [54,60]. Indeed, in such cases $\mathcal{P}(r) = -2q_e^2/r^4$ is a monotonic function (q_e is an electric charge), while $\mathcal{F}(r)$ is not since it has to vanish both at infinity and at a regular center.

III. CURING SINGULARITIES IN SCHWARZSCHILD AND REISSNER-NORDSTRÖM SOLUTIONS

A. General considerations

The regularity conditions now reduce to the requirement $A(r) = 1 + \mathcal{O}(r^2)$ at small r. The only function to be modified is $A(r)$ obeying Eq. (12), from which it follows that a regular center corresponds to a finite value of $\rho(0)$. Let us assume that, in a singular metric to be cured,

$$A(r) \approx 1 + A_1/r^m \quad \text{as } r \to 0, \qquad A_1 = \text{const}, \qquad m = \text{const} > 0. \tag{14}$$

It is then easy to verify that the substitution in the argument of $A(r)$

$$r \mapsto \frac{(r^2 + a^2)^{n+1/2}}{r^{2n}}, \qquad n = \text{const} \geq 1/m, \qquad a = \text{const} > 0 \tag{15}$$

leads to $A(r) \approx 1 + A_1 a^{-m(2n+1)} r^{2mn}$ at small r, with $2mn \geq 2$. A subtle point is that in (14) the number m is the *smallest* power in an expansion of A in powers of $1/r$, even though the singular asymptotic behavior of $A(r)$ is determined by the largest power in this expansion.

It is easy to find that, if $n = 1/m$, the resulting density $\rho(0)$ is nonzero, and the metric near $r = 0$ is asymptotically de Sitter if $\rho(0) > 0$ and anti–de Sitter (AdS) if $\rho(0) < 0$. If $n > 1/m$, then $\rho(0) = 0$, and the metric near $r = 0$ is asymptotically Minkowskian. One can notice that in the case $m = n = 1$ the substitution (15) actually coincides with the one used by Bardeen in [61] to convert a Schwarzschild black hole to a regular one. The angular part of the metric does not change, and with any value of n the metric "cured" with (15) remains at large values of r approximately the same as the original, singular one.

We see that the recipe (15) smooths out singularities of the form (14) with $m > 0$, producing different regular near-center density profiles depending on the constant n. There can be, however, a "softer" singularity characterized by $m = 0$, it requires a somewhat finer approach to be considered in the next section.

B. The Schwarzschild metric

In the Schwarzschild vacuum solution we have $A = 1\text{–}2M/r$, where $M = \text{const}$ is the mass parameter. In the above scheme it corresponds to $m = 1$, and applying (15) with $n = 1$, we obtain the regularized function $A = A_{\rm reg}(r)$ of the form suggested by Bardeen [61] in his first regular black hole model,

$$A_{\rm reg}(r) = 1 - \frac{2Mr^2}{(r^2 + a^2)^{3/2}}. \tag{16}$$

From (13) we then find the Lagrangian function of the corresponding NED source

$$\mathcal{L}(\mathcal{F}) = \frac{12a^2M}{(r^2 + a^2)^{5/2}} = \frac{12a^2M\mathcal{F}^{5/4}}{(\sqrt{2q^2} + a^2\sqrt{\mathcal{F}})^{5/2}}, \tag{17}$$

coinciding with that obtained in [62] (up to notations).

With the function (16), the metric is asymptotically de Sitter at $r = 0$. An asymptotically Minkowskian metric at $r = 0$ can be obtained with any $n > 1$ in (15). For example, choosing $n = 3/2$, we find

$$A_{\rm reg}(r) = 1 - \frac{2Mr^3}{(r^2 + a^2)^2},$$
$$\mathcal{L}(\mathcal{F}) = \frac{16a^2Mr}{(r^2 + a^2)^3} = \frac{16a^2M(2q^2)^{1/4}\mathcal{F}^{5/4}}{(\sqrt{2q^2} + a^2\sqrt{\mathcal{F}})^3}. \tag{18}$$

Note that the function $\mathcal{L}(\mathcal{F})$ tends to a finite limit as $\mathcal{F} \to \infty$ (to zero if $n > 1$), as should be the case at a regular center [54], but does not have a correct Maxwell limit ($\mathcal{L} \sim \mathcal{F}$) at small $\mathcal{F}$.

The behavior of $A_{\rm reg}(r)$ is quite generic for regular NED-GR solutions, as can be seen in Fig. 1: it is a solitonlike structure at small masses, at some critical value of M emerges an extremal horizon, and at larger M there is a black hole with two horizons and a global structure similar to the Reissner-Nordström one. Quite naturally, a plot of

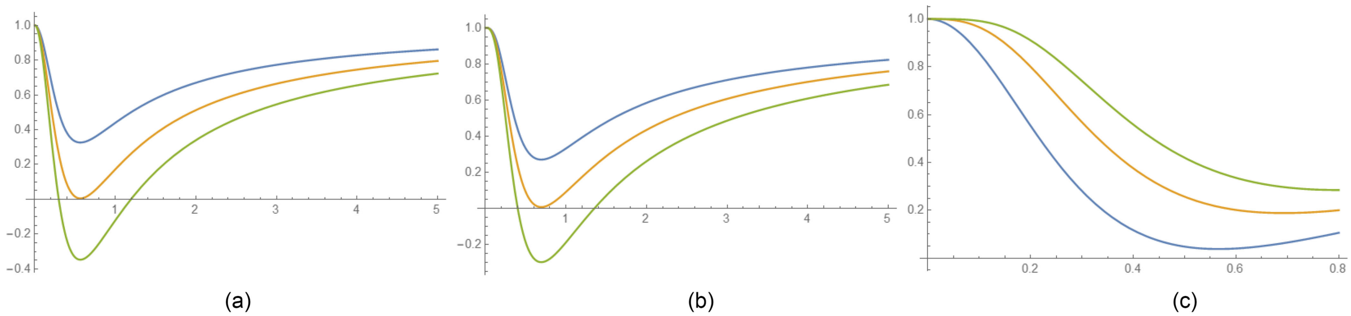

FIG. 1. Plots of $A(x)$ for the regularized Schwarzschild metric with $a = 0.4$: (a) the function (16) with $M = 0.35, 0.517, 0.7$ (upside down); (b) the function (18) with $M = 0.45, 0.612, 0.8$ (upside down); (c) comparison of $A_{\rm reg}(x)$ with the same $M = 0.5$ and $n = 1, 3/2, 2$; flattened plots at $r = 0$ correspond to asymptotically Minkowski metrics.

$A_{\rm reg}(r)$ for an asymptotically Minkowski (at $r = 0$) metric has a flattened summit, unlike the one for an asymptotically de Sitter metric, as illustrated in Fig. 1(c).

C. The Reissner-Nordström metric

The Reissner-Nordström electrovacuum solution corresponds to the metric function

$$A = 1 - 2M/r + Q^2/r^2, \quad M = \text{mass}, \quad Q = \text{charge}. \tag{19}$$

We have again $m = 1$, and using the Bardeen replacement (15) with $n = 1$, we obtain

$$A_{\rm reg}(r) = 1 - \frac{2Mr^2}{(r^2+a^2)^{3/2}} + \frac{Q^2 r^4}{(r^2+a^2)^3}, \tag{20}$$

with a de Sitter behavior near the regular center $r = 0$. For the corresponding NED source we have according to (13),

$$\mathcal{L}(r) = \frac{2[6a^6M + 12a^4Mr^2 + Qr^4\sqrt{r^2+a^2} + a^2(6Mr^4 - 5Qr^2\sqrt{r^2+a^2})]}{(r^2+a^2)^{9/2}}; \tag{21}$$

here and henceforth in similar formulas, an expression for $\mathcal{L}(\mathcal{F})$ is readily obtained by substituting $r^2 \mapsto \sqrt{2q^2/\mathcal{F}}$ (please note that the charge q refers to the NED source and has nothing to do with the "original" charge Q in the Reissner-Nordström metric). For convenience, to avoid writing $|Q|$ in many relations, we assume $Q > 0$.

At small r the function (20) behaves as

$$A_{\rm reg}(r) = 1 - \frac{2Mr^2}{a^3} + \frac{3aM + Q}{a^6} r^4 + \mathcal{O}(r^6), \tag{22}$$

so that the central asymptotic is de Sitter as long as $M > 0$. The case $M = 0$ is treated separately in Sec. III E since it requires another substitution for $A(r)$.

As a whole, the behavior of $A_{\rm reg}(r)$ is rather diverse, as shown in Fig. 2 for a particular value of $a = 0.4$, taken to be sufficiently large for illustration purposes. Two special cases deserving separate attention are discussed below.

Taking $n > 1$, we would obtain regularizations with a Minkowski central asymptotic, with slightly other properties than (20) and (21), not to be described here in detail.

D. The extreme Reissner-Nordström metric

In the case $Q = M$, the function (19) is a full square, $A(r) = (1 - M/r)^2$ (the same line element also belongs to a black hole with a conformal scalar field [63–65]). As before, to obtain a regular metric with de Sitter behavior near $r = 0$ we take $n = 1$ and obtain

$$A_{\rm reg}(r) = \left[1 - \frac{Mr^2}{(r^2+a^2)^{3/2}}\right]^2, \quad \mathcal{L}(\mathcal{F}) = \frac{2M[Mr^4 + a^2r^2(-5M + 6\sqrt{a^2+r^2})] + 6a^4\sqrt{a^2+r^2}}{(a^2+r^2)^4}, \tag{23}$$

while an asymptotically Minkowski center can be formed by choosing $n = 3/2$, which leads to

$$A_{\rm reg}(r) = \left[1 - \frac{Mr^3}{(r^2+a^2)^2}\right]^2, \qquad \mathcal{L}(\mathcal{F}) = \frac{2Mr(Mr^5 + a^2r^3(-7M + 8r) + 16a^4r^2 + 8a^6)}{(a^2+r^2)^5}. \tag{24}$$

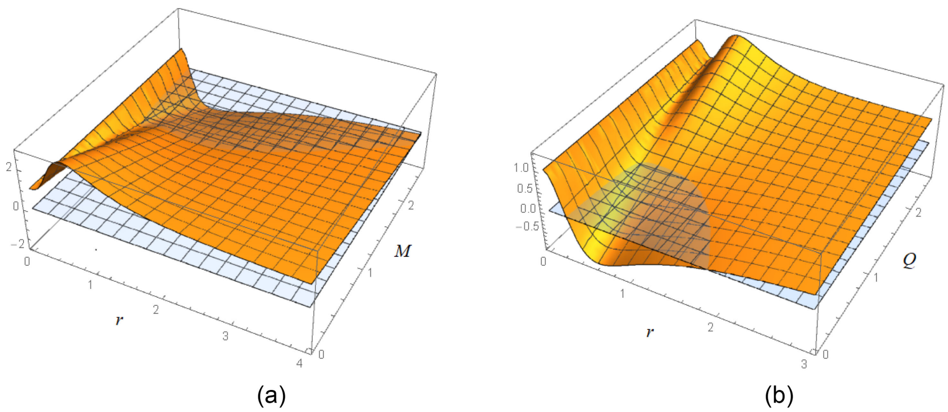

FIG. 2. Plots of $A_{\rm reg}(r)$ for the regularized Reissner-Nordström metric (20) with $a = 0.4$: (a) for fixed $Q = 2$ and different M, (b) for fixed $M = 1$ and different Q. The plane $A = 0$ is shown in each panel to visualize the regions where $A_{\rm reg} < 0$ corresponding to black hole interiors.

In both cases $A_{\rm reg}(r)$ is non-negative, but, depending on M (at fixed a), it can have up to two zeros corresponding to extremal horizons, as illustrated in Fig. 3. The Carter-Penrose global structure diagram in Fig. 3(c) for a space-time with two extremal horizons indefinitely extends up and down; region I and its analogs are those near the center, regions II and III are those between the horizons, and regions like IV are the external asymptotically flat ones.

E. The massless Reissner-Nordström metric

Consider the only case of Reissner-Nordström metric corresponding to (14) with $m = 2$, with

$$A(r) = 1 + Q^2/r^2, \qquad Q = \text{const} = \text{charge}. \tag{25}$$

Thus everywhere $A > 1$, and there is a naked repulsive singularity at $r = 0$.

In (15), to obtain an (A)dS behavior near $r = 0$, we must take $n = 1/2$, with the results

$$A_{\rm reg}(r) = 1 + \frac{Q^2 r^2}{(r^2 + a^2)^2}, \quad \mathcal{L}(\mathcal{F}) = \frac{2Q^2(r^2 - 3a^2)}{(r^2 + a^2)^3}. \tag{26}$$

Near $r = 0$ we have $\mathcal{L}(\mathcal{F}) = 2\rho < 0$, corresponding to an anti–de Sitter center, as illustrated in Fig. 4(a). A Minkowski asymptotic behavior near $r = 0$ is obtained, for example, with $n = 1$, which leads to

$$A_{\rm reg}(r) = 1 - \frac{Q^2 r^4}{(r^2 + a^2)^3}, \quad \mathcal{L}(\mathcal{F}) = \frac{2Q^2 r^2 (r^2 - 5a^2)}{(r^2 + a^2)^4}, \tag{27}$$

and again $\mathcal{L}(\mathcal{F}) = 2\rho < 0$ near $r = 0$, but it is zero at the center itself due to the factor r^2 [Fig. 4(b)]. It is the case where $A_{\rm reg}(r)$ has the shape of a pure barrier, while barriers along with depressions are observed in the more general pictures in Fig. 2.

IV. CURING SINGULARITIES IN SOME EINSTEIN-NED SOLUTIONS

The metric (14) with $m = 0$ and finite $A(0) \neq 1$, also has a singularity at $r = 0$, such that $\rho \sim 1/r^2$ at small r, and the integral in (12) taken from zero to small r behaves as $\text{const} \cdot r$ and adds a constant to A according to (12). In this case, replacements like (15) do not work because there is no particular r dependence of A to be modified. Instead,

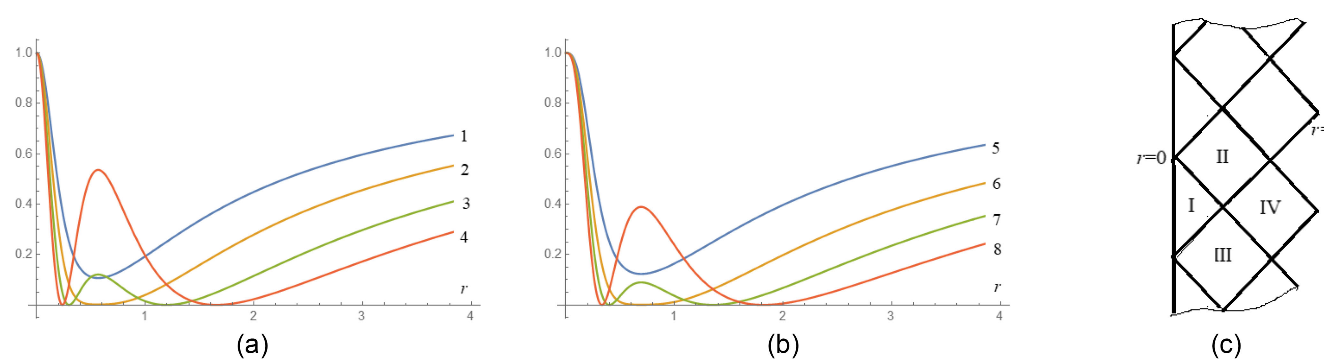

FIG. 3. Plots of $A_{\rm reg}(r)$ for the regularized extremal Reissner-Nordström metric (20) with $a = 0.4$: (a) for the function (23) with $M = 0.7$, 1, 1.4, 1.8, curves 1–4; (b) for the function (24) with $M = 0.8$, 1.2, 1.6, 2, curves 5–8. (c) The Carter-Penrose diagram for a configuration with two extremal horizons.

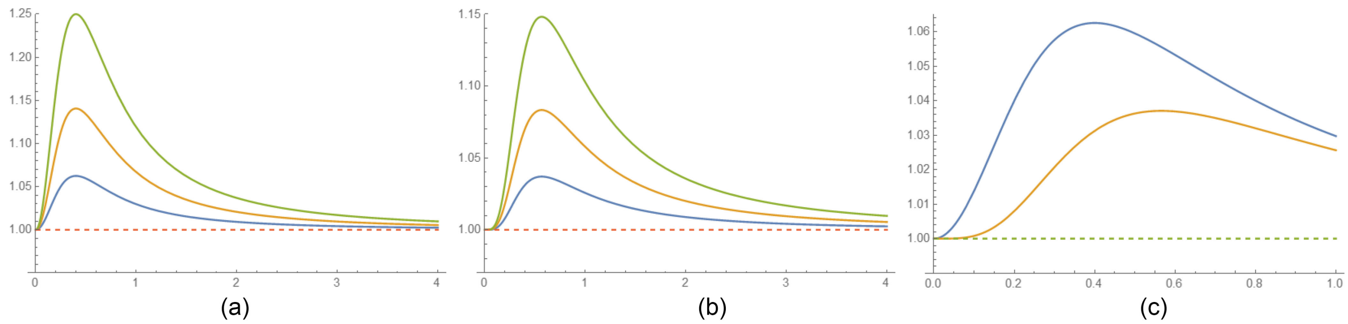

FIG. 4. Plots of $A_{\rm reg}(r)$ for the regularized massless Reissner-Nordström metric with $a = 0.4$ and $Q = 0.2, 0.3, 0.4$ (bottom-up in each panel): (a) Eq. (26) with an AdS central asymptotic, (b) Eq. (27) with a Minkowski central asymptotic; (c) a comparison of two curves for $Q = 0.2$ at small r, the flattened curve near $r = 0$ corresponding to a Minkowski asymptotic.

one can either directly modify $\mathcal{L}(\mathcal{F})$ to lead it to a finite limit at large $\mathcal{F}$ or make finite the density $\rho(r)$, for example, replacing there $1/r^2$ with $1/(r^2+a^2)$. Let us consider two such examples.

A. The Einstein-Born-Infeld solution

The famous Born-Infeld NED Lagrangian, which for pure electric or magnetic fields reads

$$\mathcal{L}(\mathcal{F}) = \beta\left(-1+\sqrt{1+2\mathcal{F}/\beta^2}\right), \qquad \beta = \mathrm{const} > 0, \tag{28}$$

is known to make finite the electromagnetic field strength and energy in its Coulomb-like spherically symmetric solution while, being coupled to GR, it weakens but does not remove the curvature singularity at $r = 0$. Thus, with (28), integration in (12) with substituted $\mathcal{F} = 2Q^2/r^4$ (assuming a radial magnetic field with charge Q) leads to the expression (see, e.g., [66–69])

$$A(r) = 1 - \frac{1}{6}\left(\sqrt{4Q^2+\beta^2 r^4} - \beta r^2\right) - \frac{2}{3r}\sqrt{\frac{|Q|^3}{\beta}}(1+i)F\left(i\sinh^{-1}\left[(1+i)\frac{r\sqrt{\beta}}{2\sqrt{|Q|}}\right], -1\right), \tag{29}$$

where F is an elliptic function. The asymptotic behavior of $A(r)$ at small r is

$$A(r) = 1 - \beta|Q| + \beta^2 r^2/6 + \mathcal{O}(r^4) \tag{30}$$

and can be associated with (14) at $m = 0$. Examples of the behavior of $A(r)$ with $\beta = 2$ (used for convenience), both corresponding to a black hole (if $|Q| > 1$) and a naked singularity (otherwise), are shown in Fig. 5(a).

As discussed above, to regularize the metric in this case, it is necessary to modify the Lagrangian function for having its finite limit as $\mathcal{F} \to \infty$. Such a simple way, preserving the Born-Infeld behavior (28) of $\mathcal{L}(\mathcal{F})$ at moderate values of $\mathcal{F}$, can be proposed as

$$\mathcal{L}(\mathcal{F}) = \beta\left(-1+\sqrt{1+\frac{2\mathcal{F}}{\beta^2+\gamma\mathcal{F}}}\right), \qquad \beta, \gamma = \mathrm{const} > 0, \tag{31}$$

assuming sufficiently small values of γ. For $A(r)$ we then obtain a long expression with several Appell functions of six arguments, not to be presented here, with the near-center behavior

$$A_{\rm reg}(r) = 1 - \frac{1}{6}\left(-\beta + \sqrt{\beta^2 + 2/\gamma}\right) r^2 + \mathcal{O}(r^4), \tag{32}$$

indicating a de Sitter asymptotic. The plots of $A_{\rm reg}(r)$, drawn in Fig. 5(b) for particular values of the parameters, show the characteristic form of the metric functions known in regular NED-GR black hole solutions.

Let us, for methodological purposes, introduce a NED theory, whose solutions behave similarly to those of the

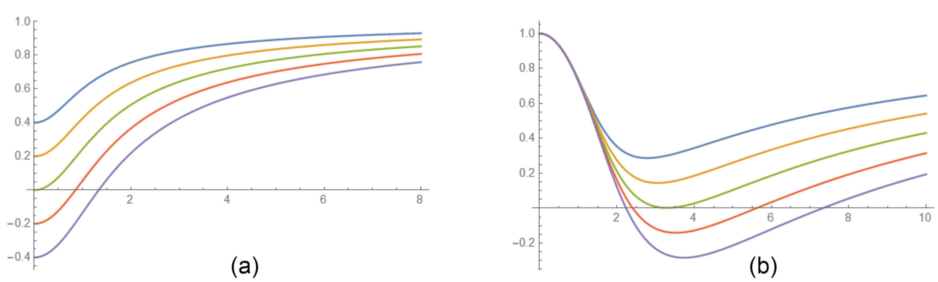

FIG. 5. Plots of $A(r)$ in solutions with the original (a) and modified (b) Born-Infeld theory as a source of gravity. The parameters are (a) $\beta = 2$, $Q = 0.6, 0.8, 1, 1.2, 1.4$ (upside down); (b) $\beta = 2, \gamma = 0.5, q = 2.5, 3, 3.5, 4, 4.5$ (upside down).

Born-Infeld theory but are expressed analytically in a much simpler way.

B. A simpler NED with Born-Infeld-like behavior

Consider the NED Lagrangian (from a family of theories considered in [70])

$$\mathcal{L}(\mathcal{F}) = \frac{\mathcal{F}}{1 + h\sqrt{\mathcal{F}/2}}, \qquad h = \text{const} > 0. \tag{33}$$

Like the Born-Infeld theory, this one takes a correct Maxwell form, $\mathcal{L} = \mathcal{F}$, at small $\mathcal{F}$. With $\mathcal{F} = 2Q^2/r^4$, Eq. (12) leads to the following expression for $A(r)$:

$$A(r) = 1 - \frac{Q^2}{Hr}\arctan\frac{r}{H}, \qquad H \coloneqq (Q^2h^2)^{1/4}, \tag{34}$$

with the near-center behavior similar to (30),

$$A(r) = 1 - \frac{|Q|}{h} + \frac{r^2}{3h^2} + \mathcal{O}(r^4), \tag{35}$$

and the r dependence of the Lagrangian function (equal to 2ρ) is

$$\mathcal{L}(\mathcal{F}(r)) = \frac{2Q^2}{r^2(r^2 + H^2)}. \tag{36}$$

It is now easy to regularize the system by simply replacing $1/r^2 \mapsto 1/(r^2 + c^2)$ in this $\mathcal{L}(r)$, $c > 0$ being a small regularization parameter,

$$\begin{aligned}\mathcal{L}(\mathcal{F}) \mapsto \mathcal{L}_{\rm reg} &= \frac{2Q^2}{(r^2 + c^2)(r^2 + H^2)} \\ &= \frac{\mathcal{F}}{(1 + h\sqrt{\mathcal{F}/2})(1 + c^2|Q|^{-1/2}\sqrt{\mathcal{F}/2})}.\end{aligned} \tag{37}$$

This leads to the regularized metric function

$$A_{\rm reg}(r) = 1 - \frac{Q^2}{(H^2 - c^2)r}\left(H\arctan\frac{r}{H} - c\arctan\frac{r}{c}\right), \tag{38}$$

with the near-center asymptotic behavior

$$A_{\rm reg}(r) = 1 - \frac{|Q|r^2}{3c^2h^{3/2}} + \mathcal{O}(r^4). \tag{39}$$

It can be verified that the functions (34) and (38) behave qualitatively in the same way as solutions for the Born-Infeld theory and its modification (31), shown in Fig. 5.

One can notice that in both theories (28) and (33) the solution has either a naked singularity or a single horizon, whereas the regularized solution is either solitonlike or represents a regular black hole with one or two horizons (a single, extremal horizon emerges in the intermediate case). This picture is common to many solutions with a regular center and is related to the fact that the regularity requires $A(0) = 1$, hence such a center can only exist in a static region with $A > 0$.

V. CONCLUDING REMARKS

Let us enumerate and discuss some results and observations made in this paper.

(i) A way of singularity removal is proposed for static, spherically symmetric space-times satisfying the condition $R^t_t = R^r_r$ by properly changing a neighborhood of a singularity at $r = 0$ so that it becomes a regular center. This is achieved by a Bardeen-like replacement in the metric function $A(r)$ containing a small regularization parameter; the condition $R^t_t = R^r_r$ is preserved, and the resulting metric can be interpreted as an Einstein-NED solution with a radial magnetic field, by analogy with studies devoted to regular NED-sourced black holes, see, e.g., [54,71] and references therein.

(ii) More involved are cases where the original singularity is of comparatively soft nature [for instance, those with a finite value of $A(0) \neq 1$]; its removal requires a direct modification in the original Lagrangian or the expression for the energy density, as is seen in the example of the Einstein-Born-Infeld solution. A much simpler version of NED is considered, given by Eq. (33), which leads to similar properties of the solution near the singularity, while its modified version (37) leads to a regular solitonic or black hole solution.

(iii) We have dealt here with examples of asymptotically flat space-times, with Reissner-Nordström behavior at large r. There is a straightforward extension of these solutions with a nonzero cosmological constant Λ, similar to that described, e.g., in [72]. It is achieved by adding the term $-\Lambda r^2/3$ to all functions $A(r)$ mentioned in this paper, and this term even does not need any modification at singularity removal since it does not spoil the metric behavior near a center.

Black-bounce space-times contain regular minima of the spherical radius r and therefore require phantom matter as a source in the framework of GR. Obtaining regular centers instead of singularities does not require such exotic matter, and, in particular, all metrics with $R^t_t = R^r_r$ can be sourced by nonlinear electromagnetic fields which (though marginally) satisfy the null energy condition.

More general space-times, not restricted by the above condition for the Ricci tensor, require other regularization methods, and their formulation can be a next task. Concerning their possible material sources, one may recall that, as shown in [44], any static, spherically symmetric metric may be described as a solution of GR with a combined source consisting of a magnetic field obeying NED and a scalar field with a certain self-interaction potential, and one may hope that, unlike black-bounce space-times, those with a regular center will not require a phantom field as a source.

Various aspects of the new proposed regular metrics can be further studied, such as geodesics, lensing, stability, quasinormal modes, etc. In the context of finding sources of gravity for such metrics, one can note that the stability of a particular geometry essentially depends on the dynamics of its source. Thus, it has been found that the Ellis simple wormhole geometry [65,73] can be stable or unstable, depending on the source to which it is ascribed: a phantom scalar field, a perfect fluid, or a k-essence field [74–76].

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